Orthonormal bases and quasi-splitting subspaces in pre-Hilbert spaces

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\textbf{Abstract}

Let $S$ be a pre-Hilbert space. We study quasi-splitting subspaces of $S$ and compare the class of such subspaces, denoted by $\mathcal{E}_q(S)$, with that of splitting subspaces $\mathcal{E}(S)$. In [D. Buhagiar, E. Chetcuti, Quasi splitting subspaces in a pre-Hilbert space, Math. Nachr. 280 (5–6) (2007) 479–484] it is proved that if $S$ has a non-zero finite codimension in its completion, then $\mathcal{E}_q(S) \neq \mathcal{E}(S)$. In the present paper it is shown that if $S$ has a total orthonormal system, then $\mathcal{E}_q(S) = \mathcal{E}(S)$ implies completeness of $S$. In view of this result, it is natural to study the problem of the existence of a total orthonormal system in a pre-Hilbert space. In particular, it is proved that if every algebraic complement of $S$ in its completion is separable, then $S$ has a total orthonormal system.

\section{1. Introduction}

In what follows $S$ is a real or complex pre-Hilbert space (= inner product space) and $H$ is its completion, i.e. a Hilbert space containing $S$ as a dense subspace. For any subset $A \subseteq S$ we write $\bar{A}$ to denote the closure of $A$ in $H$ and $A^\perp_S$ the orthogonal complement of $A$ in $S$, i.e. $A^\perp_S = \{x \in S \mid \langle x, a \rangle = 0, \forall a \in A\}$. Let us recall that the orthogonal (Hilbert) dimension $\dim_S S$ is the cardinality of any maximal orthonormal system of $S$. For a vector space $K$ we denote by $\dim K$ the linear (Hamel) dimension of $K$.

In the Hilbert space model for quantum mechanics, the events of a quantum system can be identified with projections on a Hilbert space or, equivalently, a collection of closed subspaces of a Hilbert space [2,5,9,11,13]. Two classes of closed subspaces of $S$ that can naturally replace the lattice of projections in a Hilbert space are those of orthogonally closed, and splitting subspaces. We recall that a subspace $M$ of $S$ is orthogonally closed if $M^\perp_S = M$, and is splitting if $S = M \oplus M^\perp_S$. It is not difficult to check that every splitting subspace is orthogonally closed. By the classical Amemiya–Araki–Piron Theorem, equality between these two classes holds if and only if $S$ is complete [1,10] (see also [5,9]). When endowed with the partial ordering of set-theoretical inclusion $\subseteq$ and orthocomplementation $\perp_S$, the set of orthogonally closed subspaces $\mathcal{F}(S)$ and the set of splitting subspaces $\mathcal{E}(S)$ carry an algebraic structure with orthocomplementation. It is not difficult to check that $\mathcal{E}(S) \subseteq \mathcal{F}(S)$. In general, the algebraic structures of these two orthoalgebras are different; $\mathcal{F}(S)$ is a complete lattice whereas $\mathcal{E}(S)$ is an orthomodular poset and the following three statements are equivalent:

(i) $F(S)$ is orthomodular;
(ii) $E(S)$ is a complete lattice;
(iii) $S$ is complete.

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The class $E_Q(S)$ of quasi-splitting subspaces of $S$ was introduced in [4] as an intermediate between $E(S)$ and $F(S)$. A subspace $M$ of $S$ is quasi-splitting if it is closed in $S$ and $M \oplus M^\perp$ is a dense subspace of $S$. Equivalently, a closed subspace $M$ of $S$ is quasi-splitting if $\overline{M^\perp} = M^\perp$ [4, Proposition 2.2].

If $S$ is complete, then $E(S) = E_Q(S) = F(S)$. The inclusions

$$E(S) \subseteq E_Q(S) \subseteq F(S)$$

hold though, in general, they are proper. Motivated by the Amemiya–Araki–Piron Theorem, the authors of [4] conjectured that: $E_Q(S) = E(S)$ if and only if $S$ is a Hilbert space and also have settled this in the affirmative for the case when $d(H) / S$ is finite. As will be seen further on, this question is closely related to the problem of characterizing those pre-Hilbert spaces that admit an orthonormal basis, i.e. an orthonormal system (ONS) that is total. It is known that in a Hilbert space every maximal orthonormal system (MONS) is an orthonormal basis (ONB). This fact distinguishes Hilbert spaces completely [6–8]. Moreover, it is possible to exhibit pre-Hilbert spaces admitting no ONB. Indeed, it can happen that $\dim S \neq \dim H$ [3,6].

The main result of Section 2 says that if $S$ has an ONB, then $E_Q(S) = E(S)$ if and only if $S$ is complete. (In particular, this means that $E(S) \neq E_Q(S)$ when $S$ is an incomplete separable pre-Hilbert space.) In Section 3 of the paper we investigate when a pre-Hilbert space has an ONB. It is shown that when every linear complement of $S$ in $H$ is separable, then $S$ has an ONB. This means, for example, that $S$ admits an ONB when $d(H) / S \leq \aleph_0$. (In particular, all hyperplanes have an ONB.) The relation between $\dim S$, $\dim H$ and $d(S)$ is also studied in Section 3.

2. Pre-Hilbert spaces in which every quasi-splitting subspace is splitting

In this section it is proved that if $S$ is a pre-Hilbert space with an ONB, then $E_Q(S) = E(S)$ if and only if $S$ is complete. This result is first proved for the case when $S$ is separable. (In such a case $S$ always has an ONB; see for example [3, V.24] or [12].) The proof is divided in shorter lemmas. The main lemma, which is also referred to in Section 3, is Lemma 1. We denote the set $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In the proof of the following lemma we use the result that $M \cap S$ is dense in $M$ whenever $M$ is a closed subspace of $H$ with finite $\dim M \perp S$ ([5, Theorem 4.1.2], [9, Lemma 4.2.3]).

**Lemma 1.** Let $(z_n)_{n \in \mathbb{N}_0}$ be a sequence of vectors in $H$ such that $\|z_0\| = 1$ and $z_0 \perp z_1$. There is a double sequence $(y_{nm})_{m, n \in \mathbb{N}_0}$ in $S$ and a sequence $(y_{n})_{n \in \mathbb{N}_0}$ in $H$ such that $y_0 = z_0$ and $y_1 = z_1$, with the following properties:

(i) $\|y_{nm} - y_n\| \leq 1/2^m$ for $m, n \in \mathbb{N}_0$ with $n \leq m$;

(ii) $(y_{nm}, y_{pq}) = (y_n, y_q) = 0$ for $m, n, p, q \in \mathbb{N}_0$ with $q < n, q \leq p, n \leq m$;

(iii) $y_n - z_0 \in \text{span } Y_n$ where

$$Y_n = \{y_k \mid 0 \leq k < n\} \cup \{y_{ij} \mid 0 \leq j \leq i < n\}.$$

**Proof.** By induction on $n \in \mathbb{N}_0$ we define vectors $y_n, y_{n0}, \ldots, y_{nm}$ such that condition (i)–(iii) are satisfied by the vectors of $Y_{n+1}$. Set $y_0 = z_0$ and $y_{00} = 0$. In the $n$th induction step we proceed as follows. Define $y_n$ as the orthoprojection of $z_n$ in $Y_n^{\perp S}$. Condition (iii) is satisfied since $y_n \in z_0 + \text{span } Y_n$.

Let us further observe that for $n = 1$ we have $Y_1 = \{0, z_0\}$ and therefore $y_1 = z_1$.

We now construct inductively the vectors $y_{nl}$ ($l = 0, \ldots, n$) such that $y_{nl} \perp A_l$, where $A_l$ is the set

$$\{y_j \mid 0 \leq j \leq n, j \neq l\} \cup \{y_{ij} \mid 0 \leq j \leq i < n, j \neq l\} \cup \{y_{ij} \mid 0 \leq j < i\}.$$

Suppose that $y_{nl}$ ($l < n$) are constructed. Since $y_1 \perp A_1$, by the above stated result we can find $y_{nl} \in S$ such that $y_{nl} \perp A_l$ and $\|y_{nl} - y_l\| \leq 1/2^l$. It is clear then that conditions (i) and (ii) are satisfied by the vectors of $Y_n$. \qed

**Lemma 2.** With the assumption and notations of Lemma 1, let $M := \{y_{m0} \mid m \in \mathbb{N}_0\}^{\perp S}$. Then $M \notin E(S)$.

(i) If $z_0, z_1 \in H \setminus S$ and $\text{span } \{z_0, z_1\} \cap S \neq \{0\}$, then $M \notin E(S)$.

(ii) If $u \in H$, $u \perp M$ and $u \perp M^{\perp S}$, then $u \perp z_n$ for all $n \in \mathbb{N}_0$.

**Proof.** (i) Suppose that $z_0, z_1 \notin S$, $\text{span } \{z_0, z_1\} \cap S \neq \{0\}$ and $M \in E(S)$ for contradiction. Then there is a non-zero vector $h \in S$ such that

$$h = \alpha z_0 + \beta z_1 = h_1 + h_2,$$

where $h_1 \in M$ and $h_2 \in M^{\perp S}$. Then $h_1 = \alpha z_0$ and $h_2 = \beta z_1$ because $z_0 \in M$ and $z_1 \in M^{\perp S}$. This contradicts the assumption that $z_0$ and $z_1$ are not elements of $S$.

(ii) Given any $m \in \mathbb{N}_0$, observe that $y_{m0} \in M$ and $y_{mn} \in M^{\perp S}$ ($n \neq 0$), i.e. $u \perp y_{mn}$ and $u \perp y_n$ for all $0 \leq n \leq m$. In view of property (iii) of Lemma 1, $u \perp z_n$ for all $n \in \mathbb{N}_0$. \qed
Theorem 3. A separable pre-Hilbert space $S$ is complete if and only if $E_q(S) = E(S)$.

Proof. We only need to prove sufficiency since the necessity is obvious. Let $(z_n)_{n \in \mathbb{N}}$ be total sequence in $H$. If we assume that $S$ is not complete, we can choose $z_1$ not to be an element of $S$.

Let $h \in S$ such that $(h, z_1) \neq 0$. Then $z'_0 := \frac{z_1}{(z_1, h)} - \frac{h}{(h, h)}$ is not in $S$ since $z_1 \in H \setminus S$ and $h \in S$. Let $z_0 := \frac{z'_0}{E_q(z')}$. Observe that $z_0 \perp z_1$. Let $y_m$ and $y_n$ be chosen according to Lemma 1. We show that $M := \{y_m \mid m \in \mathbb{N}\} \perp S \subseteq E_q(S) \setminus E(S)$. By part (i) of Lemma 2, $M$ is not splitting. If $M$ is not quasi-splitting, then there exists a non-zero vector $u \in H$ such that $u \perp M$ and $u \perp M^\perp$. By part (ii) of the same lemma, it follows that $u \perp z_n$ for all $n \in \mathbb{N}$. This is a contradiction since $(z_n)_{n \in \mathbb{N}}$ is total in $H$. □

Proposition 4. Let $M, N$ be two subspaces of $S$ such that $M \subseteq N$.

(i) $M \in E(S)$ implies $M \in E(N)$.
(ii) Let $N \in E(S)$. Then $M \in E(N)$ if and only if $M \in E(S)$.
(iii) Let $N \in E_q(S)$. Then $M \in E_q(N)$ implies $M \in E_q(S)$.
(iv) Let $N \in E(S)$. Then $M \in E_q(N)$ if and only if $M \in E_q(S)$.

Proof. We leave the proofs of (i) and (ii) to the reader.

(iii) Let $u \in H$, such that $u \perp M$ and $u \perp M^\perp$. Since $M^\perp \cap N, N^\perp \subseteq M^\perp$, we have that $u \perp M^\perp \cap N$ and $u \perp N^\perp$. However, since $N \in E_q(S), u \perp N^\perp$ implies that $u \in N$. On the other-hand, $M \in E_q(N)$ and therefore $u \perp M$ and $u \perp M^\perp \cap N$ implies that $u = 0$, i.e. $M \in E_q(S)$.

(iv) We need to show only one direction since $E(S) \subseteq E(S)$ and therefore we can use (iii) to deduce that $M \in E_q(N)$ implies $M \in E_q(S)$. Let $u \in N \setminus S$ such that $u \perp M$. We show that $u \in M^\perp \setminus N$. Fix $\epsilon > 0$. Since $M \in E_q(S)$, there exists $v \in M^\perp$ such that $\|u - v\| \leq \epsilon$. Since $N \in E(S), v = v_1 + v_2$, where $v_1 \in N$ and $v_2 \in N^\perp$. Observe that $v_1 = v - v_2 \in N \cap M^\perp$ and $u - v_1 \perp v_2$. Therefore

$$\|u - v_1\| \leq \|(u - v_1) - v_2\| = \|u - v\| \leq \epsilon$$

and this completes the proof. □

Corollary 5. If $S$ has an incomplete separable quasi-splitting subspace, then $E_q(S) \neq E(S)$.

Proof. Let $M \in E_q(S)$ be incomplete and separable. If $E_q(S) = E(S)$, then by (iii) of Proposition 4 we get that $E_q(M) \subseteq E_q(S) = E(S)$. It follows from Proposition 4(i) that $E_q(M) = E(M)$ and therefore, in view of Theorem 3, that $M$ is complete, a contradiction. □

Theorem 6. If $S$ has an ONB, then $E(S) = E_q(S)$ if and only if $S$ is complete.

Proof. Let $A$ be an ONB of $S$. In what follows we use the fact that for any $B \subseteq A$ the space $\overline{\text{span}}B \cap S$ is quasi-splitting. Suppose that $S$ is not complete and let $x \in H \setminus S$. There exists a countable subset $A_0$ of $A$ such that $x \in \overline{\text{span}}A_0$. Since $\overline{\text{span}}A_0 \cap S$ is an incomplete separable quasi-splitting subspace of $S$, we have $E_q(S) \neq E(S)$ by Corollary 5. The converse is trivial. □

In Section 2, it is proved that if every algebraic complement of $S$ in $H$ is separable—particularly if $d(H/S) \leq \aleph_0$—then $S$ has an ONB. With this in mind, one deduces the following result which extends [4, Theorem 2.11] (where the result is proved for finite $d(H/S)$).

Corollary 7. If every algebraic complement of $S$ in $H$ is separable, then $E(S) \neq E_q(S)$. In particular, if $0 < d(H/S) \leq \aleph_0$, then $E(S) \neq E_q(S)$.

3. Orthonormal bases in pre-Hilbert spaces

The starting point of this section is the known fact that there are pre-Hilbert spaces having their orthogonal dimension not equal to that of the completion, i.e. $\dim S \neq \dim H$. Such pre-Hilbert spaces admit no ONB, i.e. no MONS of $S$ is a MONS of $H$. Conditions forcing a pre-Hilbert space $S$ to have an ONB are studied in this section.

Theorem 8. Let $M$ be a subspace of $H$ containing $S$ and $A$ a MONS in $S$ such that $\dim A^\perp M \leq \aleph_0$. Then $S$ contains an ONS that is maximal in $M$. 
Proof. Let \((z_n)_{n \in \mathbb{N}_0}\) be a MONS in \(A^{\perp_M}\). Observe that \(A \cup \{z_n \mid n \in \mathbb{N}_0\}\) is a MONS in \(M\). We can use Lemma 1 to obtain a double sequence \((y_{mn})_{m,n \in \mathbb{N}_0, n \leq m}\) in \(S\) and a sequence \((y_n)_{n \in \mathbb{N}_0}\) in \(H\) with properties (i)-(iii) of same lemma. The set
\[ A_0 := \{a \in A \mid \exists m, n \in \mathbb{N}_0, n \leq m, \text{ with } y_{mn} \perp a\} \]
is countable. Let \(B\) be an ONB of
\[ \text{span}\{(y_{mn} \mid m, n \in \mathbb{N}_0, n \leq m) \cup A_0\}. \]
Then \(C := (A \setminus A_0) \cup B\) is an ONS of \(S\) such that \(A \subseteq \text{span}C\) and \(z_n \in \text{span}C\) for all \(n \in \mathbb{N}_0\), i.e. \(C\) is maximal in \(M\). \(\square\)

**Corollary 9.** If every algebraic complement of \(S\) in \(H\) is separable, then \(S\) has an ONB. In particular, if \(\dim(H/S) \leq \aleph_0\), then \(S\) has an ONB.

**Proof.** Let \(A\) be a MONS in \(S\), Since \(A^{\perp_H}\) is contained in an algebraic complement of \(S\), \(A^{\perp_H}\) is separable and therefore \(\dim A^{\perp_H} \leq \aleph_0\). We can now apply Theorem 8 and deduce that \(S\) contains an ONS that is maximal in \(H\); i.e. an ONB of \(H\). \(\square\)

Example 12 below shows that in Corollary 9 the assumption that every algebraic complement of \(S\) in \(H\) is separable cannot be weakened to the assumption that \(S\) has one algebraic complement in \(H\) which is separable.

In what follows, for a closed subspace \(M\) of a Hilbert space \(H\), we denote by \(P_M\) the orthogonal projection of \(H\) onto \(M\).

**Lemma 10.** Let \(U\) be a subspace of a Hilbert space \(V\) and \(A, B\) subsets of \(U\) such that \(U = \text{span}(A \cup B)\). Let further \(V_1 = \text{span}A\) and \(V_2 = V_1^{\perp_U}\). Then the following two conditions are equivalent:

1. \(A\) is an ONS, \(P_{V_2}B\) is total in \(V_2\), \(P_{V_1}A\) is one-to-one when restricted to \(B\) and \(A \cup P_{V_1}B\) is a set of linearly independent vectors;
2. \(A\) is a MONS of \(U\), \(U = V\) and \(A \cup B\) is a set of linearly independent vectors.

**Proof.** (1) \(\Rightarrow\) (2). To show that \(U\) is dense in \(V\) one only has to note that since \(V_1 \subseteq U\) and \(B \subseteq U\), it follows that \(P_{V_2}B \subseteq U\), which in turn implies that \(V_2 \subseteq U\) since \(P_{V_2}B\) is total in \(V_2\).

We now show that \(A\) is a MONS in \(U\). Assume that \(u \in U\) with \(u \perp A\). Say
\[ u = \sum_{i=1}^{n} \lambda_i a_i + \sum_{j=1}^{m} \mu_j b_j, \]
where \(a_i \in A, b_j \in B\) and \(\lambda_i, \mu_j\) are scalars. Then \(P_{V_1}u = 0\) since \(u \perp A\), so that
\[ \sum_{j=1}^{m} \mu_j P_{V_2}(b_j) \in \text{span}\{a_i \mid 1 \leq i \leq n\}. \]
Since \(P_{V_1}\) is one-to-one when restricted to \(B\) and \(A \cup P_{V_1}B\) consists of linearly independent vectors, it follows that \(\mu_j = 0\) for all \(1 \leq j \leq m\), i.e. \(u = \sum_{i=1}^{n} \lambda_i a_i\). Consequently, \(u = 0\) because by assumption \(u \perp A\).

(2) \(\Rightarrow\) (1). To see that \(P_{V_2}B\) is total in \(V_2\), one should observe that
\[ V_1 \oplus V_2 = V = U \subseteq V_1 \oplus \text{span}(P_{V_2}B) = V_1 \oplus \text{span}(P_{V_2}B), \]
hence \(V_2 = \text{span}(P_{V_2}B)\). We now show that \(P_{V_1}\) is one-to-one when restricted to \(B\), and that the set \(A \cup P_{V_1}B\) consists of linearly independent vectors. Consider a linear combination
\[ \sum_{i=1}^{n} \lambda_i a_i + \sum_{j=1}^{m} \mu_j P_{V_1}(b_j) = 0, \]
then
\[ \sum_{i=1}^{n} \lambda_i a_i + \sum_{j=1}^{m} \mu_j b_j = \sum_{i=1}^{n} \lambda_i a_i + \sum_{j=1}^{m} \mu_j P_{V_1}(b_j) + \sum_{j=1}^{m} \mu_j P_{V_2}(b_j) = P_{V_2}\left(\sum_{j=1}^{m} \mu_j b_j\right) \subseteq U \cap V_2 = \{0\}, \]
where the last equality follows from the fact that \(A\) is a MONS of \(U\). Consequently, \(\alpha_i = \mu_j = 0\) for every \(1 \leq i \leq n, 1 \leq j \leq m\) since \(A \cup B\) consists of linearly independent vectors. \(\square\)

**Remark 11.** Observe that if any (and hence both) of the conditions of Lemma 10 are satisfied, then the following statements hold:

(i) \(V_2 \cap U = \{0\}\);
(ii) If \(V_1 = \text{span}(A \cup P_{V_1}B)\), then \(V = U + V_2\), and therefore \(V_2\) is an algebraic complement of \(U\);
(iii) If \(V_2 = \text{span}(P_{V_2}B)\), then \(V = U + V_1\).

**Example 12.** Given two cardinals \(\kappa, \tau\) satisfying \(\aleph_0 \leq \kappa < \tau \leq \kappa^{\aleph_0}\), let \(V_1, V_2\) be two Hilbert spaces with \(\dim V_1 = \kappa\), \(\dim V_2 = \tau\) and let \(V = V_1 \oplus V_2\). Then \(V\) contains a dense subspace \(U\) satisfying:
(i) \( \dim U = \kappa \) and \( \dim \overline{U} = \tau \);
(ii) \( V_2 \) is a non-separable algebraic complement of \( U \);
(iii) \( V_1 \) contains an algebraic complement of \( U \). In particular, if \( \kappa = \aleph_0 \), then \( U \) has a separable algebraic complement.

**Proof.** First observe that
\[ \kappa^{\aleph_0} \leq \tau^{\aleph_0} \leq (\kappa^{\aleph_0})^\tau = \kappa^{\aleph_0}, \]
hence \( \kappa^{\aleph_0} = \tau^{\aleph_0} \). Therefore \( d(V_1) = \kappa^{\aleph_0} = \tau^{\aleph_0} = d(V_2) \).

Let \( A \) be an ONB of \( V_1 \) and \( C \) be a Hamel basis of \( V_1 \) containing \( A \). Let further \( D \) be a Hamel basis of \( V_2 \). Since \( |D| = |C \setminus A| \), there is a bijection \( g \) from \( D \) onto \( C \setminus A \). Define
\[ B := \{ g(d) + d \mid d \in D \}. \]
From Lemma 10 ((1) \( \Rightarrow \) (2)) and Remark 11 it is clear that the subspace \( U := \text{span}(A \cup B) \) has the desired properties. \( \square \)

Let us note that in the above example we have a pre-Hilbert space whose dimension is strictly less than the dimension of its completion and therefore cannot contain an ONB. We now modify the above example to construct a pre-Hilbert space that has no ONB, but its dimension agrees with the dimension of its completion.

**Example 13.** Let \( V \) and \( U \) be as in Example 12 and \( V_0 := V \oplus H_0 \) be the direct sum of \( V \) and a Hilbert space \( H_0 \) with \( \dim H_0 = \tau \). Then \( U_0 := U \oplus H_0 \) is a dense subspace of \( V_0 \) not containing an ONB, and \( \dim U_0 = \dim V_0 \).

**Proof.** Evidently \( \dim U_0 = \dim V_0 = \tau \). We show that \( U_0 \) does not have an ONB. First observe that if \( E \) is a total subset of \( U_0 \), then \( P_V E \subseteq U \) and \( P_V E \) is total in \( V \). This means that \( |P_V E| \geq \dim V = \tau \).

On the other hand, if \( E \) is an ONS contained in \( U_0 \), then \( \dim U \geq |P_V E| \). To see this, observe that if \( A \) is a MON of \( U \), then for any \( a \in A \), the set \( E_a := \{ e \in E \mid e \not\in a \} \) is countable. Moreover, since \( A \) is a MON in \( U \),
\[ P_V \left( \bigcup_{a \in A} E_a \right) = P_V E, \]
and therefore,
\[ |P_V E| = \left| P_V \left( \bigcup_{a \in A} E_a \right) \right| \leq \left| \bigcup_{a \in A} E_a \right| \leq |A| \cdot \aleph_0 = \dim U = \kappa. \] \( \square \)

How much bigger than \( \dim S \) can \( \dim H \) be? Upper bounds for \( \dim H \) in terms of \( \dim S \) and \( d(S) \) are given in the next theorem.

**Theorem 14.** Let \( S \) be a pre-Hilbert space and \( H \) be its completion. Then
\[ \dim H \leq d(S) \leq (\dim S)^{\aleph_0}. \]

**Proof.** We may assume that \( H \) has infinite dimension. Let \( A \) be a MONS in \( S \) and \( B \) a subset of \( S \) such that \( A \cup B \) is a Hamel basis of \( S \). Define \( H_1 := \text{span} A \) and let \( C \) be an ONB of \( H_2 := H_1^H \). We first show that
\[ |C| \leq \begin{cases} d(S), \\ (\dim S)^{\aleph_0}. \end{cases} \]
For any \( b \in B \), the set
\[ C_b := \{ c \in C \mid c \not\in P_{H_2}b \} \]
is countable. By Lemma 10 we know also that \( P_{H_2}b \) is total in \( H_2 \). Hence, for every \( c \in C \) there exists \( b \in B \) such that \( c \not\in P_{H_2}b \), i.e.
\[ C = \bigcup_{b \in B} C_b. \]
Hence
\[ |C| \leq |B| \cdot \aleph_0 \leq d(S) \cdot \aleph_0 = d(S). \]

For the second inequality, observe that by Lemma 10, \( P_{H_1} \) is one-to-one when restricted to \( B \) and \( P_{H_1} \) consists of linearly independent vectors. Hence
\[ |C| \leq |B| \cdot \aleph_0 = |P_{H_1}B| \cdot \aleph_0 \leq \dim(H_1) \cdot \aleph_0 = (\dim H_1)^{\aleph_0} = (\dim S)^{\aleph_0}. \]

Consequently,
\[ \dim H \leq \dim S + |C| \leq \begin{cases} \dim S + d(S) = d(S), \\ \dim S + (\dim S)^{\aleph_0} = (\dim S)^{\aleph_0}. \end{cases} \]
and therefore,
\[ d(S) \leq d(H) = (\dim H)^{\aleph_0} \leq ((\dim S)^{\aleph_0})^{\aleph_0} = (\dim S)^{\aleph_0}. \]

**Remark 15.** In view of Theorem 14 it is natural to ask whether there exists a Hilbert space \( V \) having a dense subspace \( U \) with \( \dim V = \tau, \dim U = \kappa \) and \( d(U) = \lambda \), where \( \kappa, \tau, \lambda \) are cardinal numbers satisfying \( \aleph_0 \leq \kappa \leq \tau \leq \lambda \leq \aleph_0^{\aleph_0} \). An easy modification of Example 12 shows that the answer is in the affirmative. Indeed, choose \( V_1, V_2, V, A, C \) as in the proof of Example 12 and let \( D \) be a total, linearly independent subset of \( V_2 \) with cardinality \( \lambda \). Since \( |D| = \lambda \leq \aleph_0^{\aleph_0} = |C \setminus A| \), there is an injection \( g \) from \( D \) into \( C \setminus A \). Define \( B := \{ g(d) + d \mid d \in D \} \) and \( U := \text{span}(A \cup B) \). Then \( U \) and \( V \) have the desired properties.

**References**