Gravitational waves from domain walls moving in media

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Cosmological domain walls

DWs can be formed during phase transitions in the early universe once the discrete symmetry of the underlying gauge theory is spontaneously broken (Kobařev, Okun and Zeldovich, 1974).

After creation, their average number per a Hubble radius remains constant for some time, so, if they were stable, their energy density potentially could dominate and overclose the Universe.

Toy model:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi), \]

\[ V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2. \]

\[ \phi(z) = v \tanh \left[ \sqrt{\frac{\lambda}{2} v z} \right] \]

TEM reveals negative pression \( \rightarrow \) antigravity

\[ T_{\mu\nu}(z) = \left( \frac{d\phi(z)}{dz} \right)^2 \text{diag}(+1, -1, -1, 0) \]

In the limit of zero thickness DW is well described by the Nambu-Goto action as minimal surface, whose linearized gravity is repulsive.
To avoid this, DWs either must be unstable, what happens if the discrete symmetry was only approximate, or disappear via some other mechanism. The basic viable field model of cosmological DWs is that of real scalar field with the biased potential:

![Diagram of potential function](image)

Instability is due to graduate tunneling of Higgs to lower local minimum. In presence of gravity this corresponds to collapse of DW.

Collapsing unstable DWs generate gravitational waves, whose spectrum, sensible to particular underlying models, can be an important source of information about the early universe. This spectrum was obtained via numerical simulations, for recent review see (Saikawa, Universe 3 (2017) 40).
GW spectrum from collapsing unstable domain walls in models with biased potentials

Observable at third generation of detectors

Hiramatsu, Kawasaki, Saikawa (2014)

In hybrid models admitting both DWs and cosmic strings holes will be surrounded by cosmic strings. Evolution of punctured DWs depends on balance of tensions, four hole is enough to destroy the DW (Stojkovic et al)

Here we consider another issue associated with perforation of DWs: now by elementary particles. These do not produce “holes in the walls”, but lead to dissipation of energy through gravitational radiation. This mechanism may lead to destruction of DWs independently of their nature, and generation of GWs. Contrary to mechanism of gravitational radiation of collapsing walls, our mechanism is applicable to stable DWs too.
The model

Domain wall

\[ x^M = X^M(\sigma^\mu), \quad M = 0, 1, 2, \ldots, D - 1 \]

Point particle

\[ x^M = z^M(\tau) \]

\[ S = -\frac{\mu}{2} \int \left[ X_\mu X_\nu g_{MN} \gamma^{\mu\nu} - (D - 3) \right] \sqrt{-\gamma} d^{D-1}\sigma - \frac{1}{2} \int \left( e g_{MN} z^M \dot{z}^N + \frac{m^2}{e} \right) d\tau \]

\[ - \frac{1}{\kappa^2} \int R_D \sqrt{-g} d^D x. \]

Geometry:

DW is plane, orthogonal to z-axis (M=D-1)
Particle falls orthogonally (along z) on DW
Interaction purely gravitational

Gravitational force is repulsive

Particle decelerates and reaches DW if starts inside “piercing layer”

Perforation: particle’s momentum does not change, while its derivative changes sign

\[ \dot{z}_\ell(v) = \frac{v^2 \gamma^2}{2k(D \gamma^2 v^2 + 1)} \]

\[ k = \frac{\mu \kappa^2}{4(D - 2)} \]
Iterative solution

Expand in terms of gravitational coupling

\[ \Phi = \Phi^0 \Phi^1 \Phi^2 \Phi^3 + \ldots \]

all the variables \( \dot{z}^M(\tau) \), \( e(\tau) \), \( X^\mu(\sigma) \) and \( h_{MN}(x) \).

In the first order the superposition principle holds

\[ 1 h_{MN} = h_{MN} + \tilde{h}_{MN}. \]

**DW at rest:**

\[ 0 X^M = \Sigma^M_\mu \sigma^\mu \]

\[ \Sigma^M_\mu = \delta^M_\mu \]

\[ \Xi_{MN} = \Xi^\mu_M \Xi^\nu_N \eta_{\mu\nu} \]

\[ h_{MN} = \frac{\kappa^\mu}{2} \left( \Xi_{MN} - \frac{D - 1}{D - 2} \eta_{MN} \right) |\dot{z}| = \frac{\kappa^\mu |\dot{z}|}{2(D - 2)} \text{ diag} (-1, 1, \ldots, 1, D - 1) \]

linearly growing potentials

**particle:**

\[ \tilde{h}_{MN}(x) = -\frac{\kappa m \Gamma(D - 3)}{4 \pi \frac{D - 1}{D - 2}} \left( u_M u_N - \frac{1}{D - 2} \eta_{MN} \right) \frac{1}{\left[ \gamma^2 (z - vt)^2 + r^2 \right]^{\frac{D - 3}{2}}} \]

Particle moves from positive to negative \( z \) with constant deceleration till perforation,
Then acceleration changes sign. The velocity is continuous

\[ 1 \dot{z}^{D - 1} = k (D \gamma^2 v^2 + 1) \text{ sgn}(\tau) \]
Deformation of the wall

Perturbing Nambu-Goto equations, one gets d’Alembert equation on the wall

\[ \Pi_{MN} \Box_{D-1} \delta X^N = \Pi_{MN} J^N, \quad \Pi^{MN} = \eta^{MN} - \sum_{\mu} \sum_{\nu} \eta^{\mu\nu} \]

with projector onto one-dimensional subspace orthogonal to it. The source reads

\[ J^N = \kappa \sum_{\mu} \sum_{\nu} \eta_{\mu\nu} \left( \frac{1}{2} \eta_{PQ,N} - \bar{h}_{NP,Q} \right) \]

Define branon field \( \Phi(\sigma^\mu) \equiv \delta X^z \), obeys

\[ \eta^{\mu\nu} \frac{\partial}{\partial \sigma^\mu} \frac{\partial}{\partial \sigma^\nu} \Phi(\sigma) = J(\sigma) \]

\[ J(\sigma) = -\kappa \left[ \frac{1}{2} \eta_{\mu\nu} \bar{h}_{\mu\nu,z} - \bar{h}_{z,0,0} \right] \]

\[ \lambda = \kappa^2 \pi \frac{\gamma^2 \Gamma \left( \frac{D-1}{2} \right)}{4 \pi^{D-1} \frac{D-1}{2}} \left( \frac{\gamma^2 v^2 + \frac{1}{D-2}}{2} \right) \]

Retarded solution consist of time-asymmetricm deformation (a) and free branon wave (b), starting at the moment of perforation and propagating with the velocity of light along the wall

\[ \Phi(t, r) = -\Lambda \text{sgn}(t) I_a + 2 \Lambda \theta(t) I_b \]

\[ \Lambda = \frac{\sqrt{\pi} \lambda}{2 \frac{D-2}{2} \gamma^3 \Gamma \left( \frac{D-1}{2} \right)} \]

\[ I_a(t, r) = \frac{1}{r^{D-4}} \int_0^\infty dy \ J \left( \frac{D-4}{2} \right) y^{\frac{D-6}{2}} e^{-y \gamma v |t|} \]

\[ I_b(t, r) = \frac{1}{r^{D-4}} \int_0^\infty dy \ J \left( \frac{D-4}{2} \right) y^{\frac{D-6}{2}} \cos yt \]
Case D=5 (most transparent)

Look for retarded solution

\[
\Phi = \int G_{ret}(x - x')J(x')d^4x'
= \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{k^2 + 2i\varepsilon\omega} J(k) d^4k
\]

Fourier-transform of the source

\[
J(k) = \int e^{ikx} J(x)d^4x = \frac{2\pi^2\lambda}{\gamma} \frac{i\omega}{\omega^2 + \gamma^2 \nu^2} k^2
\]

Result is

\[
\Phi = \frac{\lambda}{2\gamma^3} \left( \frac{F_0(r, t)}{r} - \frac{F_1(r, t)}{r} \right)
\]

\[
F_0(r, t) = \frac{\pi}{2} \theta(t)[\varepsilon(r + t) + \varepsilon(r - t)]
\]

\[
F_1(r, t) = \arctan \frac{r}{\gamma \sqrt{t}}.
\]

The first part \(F_0\), proportional to the Heaviside step function of time \(\theta(t)\), is zero until the moment \(t = 0\) of perforation. It describes an expanding wave caused by the perforation shake discussed above (the first term in \(F_0\) looks contracting, but actually it is a constant, \(\theta(t) \varepsilon (r + t) = 1\)). This wave, propagating with the velocity of light, does not correlate with further motion of the particle.

The second part \(F_1\) is non-zero and smooth both before \((t < 0)\) and after \((t > 0)\) the perforation. It does correlate with position of the mass describing a continuous deformation of the brane caused by its gravitational field. It is small when the particle is far away from the brane, and grows up to the maximal absolute value equal to \(\pi/2\) when it approaches the brane.

It is easy to verify that for all \(t \neq 0, r \neq 0\) this term satisfies the inhomogeneous wave equation:

\[
\frac{1}{r} \left( \partial^2_t - \partial^2_r \right) F_1(r, t) = \frac{2\gamma^3 \nu t}{(r^2 + \gamma^2 \nu^2 t^2)^2}
\]
This split is non-trivial near the perforation point. F1 term, supposed to follow smoothly the particle’s motions, has an extra (derivative) delta-term due to change of sign of the term because of instantaneous change of sign of the gravitational force between the brane and the particle. Acting by box one gets

\[
\frac{\Box F_1(r, t)}{r} = \frac{2\gamma^3 \nu t}{(r^2 + \gamma^2 \nu^2 t^2)^2} + \frac{\pi \delta'(t)}{r}
\]

Delta-term is compensated by branon term

\[
\frac{\Box F_0(r, t)}{r} = \frac{\pi \delta'(t)}{r}
\]

The delta-derivative term describes the instantaneous shaking force exerted upon the brane at perforation. It excites the branon shock wave described by F0 term. Even more surprising that in the limit of zero particle velocity this shock wave remains nonzero:

\[
\Phi_0 = \lim_{v \to 0} \Phi = \frac{m\kappa^2}{48\pi} \frac{\epsilon(r-t)}{r}
\]

\[
\Box \Phi_0 = Q_B \delta^3(r), \quad Q_B = \frac{m\kappa^2}{12} \epsilon(t)
\]

equivalent to retarded field of a “charge” instantly changing sign.
This might seem paradoxical, since the source term in the NG equation looks to be zero for \( v = 0 \). The paradox is solved if we regard the source \( J(x) \) as distribution. It is easy to see, that in the limiting cases \( t \to 0 \) or \( v \to 0 \), \( J(x) \) exhibits properties of the three-dimensional delta-function. Denoting \( \alpha = \gamma v t \), we have:

\[
\lim_{\alpha \to \pm 0} \frac{\alpha}{\left( r^2 + \alpha^2 \right)^2} = \begin{cases} 
0, & \text{if } r \neq 0, \\
\pm \infty, & \text{if } r = 0
\end{cases}
\]

Since the box operator contains three dimensional Laplace operator, it is reasonable to consider \( J(x) \) in the sense of distributions:

\[
\lim_{\alpha \to \pm 0} J(x) = \frac{\pi^2 \lambda}{\gamma} \epsilon(t) \delta^3 (r)
\]

since

\[
\int J(x) d^3 x = \frac{4 \pi \lambda}{\gamma} \int_0^\infty \frac{r^2}{\left( r^2 + \alpha^2 \right)^2} dr = \frac{\pi^2 \lambda}{\gamma} \epsilon(\alpha)
\]

Remarkably, this limiting value is the same if we consider time in the close vicinity of the perforation moment \( t = 0 \) for any velocity \( v \) of the perforating particle, or the limit of small velocity \( v \to 0 \). In the latter case this limit holds for any \( t \), and since the coefficient \( \lambda \) remains the point-like source in might be attributed to some “shaking charge” instantaneously changing sign. The corresponding retarded solution of the box operator is not a Coulomb field, but a spherical shock wave.
Second order

In the second order one obtains the leading contribution to gravitational radiation. The effective source of radiation consists of three ingredients. The first is due to particle which has constant acceleration before and after piercing. This has certain analogy with the Weinberg's computation of gravitational radiation from the system of particles colliding at a point: in that case one has the constant momenta before and after collision which instantaneously change on a finite amount. In our case it is the (proper) time derivatives of the momenta before and after collision which are constant and opposite, changing sign at the moment of perforation. The second contribution comes from the deformation of the brane world-volume caused by varying gravitational field of the moving particle. Finally, for consistency of calculations, the gravitational stresses have to be taken into account, these are described using Weinberg's expansion of the Einstein tensor up to the second order in the gravitational constant

\[
G_{MN} = -\frac{\kappa}{2} \Box \psi_{MN} - \frac{\kappa^2}{2} S_{MN} + \sum_{n>2} \kappa^n N_{MN}^{(n)}
\]

The wave equation in the second order for the trace-reversed metric reads

\[
\Box^2 \psi_{MN} = -\kappa \tau_{MN}
\]

where

\[
\tau_{MN} = \frac{1}{2} T_{MN} + \frac{1}{2} T_{MN} + S_{MN}
\]

the last term being the bilinear form of the first order metric deviations due to DW and the particle. Only the sum of three is conserved in the Minkowski sense

\[
\partial_N \tau^{MN} = 0
\]
Traditionally, both electromagnetic and gravitational radiation is computed in terms of fluxes of the field momentum in the wave zone, which is well-defined in asymptotically flat space-time. Our space-time is not asymptotically flat, so one has to revisit the derivation. In particular, the energy-momentum flux through the lateral surface of the world-tube turns out to be non-zero. The idea of new derivation is to start with radiation reaction work which is defined locally. To perform necessary transformation one has to consider expansions of Einstein equations up to the fourth order, however the leading radiation formula involves only the product of two second order terms

$$P_M = \frac{1}{2} \int 2h_{AB,M} \Box^2 \psi^{AB} = -\frac{\kappa}{2} \int 2h_{AB,M} T^{AB} d^D x$$

which finally transforms to the standard momentum space representation

$$E_{rad} = \frac{\kappa^2}{4(2\pi)^{D-1}} \sum_P \int_0^\infty \omega^{D-2} d\omega \int_{S^{D-2}} d\Omega \left| \epsilon_P^{SN} \tau_{SN}(k) \right|^2$$

where the Fourier-transforms of the effective source are contracted with polarization tensors.

In any dimensions one can choose polarization tensors in such a way that only one on them (with two zz-legs) gives non-zero projections, so we actually deal with complex scalar amplitudes. One therefore has three complex amplitudes expressing contribution of the particle, the DW and gravitational stresses. This splitting is of course gauge dependent and refers to harmonic gauge together with certain additional specification of polarization states.
Introduce the unit space-like vector $\mathbf{n}$ on the unit sphere within the DW, and the angle $\psi$ between $\mathbf{k}$ and the z-axis (the line of particle motion). The graviton wave-vector is then parametrized as $\mathbf{k} = \omega (\mathbf{n} \sin \psi; \cos \psi)$

**The wall amplitude:**

$$T_z(k) = -\sqrt{\frac{D - 3 \varepsilon^2 \mu m}{D - 2}} \frac{\gamma \nu \sin^2 \psi}{1 + \gamma^2 \nu^2 \sin^2 \psi} \times \left[ \frac{\cos \psi}{\nu [\cos^2 \psi + 2i\epsilon k^0]} \left( \gamma^2 \nu^2 + \frac{1}{D - 2} \right) + \gamma^2 \nu^2 - \frac{1}{D - 2} \right]$$

Note the infrared divergence of this amplitude which is not surprising since our procedure did not take into account the finite depth of the piercing layer. Another interesting feature is that the amplitude remains non-zero in the limit $\nu \to 0$. This is related to branon shock wave which emerges independently of velocity. Also, the amplitude diverges at $\psi = \pi/2$, i.e. along the DW. This divergence is due to the fact that the shock wave excitation which in our approach propagates to infinity without damping. Removing this contribution (this may correspond to $Z_2$-symmetric braneworld models or to the case of two mirror particles impinging upon the wall), we obtain

$$\left. 1^1 T_z(k) \right|_{\Phi = 0} = -\sqrt{\frac{D - 3 \varepsilon^2 \mu \varepsilon}{D - 2}} \frac{\nu \sin^2 \psi}{1 + \gamma^2 \nu^2 \sin^2 \psi} \left[ \gamma^2 \nu^2 - \frac{1}{D - 2} \right]$$

In this case the amplitude does not blow up at $\psi = \pi/2$ and the angular distribution is finite.
The particle amplitude

\[ 1 \bar{T}_z(k) = - \left[ \frac{D - 3}{4(D - 2)^3} \right]^{1/2} \frac{\sqrt{\mu m} \nu}{\gamma \omega^2} \frac{(D - 2) \gamma^2 \nu^2 - 1}{(1 - \nu \cos \psi)^3} \nu \cos \psi + 2 \sin^2 \psi \]

This amplitude, apart from the infrared, has also the angular divergence at \( \psi = 0 \) in the case of the massless particle \( \nu = 1 \). This is the well-known collinear divergence encountered in quantum perturbation theory for interacting massless particles. In classical theory this is the line divergence of the retarded potentials

The stress contribution

\[ S_z(k) = \sqrt{\frac{D - 3}{D - 2}} \frac{\sqrt{\mu m} \nu}{\gamma \omega^2} \frac{\sin^2 \psi}{(1 - \nu \cos \psi)^3(1 + \nu \cos \psi)} \left[ \gamma^2 \nu^4 \cos^2 \psi - \frac{(1 - \nu \cos \psi)^2}{D - 2} \right] \]

Here one also observes both the infrared and the angular divergences.
The destructive interference in the ultrarelativistic limit

In the ultrarelativistic limit both the particle and the stress amplitudes have similar behavior near the forward direction which could give the leading contribution to radiation. However, keeping the common singular factors and expanding the rest as

\[
\sin \psi \approx \psi, \quad 1 - v \cos \psi \approx \frac{\psi^2 + \gamma^{-2}}{2}
\]

one finds for large $\gamma$

\[
1 \tilde{T}_z(k) = -\frac{1}{2} \sqrt{\frac{D - 3}{D - 2}} \frac{\kappa^2 \mu \mathcal{E} \sin^2 \psi}{\omega^2 (1 - v \cos \psi)^3} \left(1 + \mathcal{O}(\gamma^{-2})\right)
\]

\[
S_z(k) = \frac{1}{2} \sqrt{\frac{D - 3}{D - 2}} \frac{\kappa^2 \mu \mathcal{E} \sin^2 \psi}{\omega^2 (1 - v \cos \psi)^3} \left(1 + \mathcal{O}(\gamma^{-2})\right)
\]

where $E = m \gamma$ is the particle energy. So in the leading order in $\gamma$, these two amplitudes exactly cancel. This is manifestation of the destructive interference which reflects the equivalence principle in the language of flat space, which was encountered earlier in the bremsstrahlung problem for point particles. After cancelation of the leading terms, the sum of two amplitudes has two orders less in $\gamma$ than each term separately.

On the other hand, the brane amplitude in the forward direction is approximated as

\[
1 \tilde{T}_z(k) \bigg|_{\psi \ll 1} \approx -\sqrt{\frac{D - 3}{D - 2}} \frac{\kappa^2 \mu \mathcal{E}}{\omega^2} \frac{\gamma^2 \sin^2 \psi}{1 + \gamma^2 \sin^2 \psi}
\]

This is subleading in the ultrarelativistic limit.
Thus in the small-angle region the main contribution still comes from the sum of $1^\ast T(k^M)$ and $S(k^M)$. Expanding these with more accuracy and keeping the subleading terms, one finds to the main order:

$$
\tau_z(k) \bigg|_{\psi \ll 1} \approx -\frac{1}{8} \sqrt{\frac{D-3}{D-2}} \frac{x^2 \mu E \sin^2 \psi}{\gamma^2 \omega^2 (1 - v \cos \psi)^3} \left[ \frac{D+2}{D-2} + \gamma^2 \sin^2 \psi \right]
$$

The total amplitude is peaked at $\psi \sim 1/\gamma$ in any dimensions.

The angular dependence of the radiation amplitude in $D=4$ for $\gamma = 15$ in four (black), $\gamma = 30$ (red) and $\gamma = 45$ (green).
The massless case

Angular dependence of partial amplitudes for perforation of DW by photons:

- Particle – black, merging to green
- DW – red
- Stress – green

Partial amplitudes

\[ T_z(k) = -2C \left[ \frac{\cos \psi}{\cos^2 \psi + 2i \epsilon k^0} + 1 \right], \]
\[ S_z(k) = C \frac{\cos^2 \psi}{\sin^4(\psi/2)}, \]
\[ \bar{T}_z(k) = -C \frac{\cos \psi \cos^2(\psi/2)}{\sin^4(\psi/2)}, \]
\[ C \equiv \sqrt{\frac{D - 3}{D - 2}} \frac{\kappa^2 \mu \mathcal{E}}{4\omega^2}, \]

Particle and stress forward divergences partially compensate (destructive interference)

Total amplitude

\[ \tau_z(k) = -C \frac{\cos \psi}{\sin^2(\psi/2)} \]
Spectral and angular distribution of PGR

\[ E_{\text{rad}} = \frac{\kappa^2}{(4\pi)^{D/2} \Gamma \left( \frac{D-2}{2} \right)} \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \omega^{D-2} \int_{\psi_{\text{min}}}^{\pi} d\psi \sin^{D-3} \psi |\tau_z(\omega, \psi)|^2 \]

- Forward peaking is D-dependent, and may be divergent (e.g. for photons in D=4
- IR divergence for all D<6
- UV divergent for D>4

D=4 with IR cutoff

\[ E_{\text{rad}} = \frac{1}{80\pi^2} \frac{\left( \kappa_4^3 \mu \mathcal{E} \right)^2 \gamma^2}{\omega_{\text{min}}} \]
Non-relativistic velocities

Angular cutoff required

\[ \tau_z(k) \bigg|_{\psi \approx \pi/2} = - \sqrt{\frac{D-3}{D-2}} \frac{\kappa^2 \mu m}{2 \omega^2 \cos \psi} \frac{\gamma}{1 + \gamma^2 v^2} \left( \gamma^2 v^2 + \frac{1}{D-2} \right) \]
Conclusions

Piercing gravitational radiation (PGR) is novel and universal mechanism of GW emission by DWs independent on their particular nature, in particular, on the issue of their stability.

The effect is especially efficient in the massless limit (photons piercing DWs). In this case no velocity (quadrupole) factor in the gravitational radiation power.

PGR predicts an excess of relict gravitons in the low frequency region.

Present model is too simplified to make definitive quantitative predictions, but on general grounds the expected effect is not small compared to GW generation by collapsing walls.

PGR may serve a novel mechanism for DW destruction. For more definitive predictions further calculations within the cosmological setting are necessary.

Thank you for attention!