Dynamical systems approach and generic properties of $f(T)$ cosmology

Manuel Hohmann, Laur Jarv, Ulbossyn Ualikhanova

Laboratory of Theoretical Physics, Institute of Physics, University of Tartu

Phys. Rev. D 96, 043508 (2017)[arXiv:1706.02376 [gr-qc]]

European Union
European Regional
Development Fund

Investing
in your future

Abstract

We present a systematic analysis of the dynamics of flat Friedmann-Lemaître-Robertson-Walker cosmological models with radiation and dust matter in generalized teleparallel $f(T)$ gravity. We show that the cosmological dynamics of this model is fully described by a function $W(H)$ of the Hubble parameter, which is constructed from the function $f(T)$. After reducing the phase space to two dimensions we derive the conditions on $W(H)$ for the occurrence of de Sitter fixed points, accelerated expansion, crossing the phantom divide, and finite time singularities. Depending on the model parameters it is possible to have a bounce (from contraction to expansion) or a turnaround (from expansion to contraction), but cyclic or oscillating scenarios are prohibited. As an illustration of the formalism we consider power law $f(T) = T + \alpha(-T)^n$ models, and show that these allow only one period of acceleration and no phantom divide crossing.

Introduction

Teleparallel gravity (TEGR)

- employs torsion scalar instead of curvature,
- conceptually distinguishes between gravitation and inertia,
- builds up its theoretical formulation more in line with gauge theories,
- equivalent to general relativity in all physical predictions.

Action and cosmological field equations

Action functional of $f(T)$ gravity

$$S = \frac{1}{16\pi G}\int |e|\, f(T) d^4x \tag{1}$$

Torsion scalar

$$T \equiv \frac{1}{4}T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + \frac{1}{2}T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - T^{\mu}{}_{\rho\mu}T^{\nu\rho}{}_{\nu}\,. \tag{2}$$

Torsion tensor

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} = e^{\rho}_{i}\left(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu + \omega^i{}_{j\mu}e^j_\nu - \omega^i{}_{j\nu}e^j_\mu\right)\,. \tag{3}$$

For a flat Friedmann-Lemaître-Robertson-Walker universe with the tetrad $e^i_\mu = \mathrm{diag}(1, a, a, a)$ the torsion scalar reduces to $T = -6\frac{\dot{a}^2}{a^2} = -6H^2$. The matter energy-momentum tensor of a perfect fluid

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}\,. \tag{4}$$

The cosmological field equations

$$12H^2 f_T + f = 16\pi G\rho\,, \tag{5a}$$

$$48H^2\dot{H}f_{TT} - (12H^2 + 4\dot{H})f_T - f = 16\pi Gp\,. \tag{5b}$$

Matter content: dust and radiation.The density and pressure

$$\rho = \rho_m + \rho_r\,, \quad p = p_m + p_r\,. \tag{6}$$

Matter continuity equations

$$\dot{\rho}_m = -3H\rho_m\,, \quad \dot{\rho}_r = -4H\rho_r\,. \tag{7}$$

Cosmological field equations finally take the form

$$W = 16\pi G(\rho_m + \rho_r)\,, \tag{8a}$$

$$-\dot{H}\frac{W_H}{3H} = 16\pi G\left(\rho_m + \frac{4}{3}\rho_r\right)\,. \tag{8b}$$

Here we have introduced the Friedmann function

$$W(H) = 12H^2 f_T + f, \tag{9}$$

which encodes the main cosmological features of any given $f(T)$ gravity model.

Dynamical system

Physical phase space is given by $\mathcal{P} = \{(H, X) \mid -\infty < H < \infty\,, \quad 0 \le X \le 1\,, \quad 0 \le W(H) < \infty\}$ with a new variable

$$X = \frac{\rho_r}{\rho_r + \rho_m}. \tag{10}$$

Dynamics of the new variables

$$\dot{X} = HX(X-1)\,, \quad \dot{H} = -(X+3)H\frac{W}{W_H} = -\frac{(X+3)H}{(\ln W)_H}\,. \tag{11}$$

Fixed points and their stability

For a constant Hubble parameter $\dot{a}/a = H = H^\star$ the scale factor behaves as

$$a(t) \sim \exp(H^\star t)\,, \tag{12}$$

and so is constant or exponentially increasing / decreasing, depending on the sign of $H^\star$. A point $(H^\star, X^\star)$ is an irregular fixed point of the dynamical system, if it satisfies one of the following criteria:

i. In the case $X^\star = 0$, $H^\star > 0$, $W^\star = 0$ it is an isolated attractor. The corresponding solution is an expanding de Sitter vacuum solution with scale factor (12).
ii. In the case $X^\star = 0$, $H^\star < 0$, $W^\star = 0$ it is an isolated repeller. The corresponding solution is a contracting de Sitter vacuum solution with scale factor (12).
iii. In the case $X^\star = 1$, $H^\star \neq 0$, $W^\star = 0$ it is an isolated saddle point. The corresponding solution is physically equivalent to either of the two aforementioned cases.
iv. Points with $0 \le X^\star \le 1$, $H^\star = 0$, $W^\star > 0$ and $W^\star_H > 0$ are non-isolated attractors. The corresponding solution is a static universe with Minkowski geometry, but non-vanishing matter content.
v. Points with $0 \le X^\star \le 1$, $H^\star = 0$, $W^\star > 0$ and $W^\star_H < 0$ are non-isolated repellers. The corresponding solution is a static universe as in the aforementioned case.
vi. Fixed points, whose stability cannot be determined from a linearised analysis, are given by:
    a. $H^\star = 0$ and $W^\star = 0$; this is a Minkowski vacuum solution.
    b. $H^\star = 0$, $W^\star > 0$ and $W^\star_H$ diverges; also this is a static universe with non-vanishing matter content.
    c. $H^\star = 0$, $W^\star > 0$ and $W^\star_H = 0$ such that $H/W_H \to 0$; this case corresponds to a finite time singularity of type IV.

Possibility of bounce and turnaround

At $H = 0$ we have $\dot{H} \neq 0$ if and only if $W > 0$, $W_H = 0$ and $W_{HH} \neq 0$, where

i. for $W_{HH} < 0$ we have $\dot{H} > 0$ and hence a bounce,
ii. for $W_{HH} > 0$ we have $\dot{H} < 0$ and hence a turnaround.

Impossibility of cyclic and oscillating universes

Periodic and oscillating orbits in the (H, X) phase space, as well as cyclic and oscillating universe solutions are not possible.

Accelerating expansion and the phantom divide

Acceleration

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = H\left(H - (X+3)\frac{W}{W_H}\right)\,. \tag{13}$$

The effective barotropic index

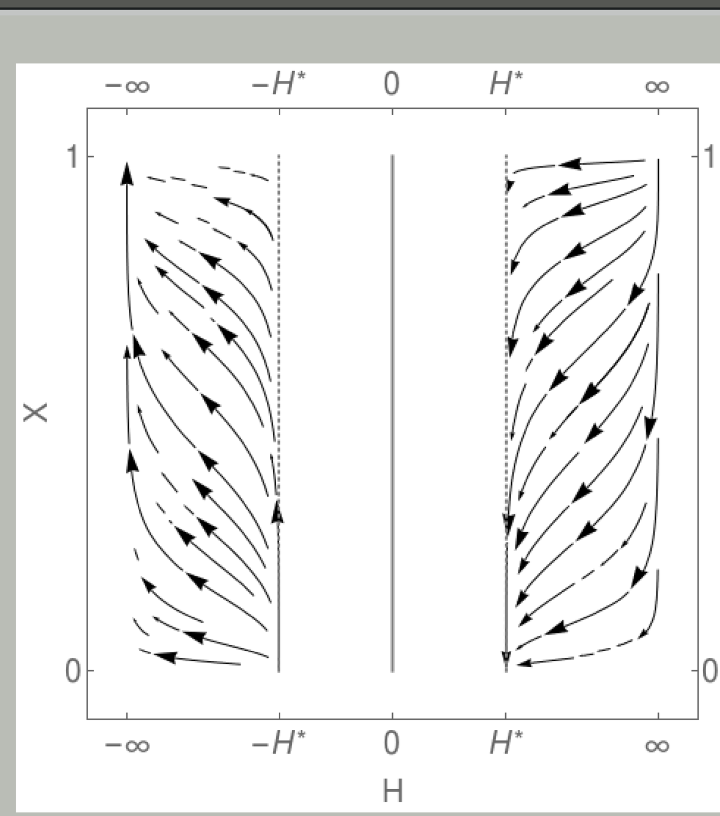
$$w_{DE} = -1 - \frac{(X+3)}{3}\left(1 - 12\frac{H}{W_H}\right)\left(1 - 6\frac{H^2}{W}\right)^{-1} = -1 - \frac{X+3}{3}\frac{[\ln|W - 6H^2|]_H}{(\ln W)_H}\,. \tag{14}$$

Example: Power law model

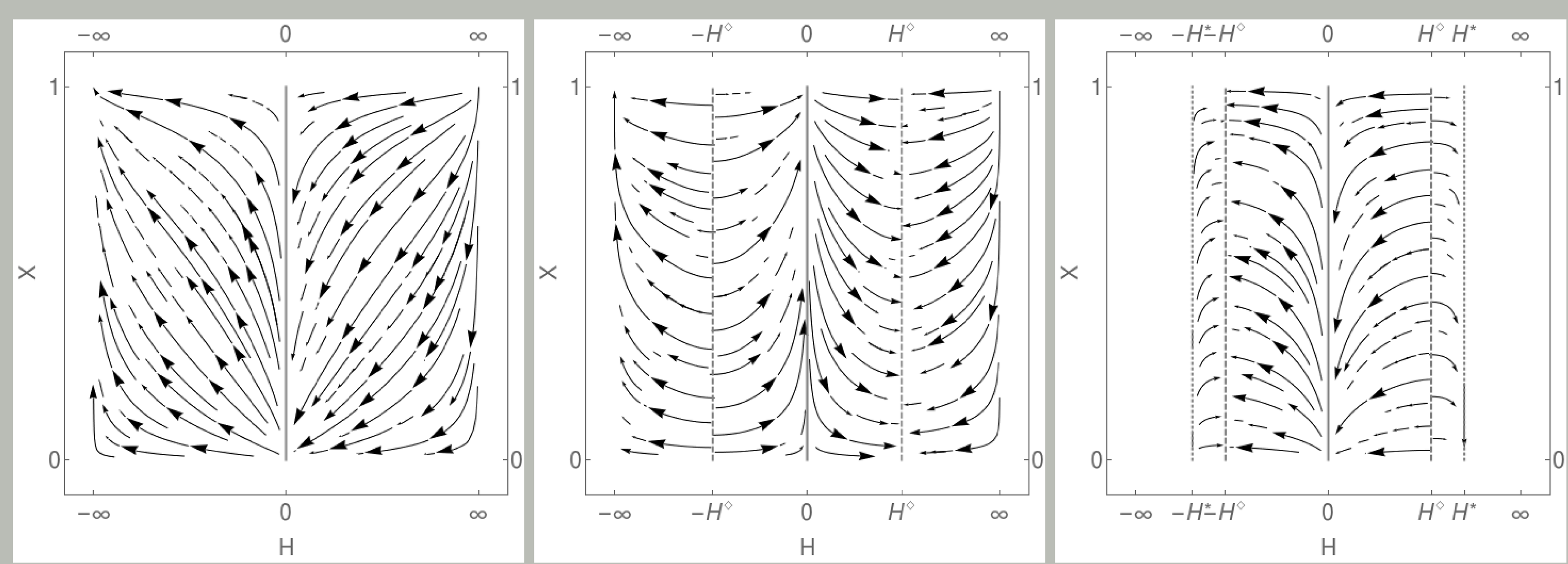
The power law model $f(T) = T + \alpha(-T)^n$, where α and n are constant parameters. We calculate the function

$$W = 6H^2 + (1-2n)\alpha(6H^2)^n\,. \tag{15}$$

In the Figure 1(a) physical trajectories in the region $H > H^\star, 0 < X < 1$ start at the Big Bang singularity $(\infty, 1)$ and end at the attractive de Sitter fixed point $(H^\star, 0)$. Also in this case there exist bounding trajectories with $X \equiv 0$ and $X \equiv 1$ going from (∞, X) to $(H^\star, X)$. Another bounding trajectory connects the de Sitter saddle point $(H^\star, 1)$ to the attractive de Sitter fixed point $(H^\star, 0)$.



(a) $\alpha < 0$, $n < \frac{1}{2}$ and $\alpha > 0, \frac{1}{2} < n < 1$



(b) $\alpha < 0, n > \frac{1}{2}$ and $\alpha > 0, 0 < n < \frac{1}{2}$ (c) $\alpha > 0$ and $n < 0$ (d) $\alpha > 0$ and $n > 1$

Figure 1: Qualitative phase diagrams for the power law model.

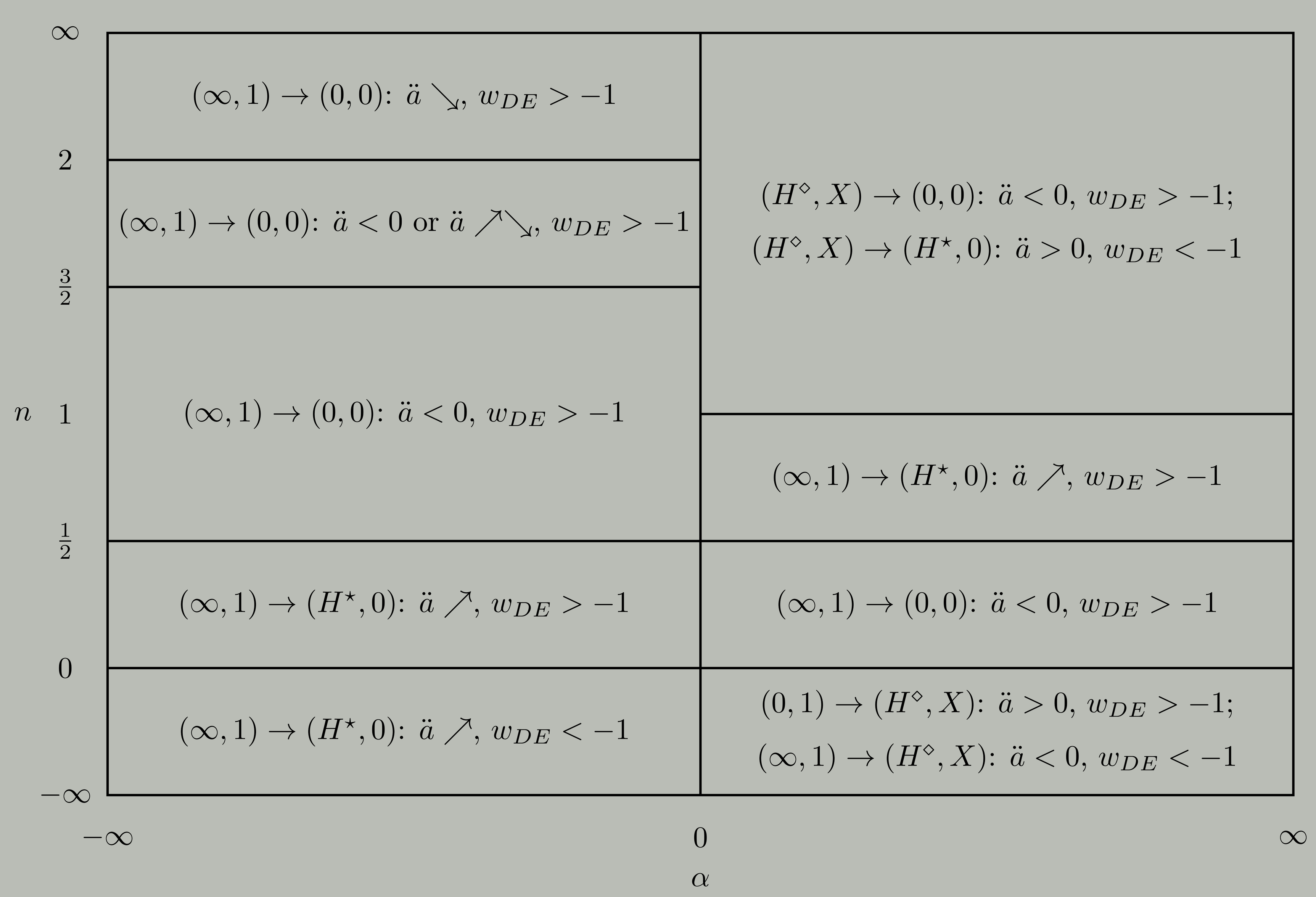


Figure 2: Properties of physical trajectories. $\ddot{a} \nearrow$ indicates a transition from deceleration to acceleration, while $\ddot{a} \searrow$ indicates a transition in the opposite direction.

Acknowledgments

The authors were supported by the Estonian Ministry for Education and Science Institutional Research Support Project IUT02-27 and Startup Research Grant PUT790, as well as by the European Regional Development Fund. The visit was further supported by the Archimedes Foundation.