Unimodular quantum gravity and the classical limit

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Unimodular Gravity

\[ \delta g_{\mu \nu} \left( \frac{1}{2c^2} \int d^4x \sqrt{-g} R + S_{\text{matter}} \right) \bigg|_{g = -1} = 0 \]

\[ \Rightarrow \ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \kappa T_{\mu \nu} - \Lambda g_{\mu \nu} \]

- Varying the action with respect to the metric under the condition \( \det g = -1 \)
- Getting Einstein's equations with an additional term
- Identifying \( \Lambda \) with the cosmological constant

Any solution of Unimodular Gravity is a solution of General Relativity with a certain cosmological constant!

Unimodular quantum cosmology: The flat Friedmann universe with a scalar field

\[ ds^2 = -N(t)^2 dt^2 + a(t)^2 d\Omega^2 \]

Lagrangian of the field \( \Phi \)

\[ L_{\text{matter}} = N a^3 \left( \frac{\dot{\Phi}^2}{2Nc^2} - V(\Phi) \right) \]

The unimodular Hamiltonian of the model reads

\[ H = \frac{c^2 R^2}{2a^4} \frac{c^2 R^2}{2a^2} + V(\Phi) \] where \( \epsilon = 3/\kappa \).

- There is no Hamiltonian constraint! (H is a constant of motion)

Quantization (with Laplace Beltrami operator ordering) yields a Schrödinger equation

\[ \hat{H} \Psi = \left( -\frac{1}{4c^4} \frac{\partial^2}{\partial a^2} \frac{\partial}{\partial a} - \frac{1}{2} \frac{\partial^2}{\partial a^2} + V(\Phi) \right) \Psi = \frac{\hbar}{c} \frac{\partial \Psi}{\partial t} \]

- The problem of time of standard canonical quantum gravity (\( \hat{H} \Psi = 0 \)) has disappeared!
- It is possible to define a scalar product.

Introducing light-cone coordinates

\[ u = \frac{a^2}{3} \Phi, \quad v = \frac{a^2}{3} \dot{\Phi} \]

we find

\[ \hat{H} = \frac{\hbar^2 c^2}{\epsilon} \frac{\partial^2}{\partial \Phi \partial \bar{\Phi}} \frac{\partial^2}{\partial \bar{\Phi} \partial \Phi} + V \left( \frac{U}{V} \right). \]

\[ H = \frac{c^2}{\epsilon} \rho_\nu \rho_\nu + V \left( \frac{U}{V} \right). \]

A superposition of eigensolutions \( \Psi_\lambda \)

\[ \frac{\partial^2}{\partial \Phi \partial \bar{\Phi}} \Psi_\lambda(u, v) = -\frac{\Lambda}{3} \Psi_\lambda(u, v), \]

yields wavepacket solutions of the form

\[ \Psi(u, v, \tau) = \int_0^\infty e^{i\hbar c^2 t} \psi(u, v) F(\Lambda) d\Lambda. \]

- It is possible to construct wavepackets with a unitary time evolution and to calculate expectation values.

The classical limit of unimodular quantum cosmology

We find for the variable \( A^2 \), related to the scalefactor by

\[ A^2 = uv = \frac{\Phi^2}{9} \]

\[ \lim_{t \to \infty} \frac{d^2}{dt^2} (A^2) = -\frac{c^2}{\epsilon} \left( -2 \hat{H} + u \frac{\partial V}{\partial \Phi} + v \frac{\partial V}{\partial \bar{\Phi}} \right) \]

There is a correspondence of classical and quantum behaviour for late times!

Results for the massless scalar field

We found explicit solutions for the special case of a massless scalar field \( V = 0 \) (corresponds to stiff matter: \( p = \rho c^2 \)) with a negative expectation value of the Hamiltonian [1].

Further questions

- In the case of the massless scalar field the uncertainty of the scaled factor increases: \( \Delta A^2 \sim t^2 \) for late times.
  - For which scalar field potentials a better quasiclassical behaviour can be expected?
  - What are the results for quantum black holes in unimodular quantum gravity? Will the black hole solutions be stable?