# Gravitational multipole moments from Noether charges

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joint work with Geoffrey Compère and Ali Seraj.

## Introduction

$\rightarrow$ Multipole Moments describe the gravitational field surrounding the source. *E.g.*, for Kerr:

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**Motivation:**

Provide a definition of *Multipole Moments* for an arbitrary theory of gravity.

# Outline

1. Multipole Symmetries
2. Noether Charges generated by Multipole Symmetries
3. Conservation law
4. Conclusion

# Multipole Symmetries

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Multipole Moments are coefficients of order $r^{-l}$ in the metric.

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**Strategy:** Fixed the harmonic gauge $\partial_\nu \sqrt{-g} g^{\mu\nu} = 0$, we look for $\xi$ such that

- $\to$ are inside the residual gauge transformations: $\Box_g \xi^\mu = 0$,
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The *Multipole Symmetries* are

$$\boxed{K_\epsilon = \epsilon\, \partial_t + t\, \nabla\epsilon, \quad L_\epsilon = -\mathbf{r} \times \nabla\epsilon, \quad P_\epsilon = \nabla\epsilon}$$

Performing an harmonic decomposition with $\epsilon \sim r^l Y_{lm}(\theta, \phi)$:

| | $l = 0$ | $l = 1$ | $l \geq 2$ |
|---|---|---|---|
| $K_{lm}$ | Time translation | Boosts | Mass multipole symmetries |
| $L_{lm}$ | $\varnothing$ | Rotations | Current multipole symmetries |
| $P_{lm}$ | $\varnothing$ | Spatial translations | Momentum multipole symmetries |

# Noether Charges generated by Multipole Symmetries

# Noether Charges

Noether Charges $Q$ are

- *uniquely defined* as a functional of the Langrangian,
- *associated* to the vector field $\xi$.

$$Q_\xi[h] = \frac{1}{8\pi G}\int_S \overbrace{\mathbf{k}_\xi[h;\eta]}^{\text{fixed by the Lagrangian}} \qquad \begin{cases} \eta : \text{Minkowski reference metric} \\ h : \text{linear perturbation around } \eta \end{cases}$$

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We are going to show that

Noether Charges associated to Multipole Symmetries

$\updownarrow$

Gravitational Multipole Moments

# Noether Charges generated by Multipole Symmetries

The metric tensor in *canonical harmonic gauge* reads as [Thorne, Blanchet-Damour]

$$g_{00} = g_{00}[I^{lm}(u)], \quad g_{0i} = g_{0i}[I^{lm}(u), S^{lm}(u)], \quad g_{ij} = g_{ij}[I^{lm}(u), S^{lm}(u)],$$

where $I^{lm}$ are the *mass moments*, $S^{lm}$ are the *current moments*, and $u = t - r$

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$$8\pi G \; Q_L^{lm} = \sum_{p=0}^{l+1} \; C_L(p, l) \; r^p \; {}^{(p)}S^{lm}(u), \quad \text{(current charge)}$$

$$8\pi G \; (Q_K^{lm} - t \, Q_P^{lm}) = \sum_{p=0}^{l+1} \; C_K(p, l) \; r^p \; {}^{(p)}I^{lm}(u). \quad \text{(mass charge)}$$

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From them, we define

→ *total conserved multipole moments* at spatial infinity,

→ *source multipole moments* in the near zone,

→ *radiative multipole moments* at null infinity.

# Noether Charges generated by Multipole Symmetries: conserved moments

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From them, we define

→ *total conserved multipole moments* at spatial infinity ($t = \text{const},\ r \to \infty$)
We assume stationarity at the past of null infinity ($u \to -\infty$):

$$I^{lm}(u) = I^{lm} + \mathcal{O}(1/u) \Longrightarrow r^{p\ (p)}I^{lm}(u) \sim u^{p\ (p)}I^{lm}(u) \sim \mathcal{O}(1/u), \quad p \geq 1$$

$$\boxed{{}^{total}Q = Q\Big|_{\substack{t=const\\ u\to-\infty}} \Longrightarrow \begin{cases} {}^{total}Q_K^{lm} - t\ {}^{total}Q_P^{lm} & \propto I^{lm} \\ {}^{total}Q_L^{lm} & \propto S^{lm} \end{cases}}$$

These are the Multipole Moments as defined by Thorne/Geroch-Hansen.

# Noether Charges generated by Multipole Symmetries: source moments

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→ *source multipole moments* in the near zone $(r \ll \lambda)$

Since $\left|\frac{r^p\; {}^{(p)}S^{lm}}{S^{lm}}\right| \sim \left|\frac{r^p\; {}^{(p)}I^{lm}}{I^{lm}}\right| \sim \left(\frac{r}{\lambda}\right)^p \ll 1,$

$$\boxed{{}^{source}Q = Q_{|r=0} \Longrightarrow \begin{cases} {}^{source}Q_K^{lm} - t\; {}^{source}Q_P^{lm} & \propto I^{lm}(t) \\ {}^{source}Q_L^{lm} & \propto S^{lm}(t) \end{cases}}$$

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From them, we define

→ *radiative multipole moments* at null infinity ($u = \text{const}$, $r \to \infty$)

$$ {}^{rad}Q = \underset{\substack{u\ \text{const} \\ r\to\infty}}{\text{FP}}\ Q \Longrightarrow \begin{cases} {}^{rad}Q_K^{lm} - u\ {}^{rad}Q_P^{lm} & \propto I^{lm}(u) \\ {}^{rad}Q_L^{lm} & \propto S^{lm}(u) \end{cases}$$

Finite Part (FP) prescription inspired by [Blanchet-Damour].

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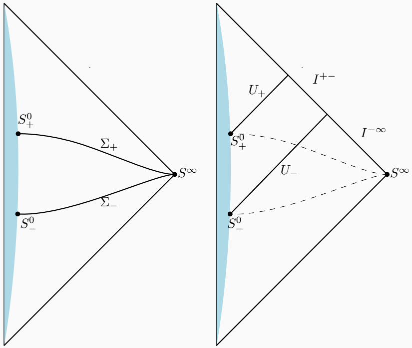


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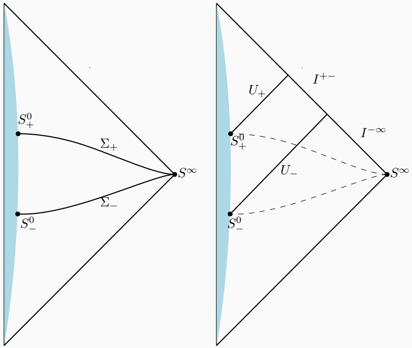


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Asymptotic stationarity at early retarded times implies that $\frac{d}{dt}\,{}^{total}Q = 0$. Then

$$\boxed{{}^{source}Q_{t^+} - {}^{source}Q_{t^-} = {}^{rad}Q_{I^{+-}} + {}^{rad}Q_{U_+} - {}^{rad}Q_{U_-}}$$

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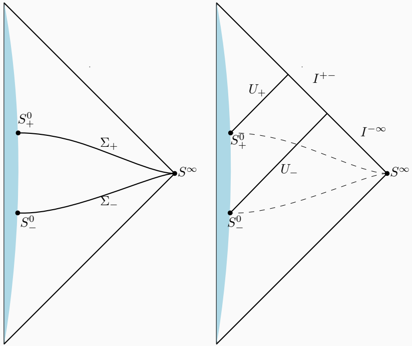


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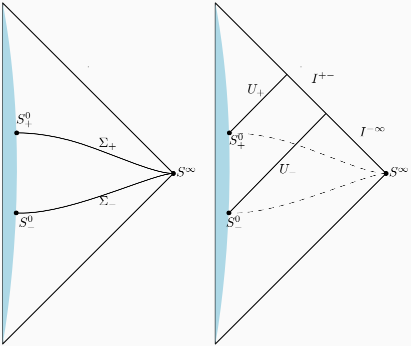


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where $F^{+-} \equiv \underset{\substack{u\ \text{const} \\ r\to\infty}}{\text{FP}}\ {}^{rad}Q_{I^{+-}}$ is the radiation flux between $u^+$ and $u^-$.

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We have

- $\rightarrow$ introduced and defined *Multipole Symmetries*;
- $\rightarrow$ reformulated Thorne's Multipole Moments in terms of Noether charges;
- $\rightarrow$ derived the conservation law relating source moments and radiation flux.

This reformulation extends to an arbitrary generally covariant theory of gravity, *e.g.*, to determine the multipole structure of BH with scalar hair [Herdeiro-Radu].

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THANK YOU FOR YOUR ATTENTION!