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Extended Teleparallel Cosmology

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Statement of Originality

Chapter 1: In this chapter the motivation for the dissertation is given. In addition, the nature of the problem is defined and how it naturally leads to the work done.

Chapter 2: In this chapter, the scientific context in which the research of this dissertation is being conducted. This is an overview of the literature of teleparallel gravity.

Chapter 3: This chapter is based on the publication [1]. The tensor perturbations of the BDLS theory are calculated thus obtaining the gravitational wave propagation equation which is further discussed as of how is affected by recent cosmological observations. A discussion about possible models being revived from standard Horndeski gravity follows.

Chapter 4: This chapter is based on the publication [2]. This is also an extension of [1]. The perturbations around the Minkowski spacetime for the BDLS theory are calculated for all scalar-vector-tensor components. The number of propagating degrees of freedom is calculated. Using these results the polarizations modes are also probed for the BDLS theory.

Chapter 5: This chapter is based on the publication [3]. The $f(T,B)$ gravity is studied against the cosmological background. More specifically, the scalar-vector-tensor perturbations are probed along with the matter density equation at the sub-horizon limit. This lead to a branching expression of the effective gravitational constant.

Chapter 6: This chapter is based on the publication [4]. A teleparallel analogue of the generalized Proca theory is constructed in an systematic way. The Friedmann equations are also calculated for this new model as a cosmological application and contrasted against the generalised Proca theory.

Chapter 7: In this chapter the content presented in this Transfer Report, is summarized. Further, the wider impact of this work in the community is discussed. Finally, we consider a future plan as an extension of this work.

PUBLICATIONS

[1] Bahamonde S, Dialektopoulos KF, Gakis V, Levi Said J. Reviving Horndeski theory using teleparallel gravity after GW170817. Phys Rev D. 2020;101(**8**):084060.

[3] Bahamonde S, Gakis V, Kiorpelidi S, Koivisto T, Levi Said J, Saridakis EN. Cosmological perturbations in modified teleparallel gravity models: Boundary termextension. 2020 9; Eur.Phys.J.C 81 (2021) 1, 53

[2] Sebastian Bahamonde, Maria Caruana, Konstantinos F. Dialektopoulos, Viktor Gakis, Manuel Hohmann, Jackson Levi Said, Emmanuel N. Saridakis, Joseph Sultana, Gravitational-wave propagation and polarizations in the teleparallel analog of Horndeski gravity, Phys. Rev. D 104, 084082 – Published 21 October 2021

[4] Gianbattista-Piero Nicosia, Jackson Levi Said, Viktor Gakis, The European Physical Journal Plus volume 136 (2021), Eur.Phys.J.Plus 136 (2021) 2, 191

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Abstract

Over the last 100 years General Relativity (GR) has been an extremely successful theory of gravity and also considered as a fundamental theory. GR has been extensively used to study the Universe, however it turned out that it is not a complete theory as more observations were available. This resulted in GR not being able to explain Dark Energy (DE) and Dark Matter (DM), assuming that these are the missing components in explaining the late time cosmology. Incorporating DE and cold DM into GR, the ΛCDM model was created that solved most of observational cosmological problems. So far, DE and DM although kind of natural hypotheses, are not globally accepted concepts and have not been directly observed. On top of that there is a new sector of cosmology called gravitational wave astronomy which is based on the dynamics of gravitational waves and the plethora of relevant datasets that can be immediately used to constrain gravitational models. All the aforementioned problems serve only part of the reasons that lead to modify GR. The process of modifying GR is not that simple. At times this can lead to solving problems but also creating new ones. As such in depth studies of these modified theories must be always performed. From these studies a lot of useful insights were gained regarding the foundations and cosmology of GR. It should be noted that Einstein himself first modified GR in an attempt to unify electromagnetism with gravity via torsion by setting curvature to zero. This is exactly the birth of teleparallel theories of gravity (TG). In this thesis, the TG framework is introduced starting from its motivation to its technical details and how to modify it. Moreover, the teleparallel analogue of Horndeski theory is presented and probed against multimessenger events of GW170817 [5] and GRB170817A [6]. Its polarization modes and degrees of freedom are also extensively studied and compared with Horndeski theory. In this direction the $f(T,B)$ gravity is also probed against the multimessenger events and in general studied perturbatively in the cosmological background. Finally, a version of a teleparallel equivalent of the generalised Proca theories is constructed and its Friedmann equations are calculated as a first application to cosmology.

Acronyms

BDLS Bahamonde, Dialektopoulos, Levi Said. 7, 21, 44, 54, 58, 61, 62, 157–160, 162

CDM Cold Dark Matter. 3

dof Degrees of Freedom. 4, 12, 16, 27, 28, 34–37, 63–65, 74, 76, 78, 80, 86, 102, 104, 106, 107, 109–113, 117, 118, 120, 123, 138, 139, 144, 154–156

EMT Energy Momentum Tensor. 4, 27, 36, 121, 162

FLRW Friedmann–Lemaître–Robertson–Walker. 31–35, 38, 40, 44, 57, 58, 69, 113, 115, 116, 119, 139, 151, 157–159, 161

GP Generalized Proca. 8, 138–140, 143, 144, 151–155, 157, 160–162

GR General Relativity. 1–7, 9, 11, 18, 19, 22, 24, 27, 28, 31, 34, 41, 43, 45–47, 57, 59, 62–64, 76, 77, 102, 107, 110, 138, 156–158, 162

GW Gravitational Wave(s). vi, 3, 5–8, 40–44, 58, 59, 62–64, 74, 76, 104, 105, 107, 109, 111, 113, 117, 118, 134, 159, 161, 162

GWPE Gravitational wave Propagation Equation. 40, 41, 43, 44, 57, 58, 113, 117, 118, 135, 157–159, 162

LC Levi Civita. 10, 11, 18–20, 22, 24, 34, 43, 45, 46, 57, 62, 142, 156, 157

LLT Local Lorentz Transformation. 15–17, 27, 54, 61, 156

pdof Propagating Degrees of Freedom. vii, 7, 8, 28–32, 57, 63, 64, 69, 78, 80, 83, 86, 100, 102–104, 112, 113, 119, 135, 158–161

RHS Right hand side. 18, 19, 49, 122

SVT Scalar-Vector-Tensor. 35–37, 63, 70, 72, 104, 106, 110, 117, 120, 134, 135, 158, 159

TEGR Teleparallel Equivalent of General Relativity. 4, 20, 22, 24, 27, 28, 35, 114, 115, 156, 157

TG Teleparallel Gravity. 3–9, 14, 16, 17, 19, 20, 27, 40, 44, 50, 54, 57, 107, 109–111, 113, 114, 134, 137–139, 143, 144, 154–158

wrt with respect to. 5, 10, 13, 16–18, 24–27, 35, 41, 46, 48–50, 52, 54, 56, 62, 106, 114, 118, 145, 157, 160

Chapter 1

Introduction

The General Theory of Relativity [7] along with quantum field theory [8], are accepted to be two fundamental theories that serve as foundations for modern physics. General Relativity (GR), since its conception by Einstein in 1915 [9], was founded upon differential geometry, which at the time was not well understood, or that popular in physics. Nevertheless, this influenced the way fundamental theories were founded later on. As a consequence, differential geometry, or rather geometry in general, was more and more involved in setting up physical theories. Two such great examples are gauge theory and string theory. In a way, GR has not only allowed for a deeper understanding of gravity but also revolutionzed field theories in their totality.

The introduction of GR was immediately and soon after, followed by alternatives and modifications in order to further extend its validity and even unify the forces. A few major examples include Weyl's scale independent theory [10] and Kaluza and Klein theory [11] that includes higher dimensional spacetimes. All of these examples, had also influenced Einstein himself who also tried to unify electromagnetism with gravity under the same geometrical framework as a potential groundwork for the unification of all forces. Hence, there were already quite a few theoretical reasons why GR should be extended already.

Nowadays, the validity of GR has been revisited due to new observations. In particular, if

Einstein's theory is to hold at cosmological scales then the Universe should be filled, to a substantial degree, with dark matter on top of the regular matter that constitutes galaxies. In addition, in order to explain the observed accelerating expansion of the Universe, dark energy has been called as the agent. Both dark matter and dark energy do not interact with known forces except gravity. In fact, if dark energy is the correct explanation then, 96% of the Universe must be occupied by dark energy densities that are invisible to electromagnetism. As a matter of fact, assuming ΛCDM cosmology, the late-Universe parameters are: Hubble constant $H_0 = (67.4 \pm 0.5)\mathrm{kms^{-1}Mpc^{-1}}$; matter density parameter $\Omega_m = 0.315 \pm 0.007$; and matter fluctuation amplitude $\sigma_8 = 0.811 \pm 0.006$ as calculated by the Planck 18 collaboration [12]. This whole scenario of the dark constituents is called the dark universe scenario and if it holds then GR needs to be extended in order to incorporate it. Henceforth, this is an observational indication that extensions or modifications to GR are a way of solving this issue.

On top of that, there is the quantum aspect of gravity which is yet to be fully understood. At very small scales, GR seems to break down which leads to not being able to explain singularities caused by black holes. Ultimately, this is attributed to the fact that gravity is not renormalizable which means that infinities in the Feynman loops cannot be absorbed by re-definitions. Failure of renormalizability, in more physical terms, means that the theory cannot produce well defined quanta in order to describe forces properly. This is a fundamental disagreement with the framework of quantum field theory where forces act locally via the interchange of well-defined quanta.

If the theoretical, observational and quantum aspects are taken into account in a unified manner it is clear that GR although highly successful so far, is not the theory of gravity. This does not mean it is wrong but is rather a very good approximation of how gravity should behave at low energies and intermediate scales which lie between the quantum world and the Universe. Using this idea as a compass, modifications or extensions to GR can be made in a much more accurate and specific manner in order to probe different aspects of gravity or solve a problem at time.

Nevertheless, for cosmology which is the main focus of this dissertation, GR seems to be in good accordance with the phenomenology just by adding slight modifications to the matter sector. The globally accepted models in this direction, include the concordance model flat ΛCold Dark Matter (CDM) and inflation. In the ΛCDM model, the Universe is 13.7 billion years old and made up of 4% baryonic matter, 23% cold dark matter (abbreviated CDM) and 73% dark energy which is attributed to the cosmological constant Λ. The Hubble constant for this model is $67,4 \pm 0.5$ km/s/Mpc and the density of the Universe is very close to the critical value for re-collapse. These values were derived from the Planck 18 measurements of the CMB anisotropies [12] that GR is the correct theory of gravity at cosmological scales. Accordingly, inflation is a theory of exponential expansion of space during the early Universe. In this theory, the inflationary epoch lasted from 10^{-36} seconds after the conjectured Big Bang singularity to some time between 10^{-33} and 10^{-32} seconds after the singularity. Succeeding the inflationary period, the Universe continued expanding, but at a slower rate. The acceleration of this expansion began after the universe was already over 7.7 billion years old, due to dark energy.

These cosmological theories have the potential downside of being accompanied by extra exotic particles in order to be valid,i.e. dark matter and the inflaton which have not been observed in particle physics experiments. The exact particle mechanism behind the postulated particles is still unknown. These particles still remain elusive to observations in spite the great advances in technology and in particle physics. This may indicate, that after all, this is not a problem of introducing extra particles but rather modifying the gravitational sector in a different way.

Modifying, extending or even going beyond GR has been explored over the last decades with high intensity [13, 14, 15] also due to the recent observation of GW. The motivation for the plethora of modifications is quite rich and it includes from phenomenological to theoretical reasons and observations on top of that. One particular approach that will be the focus of this dissertation is the so called teleparallel gravity (Teleparallel Gravity (TG)), which has gained a lot of momentum the last decade. The origin of TG dates back to Einstein's first steps in force unification. The main idea behind TG is that gravity is expressed through torsion only and the fundamental variables are the frame fields instead of the metric. The geometry by such configuration is assumed to be based on the absolute parallelism [16], where the frames can be transported in a parallel manner **globally**. This property is only local if there is curvature. In fact, the simplest theory based on this, torsionful geometry that exactly reproduces GR dynamically, is the **teleparallel equivalent of general relativity** (Teleparallel Equivalent of General Relativity (TEGR)). This means that TEGR is indistinguishable from GR regarding any experiment of classical origin. Hence, TG is an alternative geometrical framework that can reproduce GR in the form of TEGR and can be further modified in order to probe cosmology.

The initial motivation of Einstein in order to conceive the first form of TG was not observational but rather purely theoretical, i.e, unifying the forces. Although the idea of absolute parallelism was very promising it could not work even at the level of unifying just electromagnestism and gravity. This is due to the fact that the maximum available Degrees of Freedom (dof) in TG are sixteen, exactly as much as the components of the tetrad field since the connection is a pure gauge choice. Within this scheme, gravity was supposed to be represented by ten components of the tetrad, which are the metric dof, and the rest six would correspond via a connection form to the electromagnetism. Nevertheless, after three years of research between 1928 and 1931, Einstein realized that this was not possible. In the end, the relation between the connection, that contained the six dof, and electromagnetism was not feasible. The problem was deeply rooted to the fact that these specific six dof were actually related to the Lorentz group which is directly linked to the local Lorentz freedom of gravity. After this realization of Einstein, the idea of teleparallism was abandoned for quite some time, until 1961 by Møller [17], Pellegrini and Plebaski [18] where the idea of teleparallel Lagrangians was revisited. Specifically, Møller on a follow up work [19] a few years later introduced the first notion of well defined Energy Momentum Tensor (EMT) of gravitation which eventually led to the realization that the standard EMT of GR is only quasi-local [20].

The modern status of TG started to form in the subsequent years by works of Hehl, Heyde and Kerlick in Ref. [21] and also from Hayashi and Nakano in Ref. [22], where they implemented the idea of teleparallelism in the form of a gauge theory of translations of the Local Lorentz Group. These early works on TG, were of foundational nature and as such, they did not reach a wide audience at the time.

The idea of teleparallelism in its modern formulation, was shaped in 1995, among others, by Aldrovandi and Pereira [23] in a comprehensive manner. In their work all the geometrical ideas of TG were revisited, from elementary considerations of its gauge theoretic nature up to quantum aspects. This work served as the modern foundation for TG from 1995 onward.

The rise of the modified theories of gravity, in the last decade, revived TG framework. The modifications were mostly needed in order to incorporate the concepts of dark energy and dark matter [24, 25, 26] which were supported by the discoveries of late-time accelerating expansion of the Universe and the excess mass found in galactic rotation curves. The most popular and direct modification of GR, which is described just by the Ricci scalar $\mathring{R}$, is the $f(\mathring{R})$ model [27, 28]. The most popular model of modified TG theories of gravity was $f(T)$, which was introduced in Ref. [29] by Ferrari and Fiorini. This exact form was also revisited by Linder in Ref. [30].

The $f(T)$ models served the same purpose as the well known $f(\mathring{R})$ models [31, 32, 27], with the difference that the field equations were of second order with respect to (wrt) the tetrad field. This lower order nature quickly made the $f(T)$ theories very attractive, since the counterpart $f(\mathring{R})$ models are of fourth order. This was the status regarding the most famous modified theories up to 2010, since then the range of modified TG theories has since expanded by including all sorts of potential ingredients like nonminimal couplings to matter, scalar fields, vector fields, tensor fields and even boundary terms (for an exhaustive review see [33]). The concept of modified theories gained a lot of attention around 2010, mostly regarding the dark universe scenario and the observation of GW.

In 2015, GW were observed for the first time [34] verifying once again GR. This has also set bounds on the polarization states on these GW that occur from specific known sources [35]. By using high precision measurements, the propagation speed of these waves is equal to the speed of light [36, 37, 38, 39, 40], verifying again GR. In contrast, modified theories of gravity need not predict light speed propagation for their GWs without imposing further conditions on their form. It particular, it is known that $f(T)$ gravity trivially satisfies these constraints [41, 42, 43, 44, 45, 46, 47, 48]. Nonetheless, there has not been performed a systematic investigation of GWs predicted by the rest modified \TG theories.

Nowadays, considering observational astrophysics and cosmology there has been considerable progress. This is due to the fact that there are currently available cosmological data sets measured by unprecedented statistical precision which can be used to test or further constrain gravitational theories. The framework upon this process happens is called precision cosmology, which has been around since the 90s. Due the plethora of different data sets, from different sources, precision cosmology has gained a lot of attention in recent years. This attention will only be amplified in the years to come due to the more data sets from new detectors and observatories.

The plethora of data sets, has showed that there are tensions between measurements in the early and the late universe. This might as well be a first hint of new physics beyond the cosmological standard model. In particular, the clustering of large scale structure and the current value of the Hubble parameter show intriguing discrepancies between measurements in the early and late universe. In particular, the most famous tensions at the moment are the H_0, S_8 problems [49, 50, 51, 52, 53]. Using precision cosmology, allows testing a model if it provides a description of the observed and well-tested late-time dynamics [54, 55]. GR, currently, cannot provide adequate answers for either the tensions nor late-time dynamics. Modified TG theories are able to alleviate tensions with the tensions, the late-time dynamics as well as ambiguities in early Universe physics [33].

Another immensely useful tool that is heavily used currently, in cosmology, is perturbation theory. The idea behind it, is that small deviations are applied in the background solutions of a theory in order to study their behaviour compared to the background state of the solution. The first consistent formulation was introduced in Ref. [56, 57]. Throughout the years, it became a standard tool in probing cosmology [13]. Specifically, the cosmological perturbations offer a much deeper understanding of the evolution of the Universe [58], since it is now possible to compare the theoretical aspects of the perturbed theory against observational data. This data can be linked with either the cosmic evolution or even with properties of the GW in order to constrain models and ensure that there are healthy pdof [13].

One of the most important and well-studied theories to date is the Horndeski gravity due to its potential of solving fundamental cosmological problems like the dark universe scenario [59, 60, 61], the H_0, S_8 tensions and even inflation [62, 63, 15]. Nevertheless, it was severely constrained by [5] and [6], forcing a very wide class of models to be abandoned. Extending Horndeski gravity by using the framework of TG, the Bahamonde, Dialektopoulos, Levi Said (BDLS) theory was introduced in [64]. In this dissertation, it will be shown that the BDLS theory is not severely affected by these constraints, in the cosmological background. On top of that, a full study of the GW predicted by BDLS theory will be performed, in the Minkowski spacetime in order to obtain information such as the speed of propagation and the polarization content.

In fact, $f(T)$ gravity has been well studied perturbatively in Minkowski and cosmological backgrounds [44, 65, 66, 67], nonetheless there is still an open problem with the number of pdof since only two are found in these backgrounds. In general, from Hamiltonian analyses one should expect at least five pdof [68, 69, 70]. Hence, only having two pdof in $f(T)$ gravity, which is a highly non-linear theory, the theory just reduces to GR which signals to missing pdof. This is why, in this dissertation more general theories like $f(T, B)$ gravity [33] will be probed perturbatively in cosmological backgrounds. These type of theories serve also as a medium between teleparallel theories and curvature based theories since $f(T, B) = f(-T + B) = f(\mathring{R})$ [27]. Thus, $f(T, B)$ will be probed perturbatively in a cosmological background, in order to extract information about the number and nature of pdof. Knowledge about the pdof leads to complete understanding about the GW and the effective gravitational constant G_{eff} predicted from the theory.

Inspired from the standard model of particle physics, which is founded upon abelian and non-abelian vector fields that carry the gauge interactions, the idea of scalar tensor theories can be generalized by including vector fields. The prototype theory of this type is Maxwell's theory, where the vector potential carries the electromagnetic interactions. Generalizing further by including mass interactions the Proca theories are generated. Adding also self-interactions of the vector field, the most general theory including a vector field is created which is called Generalized Proca. This theory can also be considered as a next step generalization of Horndeski theory. However, not all vector interactions have scalar counterparts which means that there are pure vector interactions. Hence adding vector fields actually adds new unique interactions which cannot be generated from scalar fields. This apparently has an important effect at the quantum level, since only the vector interactions with scalar counterparts are stable.

The concept of the vector field can also be used as generic means of shedding more light in describing the cosmological evolution of the Universe. So far, scalar fields were more favoured due to their simplicity [71, 72], as external field candidates. Nevertheless, vector fields are the first step towards generalizing the scalar framework and as such the cosmic evolution can be studied through bosonic vector fields. In this way, vector fields could impact the relation of ΛCDM with particle physics in a more straightforward manner than scalars. In addition, these vector-tensor type of frameworks have been proven to support isotropic solutions along with screening mechanisms [73, 74]. In this sense, it is natural to extend the Generalized Proca (GP) theories using TG as the basis, just like it in the case of Horndeski gravity. To this end, a detailed construction of the teleparallel analogue of TG theory will be probed and as a first application of its cosmology the Friedmann equations will be calculated.

Chapter 2

New physics beyond TEGR

In this section, the foundations of the so called metric affine geometry [75] will be introduced. Then, the focus will be shifted to the specific subcase of the TG. In this teleparallel type of geometry, although, curvature is zero it is not really flat. This comes with various intricacies which will be explored from first principles. These intricacies have shaped the way the modern theories of TG work.

GR as a starting point [7], can be considered as the prototype framework of a spacetime structure. The dynamics of the spacetime structure are studied using geometrodynamics, which is a framework that considers the dynamics of spacetime via its geometrical point of view. On top of that, there is also the causal structure that needs to be taken into account [76]. Causality must be a part of the overall mathematical structure since it must be ensured that, an effect cannot occur from a cause that is not in the back (past) light cone of that event. Similarly, a cause cannot have an effect outside its front (future) light cone. The causal structure is comprised by the causality (set theoretical) and the time orientation on the tangent space (arrow of time). Endowing the manifold with a way to measure distances of points and inner products of vectors, then these operations are solely related to the metric g. The metric in general is a completely separate structure from the causality. In what follows, only the metric tensor $g_{\mu\nu}$ will be of interest which is the inner product of vector fields.

In order to covariantize the theory and work with tensors which are objects that are well posed, i.e. do not depend on coordinate or frame choices then the introduction of the so called connection is vital. This covariantization scheme is achieved by relating the neighbourhood tangent spaces that correspond to each point of the manifold [7]. The way the tangent spaces are related to each other has a direct effect on the shape of the manifold. For example what we call as the standard 3 dimensional sphere attains the standard spherical shape only using the Levi Civita connection. Otherwise, it is only a sphere as an algebraic relation.

In practice, the connection upgrades the usual operation of partial differentiation into a covariant procedure carried out by a new operator called the covariant derivative and denoted as $\hat{\nabla}$. A geometric intuition of the role of the connection is that, it is responsible for the shape of the manifold. The connection as a structure is completely independent of the metric or causality. A general linear connection denoted as $\hat{\Gamma}$ can be fully specified by its coefficients $\hat{\Gamma}^{\lambda}{}_{\mu\nu}$ [75]. This connection induces its covariant derivative $\nabla_\mu \rightarrow \partial_\mu \pm \hat{\Gamma}^{\lambda}{}_{\mu\nu}$ where we add one correction coefficient $\hat{\Gamma}^{\lambda}{}_{\mu\nu}$ for each of the indices of the tensor that it is applied to.

As a matter of fact the most famous linear connection is the Levi Civita (LC) connection for which the connection coefficients are uniquely called the Christoffel symbols and denoted as $\mathring{\Gamma}^{\lambda}{}_{\mu\nu}$. Specifically, they are defined as

$$\mathring{\Gamma}^{\mu}{}_{\nu\rho} := \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right). \tag{2.1}$$

Any quantity X that is calculated wrt the LC connection will be denoted with an overcircle as $\mathring{X}$. This connection is the most famous one due to its simplicity and uniqueness. It is the only connection that can be expressed uniquely through the metric completely. This is achieved by demanding that the torsion tensor is trivial which is translated as $\mathring{\Gamma}^{\lambda}{}_{\mu\nu} \equiv \mathring{\Gamma}^{\lambda}{}_{(\mu\nu)}$ and it also also metric compatible $\mathring{\nabla} g \equiv 0$. It should be stressed that the LC is the unique connection that completely depends on the metric tensor along with the rest aforementioned properties.

2.1 Affine Connections

The most general affine connection $\hat{\nabla}$ [75], in a 4 dimensional manifold M, is fully described by its 4^3 = coefficients $\hat{\Gamma}^{\lambda}{}_{\mu\nu}$. In addition, the metric tensor $g_{\mu\nu}$, in 4 dimensions, since it is symmetric it is fully described by 10 components. The metric and the connection are enough in order to fully describe the geometry of M.

The connection coefficients $\hat{\Gamma}^{\lambda}{}_{\mu\nu}$ can be used to define 3 important tensors, the Riemann tensor [7]

$$\hat{R}^{\mu}{}_{\nu\rho\sigma} := 2\partial_{[\rho}\hat{\Gamma}^{\mu}{}_{|\nu|\sigma]} + 2\hat{\Gamma}^{\mu}{}_{\beta[\rho}\hat{\Gamma}^{\beta}{}_{|\nu|\sigma]} = \partial_{\rho}\hat{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\hat{\Gamma}^{\mu}{}_{\nu\rho} + \hat{\Gamma}^{\mu}{}_{\tau\rho}\hat{\Gamma}^{\tau}{}_{\nu\sigma} - \hat{\Gamma}^{\mu}{}_{\tau\sigma}\hat{\Gamma}^{\tau}{}_{\nu\rho}\,, \tag{2.2}$$

the torsion tensor

$$\hat{T}^{\lambda}{}_{\mu\nu} := -2\hat{\Gamma}^{\lambda}{}_{[\mu\nu]} = \hat{\Gamma}^{\lambda}{}_{\nu\mu} - \hat{\Gamma}^{\lambda}{}_{\mu\nu}\,, \tag{2.3}$$

and the non-metricity tensor

$$\hat{Q}_{\mu\nu\rho} := \hat{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \hat{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \hat{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}\,. \tag{2.4}$$

In general there is no, a priory, reason that one of the tensors (2.2)-(2.4) is trivial, unless we specifically impose further conditions. The most important combinations one deduce are

- GR spacetime$\Longleftrightarrow$ $\{\mathring{R}^{\mu}{}_{\nu\rho\sigma} \neq 0,\, \mathring{T}^{\lambda}{}_{\mu\nu} \equiv 0,\, \mathring{Q}_{\lambda\mu\nu} \equiv 0\}$ which describes the LC connection $\mathring{\nabla}$ whose coefficients are in Eq. (2.1)
- General Teleparallelism $\Longleftrightarrow$$\{\, R^{\mu}{}_{\nu\rho\sigma} \equiv 0,\, T^{\lambda}{}_{\mu\nu} \neq 0,\, Q_{\lambda\mu\nu} \neq 0\,\}$ for a general teleparallel connection ∇.

– Torsional Teleparallelism with $Q_{\lambda\mu\nu} \equiv 0 \Rightarrow$ Teleparallel connection [77].

– Non-Metricty Teleparallelism with $T^{\lambda}{}_{\mu\nu} \equiv 0 \Rightarrow$ STG[1] connection [78].

- Minkowski space$\Longleftrightarrow$\{ $\hat{R}^{\mu}{}_{\nu\rho\sigma} \equiv 0$, $\hat{T}^{\lambda}{}_{\mu\nu} \equiv 0$, $\hat{Q}_{\lambda\mu\nu} \equiv 0$ \}. In this case the connection is fixed up to diffeomorphisms and hence it does not carry any gravitational dof.

If there is a metric tensor the connection coefficients $\hat{\Gamma}^{\rho}{}_{\mu\nu}$ can be split into 3 parts as follows [76]

$$\hat{\Gamma}^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} + \hat{K}^{\rho}{}_{\mu\nu} + \hat{L}^{\rho}{}_{\mu\nu} \,, \tag{2.5}$$

where the contortion tensor has been defined as

$$\hat{K}^{\mu}{}_{\nu\rho} := \frac{1}{2}\left(\hat{T}_{\nu}{}^{\mu}{}_{\rho} + \hat{T}_{\rho}{}^{\mu}{}_{\nu} - \hat{T}^{\mu}{}_{\nu\rho}\right) \,, \tag{2.6}$$

and the disformation tensor

$$\hat{L}^{\mu}{}_{\nu\rho} := \frac{1}{2}\left(\hat{Q}^{\mu}{}_{\nu\rho} - \hat{Q}_{\nu}{}^{\mu}{}_{\rho} - \hat{Q}_{\rho}{}^{\mu}{}_{\nu}\right) \,. \tag{2.7}$$

The metric tensor is actually vital in the definition of the contortion tensor (2.6) although it is only constructed superficially from the torsion tensor.

2.2 Tetrad field

The matrix representation of metric tensor, $g_{\mu\nu}$ in 4 dimensions, is invertible and smooth which allows for it to be diagonalizable. This effectively means that there is always a frame in which the metric can be diagonalized, at least locally. Such general frames are introduced by the tetrad field acted as a map between $\mathbb{R}^n$ and the tangent bundle of the manifold M(the space of all tangent spaces of M). The difference in the case of gravity is that $\mathbb{R}^n$ is upgraded to Minkowski space.

[1] Symmetric Teleparallel Gravity

Since the diagonalized metric can have any form, it will be restricted to only coordinate systems or frames that produce specifically the Minkowski metric (this is just a convention) [7]. The components of the coordinate/non-coordinate frame will be denoted as $e^A{}_\mu$, where Latin letters $A, B, C, ..$ run from 0 to 3 and denote the Lorentz indices. Hence, diagonalizing the metric

$$g_{\mu\nu} \equiv \eta_{AB} e^A{}_\mu e^B{}_\nu \,, \tag{2.8}$$

where

$$\eta_{AB} := \mathrm{diag}(1, -1, -1, -1) \,, \tag{2.9}$$

denotes the Minkowski metric. There is also the dual space picture of the inverse metric $g^{\mu\nu}$ which is generated from the inverse tetrad field $E_A{}^\mu$

$$E_A{}^\mu e^A{}_\nu \equiv \delta^\mu{}_\nu \,, \tag{2.10}$$

$$E_A{}^\mu e^B{}_\mu \equiv \delta^A{}_B \,, \tag{2.11}$$

which in turn generates the inverse metric as

$$g^{\mu\nu} \equiv \eta^{AB} E_A{}^\mu E_B{}^\nu \,. \tag{2.12}$$

Moreover, there is a condition that allows for a splitting of an affine connection $\hat{\Gamma}^\rho{}_{\nu\mu}$ into the tetrad and the spin connection $\hat{\omega}^A{}_{B\mu}$. This is dubbed the "tetrad postulate" [23]

$$\partial_\mu e^A_\nu + \hat{\omega}^A{}_{B\mu} e^B{}_\nu - \hat{\Gamma}^\rho{}_{\nu\mu} e^A{}_\rho \equiv 0 \,. \tag{2.13}$$

In geometrical terms states that the frame fields are constant along any path wrt the total covariant derivative that takes into account all types of indices. The important take away message from this is the unique solution of Eq. (2.13) in terms of $\hat{\Gamma}^{\rho}{}_{\mu\nu}$(or $\hat{\omega}^{A}{}_{B\nu}$) as

$$\hat{\Gamma}^{\rho}{}_{\mu\nu} = E^{\rho}_{A}\left(\partial_{\nu}e^{A}{}_{\mu} + \hat{\omega}^{A}{}_{B\nu}e^{B}{}_{\mu}\right) , \tag{2.14}$$

On the whole, it should be stressed that the invertibility of the $g_{\mu\nu}$, $g^{\mu\nu}$ is in general only a local property. As a direct consequence, Eqs. (2.8) – (2.12) should only really be considered valid only locally. Nevertheless, there is a specific family of manifolds dubbed parallelizable which allow for the existence of globally defined frames which extends the validity of Eqs. (2.8) – (2.12) to the whole manifold. This is due to the fact that these manifolds admit connections which are defined by having zero Riemann tensor as is the case for TG.

2.3 Tetrad - Spin Connection formulation

The spin connection in contrast to the usual affine connection representation can be at times more flexible. This is due to the fact that it can carry explicitly the metric compatibility condition as an antisymmetry property on its indices as $\hat{\omega}_{AC\mu} \equiv -\hat{\omega}_{CA\mu}$ [75]. Also there is the possibility on working only with Lorentz indices and the Minkowski metric hence index manipulation becomes a bit easier. For these reasons it is instructive to go through some mixed tensors and how index manipulation works.

$$\hat{R}^{A}{}_{B\mu\nu} = \partial_{\mu}\hat{\omega}^{A}{}_{B\nu} - \partial_{\nu}\hat{\omega}^{A}{}_{B\mu} + \hat{\omega}^{A}{}_{C\mu}\hat{\omega}^{C}{}_{B\nu} - \hat{\omega}^{A}{}_{C\nu}\hat{\omega}^{C}{}_{B\mu} , \tag{2.15}$$

$$\hat{T}^{A}{}_{\mu\nu} = \partial_{\mu}e^{A}{}_{\nu} - \partial_{\nu}e^{A}{}_{\mu} + \hat{\omega}^{A}{}_{B\mu}e^{B}{}_{\nu} - \hat{\omega}^{A}{}_{B\nu}e^{B}{}_{\mu} , \tag{2.16}$$

$$\hat{Q}_{\mu AB} = -\eta_{AC}\hat{\omega}^{C}{}_{B\mu} - \eta_{CB}\hat{\omega}^{C}{}_{A\mu} . \tag{2.17}$$

These mixed index forms are related to the standard ones which have only spacetime indices as

$$\hat{R}^{A}{}_{B\mu\nu} = e^{A}{}_{\rho} E_{B}{}^{\sigma} \hat{R}^{\rho}{}_{\sigma\mu\nu}\,, \qquad \hat{R}^{\mu}{}_{\nu\rho\sigma} = E_{A}{}^{\mu} e^{B}{}_{\nu} \hat{R}^{A}{}_{B\rho\sigma}\,, \tag{2.18a}$$

$$\hat{T}^{A}{}_{\mu\nu} = e^{A}{}_{\rho} \hat{T}^{\rho}{}_{\mu\nu}\,, \qquad \hat{T}^{\mu}{}_{\nu\rho} = E_{A}{}^{\mu} \hat{T}^{A}{}_{\nu\rho}\,, \tag{2.18b}$$

$$\hat{Q}_{\mu AB} = E_{A}{}^{\nu} E_{B}{}^{\rho} \hat{Q}_{\mu\nu\rho}\,, \qquad \hat{Q}_{\mu\nu\rho} = e^{A}{}_{\nu} e^{B}{}_{\rho} \hat{Q}_{\mu AB}\,, \tag{2.18c}$$

thus the tetrad field really is an isomorphism between the spacetime tangent space and the Minkowski tangent space. In essence, conversion between spacetime – Minkowski indices can be facilitated through the introduction of the tetrad fields.

One can further decompose an arbitrary spin connection in an analogous way as the affine connection [23] in Eq. (2.5)

$$\hat{\omega}^{A}{}_{B\mu} = \mathring{\omega}^{A}{}_{B\mu} + \hat{K}^{A}{}_{B\mu} + \hat{L}^{A}{}_{B\mu}\,. \tag{2.19}$$

The Eqs. (2.5) – (2.3) are the same equation expressed with different indices assuming that the tetrad postulate (2.13) holds. In the rest of this dissertation, only teleparallel spacetimes will be consired endowed with the teleparallel connection denoted as $\Gamma^{\lambda}{}_{\mu\nu}$.

2.4 Local Lorentz transformations

One more important property of the metric tensor is that it is Local Lorentz Transformation (LLT) invariant by default although the tetrad and the spin connection are not. As a consequence, one can generate the same metric from an infinite amount of tetrad choices but a single tetrad choice can only generate a specific metric [76]. Thus a tetrad is unique up to LLT. More specifically

$$e^A{}_\mu \mapsto e'^A{}_\mu = \Lambda^A{}_B e^B{}_\mu \,, \tag{2.20}$$

where $\Lambda^A{}_B = \Lambda^A{}_B(t,x)$ is a LLT, i.e., it must satisfy

$$\eta_{AB}\Lambda^A{}_C\Lambda^B{}_D = \eta_{CD} \,. \tag{2.21}$$

In general the spin connection transforms as

$$\omega^A{}_{B\mu} \mapsto \omega'^A{}_{B\mu} = \Lambda^A{}_C(\Lambda^{-1})^D{}_B\omega^C{}_{D\mu} + \Lambda^A{}_C\partial_\mu(\Lambda^{-1})^C{}_B \,. \tag{2.22}$$

hence it is not covariant under the LLT transformations unless they are global, i.e $\partial_\mu\Lambda^A{}_B \equiv 0$. This is to be expected since it is a connection after all.

The transformation of tetrad (2.20) and spin connection (2.22) under the LLT group suggests that they should be really considered as a pair and not individually. In this sense, a tetrad – spin connection pair is unique up to LLT. A way to, exploit this extra LLT is to single out a transformation $\Lambda^A{}_B$ that trivializes the spin connection. This is exactly what we will call the Weitzenböck gauge. In the Weitzenböck gauge the frame that is used is one that the spin connection is zero, hence simplifying for the most part any calculation. One can re-introduce the spin connection at any point by performing another LLT as

$$\omega'^A{}_{B\mu} = \Lambda^A{}_C\partial_\mu(\Lambda^{-1})^C{}_B \,. \tag{2.23}$$

The spin connection is just a pure gauge dof in TG meaning that it does not carry any physical significance. In other words, it serves as a non-dynamical dof that ensures covariantization wrt the Local Lorentz group. On the other hand, the teleparallel connection in its affine representation form in the Weitzenböck gauge becomes

$$\Gamma^\rho{}_{\mu\nu} = E_A{}^\rho\partial_\nu e^A{}_\mu \,. \tag{2.24}$$

It should be stressed that the introduction of the connection in any type of geometry serves as a median in order to attain covariance of the action wrt a group of transformations. Such groups of transformations include in general the spacetime diffeomorphisms (most general change of coordinates) and LLT [79]. The role of the connection becomes more important if it is actually a dynamical variable. This is realized in theories were the connection is completely independent from the metric/tetrad and it assumes non-trivial Riemann tensor. Contrastingly, in frameworks like that of TG where the connection is non-dynamical by construction the connection is irrelevant for studies of purely dynamical content.

In addition, covariance in the action is realized by demanding that both fundamental variables of the theory transform properly. For example, in theories were the metric and the affine connection play the role of the fundamental variables in order to attain diffeomorphism covariance we must demand that, under a change of coordinates of the form $x \to x'(x)$

$$g_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(x) \,, \tag{2.25}$$

$$\Gamma^{\lambda}{}_{\mu\nu}(x') = \frac{\partial x'^{\lambda}}{\partial x^{\tau}} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\tau}{}_{\rho\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\tau}} \frac{\partial^2 x'^{\tau}}{\partial x'^{\mu} \partial x'^{\nu}} \,. \tag{2.26}$$

This is just demanding that the components $g_{\mu\nu}$ comprise a (0,2) tensor and the components $\Gamma^{\lambda}{}_{\mu\nu}$ are connection coefficients. The other choice is to consider the tetrad – spin connection pair as fundamental variables for which case LLT covariance is attained by demanding for Eqs. (2.20)–(2.22) to hold.

2.5 Covariantization and Coupling Prescription

In order to activate gravity geometrically, starting from Minkowski spacetime where there is no gravity, the standard partial derivative operator must be promoted to a more general covariant derivative. That is in order to account for the non-flatness of any spacetime other than Minkowski which is the unique fully flat spacetime. By unique fully flat it meant that curvature, torsion and non-metricity are all trivial. Truly flat spacetime means that it is equipped with the trivial connection which by definition assumes no curvature, torsion or non-metricity. For example there are partially flat spacetimes such as spatially flat which are used in cosmology. Thus, introducing a non-trivial connection is a geometric deformation of Minkowski spacetime to some non-flat manifold. This process in general, is called covariantization. In practice the standard covariantization procedure is enforced by promoting the Minkowski metric to a general metric on the spacetime manifold M and the partial derivative operator to the covariant derivative of the LC connection

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x) \tag{2.27}$$

$$\partial_\mu \rightarrow \mathring{\nabla}_\mu\,, \tag{2.28}$$

and this introduces the so-called geodesic equation

$$u^\mu \mathring{\nabla}_\mu u^\nu = \frac{d^2 x^\nu}{d\lambda^2} + \mathring{\Gamma}^\nu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0\,, \tag{2.29}$$

where the overcircle denotes quantites calculated wrt the LC connection. This equation generalizes the usual straight lines from flat spacetime into geodesic curves of the LC connection. Another interpretation of this equation is that gravity is expressed directly through geometry since its generalized acceleration vanishes ($\mathring{a}^\nu = 0$). Thus there is no leftover force term to counter to drive the acceleration since the Right hand side (RHS) is zero. In this sense gravity is not a force anymore within the framework of the Lorentzian geometry that utilizes the LC but rather absorbed into the geometry itself. The most trivial example of a model in this category is GR. The Lorentzian geometry though is a much more widely used framework that encompasses all types of curvature only based models, thus the geodesic Eq. (2.29) is not affected by the choice of Lagrangian at all. Hence when the connection is fixed to the LC one then the geodesics are fixed.

The situation is different if another connection is chosen in order to covariantize the geometrical framework. In TG where the connection is only torsionful the geodesic Eq. (2.29) now becomes the autoparallel equation for the teleparallel connection

$$u^{\mu}\nabla_{\mu}u^{\nu} = \frac{d^2x^{\nu}}{d\lambda^2} + \Gamma^{\nu}{}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = K^{\nu}{}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda}\,, \tag{2.30}$$

where in contrast to Eq. (2.29) the acceleration is now driven by the force term $K^{\nu}{}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda}$ on the RHS. Hence, using the teleparallel connection gravity is not fully absorbed into the geometry and it also manifests a force term proportional to the contorsion. This force term means that gravity is split into a geometrical component and a force component. This is obviously different than GR where gravity is fully absorbed into the geometrical structure of spacetime which is the LC. Nevertheless, both Eq. (2.29) – (2.30) are completely equivalent [23] if one considers the fundamental relation of Eq. (2.5).

Hence, in the TG framework the covariantization can be realized by either using the teleparallel connection or the LC. On top of that, since the tetrad is the fundamental variable in TG one needs to also promote the tetrad to some general function on the spacetime manifold M. Thus, the covariantization procedure can be condensed as

$$e^{A}{}_{\mu}\big|_{flat} \rightarrow e^{A}{}_{\mu}(x)\,, \tag{2.31}$$

$$\partial_{\mu} \rightarrow \mathring{\nabla}_{\mu}\,. \tag{2.32}$$

The covariantization procedure is in general of outmost importance since it allows to translate Lagrangians from purely field theoretic settings (in Minkowski spacetime) to more general non-flat spacetimes.

Finally, the choice of the covariantization procedure also affects the coupling to matter which also depends on the form of the matter action, i.e, if it includes partial derivatives acting on (higher than one order) tensors or not. Nevertheless, within this dissertation only matter Lagrangians minimally coupled to the metric will be used of the form $\mathcal{L}_m = \mathcal{L}_m(g_{\mu\nu}, \psi)$ where ψ denotes any matter related field [76].

2.6 Torsion Decomposition and Torsion Scalar

The simplest theory in TG dubbed TEGR is built from the torsion scalar T which assumes three completely equivalent representations. The standard and first representation is

$$T = \frac{1}{4}T_1 + \frac{1}{2}T_2 - T_3 \,. \tag{2.33}$$

where

$$T_1 = T^{\mu\nu\rho}T_{\mu\nu\rho}\,, \quad T_2 = T^{\mu\nu\rho}T_{\rho\nu\mu}\,, \quad T_3 = T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}\,. \tag{2.34}$$

As for the second representation being a bit more involved, the torsion tensor needs to be split into its irreducible components under the action of the Lorentz group [23]. The irreducible notion means that the contraction of any different irreducible components is zero [23]. To realise this representation the torsion tensor must be split as

$$a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}T^{\nu\sigma\rho}\,, \tag{2.35}$$

$$v_\mu = T^{\sigma}{}_{\sigma\mu}\,, \tag{2.36}$$

$$t_{\sigma\mu\nu} = \frac{1}{2}\left(T_{\sigma\mu\nu} + T_{\mu\sigma\nu}\right) + \frac{1}{6}\left(g_{\nu\sigma}v_\mu + g_{\nu\mu}v_\sigma\right) - \frac{1}{3}g_{\sigma\mu}v_\nu\,, \tag{2.37}$$

where $\epsilon_{\mu\nu\sigma\rho}$ is the totally antisymmetric LC tensor associated to the metric $g_{\mu\nu}$.

The pseudo tensor a_μ is the axial vector, v_μ is a vector and $t_{\sigma\mu\nu}$ is a rank 3 tensor with the following properties

$$t_{\alpha\mu\nu} = t_{\mu\alpha\nu}, \tag{2.38}$$

$$t_{\alpha\mu\nu} + t_{\nu\alpha\mu} + t_{\mu\nu\alpha} = 0, \tag{2.39}$$

$$t^{\alpha\mu}{}_{\alpha} = t^{\alpha}{}_{\alpha}{}^{\mu} = t^{\mu\alpha}{}_{\alpha} = 0. \tag{2.40}$$

Using the irreducible torsion components the following scalars can be generated

$$T_{\rm axi} = a_\mu a^\mu = \frac{1}{18}\left(T_{\sigma\mu\nu}T^{\sigma\mu\nu} - 2T_{\sigma\mu\nu}T^{\mu\sigma\nu}\right), \tag{2.41a}$$

$$T_{\rm vec} = v_\mu v^\mu = T^{\sigma}{}_{\sigma\mu}T_{\rho}{}^{\rho\mu}, \tag{2.41b}$$

$$T_{\rm ten} = t_{\sigma\mu\nu}t^{\sigma\mu\nu} = \frac{1}{2}\left(T_{\sigma\mu\nu}T^{\sigma\mu\nu} + T_{\sigma\mu\nu}T^{\mu\sigma\nu}\right) - \frac{1}{2}T^{\sigma}{}_{\sigma\mu}T_{\rho}{}^{\rho\mu}, \tag{2.41c}$$

hence the second representation of the torsion scalar assumes the form

$$T = \frac{1}{4}T_1 + \frac{1}{2}T_2 - T_3 = \frac{3}{2}T_{\rm axi} + \frac{2}{3}T_{\rm ten} - \frac{2}{3}T_{\rm vec}. \tag{2.42}$$

The irreducible representation of the torsion scalar will be of use in the formulation of the BDLS theory [64] which is investigated in Secs. 3–4.

Finally a more standard representation of the torsion scalar

$$T = \frac{1}{2}S_{\rho}{}^{\mu\nu}T^{\rho}{}_{\mu\nu}, \tag{2.43}$$

is obtained via the use of the superpotential defined as

$$S_{\rho}{}^{\mu\nu} := K^{\mu\nu}{}_{\rho} - \delta^{\mu}_{\rho}T_{\sigma}{}^{\sigma\nu} + \delta^{\nu}_{\rho}T_{\sigma}{}^{\sigma\mu} = -S_{\rho}{}^{\nu\mu}. \tag{2.44}$$

This tensor is very suitable in compactifying the calculations and it is also the conjugate momentum of the torsion tensor in the sense of Lagrangian field theory.

In whichever representation, the action of TEGR is defined as

$$\mathcal{S}_{\mathrm{TEGR}} = -\frac{1}{2\kappa^2}\int \mathrm{d}^4 x e T\,, \tag{2.45}$$

where e denotes the determinant of the tetrad field and it is completely equivalent in any of the three representations. The use of each representation depends on the context of the analysis. In the next section a proof of why TEGR and GR are dynamically equivalent will be given.

2.7 TEGR action and field equations

The action of GR, dubbed the Einstein-Hilbert action depends only one one single and fundamental variable the metric [7]. This is due to the unique form of the LC connection which is ultimately a function of the metric and its first derivatives. Consequently, anything built from the Riemann tensor

$$\mathring{R}^{\rho}{}_{\lambda\nu\mu} = \mathring{\Gamma}^{\rho}{}_{\lambda\mu,\nu} - \mathring{\Gamma}^{\rho}{}_{\lambda\nu,\mu} + \mathring{\Gamma}^{\rho}{}_{\sigma\nu}\mathring{\Gamma}^{\sigma}{}_{\lambda\mu} - \mathring{\Gamma}^{\rho}{}_{\sigma\mu}\mathring{\Gamma}^{\sigma}{}_{\lambda\nu}\,, \tag{2.46}$$

will depend on the metric and its derivatives (up to second order). Calculating the Riemann tensor of the teleparallel connection

$$R^{\rho}{}_{\lambda\nu\mu} = \Gamma^{\rho}{}_{\lambda\mu,\nu} - \Gamma^{\rho}{}_{\lambda\nu,\mu} + \Gamma^{\rho}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\lambda\nu} \equiv 0\,, \tag{2.47}$$

which is identically zero by definition. Utilizing Eq. (2.5), the Riemann tensor of the teleparallel connection $R^{\rho}{}_{\lambda\nu\mu}$ can be related to the Riemann tensor of the LC connection $\mathring{R}^{\rho}{}_{\lambda\nu\mu}$ as [80]

$$0 \equiv R^{\rho}{}_{\lambda\nu\mu} = \mathring{R}^{\rho}{}_{\lambda\nu\mu} + P^{\rho}{}_{\lambda\nu\mu}\,, \tag{2.48}$$

where the $P^{\rho}{}_{\lambda\nu\mu}$ tensor is defined as

$$\begin{aligned}\mathring{R}^{\rho}{}_{\lambda\nu\mu} = & -\Big(K^{\rho}{}_{\lambda\mu,\nu} - K^{\rho}{}_{\lambda\nu,\mu} + \mathring{\Gamma}^{\rho}{}_{\sigma\nu}K^{\sigma}{}_{\lambda\mu} - \mathring{\Gamma}^{\rho}{}_{\sigma\mu}K^{\sigma}{}_{\lambda\nu} + \mathring{\Gamma}^{\sigma}{}_{\lambda\mu}K^{\rho}{}_{\sigma\nu} - \mathring{\Gamma}^{\sigma}{}_{\lambda\nu}K^{\rho}{}_{\sigma\mu} \\ & + K^{\rho}{}_{\sigma\nu}K^{\sigma}{}_{\lambda\mu} - K^{\rho}{}_{\sigma\mu}K^{\sigma}{}_{\lambda\nu}\Big) =: -P^{\rho}{}_{\lambda\nu\mu}\,. \end{aligned} \tag{2.49}$$

and is expressed only in terms of the teleparallel connection and contortion. This further leads to a relation between the Ricci tensors of the form

$$0 \equiv R_{\lambda\mu} = R^{\rho}{}_{\lambda\rho\mu} = \mathring{R}^{\rho}{}_{\lambda\rho\mu} + P^{\rho}{}_{\lambda\rho\mu}\,, \tag{2.50}$$

which after contracting the indices of the Ricci tensors results in the relation of the Ricci scalars of the two connection in the particular form

$$0 = R = \mathring{R} + P\,, \tag{2.51}$$

where $R = g^{\lambda\mu}R_{\lambda\mu}$, and by using the identities

$$K^{\mu}{}_{\rho\mu} = T^{\mu}{}_{\mu\rho}\,, \quad T^{\sigma}{}_{\mu\nu} = -T^{\sigma}{}_{\nu\mu}\,. \tag{2.52}$$

the equation

$$P = g^{\lambda\mu}P^{\rho}{}_{\lambda\rho\mu} = \frac{2}{e}\partial_{\rho}\left(eT^{\mu\rho}{}_{\mu}\right) + K^{\rho\sigma\mu}K_{\mu\sigma\rho} - K^{\rho}{}_{\sigma\rho}K^{\mu\sigma}{}_{\mu}\,, \tag{2.53}$$

is obtained. After a few algebraic manipulations, it is found that the first term in P is a total divergence which can be written as

$$B := \frac{2}{e}\partial_{\rho}\left(eT^{\mu}{}_{\mu}{}^{\rho}\right) \equiv -\frac{2}{e}\partial_{\rho}\left(eT^{\mu\rho}{}_{\mu}\right)\,, \tag{2.54}$$

and the remainder can be simplified in the form

$$K^{\rho\sigma\mu}K_{\mu\sigma\rho} - K^{\rho}{}_{\sigma\rho}K^{\mu\sigma}{}_{\mu} = \frac{1}{4}T^{\rho\sigma\mu}T_{\rho\sigma\mu} + \frac{1}{2}T^{\mu\sigma\rho}T_{\rho\sigma\mu} - T^{\rho}{}_{\rho\sigma}T^{\mu}{}_{\mu}{}^{\sigma}\,, \tag{2.55}$$

which is exactly what is defined as the torsion scalar

$$T = \frac{1}{4}T^{\rho\sigma\mu}T_{\rho\sigma\mu} + \frac{1}{2}T^{\mu\sigma\rho}T_{\rho\sigma\mu} - T^{\rho}{}_{\rho\sigma}T^{\mu}{}_{\mu}{}^{\sigma}\,. \tag{2.56}$$

Thus, the Ricci scalar of the LC connection can be finally written as

$$\mathring{R} = -P = -T + B\,, \tag{2.57}$$

which enforces the dynamical equivalence between the Einstein-Hilbert and Teleparallel actions which are proportional to $\mathring{R}$ and T (2.45). This simply means that the field equations of GR and TEGR are identical wrt their dynamical content but they may differ visually. The boundary term B compensates for the second derivatives of the tetrad that the $\mathring{R}$ contains that are extracted from T since it only contains up to first order derivatives of the tetrad.

Considering the TEGR action while also including a matter sector

$$\mathcal{S}_{\text{TEGR}} := -\frac{1}{2\kappa^2}\int \mathrm{d}^4x e\,T + \int \mathrm{d}^4x e\mathcal{L}_{\text{m}}\,, \tag{2.58}$$

where $\kappa^2 := 8\pi G$ and the extra minus sign has been introduced in order for this action to comply with Eq. (2.57). In order to calculate the field equations of actions containing the tetrad a few identities are needed such as

$$\frac{\partial e^{B}{}_{\nu}}{\partial e^{A}{}_{\mu}} = \delta^{B}_{A}\delta^{\mu}_{\nu}\,, \tag{2.59a}$$

$$\frac{\partial E_{B}{}^{\nu}}{\partial e^{A}{}_{\mu}} = -E_{B}{}^{\mu}E_{A}{}^{\nu}\,, \tag{2.59b}$$

$$\frac{\partial e}{\partial e^{A}{}_{\mu}} = eE_{A}{}^{\mu}\,, \tag{2.59c}$$

$$\frac{\partial g^{\alpha\beta}}{\partial e^{A}{}_{\mu}} = -g^{\mu\beta}E_{A}{}^{\alpha} - g^{\mu\alpha}E_{A}{}^{\beta}\,. \tag{2.59d}$$

which can be further combined to obtain the variations of standard tensorial quantities wrt the tetrad. Such quantities involve the inverse tetrad, $E_A{}^{\mu}$, the determinant of the tetrad, e, a general metric and its inverse, $(g_{\mu\nu}, g^{\mu\nu})$, the torsion tensor, $T^{\alpha}{}_{\mu\nu}$ and the torsion vector, T^{μ} which then all read as

$$\delta_e E_A{}^{\mu} = -E_B{}^{\mu} E_A{}^{\nu} \delta e^B{}_{\nu}\,, \tag{2.60a}$$

$$\delta_e e = \delta \det(e^A{}_{\mu}) = e\, E_A{}^{\mu} \delta e^A{}_{\mu}\,, \tag{2.60b}$$

$$\delta_e g_{\mu\nu} = \eta_{AB} \left(e^A{}_{\mu} \delta e^B{}_{\nu} + e^A{}_{\nu} \delta e^B{}_{\mu} \right)\,, \tag{2.60c}$$

$$\delta_e g^{\mu\nu} = -\left(g^{\mu\alpha} E_A{}^{\nu} + g^{\nu\alpha} E_A{}^{\mu} \right) \delta e^A{}_{\alpha}\,, \tag{2.60d}$$

$$\delta_e T^{\alpha}{}_{\mu\nu} = -E_A{}^{\alpha} T^{\beta}{}_{\mu\nu} \delta e^A{}_{\beta} + 2 E_A{}^{\alpha} \delta_e \Gamma^A{}_{[\nu\mu]} \tag{2.60e}$$

$$= -E_A{}^{\alpha} T^{\beta}{}_{\mu\nu} \delta e^A{}_{\beta} + E_A{}^{\alpha} \Big[\partial_{\mu} \delta e^A{}_{\nu} - \partial_{\nu} \delta e^A{}_{\mu} + \omega^A{}_{B\mu} \delta e^B{}_{\nu} - \omega^A{}_{B\nu} \delta e^B{}_{\mu} \Big]\,, \tag{2.60f}$$

$$\delta_e T^{\mu} = -\left(E_A{}^{\mu} T^{\lambda} + g^{\mu\lambda} T_A + T^{\lambda}{}_A{}^{\mu} \right) \delta e^A{}_{\lambda}$$
$$+ g^{\mu\nu} E_A{}^{\lambda} \left(\partial_{\lambda} \delta e^A{}_{\nu} - \partial_{\nu} \delta e^A{}_{\lambda} + \omega^A{}_{B\lambda} \delta e^B{}_{\nu} - \omega^A{}_{B\nu} \delta e^B{}_{\lambda} \right)\,. \tag{2.60g}$$

These results can be used in calculating the variation of any quantity built from the tetrad. One of the most important such quantities is the torsion scalar which is calculated as

$$\delta_e T = \frac{1}{4} \delta(T^{\mu\nu\alpha} T_{\mu\nu\alpha}) + \frac{1}{2} \delta(T^{\mu\nu\alpha} T_{\nu\mu\alpha}) - \delta(T^{\mu} T_{\mu})\,, \tag{2.61}$$

where

$$\delta(T^{\mu\nu\alpha} T_{\mu\nu\alpha}) = 4 T_{\mu}{}^{\nu\alpha} E_A{}^{\mu} (\partial_{\nu} \delta e^A{}_{\alpha} + \omega^A{}_{B\nu} \delta e^B{}_{\alpha}) - 4 T^{\mu\nu\alpha} T_{\mu\nu\beta} E_A{}^{\beta} \delta e^A{}_{\alpha}\,, \tag{2.62a}$$

$$\delta(T^{\mu\nu\alpha} T_{\nu\mu\alpha}) = 2(T^{\beta\nu\mu} - T^{\mu\nu\beta}) T_{\nu\mu\alpha} E_A{}^{\alpha} \delta e^A{}_{\beta} + 2(T^{\mu}{}_{\nu}{}^{\beta} - T^{\beta}{}_{\nu}{}^{\mu}) E_A{}^{\nu} (\partial_{\mu} \delta e^A{}_{\beta} + \omega^A{}_{B\mu} \delta e^B{}_{\beta})\,, \tag{2.62b}$$

$$\delta(T^{\mu}T_{\mu}) = -2(-T^{\beta}T^{\alpha}{}_{\beta\mu} + T^{\alpha}T_{\mu})E_{A}{}^{\mu}\delta e^{A}{}_{\alpha} - 2(T^{\mu}E_{A}{}^{\beta} - T^{\beta}E_{A}{}^{\mu})(\partial_{\mu}\delta e^{A}{}_{\beta} + \omega^{A}{}_{B\mu}\delta e^{B}{}_{\beta})\,. \tag{2.62c}$$

Using Eqs. (2.62a), (2.62b), (2.62c) in Eq. (2.61) and performing integration by parts

$$e\,\delta_{e}T = 2e\left(\frac{1}{e}\partial_{\mu}(eS_{A}{}^{\lambda\mu}) - T^{\sigma}{}_{\mu A}S_{\sigma}{}^{\mu\lambda} + \omega^{B}{}_{A\nu}S_{B}{}^{\nu\lambda}\right)\delta e^{A}{}_{\lambda}\,. \tag{2.63}$$

which can be used to vary the action (2.58) wrt the tetrad and obtain the field equations

$$W^{A}{}_{\mu} = e^{-1}\partial_{\sigma}(eS_{A}{}^{\mu\sigma}) - T^{\sigma}{}_{\nu A}S_{\sigma}{}^{\nu\mu} + \frac{1}{2}E_{A}{}^{\mu}T + \omega^{B}{}_{A\nu}S_{B}{}^{\nu\mu} = \kappa^{2}\Theta_{A}{}^{\mu}\,, \tag{2.64}$$

and contracting it with $e^{A}{}_{\beta}g_{\mu\alpha}$ in order to transform all indices to Greek covariant ones

$$e^{A}{}_{\beta}\,g_{\mu\alpha}\,e^{-1}\partial_{\sigma}(eS_{A}{}^{\mu\sigma}) - T^{\sigma}{}_{\nu\beta}S_{\sigma}{}^{\nu}{}_{\alpha} + \frac{1}{2}g_{\alpha\beta}T + \omega^{B}{}_{\beta\nu}S_{\beta}{}^{\nu}{}_{\alpha} = \kappa^{2}\Theta_{\alpha\beta}\,. \tag{2.65}$$

The field equation tensor $W_{\mu\nu}$, resulting from variation wrt the tetrad, can be always further split into symmetric and antisymmetric parts

$$W_{(\mu\nu)} = \Theta_{\mu\nu}, \quad \text{and} \quad W_{[\mu\nu]} \equiv 0\,. \tag{2.66}$$

where the symmetric part

$$W_{(\mu\nu)} := \frac{1}{2}(W_{\mu\nu} + W_{\nu\mu}), \tag{2.67}$$

entails 10 independent components while the antisymmetric part comprised of 6 independent components

$$W_{[\mu\nu]} := \frac{1}{2}(W_{\mu\nu} - W_{\nu\mu}), \tag{2.68}$$

is equivalent to the field equations coming from the variation of the spin connection [33].

The EMT is defined as

$$\Theta_{\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\delta L_{\rm m}}{\delta g^{\mu\nu}} = e^{A}{}_{\mu}\left(\frac{1}{e}\frac{\delta L_{\rm m}}{\delta e^{A}{}_{\nu'}}\right)g_{\nu\nu'} =: e^{A}{}_{\mu}\Theta_{A}{}^{\nu'}g_{\nu\nu'}\,. \tag{2.69}$$

For the case of TEGR, it turns out that $W_{[\mu\nu]} \equiv 0$ and this is rooted in the Bianchi identities. This is rather not the case for modified teleparallel theories like $f(T)$. The field equation tensor for the tetrad assumes 16 components in 4 dimensions and it can thus be decomposed into a symmetric part comprised of 10 components as which combined give

$$\underbrace{W_{\mu\nu}}_{16} = \underbrace{W_{(\mu\nu)}}_{10} + \underbrace{W_{[\mu\nu]}}_{6}. \tag{2.70}$$

In this form it is clear that the symmetric part of the field equations entails the purely metrical dof. To prove that the field equations of GR and TEGR are equivalent, notice that the Einstein tensor can be re-written in terms of teleparallel quantities as

$$\mathring{G}_{\alpha\beta} = -\Big(T^{B}{}_{\nu\beta}S_{B}{}^{\nu}{}_{\alpha} - \omega^{B}{}_{\beta\nu}S_{B}{}^{\nu}{}_{\alpha} - \frac{1}{e}g_{\mu\alpha}e^{A}{}_{\beta}\partial_{\nu}(eS_{A}{}^{\mu\nu}) - \frac{T}{2}g_{\alpha\beta}\Big). \tag{2.71}$$

The field equations (2.65) are in general covariant both under the LLT and diffeomorphism group. It should also be stressed that the tetrad and the spin connection represent different dof and they are rather determined from different field equations in general. Nevertheless, in TG the spin connection is just a gauge dof and as such its field equations are linearly dependent on those produced by the tetrad.

In theories where the metric plays the role of the only fundamental variable the resulting field equations are 10, exactly as much as the independent components of the metric. These field equations are also symmetric by construction since the are generated from variations wrt the metric tensor. On the other hand, if the only fundamental variable is the tetrad then there are 16 field equations which assume no symmetry since the tetrad has no index symmetries. As a matter of fact, 6 from these field equations belong to the trivial spin connection and only 10 of them are directly related to the tetrad [81]. Another way to understand this is by considering the fact that the Local Lorentz group has 6 dof and thus this freedom should be imprinted somehow in the field equations.

Finally, it should be noted that the teleparallel action (2.58) was chosen to assume this specific form in order to properly reproduce the field equations of GR. However, this exact formulation can be reproduced by a completely different starting point which is the gauge aspect of the theory. TEGR can be formulated as the gauge theory of translations [23] in which the field strength is expressed via the torsion tensor. As such, the action is then constructed by the quadratic contraction of the field strength and it results in the form [80]

$$\mathcal{S}_{\text{TEGR}} = \frac{1}{2\kappa^2} \int \text{tr}\Big(\boldsymbol{T} \wedge \star\boldsymbol{T}\Big), \tag{2.72}$$

where $\boldsymbol{T} = (1/2)T^{A}{}_{\mu\nu}P_A \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is the torsion 2-field and $P_A = \partial_A$ is the translation generators. It turns out that $\star\boldsymbol{T} = (1/2) \star T^{A}{}_{\mu\nu}P_A \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ with $\star T^{A}{}_{\mu\nu} \equiv (e/2)\epsilon_{\mu\nu\alpha\beta}S^{A\alpha\beta}$. This is in direct analogy to the gauge theories built from non-abelian group. Hence, replacing the $\star\boldsymbol{T}$ back in the action (2.72) and after some manipulations the Einstein-Hilbert action is recovered. Thus, just using the group of translations and building a gauge theory on top of it one can end up with TEGR. This is also without any prior knowledge of GR.

2.8 Degrees of Freedom and Stability

The single most important piece of data needed for any field theory in order to be well posed, consistent and healthy is the detailed knowledge of its pdof. The number of pdof is directly related with the wellposedness of the Cauchy problem [82] which involves the existence and uniqueness of the solutions of a theory given boundary conditions. Hence, solving a system of partial differential equations which are non-linear (self-interacting) and the variable(metric) itself describes the spacetime is a bit more subtle.

Ignorance of the pdof may lead to a lot of fatal problems in a theory. First of all, the number of pdof might introduce some uncertainty in the number of initial conditions needed to solve the field equations. Secondly, there are situations where the Hamiltonian of the system is unbounded from below for various reasons and the most common one is introduced by terms linear to momenta. These linear terms come from higher order derivatives (3rd order and onwards) and as such higher order derivatives are prone to this issue which is called Ostrogradsky instability [83]. Physically, this is translated as that some of the pdof are ghosts and can extract arbitrarily negative energy from the Hamiltonian. This issue can be remedied by either constraining the Lagrangian of the theory accordingly or demanding that the field equations are strictly of second order in the derivatives.

In general, there are other types of instabilities that can render the Hamiltonian unbounded from below. A single scalar field will be used in order to illustrate them. It is assumed that the scalar field is described by the action

$$\mathcal{S}_\phi := \int \mathrm{d}^4x\mathcal{L} = \int \mathrm{d}^4x\left[\frac{1}{2}K_t(t)\dot{\phi}^2 - \frac{1}{2}K_s(t)\eta^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}M(t)^2\phi^2\right], \tag{2.73}$$

where $K_t(t)$, $K_s(t)$ and $M(t)$ are time dependent functions to be determined. This action describes a scalar field with a standard kinetic term and a mass in a 3+1 split background [76] where Latin indices denote spatial coordinates. The Hamiltonian of this system is then calculated as

$$\begin{aligned} H &= \int \mathrm{d}^4x\,[PQ - \mathcal{L}] \\ &= \int \mathrm{d}^3x\,dt\left[\frac{1}{2K_t}P^2 + \frac{1}{2}K_sk^2Q^2 + \frac{1}{2}M^2Q^2\right] \end{aligned} \tag{2.74}$$

where $Q := \phi$ denotes the the generalized coordinate and $P := \partial\mathcal{L}/\partial\dot{\phi}$ is the the generalized momenta. The Hamiltonian is expected to be positive definite if and only if $K_t(t)$, $K_s(t)$ and $M(t)$ are positive and thus the Hamiltonian is bounded from below[84] allowing the system to be stable. Using the Hamiltonian of Eq. (2.74) as reference there are the following instabilities:

1. Tachyonic instability if $M^2 < 0$, i.e negative mass squared.
2. Gradient instability if $K_s < 0$, i.e the field has negative kinetic energy.
3. Ghost instability if $K_t < 0$, i.e the field has negative momentum squared.

These 3 fundamental types of instabilities are mathematicaly completely different since ghosts arise from negative time derivatives, gradient instabilities from negative spatial derivatives, and tachyons from negative non-derivative interactions [85]. It should be stressed that the canonical transformations that leave the Hamiltonian invariant (in the phase space) can be used in order to find relations between these types of instabilities or even remove some of them.

All these considerations are very important since the stability of a theory highly depends mostly on three factors:

1. The form of the Lagrangian itself.
2. The symmetries of the background in question.
3. The choice of background solution for the system.

In most cases the most elusive factor and most dangerous one, is the choice of background solution. A wrong choice can lead to finding less pdof than the actual true number. Hence, these missing pdof seem to be coupled to themselves or other pdof in a way that they lose their kinetic terms. This is called the strong coupling [86] issue and is one of the most common problems in modified theories of gravity.

A practical way of understanding why the choice of a proper background solution is very important, is that it controls the functions $K_t(t)$, $K_s(t)$ and $M(t)$ which dictate the behaviour of the Hamiltonian. In general though, the Hamiltonians of gravitational theories are anything but simple. Even from calculating the Hamiltonian of GR it is evident that one should expect to deal with quite evolved constraint analysis in order to derive the reduced Hamiltonian [7].

As a final remark, in order to perform the Hamiltonian analysis of a theory there are two ways:

1. The standard background Hamiltonian analysis already described.
2. Perturbative Hamiltonian analysis, where the second order expansion of the Hamiltonian is used around a fixed background.

It turns out that in highly symmetric backgrounds the perturbative Hamiltonian scheme is the most optimal choice since the analysis is highly simplified. Nevertheless, issues of potential strong coupling may still be present. This is due to the fact that using just perturbations one cannot really tell if a background solution is physical, i.e reproduces the correct number of pdof or not, although some stability conditions can still be obtained. This is directly linked with modern Cosmology which is described by geometrical backgrounds that are highly symmetric [58] and thus perturbative schemes are used heavily.

2.9 Cosmological background and Perturbations

In order to study the dynamics of the universe, an assumption regarding its geometry is needed. The spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) geometry turns out to be this choice. These geometries are defined by demanding that the Universe homogeneous and isotropic (cosmic principle [87]). In essence, the cosmological principle is that the Universe looks the same from wherever it is observed in sufficient large distances. In modern cosmology, the cosmological principle is described as the notation that the spatial distribution of the matter in the Universe is homogeneous and isotropic when observed on large enough scales. This is due to the fact that in large enough scales the forces are suppoed to act in a uniform way in the Universe. Henceforth, there should be no irregularities in the evolution of the matter field that was introduced by the Big Bang. In light of the isotropic character of the Cosmic Microwave Background the spatially flat FLRW geometry turns out to be a very good fit [88, 89, 90, 91, 92]. This choice is quite dominant even if there is a few percent non-zero spatial curvature that cannot be ruled out at high precision [93, 94, 95, 96].

Although these spatially flat FLRW geometries seem to work well so far, there are small temperature fluctuations in the formation of overdense regions which signal to the fact that the studying just the background cosmology is not enough. Note that the key point is that these deviations are small enough otherwise using the spatially flat FLRW geometry would have to be revisited. On top of this, this particular choice of geometry is valid within the current time but that is not to be assumed for any other era like the very early or the very late one [97, 98, 99].

The tool that expands the study of a system from a fixed state around small deviations is the called the perturbative framework. As a matter of fact, the cosmological perturbations are quite important and interesting because among others they constitute a way to perform Hamiltonian analysis on the cosmological background. Thus, allowing for an in-depth probe of the stability of the theory [58]. It should be stressed that Hamiltonian analysis can be performed either on the background or using the perturbative framework and if done properly, the results must be identical. The drawback of the background Hamiltonian analysis is that it is very complicated and cumbersome in general. On the other hand, performing Hamiltonian analysis via perturbations on a fixed highly symmetric background reduces the analysis dramatically. The only drawback in this case is that one needs to choose some proper background solution that corresponds to the maximum pdof of the theory due to the lingering strong coupling issue as discussed in Sec. 2.8.

2.9.1 Background Cosmology

The spatially flat FLRW geometry in terms of the metric is described by the well known form of the diagonal metric

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)\,, \tag{2.75}$$

where $a(t)$ is the scale factor. This metric could be generated by the diagonal tetrad

$$e^A{}_\mu = \text{diag}(1, a(t), a(t), a(t))\,, \tag{2.76}$$

which is in conjunction with the Weitzenböck gauge. Starting with the background values of the torsion tensor (2.16) and superpotential tensor (2.44), these are obtained by substituting the tetrad (2.76) and they result to

$$T^i{}_{0j} = H\delta^i{}_j\,, \tag{2.77}$$

$$S_i{}^{0j} = -H\delta^i{}_j\,, \tag{2.78}$$

from which any quantity based on torsion can be calculated in this background. The most common examples are the torsion scalar as illustrated in (2.80) and the boundary (2.54).

On the other hand the matter content is described via the continuity equation and is fully conserved giving the standard conservation equation for a perfect fluid

$$\mathring{\nabla}_\nu \Theta_\mu{}^\nu : \; \dot{\rho} + 3(\rho + p) = 0\,. \tag{2.79}$$

Using the tetrad (2.76), the torsion scalar (2.43) assumes the value

$$T = -6H^2\,, \tag{2.80}$$

and the boundary term (2.54) becomes

$$B = -6(3H^2 + \dot{H})\,. \tag{2.81}$$

As a crosscheck by using Eqs. (2.43) – (2.54) the correct Ricci scalar of the LC connection is recovered via Eq. (2.57)

$$\mathring{R} = -T + B = -6(\dot{H} + 2H^2)\,. \tag{2.82}$$

In general the class of FLRW geometries is based upon the assumptions homogeneity and isotropy of the Universe (cosmic principle [87]). Observations regarding the distribution of structure formation at large scales, in conjunction with the isotropic nature of Cosmic Microwave Background render spatially flat cosmological geometries well motivated and founded [88, 89, 90, 91, 92]. This is the reason that there is high priority in analysing these highly symmetric backgrounds compared to more general (less symmetric) ones.

2.9.2 Cosmological Perturbations

The perturbed metric is defined at first order as

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \tag{2.83}$$

where $|\delta g_{\mu\nu}| \ll 1$ is the first order perturbation of the metric and $g_{\mu\nu}$ represents the background value of the metric. The perturbation $\delta g_{\mu\nu}$ will carry the 10 dof which for example in GR is reduced to a massless spin-2 field that counts for 2 dof.

Along the same lines the first order perturbation of the tetrad can be defined as

$$e^a{}_\mu \to e^a{}_\mu + \delta e^a{}_\mu\,, \tag{2.84}$$

where $e^a{}_\mu$ represents the background value of the tetrad and $|\delta e^a{}_\mu| \ll 1$ represents the first order perturbation of the tetrad. The perturbation $\delta e^a{}_\mu$ entails 16 dof which in TEGR are manifested as a massless spin-2 field that counts for 2 dof just as in GR.

Imposing diffeomorphism covariance at first order allows us to remove 4 dof from the metric and the tetrad perturbations. This is achieved via the process of gauge fixing [87]. Although different choice of gauge exists, in the end physics is the same. Thus, for each situation there might be a specific choice of gauge which may greatly simplify the calculational aspect of the problem at hand.

In highly symmetric backgrounds, there are in principal more simplifications that can be done. One such, in a spatially flat FLRW background, is the ability to split any perturbation into irreducible components wrt the linearised diffeomorphism group. This is commonly dubbed as the Scalar-Vector-Tensor (SVT) decomposition of the perturbations and for the tetrad assumes the form

$$\delta e^A{}_\mu := \begin{bmatrix} -\varphi & -a\left(\partial_i\beta + \beta_i\right) \\ \delta^I{}_i\left(\partial^i b + b^i\right) & a\delta^{Ii}\left(-\psi\delta_{ij} + \partial_i\partial_j h + 2\partial_{(i}h_{j)} + \frac{1}{2}h_{ij} + \epsilon_{ijk}\left(\partial^k\sigma + \sigma^k\right)\right) \end{bmatrix}, \quad (2.85)$$

where h_{ij} is symmetric, traceless $h_{ij}\delta^{ij} = 0$, and transverse $\partial^i h_{ij} = 0$, while all the vectors are solenoidal $\partial_i b^i = \partial_i\beta^i = \partial_i h^i \equiv 0$. In total there are 5 scalars $\{\varphi, b, \beta, \psi, h\}$ plus a pseudoscalar σ, 3 vectors $\{b_i, \beta_i, h_i\}$ plus a pseudovector σ_i and the tensor h_{ij}. Note that we use the indices, A,B,C,D,.. and Greek lowercase letters $\mu, \nu, \rho, \sigma, ..$ are used as 4-D indices on the Minkowski and general manifold respectively. The σ and σ_i are pseudoscalar and pseudovector which means that they transform in the opposite way under parity transformations. The middle range Latin indices I,J,K,.. and i,i,k,.. refer to spatial 3-D indices in Minkowski and general manifold respectively.

The perturbed tetrad can reproduce the standard metric [92, 100, 101]

$$\delta g_{\mu\nu} = \begin{bmatrix} -2\,\varphi & a\,(\partial_i \mathcal{B} + \mathcal{B}_i) \\ a\,(\partial_i \mathcal{B} + \mathcal{B}_i) & 2a^2\left(-\psi\delta_{ij} + \partial_i\partial_j h + 2\partial_{(i}h_{j)} + \frac{1}{2}h_{ij}\right) \end{bmatrix}, \tag{2.86}$$

by defining $\mathcal{B} := b - \beta$ and $\mathcal{B}_i := b_i - \beta_i$. This signifies that 6 dof in the off diagonal part of the tetrad (b, b_i, β, β_i) are condensed into 3 within the metric. In addition the antisymmetric part that corresponds to 3 dof via (σ, σ_i) trivially vanishes since the metric is symmetric. Thus, in the end the perturbation of the metric assumes the correct 10 dof.

We define the perturbation of the EMT of a perfect fluid as

$$\delta\Theta_\mu{}^\nu := \begin{bmatrix} \delta\rho & (\rho + p)(v^i + \partial^i v) \\ -a^2(\rho + p)(v_i + \partial_i v) & -\delta p \delta^i_j \end{bmatrix}, \tag{2.87}$$

where ρ is the matter density, p denotes the pressure, and v, v^i denote the scalar and vector parts of the perturbation of the velocity field. The spatial part $\delta\Theta_i{}^j$, in general, can include an anisotropic stress piece $\Pi_i{}^j$ which can be further split into SVT decomposition as

$$\Pi_{ij} = \Pi^S_{ij} + \Pi^V_{ij} + \Pi^T_{ij}\,, \tag{2.88a}$$

$$\Pi^S_{ij} := \partial_i\partial_j\Pi^S\,, \tag{2.88b}$$

$$\Pi^V_{ij} := -\frac{1}{2}(\partial_j\Pi^V_i + \partial_i\Pi^V_j)\,, \tag{2.88c}$$

$$\Pi^T_{ij} \quad \text{is the tensor part}\,, \tag{2.88d}$$

where $\partial^i\Pi^V_i \equiv 0 \equiv \partial^i\Pi^T_{ij}$, $\Pi^T_{ij} \equiv \Pi^T_{(ij)}$ and $\delta^{ij}\Pi^T_{ij} \equiv 0$.

Finally, raising and lowering indices is realized as $X_0 = X^0$, $X_j = -X^j$. In addition, $\square := \partial_\mu\partial^\mu = \partial_0^2 - \partial^2$, where defined the spatial Laplacian as $\partial^2 := -\eta^{ij}\partial_j\partial_i = \delta^{ij}\partial_i\partial_j$.

The Fourier space convention will be $f(x^\mu) \to f(k^\mu)e^{-i\omega t + ik_j x^j}$ and the norm of the wave covector will be defined as $k^2 := -\eta^{ij}k_i k_j = \delta^{ij}k_i k_j$.

2.9.3 Gauge Transformations

The diffeomorphism covariance at first order allows the fixing of 4 dof from either the perturbation of the metric or the tetrad. This is the so called gauge fixing. In general, a linearised diffeomorphism can be described as a linear/infinitesimal change of coordinates of the form $\widetilde{x}^\mu \to x^\mu + \xi^\mu$. This change is completely determined by the vector field ξ^μ. Under this type of transformation the perturbed tetrad changes as

$$\widetilde{\delta e}^A{}_\mu \to \delta e^A{}_\mu + \mathcal{L}_\xi e^A{}_\mu . \tag{2.89}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the flow of the vector field ξ^μ. The vector field ξ^μ can be further split into SVT decomposition as $\xi^\mu = \left\{\xi^0, \omega(\xi^i + \delta^{ij}\partial_j\xi)\right\}$ where $\partial_i\xi^i = 0$. Hence we can unpack the components of Eq. (2.89) as

$$\widetilde{\varphi} = \varphi - \dot{\xi}^0 , \quad \widetilde{\psi} = \psi , \quad \widetilde{\beta} = \beta - \xi^0 , \quad \widetilde{\beta_i} = \beta_i , \tag{2.90a}$$

$$\widetilde{b} = b - \dot{\xi} , \quad \widetilde{b_i} = b_i + \dot{\xi}_i , \quad \widetilde{\sigma} = \sigma , \quad \widetilde{\sigma}_i = \sigma^i - \frac{1}{2}\epsilon^i{}_{jk}\partial^j\xi^k , \tag{2.90b}$$

$$\widetilde{h} = h - \xi , \quad \widetilde{h_i} = h_i + \frac{1}{2}\xi_i , \quad \widetilde{h_{ij}} = h_{ij} . \tag{2.90c}$$

It should be stressed that φ, ψ, σ and β_i are gauge invariant in a Minkowski background described by a constant tetrad. The transformation properties in Eq. (2.90) are of vital importance in properly understanding the true dof and also generating gauge invariant variables.

2.9.4 Perturbations in Teleparallel Gravity

The process of calculating the perturbation of any quantity can be realized by using only a few fundamental building blocks. Henceforth, all these blocks will be calculated and only the non-zero components will be directly presented for the various sectors. Any of the calculations are evaluated in the spatially flat FLRW background as introduced in Sec. 2.9.

The non-zero components of the torsion tensor and the superpotential are

$$\delta T^{i}{}_{0j} = \frac{1}{2}\dot{h}_{ij}\,, \tag{2.91}$$

$$\delta T^{i}{}_{jk} = \frac{1}{2}(\partial_j h_{ik} - \partial_k h_{ij})\,, \tag{2.92}$$

$$\delta S_{0}{}^{0i} = 0\,, \tag{2.93}$$

$$\delta S_{i}{}^{0j} = \frac{1}{4}\dot{h}_{ij}\,, \tag{2.94}$$

$$\delta S_{i}{}^{jk} = -\frac{1}{4a^2}(\partial_j h_{ik} - \partial_k h_{ij})\,, \tag{2.95}$$

while the scalars are

$$\delta T = 0\,, \quad \delta B = 0\,. \tag{2.96}$$

The non-zero components of the vectorial and pseudo vectorial perturbations for the torsion tensor and the superpotential are

$$\delta T^{0}{}_{0i} = a\dot{\beta}_i\,, \tag{2.97}$$

$$\delta T^{i}{}_{0j} = 2\partial_i \dot{h}_j - \frac{1}{a}\partial_j b_i - \epsilon_{kij}\dot{\sigma}_k\,, \tag{2.98}$$

$$\delta T^{0}{}_{ij} = a(\partial_i \beta_j - \partial_j \beta_i)\,, \tag{2.99}$$

$$\delta T^i{}_{jk} = 2(\partial_i\partial_j h_k - \partial_i\partial_k h_j) + \epsilon_{ijl}\partial_k\sigma_l - \epsilon_{ikl}\partial_j\sigma_l\,, \tag{2.100}$$

$$\delta S_0{}^{0i} = -\frac{1}{2a^2}\Big[2aH(b_i-\beta_i) + \epsilon_{ilk}\partial_k\sigma_l\Big]\,, \tag{2.101}$$

$$\delta S_i{}^{0j} = -\frac{1}{2a}\Big[\frac{1}{2}\Big(\partial_i(b_j+\beta_j-a\dot{h}_j) + \partial_j(b_i-\beta_i-a\dot{h}_i)\Big)\Big]\,, \tag{2.102}$$

$$\delta S_0{}^{ij} = -\frac{1}{4a^3}\Big[\partial_i(b_j-\beta_j+2a\dot{h}_j) - \partial_j(b_i-\beta_i+2a\dot{h}_i) - 2a\epsilon_{lij}\dot{\sigma}_l\Big]\,, \tag{2.103}$$

$$\begin{aligned}\delta S_i{}^{jk} = -\frac{1}{2a^2}\Big[&\delta_{im}\epsilon_{kjl}\partial_l\sigma_m + \delta_{ij}\Big(2aH(b_k-\beta_k) - a\dot{\beta}_k - 2\partial^2 h_k\Big)\\ &-\delta_{ik}\Big(2aH(b_j-\beta_j) - a\dot{\beta}_j - 2\partial^2 h_j\Big) - 2\delta_{il}\partial_k\partial_l h_j + 2\delta_{kl}\partial_i\partial_j h_l\Big]\,,\end{aligned} \tag{2.104}$$

and the perturbations related to the torsion and boundary term scalars are

$$\delta T = 0\,, \tag{2.105}$$

$$\delta B = 0\,. \tag{2.106}$$

The components of the torsion tensor and the superpotential for scalar and pseudo scalar perturbations up to first order are

$$\delta T^0{}_{0i} = \partial_i(a\dot{\beta} - \phi)\,, \tag{2.107}$$

$$\delta T^i{}_{0j} = \partial_i\partial_j(\dot{h} - a^{-1}b) - \epsilon_{lij}\partial_l\dot{\sigma} - \dot{\psi}\delta_{ij}\,, \tag{2.108}$$

$$\delta T^0{}_{ij} = 0\,, \tag{2.109}$$

$$\delta T^i{}_{jk} = \delta_{ij}\partial_k\psi - \delta_{ik}\partial_j\psi + \delta_{il}(\epsilon_{klm}\partial_j\partial_m\sigma - \epsilon_{jlm}\partial_k\partial_m\sigma)\,, \tag{2.110}$$

$$\delta S_0{}^{0i} = -\frac{H}{a}\partial_i\Big(b - \beta - (aH)^{-1}\psi\Big)\,, \tag{2.111}$$

$$\delta S_i{}^{0j} = \Big[(2H\phi + \dot{\psi})\delta_{ij} + \frac{1}{2}\partial_i\partial_j(\dot{h} - a^{-1}b) - \frac{1}{2}\partial^2(\dot{h} - a^{-1}b)\delta_{ij}\Big]\,, \tag{2.112}$$

$$\delta S_0{}^{ij} = \frac{1}{2a^2}\epsilon_{ijk}\partial_k\dot{\sigma}\,, \tag{2.113}$$

$$\delta S_i{}^{jk} = \frac{1}{2a^2}\Big[\delta_{ik}\partial_j\Big(2aH(b-\beta)+\phi-\psi-a\dot{\beta}\Big)-\delta_{ij}\partial_k\Big(2aH(b-\beta)+\phi-\psi-a\dot{\beta}\Big)\Big]\,, \tag{2.114}$$

and the perturbations up to first order to the scalar torsion and boundary term become

$$\delta T = 4H\Big(3H\phi+3\dot{\psi}+\frac{1}{a}\partial^2 b-\partial^2\dot{h}\Big)\,, \tag{2.115}$$

$$\delta B = -\Big[H\left(\frac{1}{a}\partial^2(6\beta-10b)-6(6\dot{\psi}+\dot{\phi}-2\partial^2\dot{h}+6H\phi)\right)+\frac{2}{a}\partial^2(\dot{\beta}-\dot{b})+\frac{2}{a^2}\partial^2(2\psi-\phi)$$
$$+\,2(\partial^2\ddot{h}-6\dot{H}\phi-3\ddot{\psi})\Big]\,. \tag{2.116}$$

In a very similar manner the perturbation conservation equations is calculted as

$$\mathring{\nabla}_\mu\Theta_0{}^\mu = \delta\dot{\rho}+3H(\delta p+\delta\rho)+\frac{\partial^2 v(p+\rho)}{a}-3\dot{\psi}(p+\rho)+\partial^2\dot{h}(p+\rho)=0\,, \tag{2.117}$$

$$\mathring{\nabla}_\mu\Theta_i{}^\mu = \partial_i\Big[\delta p+(\rho+p)\Big(4aH(b+v-\beta)+\phi+a(\dot{b}-\dot{\beta}+\dot{v})\Big)+a(\dot{\rho}+\dot{P})(v+b-\beta)\Big]=0\,. \tag{2.118}$$

This is concludes the most elemental blocks of calculations needed to perform any type of perturbation in TG around flat FLRW background.

2.9.5 Tensor waves and observations

In order to study the physical GW that are observed, a mathematical correspondence is needed. This is facilitated via the tensorial part of the perturbations of the tetrad/metric Eq. (2.85). The very dynamics of the tensor perturbations h_{ij} completely describes its dynamical properties. This field equations of h_{ij} will be dubbed as Gravitational wave Propagation Equation (GWPE). In Fourier space [39, 102, 103] the general form for massless waves is

$$\ddot{h}_{ij} + (3 + \alpha_M) H\dot{h}_{ij} + (1 + \alpha_T) \frac{k^2}{a^2} h_{ij} = 0\,, \tag{2.119}$$

where the tensor excess speed is defined as

$$\alpha_T := c_T^2 - 1\,, \tag{2.120}$$

and the frictional term is defined as

$$\alpha_M := \frac{1}{HM_*^2}\frac{dM_*^2}{dt}. \tag{2.121}$$

where M_* is the effective Planck mass. The quantities $\alpha_T,\ \alpha_M$ directly parametrize the GWPE wrt the theory/model under consideration. Effectively, these parameters alter the waveform both in amplitude(α_M) and phase(α_T). Comparing with the GR waveform it is evident that [104, 105]

$$h_{\text{Modified}} \sim h_{\text{GR}}\, \underbrace{e^{-\frac{1}{2}\int \alpha_M \mathcal{H} d\eta}}_{\text{Amplitude}}\, \underbrace{e^{ik\int \sqrt{\alpha_T} d\eta}}_{\text{Phase}}\,, \tag{2.122}$$

where $\eta = \int dt/a$ denotes conformal time, $\mathcal{H} = a'/a$ is the conformal Hubble parameter and primes represent derivatives with respect to conformal time.

The parameter α_{M} also leads to modifications of the luminosity distance [104, 106] related to its electromagnetic counterpart as [39]

$$\frac{d_L^{Modified}(z)}{d_L^{EM}(z)} = \exp\left[\frac{1}{2}\int_0^z \frac{\alpha_M}{1+z'}\, dz'\right]. \tag{2.123}$$

So far this constraint cannot be realized since extra data is needed from the standard sirens which belong to the next generation of GW detectors. If $\alpha_T \equiv 0$ then the tensor waves travel at the speed of light and if $\alpha_M \equiv 0$ then there is no friction which both combined give us GR.

Regarding observations, the detection of the GW event GW170817 and the γ-ray burst

GRB 170817A place a strong constraint on the speed at which GW propagate, leading to the following constraint [107]

$$-3\times 10^{-15} < \left|\frac{c_{\rm g}}{c} - 1\right| < 7\times 10^{-16}\,. \tag{2.124}$$

On top of that, observations for the mass of the graviton set an upper bound of $m_{\rm g} < 1.2\times 10^{-22}$ eV/c^2 from Ref. [34]. There is also a stronger constraint coming directly from Solar System tests from which we have the stronger upper bound

$$m_{\rm g} < 10^{-30} eV/c^2\,. \tag{2.125}$$

Hence suggesting that our current understanding of the graviton regarding its mass was not really changed from the GW170817 event [108].

Chapter 3

Reviving Horndeski Theory using Teleparallel Gravity after GW170817

After the recent observations of LIGO collaboration [5] and from multimessenger observations which involve the Gamma-Ray Bursts [109], the speed of GW was constrained to $|c_g/c - 1| \gtrsim 10^{-15}$. It should be noted that these observations are realized under conservative assumptions such as that the signals measured in these events were emitted about a millisecond after each other. Such an assumption might make the constraint even tighter in reality.

One of the most prominent models that relies on the curvature based geometries is Horndeski gravity [110]. Horndeski gravity extends GR to the most general scalar tensor theory including one additional scalar field. It turns out that the effect of this model on the GWPE compared to GR is the modification of the amplitude and speed of propagation of the GW. This modification of the GWPE led to the Horndeski gravity being highly constrained in order to comply with the speed of light propagation [5]. This also contributed in the shift of interest to more general extensions such as Beyond Horndeski gravity where the Lagrangian contains higher order derivatives and even Proca theories that utilize vector fields instead of just scalar fields. All of these extensions rely on the standard curvature based framework of the LC connection.

As a matter of fact gravity being expressible via the Riemann tensor is not the only option. As it was shown in Sec. 2.1, gravity can also be formulated in the TG geometry by using the specific teleparallel connection that is curvatureless, metric compatible and torsionful [80]. Relying on this idea and how constrained the Horndeski gravity was, the Teleparallel Analog of Horndeski gravity or BDLS gravity after Bahamonde, Dialepktopoulos and Levi Said who first introduced it [64] was born. The BDLS model in principle extends the Horndeski paradigm by adding the most general quadratic in torsion action. In this way, one can study scalar tensor theories using the standard Horndeski formulation within a teleparallel context, extending it.

As it is well known, the physical observed GW corresponds to tensor waves in terms of perturbations analysis in the spatially flat FLRW background. Hence in this section, the GWPE, which is just field equation of the tensor perturbations is of central importance. Using this equation, important quantities like the speed of propagation, the effective mass and the Planck mass run rate will be calculated. In order to calculate the GWPE for the BDLS gravity, as illustrated in Sec. 2, the analysis will unravel in a spatially flat FLRW background and perturbations will be performed specifically for the tensor sector.

It is found that the BDLS theory is more flexible against the $c_T = c_g/c = 1$ constraint compared to Horndeski theory. In contrast, in Horndeski theory the coupling functions $G_5(\phi, X)$ and $G_4(\phi, X)$ need not be trivialized (as in Horndeski). This allows for a much broader and richer selection of models and further investigation without the need to resort to more complex theories like beyond Horndeski [111].

The structure of this chapter is as follows. In Sec. 3.1 – 3.2, an overview of the Lovelock's theorem and Horndeski gravity will be given in order to introduce the motivation and ways of modifying gravity. Later on, in Sec. 3.3 the construction of the BDLS theory is revisited along with the derivation of its field equations. These field equations are then linearized in order to calculate the GWPE in Sec. 3.4 which is also compared against the observational constraints alongside with discussion of their compatibility. Subsequently, a few examples of common models that can be re-casted from Horndeski to the BDLS theory are illustrated. Finally, the core results are summarized in Sec. 3.6.

3.1 Lovelock's theorem

GR in conjunction with the standard model of particle physics has proven to be quite successful in describing the Universe on most scales [112]. In spite of all this success, there are still observations that cannot be explained within these, so far successful, models. For example, the cosmic microwave background [113] and the rotation curves of galaxies [114] imply that there is a new form type of matter that does not interact with the electromagnetic forces. This type of dark matter cannot be accounted for by neither GR nor the standard model of particle physics.

A straightforward way of incorporating these observations is by extending or modifying GR. There are a few ways of achieving this goal, nevertheless there are some tools that allow for a deeper understanding of how and why it should be done. The most important such tool is Lovelock's theorem [115, 116], that highly restricts the most general case of GR behaviour and sets a roadmap for further generalizations.

Restricting to the LC connection, i.e, curvature based gravity , there is only the Riemann tensor expressing geometry and it depends completely on the metric and its derivatives up to second order. Hence, Lagrangian densities that depend solely on the metric and its derivatives will be considered as to attain full generality. Starting from an action that depends only in the metric

$$\mathcal{S} = \int \mathrm{d}^4x \sqrt{-g}\mathcal{L}[g_{\mu\nu}]\,, \tag{3.1}$$

and assuming that the derivative dependence is up to second order in the metric, then the resulting Euler-Lagrange equations of (3.1) read as

$$\mathcal{E}^{\mu\nu}[\mathcal{L}] := \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \frac{d}{dx^{\rho}}\left[\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\rho}} - \frac{d}{dx^{\lambda}}\left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\rho\lambda}}\right)\right] - \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\,, \tag{3.2}$$

where $\mathcal{E}^{\mu\nu}$ is to be understood as the functional derivative on the functional $\mathcal{L}$. The Lovelock's theorem can be stated as: **The only possible second order field equations in a four dimensional spacetime from an action of the form** (3.1) **is**

$$W^{\mu\nu} \equiv \alpha\sqrt{-g}\left[\mathring{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathring{R}\right] + \Lambda\sqrt{-g}g^{\mu\nu}\,, \tag{3.3}$$

where α, Λ **are constants.** The theorem, in other words, states that the resulting field equations of any metric dependent action in four dimensions must be exactly (3.3). These field equations are second order and they are actually the standard Einstein equations plus a cosmological constant term. The most general Lagrangian density that produces Eq. (3.3) is [13]

$$\mathcal{L} = \alpha\sqrt{-g}\mathring{R} - 2\Lambda\sqrt{-g} + \beta\varepsilon^{\mu\nu\rho\tau}\mathring{R}^{\alpha\sigma}{}_{\mu\nu}\mathring{R}_{\alpha\sigma\rho\tau} + \gamma\sqrt{-g}\left(\mathring{R}^2 - 4c\mathring{R}^{\mu}{}_{\nu}\mathring{R}^{\nu}{}_{\mu} + \mathring{R}^{\mu\nu}{}_{\rho\tau}\mathring{R}^{\rho\tau}{}_{\mu\nu}\right)\,, \tag{3.4}$$

where β and γ are constants. This can be proven by considering that the variations of the third and fourth terms wrt the metric as

$$\mathcal{E}^{\mu\nu}\left[\varepsilon^{\alpha\beta\rho\tau}\mathring{R}^{\gamma\delta}{}_{\alpha\beta}\mathring{R}_{\gamma\delta\rho\tau}\right] = 0 \tag{3.5}$$

$$\mathcal{E}^{\mu\nu}\left[\sqrt{-g}\left(\mathring{R}^2 - 4\mathring{R}^{\alpha}{}_{\beta}\mathring{R}^{\beta}{}_{\alpha} + \mathring{R}^{\alpha\beta}{}_{\rho\tau}\mathring{R}^{\rho\tau}{}_{\alpha\beta}\right)\right] = 0 \tag{3.6}$$

where the action of $\mathcal{E}^{\mu\nu}$ on any functionals is defined as in (3.2). Also note that Eq. (3.4) is built completely from quadratic contractions of the Riemann tensor based on the LC connection, plus a constant term $-2\Lambda\sqrt{-g}$. As indicated by Eqs. (3.5)–3.6, the variations of the third and fourth terms are zero and thus they cannot contribute to the overall field equations. This concludes all the possible scenarios of generating second order equations wrt the metric only. A very important implication coming from the Lovelock's theorem is that constructing any other theory with second order field equations wrt the metric that is not GR plus a cosmological constant, requires some of the following:

1. Introducing **extra fields**, such as scalar fields, vectors fields. Any field on top of the metric tensor.

2. Keep the metric as the sole fundamental variable but introduce **higher derivatives**.

3. Introduce **extra dimensions**. Note that Eq. (3.5) is valid in any number of dimensions whereas Eq. (3.6) is valid only in four.

4. **Modify the Euler-Lagrange operator** (3.2) to include more than two indices or not being symmetric. In the same way it can be demanded that it is not divergence free.

5. **Remove locality** by introducing non-local terms like $\mathring{\Box}\mathring{R}$.

Adding an additional scalar field on top of the metric and still demanding second order field equations, introduces the scalar tensor theories. In these theories the fundamental variables are the metric and the scalar, thus there exist also field equations for the scalar field. On top of that, the resulting field equations are not just GR plus a cosmological constant anymore as described in Eq. (3.3) but rather much the most general field equations. The most general scalar tensor theory admitting second order field equations is called Horndeski theory and it will be the object of the next section.

3.2 Horndeski Gravity

One of the simplest ways to extend GR is by adding an extra propagating scalar field. This type of extension is dubbed the scalar tensor form. In four dimensions the most general type of scalar tensor theory that admits second-order field equations was derived by Horndeski himself in 1974 [110]. In his work he found that the most general Lagrangian that encompasses this behaviour is

$$\mathcal{S}_{\text{HORNDESKI}} = \frac{1}{2\kappa^2}\sum_{i=2}^{5}\int d^4x\, e\mathcal{L}_i\,, \tag{3.7}$$

where

$$\mathcal{L}_2 := G_2(\phi, X)\,, \quad \mathcal{L}_3 := G_3(\phi, X)\Box\phi\,, \tag{3.8}$$

$$\mathcal{L}_4 := G_4(\phi, X)\mathring{R} + G_{4,X}(\phi, X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right], \tag{3.9}$$

$$\mathcal{L}_5 := G_5(\phi, X)\mathring{G}_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}(\phi, X)\Big[(\Box\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}(\Box\phi)\Big], \tag{3.10}$$

where $\mathring{G}_{\mu\nu}$ is the Einstein tensor and $\Box\phi := \phi^{;\mu}{}_{;\mu}$. The total Lagrangian is divided in these $\mathcal{L}_i$'s because these represent all the possible combinations of the scalar field and its derivatives on being coupled to the metric and providing second order field equations. It is instructive to briefly sketch the derivation that lead to this Lagrangian, since it will be more transparent why the resulting field equations are of second order. The starting point is the generic action of the form

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\mathcal{L}(g_{\mu\nu}, g_{\mu\nu,\lambda_1}, \cdots, g_{\mu\nu,\lambda_1,\cdots,\lambda_p}, \phi, \phi_{,\lambda_1}, \cdots, \phi_{,\lambda_1,\cdots,\lambda_q}), \tag{3.11}$$

where $p, q \geq 2$ in four dimensions. In this theory since there are only two fundamental variables, the metric and scalar field there will be two sets of field equations. Variation of this action wrt the metric and the scalar field yields the field equations

$$W_{\mu\nu} := 2\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \equiv 0 \tag{3.12}$$

$$W_\phi := \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi} \equiv 0 \tag{3.13}$$

where $W_{\mu\nu}$ and W_ϕ include at most second derivatives of $g_{\mu\nu}$ and ϕ. Using the diffeomorphism invariance of the action

$$\nabla^\nu W_{\mu\nu} = -\nabla_\mu\phi\, W_\phi, \tag{3.14}$$

which links the two field equations $W_{\mu\nu}$ and W_ϕ. In general, since $W_{\mu\nu}$ is assumed to be of at most second order then $\nabla^\nu W_{\mu\nu}$ would be of at least third order in derivatives of $g_{\mu\nu}$ and ϕ. Nevertheless, Eq. (3.14) enforces $\nabla^\nu W_{\mu\nu}$ to be of at least second order since the RHS is of at most second order. This means that $W_{\mu\nu}$ and $\nabla^\nu W_{\mu\nu}$ must both be of at most second order wrt the derivatives of $g_{\mu\nu}$ and ϕ which constrains quite a lot the form of $W_{\mu\nu}$ itself. From there on and after some complicated calculations the final Lagrangian is found implicitly as

$$\mathcal{L} = g^{\mu\nu} W_{\mu\nu} \,, \tag{3.15}$$

whilst explicitly is expanded as Eq. (3.7). Choosing appropriately the G_i functions of the Horndeski Lagrangian (3.7) results in any other known scalar tensor theory. For example

- nonminimal coupling of the form $f(\phi)R$ can be obtained by taking $G_4 = f(\phi)$.
  - $f(R)$ is included in this branch [117].
  - In the limiting case $G_4 = \text{const} = M_p^2/2$ the Einstein-Hilbert term is recovered.
- G_2 is the familiar term used in k-inflation [118]/k-essence [119, 120].
- G_3 term was investigated more recently in the context of kinetic gravity braiding [121]/G-inflation .
- $f(\mathring{R})$ is also a subclass in its scalar tensor form.
- Nonminimal coupling of the form $G^{\mu\nu}\phi_\mu\phi_\nu$ [122] can also be achieved by either $G_4 = X$ or $G_5 = -\phi$.

Thus, the Horndeski construction encapsulates all of the well probed theories of the scalar tensor family. With appropriate choice of the G_i's one can either end up with minimal, non-minimal couplings or even derivative couplings depending on the model needed.

3.3 The Teleparallel Gravity Analog of Horndeski Gravity (BDLS theory)

As explained in Sec. 3.2, Horndeski s theory was born due to the need of the most general scalar tensor theory admitting second order field equations. After the GW 170817 and GRB 170817A events [123] Horndeski's theory was heavily constrained in its totality and thus this is served as the motivation for extending it in the TG framework. This extension is realized by adding an extra teleparallel term complying to the following criteria

- The field equations both for the tetrad and for the scalar field must be of second order,
- the scalar invariants should not be parity-violating,
- and contractions of the torsion tensor must be at most quadratic.

The first condition ensures that there is no Ostrogradsky instability in the theory as explained in Sec. 2.8. The second condition ensures that any terms added in the Lagrangian will preserve the parity as the original Horndeski Lagrangian (3.7) and so far there is no observation signaling to parity violating terms. The final condition is just the freedom of generating an infinite amount of scalars by contractions of the torsion tensor that result in second order field equations wrt the tetrad and the scalar field. In order to achieve some finite number of scalars the quadratic contraction of torsion tensors was demanded. Higher order torsion contractions have yet to be proven physically relevant or at least there are no strong arguments in their favor, yet. This is the main reason why only scalar invariants built from quadratic contractions of the torsion tensor were allowed.

The Lagrangian that satisfies the (i)-(iii) criteria is constructed by considering [124]:

1. The Lagrangian contains up to second order derivatives of the scalar field,
2. the Lagrangian is a polynomial in second order derivatives of the scalar field,

3. the corresponding field equations are at most second order in derivatives of the scalar field.

Including a scalar field ϕ with a kinetic term $X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$, which is invariant under the Galilean group $\phi \to \phi + b_\mu x^\mu + c$, assumes the form

$$\mathcal{L} = \sum_{i=1}^{5} c_i \mathcal{L}_i \,, \tag{3.16}$$

where c_i are arbitrary constants and

$$\mathcal{L}_1 := \phi \,, \tag{3.17a}$$

$$\mathcal{L}_2 := X \,, \tag{3.17b}$$

$$\mathcal{L}_3 := X\mathring{\Box}\phi \,, \tag{3.17c}$$

$$\mathcal{L}_4 := -X(\mathring{\Box}\phi)^2 + (\mathring{\Box}\phi)\phi_{,\mu}\,\phi_{,\nu}\,\phi^{,\mu\nu} + X\phi^{,\mu\nu}\phi_{,\mu\nu} - \phi_{,\mu}\,\phi^{,\mu\nu}\phi_{,\nu\rho}\,\phi^{,\rho} \,, \tag{3.17d}$$

$$\begin{aligned}\mathcal{L}_5 := & -2X(\mathring{\Box}\phi)^3 - 3(\mathring{\Box}\phi)^2\phi_{,\mu}\,\phi_{,\nu}\,\phi^{,\mu\nu} + 6X\mathring{\Box}\phi\phi_{,\mu\nu}\,\phi^{,\mu\nu} \\ & + 6\mathring{\Box}\phi\phi_{,\mu}\,\phi^{,\rho}\phi^{,\mu\nu}\phi_{,\nu\rho} - 4X\phi_{,\mu}{}^{\nu}\phi_{,\nu}{}^{\rho}\phi_{,\rho}{}^{\mu} \\ & + 3\phi_{,\mu\nu}\,\phi^{,\mu\nu}\phi_{,\rho}\,\phi_{,\lambda}\,\phi^{,\lambda\rho} - 6\phi_{,\mu}\,\phi^{,\mu\nu}\phi_{,\nu\rho}\,\phi^{,\lambda\rho}\phi_{,\lambda} \,.\end{aligned} \tag{3.17e}$$

The next step is to covariantize the action (3.16). The core process is already described in Sec. 2.5. In this instance the constants c_i must be replaceD by functions on the general manifold of the form $G_i(\phi, X)$ so that the Lagrangians (3.17a)–(3.17e) transform to

$$\mathcal{L}_2 := G_2(\phi, X) \,, \tag{3.18a}$$

$$\mathcal{L}_3 := G_3(\phi, X)\mathring{\Box}\phi \,, \tag{3.18b}$$

$$\mathcal{L}_4 := G_4(\phi, X)\,(-T + B) + G_{4,X}(\phi, X)\left[(\mathring{\Box}\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right], \tag{3.18c}$$

$$\mathcal{L}_5 := G_5(\phi, X)\mathring{G}_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}(\phi, X)\Big[\,(\mathring{\Box}\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\,(\mathring{\Box}\phi)\Big], \tag{3.18d}$$

where $G_{i,X} = \partial G_i/\partial X$ and $\phi_{;\mu\nu} = \mathring{\nabla}_\mu\mathring{\nabla}_\nu\phi$. In the $\mathcal{L}_4$ and $\mathcal{L}_5$ Lagrangians there are two new terms, $G_4(\phi, X)(-T + B)$ and $G_5(\phi, X)\mathring{G}_{\mu\nu}\phi^{;\mu\nu}$. Also $\mathring{G}_{\mu\nu}$ is the Einstein tensor formulated in teleparallel geometry, i.e., Eq. (2.71). This term coupled to the Einstein tensor is there as a counter term for higher order terms that would appear in the field equations so that it will cancel them out.

The teleparallel Lagrangian $\mathcal{L}_{\text{Tele}}$ consists of contractions of the irreducible parts of the torsion tensor with derivatives of the scalar field. Starting with the linear contractions of the torsion tensor, these read as

$$I_1 := t^{\mu\nu\sigma}\phi_{;\mu}\phi_{;\nu}\phi_{;\sigma}\,, \quad I_2 := v^\mu\phi_{;\mu}\,, \quad I_3 := a^\mu\phi_{;\mu}\,. \tag{3.19}$$

This concludes the full set of scalars that can be constructed at linear contraction order wrt the torsion tensor. This is due to the symmetry of $t_{\mu\nu\sigma}$ in its first two indices and the fact that $t^{\sigma\mu}{}_{\sigma} = t^{\sigma}{}_{\sigma}{}^{\mu} = t^{\mu\sigma}{}_{\sigma} = 0$. However, due to the fact that $t_{(\mu\nu\rho)} = 0$, it can be directly shown that I_1 vanishes. On top of this I_3 is a parity violating scalar since it contains an odd number of axial torsion vectors which are parity violating themselves. Finally, the tensorial part of the torsion tensor whilst contracted with second order derivatives of the scalar field would produce higher order derivatives in the field equations and thus it is dropped.

As for the quadratic contractions wrt the torsion tensor and the derivatives of the scalar field these read as

$$J_1 := a^\mu a^\nu\phi_{;\mu}\phi_{;\nu}\,, \quad J_2 := v^\mu v^\nu\phi_{;\mu}\phi_{;\nu}\,, \quad J_3 := v_\sigma t^{\sigma\mu\nu}\phi_{;\mu}\phi_{;\nu}\,, \quad J_4 := v_\mu t^{\sigma\mu\nu}\phi_{;\sigma}\phi_{;\nu}\,, \tag{3.20a}$$

$$J_5 := t^{\sigma\mu\nu}t_{\sigma}{}^{\bar{\mu}}{}_{\nu}\phi_{;\mu}\phi_{;\bar{\mu}}\,, \quad J_6 := t^{\sigma\mu\nu}t_{\sigma}{}^{\bar{\mu}\bar{\nu}}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar{\mu}}\phi_{;\bar{\nu}}\,, \quad J_7 := t^{\sigma\mu\nu}t^{\bar{\sigma}\bar{\mu}}{}_{\sigma}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar{\sigma}}\phi_{;\bar{\mu}}\,, \tag{3.20b}$$

$$J_8 := t^{\sigma\mu\nu}t_{\sigma\mu}{}^{\bar{\nu}}\phi_{;\nu}\phi_{;\bar{\nu}}\,, \quad J_9 := t^{\sigma\mu\nu}t^{\bar{\sigma}\bar{\mu}\bar{\nu}}\phi_{;\sigma}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar{\sigma}}\phi_{;\bar{\mu}}\phi_{;\bar{\nu}}\,, \quad J_{10} := \epsilon^{\mu}{}_{\nu\rho\sigma}a^{\nu}t^{\alpha\rho\sigma}\phi_{;\mu}\phi_{;\alpha}\,. \tag{3.20c}$$

Although none of these scalars is parity violating they are still not independent from each other. Notice that that $J_2 = I_2^2$, $J_3 = J_4$, $J_7 = -2J_6$ and just like $I_1 \to 0$, $J_9 \to 0$ due to the symmetry of the tensor part of torsion. So the only admisible scalars that comply with all the conditions are

$$I_2 = v^{\mu}\phi_{;\mu}\,, \tag{3.21}$$

$$J_1 = a^{\mu}a^{\nu}\phi_{;\mu}\phi_{;\nu}\,, \tag{3.22}$$

$$J_3 = v_{\sigma}t^{\sigma\mu\nu}\phi_{;\mu}\phi_{;\nu}\,, \tag{3.23}$$

$$J_5 = t^{\sigma\mu\nu}t_{\sigma}{}^{\bar{\mu}}{}_{\nu}\phi_{;\mu}\phi_{;\bar{\mu}}\,, \tag{3.24}$$

$$J_6 = t^{\sigma\mu\nu}t_{\sigma}{}^{\bar{\mu}\bar{\nu}}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar{\mu}}\phi_{;\bar{\nu}}\,, \tag{3.25}$$

$$J_8 = t^{\sigma\mu\nu}t_{\sigma\mu}{}^{\bar{\nu}}\phi_{;\nu}\phi_{;\bar{\nu}}\,, \tag{3.26}$$

$$J_{10} = \epsilon^{\mu}{}_{\nu\sigma\rho}a^{\nu}t^{\alpha\rho\sigma}\phi_{;\mu}\phi_{;\alpha}\,, \tag{3.27}$$

Thus, there are only seven independent scalars that satisfy all the conditions and can be condensed into

$$\mathcal{L}_{\mathrm{Tele}} := G_{\mathrm{Tele}}(\phi, X, T, T_{\mathrm{axi}}, T_{\mathrm{vec}}, I_2, J_1, J_3, J_5, J_6, J_8, J_{10})\,. \tag{3.28}$$

The Lagrangian of BDLS theory, i.e. the teleparallel analogue of Horndeski gravity is given by

$$\mathcal{S}_{\mathrm{BDLS}} := \frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\mathcal{L}_{\mathrm{Tele}} + \frac{1}{2\kappa^2}\sum_{i=2}^{5}\int \mathrm{d}^4x\, e\mathcal{L}_i + \mathcal{S}_{\mathrm{m}}\,. \tag{3.29}$$

The standard Horndeski theory as well as any scalar tensor type of teleparallel theory like e.g. $f(T)$, $f(T_{\text{axi}}, T_{\text{vec}}, T_{\text{ten}})$, teleparallel dark-energy models, Gauss-Bonnet theory in the conformal frame and more are contained in the BDLS theory.

As it is evident by setting $G_{\text{Tele}} = 0$ the Horndeski gravity, being the most general scalar tensor theory admitting second order field equations, is immediately recovered. The BDLS theory in this regard is both covariant under LLT and diffeomorphisms since G_{Tele} is a scalar build from the tetrad and the spin connection. In order, to calculate the field equations for this theory the action (3.29) is varied wrt the tetrad $e^A{}_\mu$ (in the Weitzenböck gauge) and the scalar field ϕ. The variation wrt the tetrad part gives

$$\begin{aligned}\delta_e \mathcal{S}_{\text{BDLS}} &= e\mathcal{L}_{\text{Tele}} e_a{}^\mu \delta e^a{}_\mu + e\delta_e \mathcal{L}_{\text{Tele}} \\ &\quad + e\sum_{i=2}^{5} \mathcal{L}_i e_a{}^\mu \delta e^a{}_\mu + e\delta_e \sum_{i=2}^{5} \mathcal{L}_i + 2\kappa^2 e \Theta_a{}^\mu \delta e^a{}_\mu \,, \end{aligned} \tag{3.30}$$

The contributions $\delta_e \sum_{i=2}^{5} \mathcal{L}_i$ are identified with standard Horndeski theory whereas $\delta_e \mathcal{L}_{\text{Tele}}$ is solely related with the TG sector. After some intense algebra the final form of the field equations is

$$\begin{aligned}& 4(\partial_\lambda G_{\text{Tele,T}}) S_a{}^{\lambda\mu} + 4e^{-1}\partial_\lambda(eS_a{}^{\lambda\mu}) G_{\text{Tele,T}} - 4G_{\text{Tele},T} T^\sigma{}_{\lambda a} S_\sigma{}^{\mu\lambda} + 4G_{\text{Tele,T}} \omega^b{}_{a\nu} S_b{}^{\nu\mu} \\ & - \phi_{;a}\Big[G_{\text{Tele,X}} \phi^{;\mu} - G_{\text{Tele,I}_2} v^\mu - 2G_{\text{Tele,J}_1} a^\mu a_j \phi^{;j} + G_{\text{Tele,J}_3} v_i t_k{}^{\mu i} \phi^{;k} - 2G_{\text{Tele,J}_5} t^{i\mu k} t_{ijk} \phi^{;j} \\ & + 2G_{\text{Tele,J}_6} t_{ilk} t^\mu{}_M{}^i \phi^{;k} \phi^{;l} \phi^{;m} - 2G_{\text{Tele,J}_8} t_{ijk} t^{ij\mu} \phi^{;k} - G_{\text{Tele,J}_{10}} a^j \phi^{;i}\Big(\epsilon_{\mu jcd} t_i{}^{cd} + \epsilon_{ijcd} t^{\mu cd}\Big)\Big] \\ & + \frac{1}{3}\Big[M^i(\epsilon_{ib}{}^{cd} e_c{}^\mu T^b{}_{ad} - \epsilon_{ib}{}^{cd} e_d{}^\mu \omega^b{}_{ac}) + e^{-1}\partial_\nu\Big(eM^i \epsilon_{ia}{}^{cd} e_c{}^\nu e_d{}^\mu\Big)\Big] \\ & - N^i(e_i{}^\mu \omega^\rho{}_{a\rho} - \omega^\mu{}_{ai} - T^\mu{}_{ai} - v_a e_i{}^\mu) + e^{-1}\partial_\nu\Big(eN^i(e_a{}^\nu e_i{}^\mu - e_a{}^\mu e_i{}^\nu)\Big) \\ & - O^{ijk} H_{ijka}{}^\mu + e^{-1}\partial_\nu\Big(eO^{ijk} L_{ijka}{}^{\mu\nu}\Big) - \mathcal{L}_{\text{Tele}} e_a{}^\mu + 2e_a{}^\nu g^{\mu\alpha} \sum_{i=2}^{5} \mathring{G}^{(i)}{}_{\alpha\nu} = 2\kappa^2 \Theta_a{}^\mu \,, \end{aligned} \tag{3.31}$$

where

$$M^i = 2G_{\text{Tele},\text{T}_{\text{ax}}} a^i + 2G_{\text{Tele},\text{J}_1} \phi^{;i}\phi^{;j} a_j + G_{\text{Tele},\text{J}_{10}} \epsilon_a{}^i{}_{cd} \phi^{;a}\phi^{;j} t_j{}^{cd} \,, \tag{3.32}$$

$$N^i = 2G_{\text{Tele},\text{T}_{\text{vec}}} v^i + G_{\text{Tele},\text{I}_2} \phi^{;i} + 2G_{\text{Tele},\text{J}_2} \phi^{;i}\phi^{;j} v_j + G_{\text{Tele},\text{J}_3} \phi^{;k}\phi^{;j} t^i{}_{kj} \,, \tag{3.33}$$

$$\begin{aligned} O^{ijk} = {} & G_{\text{Tele},\text{J}_3} \phi^{;j}\phi^{;k} v^i + 2G_{\text{Tele},\text{J}_5} \phi^{;l}\phi^{;j} t^i{}_l{}^k + 2G_{\text{Tele},\text{J}_6} \phi^{;j}\phi^{;k}\phi^{;l}\phi^{;m} t^i{}_{lm} + 2G_{\text{Tele},\text{J}_8} \phi^{;l}\phi^{;k} t^{ij}{}_l \\ & + G_{\text{Tele},\text{J}_{10}} \epsilon_{ab}{}^{jk} \phi^{;a}\phi^{;b}\phi^{;i} \,, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} H_{ijka}{}^{\mu} := \frac{\partial t_{ijk}}{\partial e^a{}_{\mu}} = {} & \\ & \frac{1}{2}\Big[\omega_{iaj} e_k{}^{\mu} - \omega_{iak} e_j{}^{\mu} - T_{ija} e_k{}^{\mu} - T_{iak} e_j{}^{\mu} + \omega_{jai} e_k{}^{\mu} - \omega_{jak} e_i{}^{\mu} - T_{jia} e_k{}^{\mu} - T_{jak} e_i{}^{\mu}\Big] \\ & + \frac{1}{6}\Big[\eta_{ki} C_{ja}{}^{\mu} - \eta_{kj} C_{ia}{}^{\mu} - 2\eta_{ij} C_{ka}{}^{\mu} + v_j D_{kia}{}^{\mu} - v_i D_{kja}{}^{\mu} - 2v_k D_{ija}{}^{\mu}\Big] . \end{aligned} \tag{3.35}$$

$$\begin{aligned} L_{ijka}{}^{\mu\nu} := \frac{\partial t_{ijk}}{\partial e^a{}_{\mu,\nu}} = {} & \\ & \frac{1}{2}\Big[\eta_{ai}(e_j{}^{\nu} e_k{}^{\mu} - e_j{}^{\mu} e_k{}^{\nu}) + \eta_{aj}(e_i{}^{\nu} e_k{}^{\mu} - e_i{}^{\mu} e_k{}^{\nu})\Big] + \frac{1}{6}\Big[\eta_{ki}(e_a{}^{\nu} e_j{}^{\mu} - e_a{}^{\mu} e_j{}^{\nu}) \\ & - \eta_{kj}(e_a{}^{\nu} e_i{}^{\mu} - e_a{}^{\mu} e_i{}^{\nu}) - 2\eta_{ij}(e_a{}^{\nu} e_k{}^{\mu} - e_a{}^{\mu} e_k{}^{\nu})\Big] , \end{aligned} \tag{3.36}$$

$$C_{ia}{}^{\mu} := \frac{\partial v_i}{\partial e^a{}_{\mu}} = e_i{}^{\mu} \omega^{\rho}{}_{a\rho} - \omega^{\mu}{}_{ai} - T^{\mu}{}_{ai} - v_a e_i{}^{\mu} \,, \tag{3.37}$$

$$D_{kia}{}^{\mu} := \frac{\partial \eta_{ki}}{\partial e^a{}_{\mu}} = \delta_i^b \eta_{ab} e^{\mu}{}_k + \delta_k^b \eta_{ab} e^{\mu}{}_i - \eta_{ai} e^{\mu}{}_k - \eta_{ka} e^{\mu}{}_i \,. \tag{3.38}$$

Regarding the terms $\mathring{G}^{(i)}{}_{\alpha\nu}$, $\sum_{i=2}^{5} \mathring{G}^{(i)}{}_{\mu\nu}$ they were calculated in [125] (see Eqs. (13a)-(13d) therein). On the other hand, the field equations of the scalar field ϕ, which result form the variation of the action wrt the scalar field are

$$\mathring{\nabla}^{\mu}\Big(J_{\mu-\text{Tele}} + \sum_{i=2}^{5} J^{i}_{\mu}\Big) = P_{\phi-\text{Tele}} + \sum_{i=2}^{5} P^{i}_{\phi}\,, \tag{3.39}$$

where $J_{\mu-\text{Tele}}$ and $P_{\phi-\text{Tele}}$ are defined as

$$\begin{aligned}
J_{\mu-\text{Tele}} \;&=\; -G_{\text{Tele,X}}(\mathring{\nabla}_{\mu}\phi) + G_{\text{Tele,I}_2} v_{\mu} + 2G_{\text{Tele,J}_1} a_{\mu} a^{\nu}\mathring{\nabla}_{\nu}\phi - G_{\text{Tele,J}_3} v_{\alpha} t_{\mu}{}^{\nu\alpha}(\mathring{\nabla}_{\nu}\phi) \\
&\quad -2G_{\text{Tele,J}_5} t^{\beta\nu\alpha} t_{\beta\mu\alpha}(\mathring{\nabla}_{\nu}\phi) + 2G_{\text{Tele,J}_8} t^{\alpha\nu}{}_{\mu} t_{\alpha\nu}{}^{\beta}(\mathring{\nabla}_{\beta}\phi) \\
&\quad -2G_{\text{Tele,J}_6} t^{\nu\alpha\beta} t_{\mu}{}^{\sigma}{}_{\nu}(\mathring{\nabla}_{\alpha}\phi)(\mathring{\nabla}_{\beta}\phi)(\mathring{\nabla}_{\sigma}\phi)\,, \\
&\quad -G_{\text{Tele,J}_{10}} a^{\nu}(\mathring{\nabla}_{\alpha}\phi)(\epsilon^{\mu}{}_{\nu\rho\sigma} t^{\alpha\rho\sigma} + \epsilon^{\alpha}{}_{\nu\rho\sigma} t^{\mu\rho\sigma})\,,
\end{aligned} \tag{3.40}$$

$$P_{\phi-\text{Tele}} \;=\; G_{\text{Tele},\phi}\,. \tag{3.41}$$

By utilizing the identity in Eq. (2.57), it follows that P^{i}_{ϕ} is [125]

$$P^{2}_{\phi} = G_{2,\phi}\,, \tag{3.42a}$$

$$P^{3}_{\phi} = \mathring{\nabla}_{\mu} G_{3,\phi} \mathring{\nabla}^{\mu}\phi\,, \tag{3.42b}$$

$$P^{4}_{\phi} = G_{4,\phi}(-T + B) + G_{4,\phi X}\left[(\mathring{\Box}\phi)^2 - (\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\phi)^2\right]\,, \tag{3.42c}$$

$$P^{5}_{\phi} = -\mathring{\nabla}_{\mu} G_{5,\phi} \mathring{G}^{\mu\nu}\mathring{\nabla}_{\nu}\phi - \frac{1}{6} G_{5,\phi X}\left[(\Box\phi)^3 - 3\Box\phi(\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\phi)^2 + 2(\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\phi)^3\right]\,, \tag{3.42d}$$

and J^{i}_{μ} are defined as

$$J^{2}_{\mu} = -\mathcal{L}_{2,X}\mathring{\nabla}_{\mu}\phi\,, \tag{3.43a}$$

$$J^{3}_{\mu} = -\mathcal{L}_{3,X}\mathring{\nabla}_{\mu}\phi + G_{3,X}\mathring{\nabla}_{\mu}X + 2G_{3,\phi}\mathring{\nabla}_{\mu}\phi\,, \tag{3.43b}$$

$$J^{4}_{\mu} = -\mathcal{L}_{4,X}\mathring{\nabla}_{\mu}\phi + 2G_{4,X}\mathring{R}_{\mu\nu}\mathring{\nabla}^{\nu}\phi - 2G_{4,XX}\left(\mathring{\Box}\phi\mathring{\nabla}_{\mu}X - \mathring{\nabla}^{\nu}X\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\phi\right)$$

$$
\begin{aligned}
&- 2G_{4,\phi X}(\mathring{\Box}\phi\mathring{\nabla}_{\mu}\phi + \mathring{\nabla}_{\mu}X)\,, && (3.43c)\\
J^5_{\mu} &= -\mathcal{L}_{5,X}\mathring{\nabla}_{\mu}\phi - 2G_{5,\phi}\mathring{G}_{\mu\nu}\mathring{\nabla}^{\nu}\phi\\
&- G_{5,X}\left[\mathring{G}_{\mu\nu}\mathring{\nabla}^{\nu}X + \mathring{R}_{\mu\nu}\Box\phi\mathring{\nabla}^{\nu}\phi - \mathring{R}_{\nu\lambda}\mathring{\nabla}^{\nu}\phi\mathring{\nabla}^{\lambda}\mathring{\nabla}_{\mu}\phi - \mathring{R}_{\alpha\mu\beta\nu}\mathring{\nabla}^{\nu}\phi\mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\phi\right]\\
&+ G_{5,XX}\Big\{\frac{1}{2}\mathring{\nabla}_{\mu}X\left[(\mathring{\Box}\phi)^2 - (\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\beta}\phi)^2\right] - \mathring{\nabla}_{\nu}X\left(\mathring{\Box}\phi\mathring{\nabla}_{\mu}\mathring{\nabla}^{\nu}\phi - \mathring{\nabla}_{\alpha}\mathring{\nabla}_{\mu}\phi\mathring{\nabla}^{\alpha}\mathring{\nabla}^{\nu}\phi\right)\Big\}\\
&+ G_{5,\phi X}\Big\{\frac{1}{2}\mathring{\nabla}_{\mu}\phi\left[(\mathring{\Box}\phi)^2 - (\mathring{\nabla}_{\alpha}\mathring{\nabla}_{\beta}\phi)^2\right] + \mathring{\Box}\phi\mathring{\nabla}_{\mu}X - \mathring{\nabla}^{\nu}X\mathring{\nabla}_{\nu}\mathring{\nabla}_{\mu}\phi\Big\}\,. && (3.43d)
\end{aligned}
$$

Finally, these terms can be expressed in terms of teleparallel quantities by relating the LC connection with the TG connection as

$$\mathring{R}^{\lambda}{}_{\mu\sigma\nu} = \mathring{\nabla}_{\nu}K_{\sigma}{}^{\lambda}{}_{\mu} - \mathring{\nabla}_{\sigma}K_{\nu}{}^{\lambda}{}_{\mu} + K_{\sigma}{}^{\rho}{}_{\mu}K_{\nu}{}^{\lambda}{}_{\rho} - K_{\sigma}{}^{\lambda}{}_{\rho}K_{\nu}{}^{\rho}{}_{\mu}\,, \tag{3.44}$$

$$\mathring{R}_{\mu\nu} = \mathring{\nabla}_{\nu}K_{\lambda}{}^{\lambda}{}_{\mu} - \mathring{\nabla}_{\lambda}K_{\nu}{}^{\lambda}{}_{\mu} + K_{\lambda}{}^{\rho}{}_{\mu}K_{\nu}{}^{\lambda}{}_{\rho} - K_{\lambda}{}^{\lambda}{}_{\rho}K_{\nu}{}^{\rho}{}_{\mu}\,, \tag{3.45}$$

$$\mathring{G}_{\mu\nu} = e^{-1}e^{a}{}_{\mu}g_{\nu\rho}\partial_{\sigma}(eS_{a}{}^{\rho\sigma}) - S_{b}{}^{\sigma}{}_{\nu}T^{b}{}_{\sigma\mu} + \frac{1}{4}Tg_{\mu\nu} - e^{a}{}_{\mu}\omega^{b}{}_{a\sigma}S_{b\nu}{}^{\sigma}\,. \tag{3.46}$$

This concludes the sketch of the derivation of the field equations of the BDLS theory. Having the field equations at hand, the next step will be linearizing them in order to derive the GWPE.

3.4 The Gravitational Wave Propagation Equation

GR assumes two pdof that correspond to a massless spin 2 particle which is a bosonic force carrier. The way this is manifested is by calculating the tensor perturbations in flat FLRW [58]. With the GWPE the theory can be directly confronted against observations like the ones leading to the constraints of Eqs. (2.124) – (2.125) which are of vital importance [5, 34].

For this reason, the calculations of the GWPE of the BDLS theory is of major importance for its viability. Utilizing the tensor perturbations in a spatially flat FLRW background, the tensor part of the perturbed tetrad reads as Sec. 2.9.2

$$\delta e^I{}_j = \frac{a}{2}\delta^{Ii}h_{ij}\,. \tag{3.47}$$

If one replaces this in the perturbed metric, then the standard tensor perturbations of the metric are reproduced [58]. Linearizing the field equations by performing the substitution of (2.84) in the field equations (3.3) thus splitting the field equations into zeroth and first order parts. Replacing (3.47) then the first order part of the field equations represents the GWPE of the BDLS theory which assumes the general form of Eq. (2.119) where α_T is calculated as

$$\alpha_T = \frac{2X}{M_*^2}\left(2G_{4,X} - 2G_{5,\phi} - G_{5,X}(\ddot{\phi} - \dot{\phi}H) - 2G_{\mathrm{Tele,J_8}} - \frac{1}{2}G_{\mathrm{Tele,J_5}}\right), \tag{3.48}$$

the effective Planck mass is given by

$$M_*^2 = 2\Big(G_4 - 2XG_{4,X} + XG_{5,\phi} - \dot{\phi}XHG_{5,X} + 2XG_{\mathrm{Tele,J_8}} + \frac{1}{2}XG_{\mathrm{Tele,J_5}} - G_{\mathrm{Tele,T}}\Big), \tag{3.49}$$

and the α_M is given by replacing Eq. (3.49) in Eq. (2.121). The only surviving scalars to the G_{Tele} term are $T = 6H^2$, $T_{\mathrm{vec}} = -9H^2$, and $I_2 = 3H\dot{\phi}$, while the other scalars all vanish up to first order.

The existence of the of the G_{Tele} term in Eq. (2.119) directly differentiates between the BDLS and Horndeski theories, thus leading to a revision of the propagation speed of the GW since $G4$ and $G5$ can be reconsidered while still respecting the observational constraint. In Horndeski gravity, $G4$ and $G5$ are trivialized which leads to the exclusion of many important models like those described in Sec. 3.2.

This revision of the speed of propagation is what allows for models previously rejected to reemerge as solutions of $\alpha_T = 0$. For $G_{\mathrm{Tele}} = 0$, as a consistency check, we recover all the usual results [59]. Using Eq. (3.48), the number of available model solutions can be refined further whilst solving for for G_{Tele}. It is expected that further constraints coming from observational data will help further in constraining the form of G_{Tele}.

It should be noted that the existence of non-trivial values for the tensor excess speed $\alpha_T \neq 0$ and frictional term $\alpha_M \neq 0$ affect also the waveform itself. In fact, the BDLS waveform and the standard GR waveform are related through Eq. (2.122) as

$$h_{\text{BDLS}} \sim h_{\text{GR}} \underbrace{e^{-\frac{1}{2}\int \alpha_M \mathcal{H} d\eta}}_{\text{Amplitude}} \underbrace{e^{ik\int \sqrt{\alpha_T} d\eta}}_{\text{Phase}}, \tag{3.50}$$

where α_T controls the deviation of the amplitude and α_M controls the phase difference. Thus, even at this level of waveforms it is evident that the BDLS theory is a much more general theory. Finally, $\alpha_M \neq 0$ offers another way of differentiating from GR via the luminosity distance as illustrated in Eq. (2.123) which for the BDLS is translated as

$$\frac{d_L^{BDLS}(z)}{d_L^{EM}(z)} = \exp\left[\frac{1}{2}\int_0^z \frac{\alpha_M}{1+z'}\, dz'\right]. \tag{3.51}$$

Nevertheless, this relation cannot be tested yet in order to further constrain α_M since the data needed will be available by the next GW detectors.

3.5 Reviving Horndeski using Teleparallel gravity

In standard Horndeski gravity, enforcing the constraint $\alpha_T = 0$, results in $G_4(\phi, X) = G_4(\phi)$ and $G_5(\phi, X) = \text{const}$. This severely constrains Horndeski theory. More specifically, quartic and quintic Galileon models [126, 127], de-Sitter Horndeski [128], the Fab Four [129] and the purely kinetic coupled models [130] become acutely constrained by imposing $\alpha_T = 0$.

Using, as an example, the quartic Galileon model defined by the action

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\left[\frac{\mathring{R}}{2\kappa^2} + \frac{1}{2\kappa^2}\sum_{i=2}^{5}\mathcal{L}_i\right], \tag{3.52}$$

where the $\mathcal{L}_i$ are defined in Eqs. (3.8) – (3.10). This is a well studied model and in [131] the authors found self-accelerating solutions, studied their stability and reported also spherically symmetric solutions. Additionally, in Ref.[132] it was shown that the Vainshtein mechanism, which allows to hide the effect of some degrees of freedom due to non-linear effects in appropriate distances, suppresses the variations in such a way that they are a good fit in CMB and BAO data. Moreover, in [133] it was reported that the model Eq. (3.52), via a shift symmetry of the scalar and the fermionic sector is super-symmetrizable. This model was trivialized after the constraint $\alpha_T = 0$. Within the BDLS framework this quartic model could be revived by rewriting its action as

$$\mathcal{S} = \int d^4x\, e\left[\frac{G_{\rm Tele}}{2\kappa^2} + \frac{1}{2\kappa^2}\sum_{i=1}^{4}\mathcal{L}_i\right]. \tag{3.53}$$

In this context, $G_{\rm Tele} \neq 0$ also compensates about the $\alpha_T = 0$ constraint. More precisely, we find the functions $G_{\rm Tele}$, G_4 and G_5 that satisfy $\alpha_T = 0$. This condition is solved by $G_5 = G_5(\phi)$ and $G_{\rm Tele} = \tilde{G}_{\rm tele}(\phi, X, T, T_{\rm vec}, T_{\rm ax}, I_2, J_1, J_3, J_6, J_8 - 4J_5, J_{10})$. Taking these solutions into account the Lagrangian density can be written as

$$\begin{aligned}\mathcal{L} &= \tilde{G}_{\rm tele}(\phi, X, T, T_{\rm vec}, T_{\rm ax}, I_2, J_1, J_3, J_6, J_8 - 4J_5, J_{10}) + G_2(\phi, X) + G_3(\phi, X)\Box\phi\,,\\ &+ G_4(\phi, X)\,(-T + B) + G_{4,X}\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu} + 4J_5\right] + G_5(\phi)\mathring{G}_{\mu\nu}\phi^{;\mu\nu} - 4J_5G_{5,\phi}\,.\end{aligned} \tag{3.54}$$

The essence of this result is that $G_4(\phi, X)$ and $G_5(\phi)$ are not trivialized compared to standard Horndeski gravity. Notice that the Lagrangian densities $\mathcal{L}_4$ and $\mathcal{L}_5$ are modified by a term proportional to J_5 in order to satisfy the constraint $\alpha_T = 0$. This new adapted Lagrangian (3.54) practically revives the quartic model introduced in Eq. (3.52).

In the same manner one could also consider the non-minimally coupled model

$$\mathcal{S} = \int d^4x \sqrt{-g} \Big\{ \frac{\mathring{R}}{2\kappa^2} - \Big[w\, g_{\mu\nu} + z\, \mathring{G}_{\mu\nu} \Big] \phi^{;\mu} \phi^{;\nu} - 2V(\phi) \Big\} + \mathcal{S}_{\text{matter}}\,, \tag{3.55}$$

where w and z are two coupling constants. This model is also heavily studied in the literature [134, 135, 136, 137, 138, 139, 130, 125] since it serves as a realistic cosmological framework due to its higher-order coupling. It turned out that, this model also suffered the same fate as (3.52) after the constrain. Hence in the same manner one could revive it in the BDLS framework as

$$\mathcal{S} = \int d^4x\, e \Big\{ \frac{G_{Tele}}{2\kappa^2} - \Big[\epsilon\, g_{\mu\nu} + \eta\, \mathring{G}_{\mu\nu} \Big] \phi^{;\mu} \phi^{;\nu} - 2V(\phi) \Big\} + \mathcal{S}_{\text{matter}}\,. \tag{3.56}$$

This Lagrangian is to be understood in conjunction with Eq. (3.54) just like in the previous quartic model (3.52).

3.6 Conclusions

The introduction of the BDLS model in [64] allowed for an extension of the standard Horndeski model by a purely teleparallel term G_{Tele}. Naturally for $G_{\text{Tele}} \equiv 0$ the standard Horndeski theory is retrieved. This teleparallel term in order to be formed a few conditions needed to be imposed: (i) field equations must be at most second order in tetrad and scalar field derivatives; (ii) the theory must be not parity violating (iii) the contractions of the torsion tensor must be only up to quadratic terms. The first two conditions are self explained more or less, (iii) was specifically imposed to limit the available pool of scalars since in principle one could have infinite torsion-self contractions in a scalar. This would lead to an infinite number of scalars. The theory altogether is both LLT and diffeomorphism covariant since the tetrad must be explicitly used as a dynamical variable instead of the metric.

Horndeski gravity is the most general scalar tensor theory build using the LC connection that assumes second order field equations wrt both the metric and the scalar field. Due to confrontation with the event GW170817 it was markedly constrained rendering its most popular models unusable for cosmology. By extending Horndeski gravity in the framework of BDLS theory it was shown that the revival of a big class of models was achievable by slightly modifying the G_{Tele} term as in Eq. (3.54) discussed in detail in Sec. 3.5. This modification was indicated by demanding well behaved tensor perturbations in relation to the constraint of Eq. (2.124) or $\alpha_T = 0$.

As, also discussed in Sec. 2.9.5 the fact that the BDLS theory predicts $\alpha_T \neq 0$ and $\alpha_M \neq 0$ leads to modifications compared to the standard GR waveform. The difference is both in the amplitude and the phase as generated by α_T and α_M and for the BDLS theory this is shown in Eq. (3.50). Hence, in general, the BDLS waveform will be modified both in amplitude and phase compared to GR. On top of that, α_M is also responsible for modifying the ratio of luminosity distances of gravity and electromagnetism as in Eq. (3.51). Hence, the BDLS theory is clearly distinguished from GR regarding its GW properties. Of course, all these considerations are highly model dependent properties but in the most general case of the BDLS theory it is expected that all these modifications will be active.

To conclude, the core result of this section resides in the Eq. (3.48) which has to be constrained to $\alpha_T = 0$ in order to comply with Eq. (2.124). In other words, the tensor waves or just physical observed GW is demanded to travel at the speed of light. Subsequently, imposing $\alpha_T = 0$ leads to a modified Lagrangian Eq. (3.54) that effectively revives the previously discarded models of Horndeski gravity like Eqs. (3.52) and (3.55). The modification described in Eq. (3.54) is just extending G_4 and G_5 by a term proportional to J_5. This was discussed in the context of reviving the theories described in Eq. (3.52) and (3.55). Nevertheless, this is a much general result that holds for any model within the BDLS theory.

CHAPTER 4

DEGREES OF FREEDOM AND POLARIZATIONS IN THE TELEPARALLEL ANALOGUE OF HORNDESKI.

4.1 Introduction

Having observed GW first by the LIGO collaboration [34] and then by the Virgo collaboration [35], enabled testing of GR in strong field regime via template matching [140]. Compared to GR, modified theories of gravity require much more theoretical and numerical work before any kind of competing templates can be produced even for the simplest available scenarios like the binary black hole coalescence events. In addition, having accurate knowledge of the polarization state of GW of a model, reveals a great deal about its dynamical content [104, 141].

GR propagates only 2 dof [7] which are represented by the tensorial part of the metric once split into an SVT decomposition. This also means that only tensor polarizations will be expected from any observations [142] that use GR as their template and thus are model dependent. It is also a very special case of model and normally there could be more than 2 pdof in a gravitational theory. A way to extract crucial information about the pdof of a theory observationally is via its polarizations. In general, the polarizations of a gravitational theory are calculated from the electric components of the Riemann tensor which has 6 independent components due to its symmetry. Geometrically, these components control the response of the geodesic deviation equation which describes how free falling test particles move in a gravitational field. These 6 components of the electric part of the Riemann tensor can be further split into 2 scalar ones (breathing and longitudinal), 2 vectorial ones (x and vector y) and 2 tensorial ones (+ and $\times$) [142], which are exactly all the polarization modes. For example, in $f(\mathring{R})$ gravity there are 3 pdof, just an extra massive scalar on top of tensorial ones also found in GR [27, 143, 144]. This extra scalar introduces both scalar polarizations in the massive sector. On the other hand, modified theories of gravity can be constructed that still propagate only 2 dof. Thus, depending on the problem, gravitational theories could be endowed with more dof as long as they are healthy as explained in Sec. 2.8.

The polarizations of GW present a fundamental way of discriminating between GR and other modified gravity theories. The measurements regarding polarizations will become more and more accurate with the next generation of detectors such as LISA [106] and the Einstein Telescope [145], amongst others. In this section, the polarization content of the BDLS theory will be probed. This theory, as explained in detail in Sec. 3.3, can be considered as a natural extension of the standard Horndeski theory by switching to the teleparallel connection instead. As we showed in Sec. 3, the BDLS theory is not highly constrained by the GW propagation speed like the Horndeski theory. This means that a huge class of rejected models, due to compliance with this constraint, are now available in the teleparallel extension. Hence in a way BDLS theory revived the Horndeski theory by extending it in the teleparallel realm. This indication is yet another reason to delve deeper into the properties of GW in the BDLS theory and acquire more information about the various branches of the theory and its dof.

The polarizations of Horndeski models were explored in In Ref. [146] where they found only a massive scalar sector that included all scalar polarizations on top of the usual tensorial modes of GR. This section will be devoted to the in-depth calculation of the dof and polarization modes of BDLS theory. In Sec. 4.2 the specific method of perturbations will be laid out and utilized to in order to calculate the linearized field equations of the BDLS theory in Minkowski spacetime. Then, in Sec.4.3 the linearized field equations will be analyzed in order to find the propagating dof of the theory for the various scenarios arising by the branching of the theory. Finally, in Sec. 4.6 the results of Sec.4.3 will be used in order to fully calculate and classify the polarization content of the BDLS theory. Finally, a discussion along with an overview of the results will take place in Sec. 4.7.

4.2 Perturbations in Teleparallel Gravity

The perturbative framework introduced in Sec. 2 is adapted to the Minkowski spacetime which serves as the simplest spacetime in which one can study gravity. The perturbed tetrad $\tilde{e}^A{}_\mu$ and the perturbed scalar field ϕ can be expanded up to first order as

$$\tilde{e}^A{}_\mu = \delta^A_\mu + \epsilon\,\delta e^A{}_\mu = \delta^A_\mu + \epsilon\,\delta^\nu_B \eta^{AB}\tau_{\mu\nu}\,, \tag{4.1}$$

$$\phi = \phi_0 + \epsilon\,\delta\phi\,, \tag{4.2}$$

where we introduced ϵ ($|\epsilon| << 1$) as the parameter of the perturbations denoting the order, $e^B{}_\mu$ is the background value of the tetrad which is just $\delta^B{}_\mu$ in Minkowski, and ϕ_0 is the background value of the scalar field ϕ in Minkowski.

A very useful variable, the spacetime indexed version of the perturbation of the tetrad $\delta e^A{}_\nu$, will also be used in order to simplify the calculations later on. This variable is defined as

$$\tau_{\mu\nu} := \eta_{AB} e^B{}_\mu \delta e^A{}_\nu\,, \tag{4.3}$$

In general, the background value of ϕ can be time dependent but we will not pursue this route in this analysis. The perturbed metric up to second order can be expanded as

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu} + \frac{1}{2}\epsilon^2\, \delta^2 g_{\mu\nu} = \eta_{\mu\nu} + 2\epsilon\, \tau_{(\mu\nu)} + \epsilon^2\, \tau_{\alpha\mu}\tau^{\alpha}{}_{\nu}\,. \tag{4.4}$$

The goal is obtaining the linearized field equations of the BDLS theory. There are two ways in order for this goal to be realized. The most common way is to directly perturb the field Eqs. (3.3) and (3.39) as it was done in Sec. 3. The other way is to linearize the action (3.29) by expanding it up to second order in the tetrad, scalar field and then varying it wrt to these perturbed variables. In this way the linearized field equations for each variable are obtained. The linearization of the action will be used in this instance in order to derive the perturbations of the BDLS theory in the Minkowski spacetime since it turns out to be the simpler approach in this highly symmetric background.

The perturbative expansion of the field equations is obtained by first expanding the functions in the Lagrangian up to second order. This particular choice of order, is due to the fact that at the level of the action, this is the first non-trivial order that will result in first order field equations. The second order expansion of the action is then achieved by performing a Taylor expansion of the background functions. Consider a function of scalars, say $G(\alpha,\beta)$, such that the parameters of the function are expanded as follows

$$\alpha = \alpha^{(0)} + \epsilon\, \alpha^{(1)} + \epsilon^2 \alpha^{(2)}\,, \tag{4.5}$$

$$\beta = \beta^{(0)} + \epsilon \beta^{(1)} + \epsilon^2 \beta^{(2)} + \epsilon^3 \beta^{(3)}\,, \tag{4.6}$$

where $\alpha^{(0)}$ and $\beta^{(0)}$ are constants at the zeroth order which represent the background parameters. Taylor expanding function G about the zeroth order results in the following equation:

$$G(\alpha,\beta) = G(0) + G_{,\alpha}(0)\left[\alpha - \alpha^{(0)}\right] + G_{,\beta}(0)\left[\beta - \beta^{(0)}\right] + G_{,\alpha\beta}(0)\left[\alpha - \alpha^{(0)}\right]\left[\beta - \beta^{(0)}\right] \tag{4.7}$$

$$
\begin{aligned}
&+\frac{1}{2}G_{,\alpha\alpha}(0)\left[\alpha-\alpha^{(0)}\right]^2+\frac{1}{2}G_{,\beta\beta}(0)\left[\beta-\beta^{(0)}\right]^2+...\\
&=G(0)+\epsilon\left[G_{,\alpha}(0)\alpha^{(1)}+G_{,\beta}(0)\beta^{(1)}\right]\\
&+\epsilon^2\left[G_{,\alpha}(0)\alpha^{(2)}+G_{,\beta}(0)\beta^{(2)}+G_{,\alpha\beta}(0)\alpha^{(1)}\beta^{(1)}+\frac{1}{2}\left(G_{,\alpha\alpha}\alpha^{(1)^2}+\frac{1}{2}G_{,\beta\beta}(0)\beta^{(1)^2}\right)\right]\\
&+\mathcal{O}(\epsilon^3)\,, \qquad (4.8)
\end{aligned}
$$

where $G(0)=G\left(\alpha^{(0)},\beta^{(0)}\right)$ stands for the background of the function G. This equation shows that scalars with contributions to perturbation orders higher than two, such as β, will not appear in the second order expansion of the function. In the case of BDLS theory, the scalar invariants J_1, J_3, J_5, J_6, J_8 and J_{10} do not appear in the expansion as seen in Eq. (4.9) since they do not have contributions that are smaller than the perturbative order of 3. Moreover, in the case of X, there is only a second order contribution which leads to no second order derivative terms in the function expansion.

Expanding the G_{Tele} function up to second order around the background values $e^A{}_\mu=\delta^B{}_\mu$ and $\phi=\phi_0$.

$$
\begin{aligned}
&G_{\text{Tele}}\left(\phi,X,T,T_{\text{ax}},T_{\text{vec}},I_2,J_1,J_3,J_5,J_6,J_8,J_{10}\right)=G_{\text{Tele}}+\epsilon\, G_{\text{Tele},\phi}\delta\phi\\
&+\epsilon^2\Big[\frac{1}{2}G_{\text{Tele},\phi\phi}\delta\phi^2+G_{\text{Tele},X}X+G_{\text{Tele},T}T+G_{\text{Tele},T_{\text{ax}}}T_{\text{ax}}+G_{\text{Tele},T_{\text{vec}}}T_{\text{vec}}+G_{\text{Tele},I_2}I_2\Big]\\
&+\mathcal{O}(\epsilon^3)\,, \qquad (4.9)
\end{aligned}
$$

$$
G_j(\phi,X)=G_j+\epsilon\, G_{j,\phi}\,\delta\phi+\epsilon^2\left[\frac{1}{2}G_{j,\phi\phi}\delta\phi^2+G_{j,X}X\right]+\mathcal{O}(\epsilon^3)\,, \qquad (4.10)
$$

where $G_{\text{Tele},i}=G_{\text{Tele},i}(\phi_0,0,0,0,0,0,0,0,0,0,0,0)$ for $i=\{\phi,\phi\phi,X,T,T_{\text{ax}},T_{\text{vec}},I_2\}$ such that $\phi\phi$ denotes the second order derivative wrt ϕ and $G_{j,k}=G_{j,k}(\phi_0,0)$ for $j=\{2,3,4,5\}$. Note that there are no J_i terms since they are zero. Having laid out the method of expansion the last step is to perform variation wrt to the variables $\tau_{\mu\nu}$ and $\delta\phi$ in order to obtain their linearized field equations $W_{\mu\nu} := \delta L^{(2)}/\delta\tau^{\mu\nu}$ and $W_{\delta\phi} := \delta L^{(2)}/\delta(\delta\phi)$ as

$$
\begin{aligned}
W_{\mu\nu} &= (G_{\text{Tele}} + G_2)\,\eta_{\mu\nu} \\
&\quad + \epsilon\Big[\,(G_{\text{Tele}} + G_2)\Big(\tau\,\eta_{\mu\nu} - \tau_{\mu\nu}\Big) - 2G_{\text{Tele},T_{\text{vec}}}\Big(\partial^\lambda\partial_\mu\tau_{\lambda\nu} - \partial_\mu\partial_\nu\tau - \partial_\lambda\partial_\beta\tau^{\lambda\beta}\eta_{\mu\nu} + \Box\tau\,\eta_{\mu\nu}\Big) \\
&\quad + (-2G_{\text{Tele},T} + 2G_4)\Big(\Box\tau_{(\mu\nu)} - \partial^\lambda\partial_\mu\tau_{(\nu\lambda)} - \partial^\lambda\partial_\nu\tau_{(\mu\lambda)} + \partial_\mu\partial_\nu\tau + \partial_\lambda\partial_\beta\tau^{\lambda\beta}\eta_{\mu\nu} - \Box\tau\,\eta_{\mu\nu}\Big) \\
&\quad + \frac{4}{9}G_{\text{Tele},T_{\text{ax}}}\Big(\Box\tau_{[\mu\nu]} - \partial^\lambda\partial_\nu\tau_{[\mu\lambda]} + \partial^\lambda\partial_\mu\tau_{[\nu\lambda]}\Big) + \Big(-G_{\text{Tele},I_2} + 2G_{4,\phi}\Big)\Big(\partial_\mu\partial_\nu\delta\phi - \eta_{\mu\nu}\Box\delta\phi\Big)\Big],
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
W_{\delta\phi} &= G_{\text{Tele},\phi} + G_{2,\phi} + \epsilon\Big[\Big(G_{\text{Tele},\phi} + G_{2,\phi}\Big)\,\tau + \Big(G_{\text{Tele},I_2} - 2G_{4,\phi}\Big)\Big(\Box\tau - \partial_\lambda\partial_\beta\tau^{\lambda\beta}\Big) \\
&\quad + \Big(G_{\text{Tele},\phi\phi} + G_{2,\phi\phi}\Big)\,\delta\phi + \Big(G_{\text{Tele},X} + G_{2,X} - 2G_{3,\phi}\Big)\Box\delta\phi\Big],
\end{aligned}
\tag{4.12}
$$

where $\tau := \eta^{\mu\nu}\tau_{\mu\nu}$. Imposing the background field equations, which are just the zeroth order parts of Eqs. (4.11) and (4.12)

$$0 = G_{\text{Tele}} + G_2\,, \tag{4.13}$$

$$0 = G_{\text{Tele},\phi} + G_{2,\phi}\,. \tag{4.14}$$

the on-shell linearized field equations are obtained, which read as

$$
\begin{aligned}
W_{\mu\nu} &= -2G_{\text{Tele},T_{\text{vec}}}\Big(\partial^\lambda\partial_\mu\tau_{\lambda\nu} - \partial_\mu\partial_\nu\tau - \partial_\lambda\partial_\sigma\tau^{\lambda\sigma}\eta_{\mu\nu} + \Box\tau\,\eta_{\mu\nu}\Big) \\
&\quad + (-2G_{\text{Tele},T} + 2G_4)\Big(\Box\tau_{(\mu\nu)} - \partial^\lambda\partial_\mu\tau_{(\nu\lambda)} - \partial^\lambda\partial_\nu\tau_{(\mu\lambda)} + \partial_\mu\partial_\nu\tau + \partial_\lambda\partial_\sigma\tau^{\lambda\sigma}\eta_{\mu\nu} - \Box\tau\,\eta_{\mu\nu}\Big) \\
&\quad + \frac{4}{9}G_{\text{Tele},T_{\text{ax}}}\Big(\Box\tau_{[\mu\nu]} - \partial^\lambda\partial_\nu\tau_{[\mu\lambda]} + \partial^\lambda\partial_\mu\tau_{[\nu\lambda]}\Big) + \Big(-G_{\text{Tele},I_2} + 2G_{4,\phi}\Big)\Big(\partial_\mu\partial_\nu\delta\phi - \eta_{\mu\nu}\Box\delta\phi\Big)\,,
\end{aligned}
\tag{4.15}
$$

$$W_{\delta\phi} = \Big(G_{\text{Tele},I_2} - 2G_{4,\phi}\Big)\Big(\Box\tau - \partial_\lambda\partial_\sigma\tau^{\lambda\sigma}\Big) + \Big(G_{\text{Tele},\phi\phi} + G_{2,\phi\phi}\Big)\,\delta\phi$$

$$+ \left(G_{\text{Tele},X} + G_{2,X} - 2G_{3,\phi}\right) \Box\delta\phi \,. \tag{4.16}$$

These field equations (4.15) can be further decomposed into symmetric and antisymmetric parts as follows

$$\begin{aligned} W_{(\mu\nu)} = & -2G_{\text{Tele},T_{\text{vec}}} \left(\partial^{\lambda}\partial_{(\mu}\tau_{|\lambda|\nu)} - \partial_{\mu}\partial_{\nu}\tau - \partial_{\lambda}\partial_{\beta}\tau^{\lambda\beta}\eta_{\mu\nu} + \Box\tau\eta_{\mu\nu}\right) \\ & + \left(-G_{\text{Tele},I_2} + 2G_{4,\phi}\right)\left(\partial_{\mu}\partial_{\nu}\delta\phi - \eta_{\mu\nu}\Box\delta\phi\right) \\ & + 2\left(-G_{\text{Tele},T} + G_4\right)\left(\Box\tau_{(\mu\nu)} - \partial^{\lambda}\partial_{\mu}\tau_{(\nu\lambda)} - \partial^{\lambda}\partial_{\nu}\tau_{(\mu\lambda)} + \partial_{\mu}\partial_{\nu}\tau + \partial_{\lambda}\partial_{\beta}\tau^{\lambda\beta}\eta_{\mu\nu} - \Box\tau\eta_{\mu\nu}\right) \,, \end{aligned} \tag{4.17}$$

$$W_{[\mu\nu]} = -2G_{\text{Tele},T_{\text{vec}}}\partial^{\lambda}\partial_{[\mu}\tau_{|\lambda|\nu]} + \frac{4}{9}G_{\text{Tele},T_{\text{ax}}}\left[\Box\tau_{[\mu\nu]} - \partial^{\lambda}\partial_{\nu}\tau_{[\mu\lambda]} + \partial^{\lambda}\partial_{\mu}\tau_{[\nu\lambda]}\right] \,, \tag{4.18}$$

thus stressing the fact that $\tau_{\mu\nu}$ in contrast to the metric is not symmetric but rather has no symmetry at all. As a matter of fact the antisymmetric part (4.18) coincides with the field equations one would have obtained by varying the linearized action wrt to the a non-trivial inertial and Lorentzian spin connection as also introduced in Sec. 2.3.

4.3 Scalar–Vector–Tensor Decomposition and Propagating Degrees of Freedom Analysis

In order to explicitly probe the pdof of the tetrad one needs to be aware of its constituents in the background under investigation. To that end, in spatially flat FLRW backgrounds it is already known that the tetrad can be split as Eq. (2.85). This split in conjunction with the definition of $\tau_{\mu\nu}$ (4.3) can generate the SVT decomposition of $\tau_{\mu\nu}$ as

$$\tau_{\mu\nu} = \begin{bmatrix} -\varphi & -(\partial_i\beta + \beta_i) \\ (\partial_i b + b_i) & \left(-\psi\delta_{ij} + \partial_i\partial_j h + 2\partial_{(i}h_{j)} + \frac{1}{2}h_{ij} + \epsilon_{ijk}\left(\partial^k\sigma + \sigma^k\right)\right) \end{bmatrix}, \tag{4.19}$$

which can also produce the SVT decomposition of the perturbations of the metric through their relation $\delta g_{\mu\nu} = 2\tau_{(\mu\nu)}$ as

$$\delta g_{\mu\nu} = \begin{bmatrix} -2\,\varphi & (\partial_i\mathcal{B} + \mathcal{B}_i) \\ (\partial_i\mathcal{B} + \mathcal{B}_i) & 2\left(-\psi\delta_{ij} + \partial_i\partial_j h + 2\partial_{(i}h_{j)} + \frac{1}{2}h_{ij}\right) \end{bmatrix}, \tag{4.20}$$

which completely agrees with the one stated in Eq. (2.86). On top of this in order to deal with the gauge freedom of the coordinate transformations the perturbative variables will be replaced by gauge invariant ones. In this manner, the field equations will be written in terms of the gauge invariant variables. Starting from the general gauge transformations as introduced in Eq. (2.90) the following gauge invariant variables can be generated

$$\chi := b - \dot{h}\,, \qquad \Phi := \varphi - \dot{\beta}\,, \tag{4.21}$$

$$\Sigma_j := h_j k^2 + i\varepsilon_{jlp}k^l\sigma^p\,, \quad \Xi_j := ik^2 b_j - 2\varepsilon_{jlp}k^l\dot{\sigma}^p\,,$$

$$\Lambda_j := -b_j + 2\dot{h}_j\,. \tag{4.22}$$

Having all the needed tools laid out, the linearized field equations (4.15)–(4.16) can be probed. As a next step they are split in a 3+1 manner and then the gauge invariant variables are introduced by replacing Eq. (4.19) with (4.21)–(4.22). After having split the linearized field equations in an SVT manner the scalar sector consists of 5 linearly independent equations, for the gauge invariant scalar fields $(\delta\phi, \psi, \Phi, \chi, \sigma)$, as

$$W_{00} = k^2\Big((G_{\mathrm{Tele},I_2} - 2G_{4,\phi})\delta\phi + 2G_{\mathrm{Tele},T_{\mathrm{vec}}}\Phi$$

$$- 4(G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}})\psi\Big)\,, \tag{4.23}$$

$$k^j W_{j0} = ik^2((G_{\text{Tele},I_2} - 2G_{4,\phi})\delta\dot{\phi} + 2G_{\text{Tele},T_{\text{vec}}}\chi k^2$$

$$- 2(2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}})\dot{\psi})\,, \tag{4.24}$$

$$\eta_{jl}W^{jl} = 2k^2\Big((2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}})\dot{\chi}$$

$$- 2(G_4 - G_{\text{Tele},T} + 2G_{\text{Tele},T_{\text{vec}}})\psi\Big) \tag{4.25}$$

$$+ 3(G_{\text{Tele},I_2} - 2G_{4,\phi})\delta\ddot{\phi} + 2(G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}})\Phi \tag{4.26}$$

$$- 6\Big(2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}}\Big)\ddot{\psi} + (G_{\text{Tele},I_2} - 2G_{4,\phi})\delta\phi\,, \tag{4.27}$$

$$k_l\epsilon^{ljk}W_{jk} = -\frac{8}{3}iG_{\text{Tele},T_{\text{ax}}}k^2(\ddot{\sigma} + k^2\sigma)\,, \tag{4.28}$$

$$W_\phi = \delta\phi\Big(G_{\text{Tele},\phi\phi} + G_{2,\phi\phi} + (G_{\text{Tele},X} + G_{2,X} - 2G_{3,\phi})k^2\Big)$$

$$+ (G_{\text{Tele},X} + G_{2,X} - 2G_{3,\phi})\delta\ddot{\phi}$$

$$+ (G_{\text{Tele},I_2} - 2G_{4,\phi})(-k^2(\dot{\chi} + \Phi - 2\psi) + 3\ddot{\psi})\,. \tag{4.29}$$

In a similar manner the vector sector consists of 3 linearly independent equations, for the gauge invariant variables $(\beta_i, \Sigma_i, \Lambda_i)$, as

$$W_{0j} = \frac{1}{9}\Big\{18G_{\text{Tele},T_{\text{vec}}}\ddot{\beta}_j + k^2\Big(2G_{\text{Tele},T_{\text{ax}}} - 9(G_4 - G_{\text{Tele},T})\Big)\Lambda_j$$

$$- k^2\Big(9(G_4 - G_{\text{Tele},T}) + 2G_{\text{Tele},T_{\text{ax}}}\Big)\beta_j - 2\Big(9G_{\text{Tele},T_{\text{vec}}} + 2G_{\text{Tele},T_{\text{ax}}}\Big)\dot{\Sigma}_j\Big\}\,, \tag{4.30}$$

$$W_{j0} = \frac{1}{9}\Big\{k^2\Big[\Big(2G_{\text{Tele},T_{\text{ax}}} - 9(G_4 - G_{\text{Tele},T})\Big)\beta_j - \Big(9(G_4 - G_{\text{Tele},T}) + 2G_{\text{Tele},T_{\text{ax}}}\Big)\Lambda_j\Big]$$

$$+ 4G_{\text{Tele},T_{\text{ax}}}\dot{\Sigma}_j\Big\}\,, \tag{4.31}$$

$$k^l W_{lj} = -\frac{1}{9} i \Big\{ k^2 \Big[\Big(9(G_4 - G_{\text{Tele},T}) + 2(9G_{\text{Tele},T_{\text{vec}}} + G_{\text{Tele},T_{\text{ax}}}) \Big) \dot{\beta}_j + \Big(9(G_4 - G_{\text{Tele},T}) - 2G_{\text{Tele},T_{\text{ax}}} \Big) \dot{\Lambda}_j - 18 G_{\text{Tele},T_{\text{vec}}} \Sigma_j \Big] + 4 G_{\text{Tele},T_{\text{ax}}} \ddot{\Sigma}_j \Big\} , \tag{4.32}$$

while the tensor field is described by just one equation

$$W_{ij} = (G_4 - G_{\text{Tele},T})(\ddot{h}_{ij} + k^2 h_{ij}) . \tag{4.33}$$

The use of Fourier transformations has been used through out and the final results should be considered as the real parts of each component. Nonetheless, they are keept in the their full complex form for convenience in the calculations. Having split the total system into the SVT decomposition, each sector can be solved independently and then all the solutions combined will represent the solution of the total system. The process of solving this type of systems is quite complicated in general, nevertheless in this case solutions can be found by using linear algebraic schemes. As a first step all the individual sectors will be written in terms of their matrix representations as:

Scalar sector:

$$\begin{pmatrix} M_{s11} & M_{s12} & M_{s13} & 0 & 0 \\ M_{s21} & M_{s22} & 0 & M_{s24} & 0 \\ M_{s31} & M_{s32} & M_{s33} & M_{s34} & 0 \\ 0 & 0 & 0 & 0 & M_{s45} \\ M_{s51} & M_{s52} & M_{s53} & M_{s54} & 0 \end{pmatrix} \begin{pmatrix} \delta\phi \\ \psi \\ \Phi \\ \sigma \\ \chi \end{pmatrix} =: M_s Y_s = 0 , \tag{4.34}$$

where the components of the matrix M_s are

$$M_{s11} = (G_{\text{Tele},I_2} - 2G_{4,\phi}) k^2 , \tag{4.35a}$$

$$M_{s13} = M_{s12} - \frac{M_{s22}}{\omega} , \quad M_{s12} = -4(G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}}) k^2 , \tag{4.35b}$$

$$M_{s21} = \omega M_{s11}\,, \quad M_{s22} = -2(2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}})\omega k^2\,, \tag{4.35c}$$

$$M_{s24} = \frac{ik^2}{\omega}(\omega M_{s12} - M_{s22})\,, \tag{4.35d}$$

$$M_{s31} = (2 - \frac{3\omega^2}{k^2})M_{s11}\,, \quad M_{s32} = -M_{s12} + \frac{M_{s22}}{\omega}(2 - \frac{3\omega^2}{k^2})\,, \tag{4.35e}$$

$$M_{s34} = iM_{s22}\,, \quad M_{s45} = \frac{8}{3}i\,G_{\text{Tele},T_{\text{ax}}}k^2(\omega^2 - k^2)\,, \quad M_{s33} = -M_{s12}\,, \tag{4.35f}$$

$$M_{s51} = G_{\text{Tele},\phi\phi} + G_{2,\phi\phi} - (G_{\text{Tele},X} + G_{2,X} - 2G_{3,\phi})(\omega^2 - k^2)\,, \tag{4.35g}$$

$$M_{s53} = -M_{s11}\,, \quad M_{s54} = i\omega M_{s11}\,. \quad M_{s52} = (2 - \frac{3\omega^2}{k^2})M_{s11}\,, \tag{4.35h}$$

where all the matrix elements have been represented wrt to the unique elements $M_{s11}, M_{s12}, M_{s22}$ and M_{s51}.

Vector sector:

$$\begin{pmatrix} M_{V11} & M_{V12} & M_{V13} \\ M_{V21} & M_{V22} & M_{V23} \\ M_{V31} & M_{V32} & M_{V33} \end{pmatrix} \begin{pmatrix} \beta_j \\ \Sigma_j \\ \Lambda_j \end{pmatrix} =: M_V Y_V = 0\,, \tag{4.36}$$

where the components of the vectorial matrix M_V are

$$M_{V11} = -2G_{\text{Tele},T_{\text{vec}}}\omega^2 - \tfrac{1}{9}(9G_4 - G_{\text{Tele},T} + 2G_{\text{Tele},T_{\text{ax}}})k^2\,, \tag{4.37a}$$

$$M_{V12} = \frac{i}{k^2}(\omega M_{V13} - M_{V31})\,, \quad M_{V13} = \tfrac{1}{9}(-9G_4 - G_{\text{Tele},T} + 2G_{\text{Tele},T_{\text{ax}}})k^2\,, \tag{4.37b}$$

$$M_{V21} = M_{V13}\,, \quad M_{V22} = \frac{-i\omega}{k^2(\omega^2 - k^2)}\Big(k^2 M_{V11} + (\omega^2 - k^2)M_{V13} - \omega M_{V31}\Big)\,, \tag{4.37c}$$

$$M_{V23} = \frac{1}{\omega^2 - k^2}(-k^2 M_{V11} + \omega M_{V31})\,, \tag{4.37d}$$

$$M_{V31} = -\tfrac{1}{9}(9G_4 - G_{\text{Tele},T} + 2(9G_{\text{Tele},T_{\text{vec}}} + G_{\text{Tele},T_{\text{ax}}}))\omega k^2\,, \tag{4.37e}$$

$$M_{V32} = \frac{i}{\omega k^2}\Big(\omega(k^2 M_{V11} + \omega^2 M_{V13}) - (\omega^2 + k^2)M_{V31}\Big), \quad M_{V33} = \omega M_{V13}\,, \tag{4.37f}$$

where again all the matrix elements have been represented wrt to the unique elements M_{V11}, M_{V13} and M_{V31}.

Tensor sector:

$$-(G_4 - G_{\text{Tele},T})(\omega^2 - k^2) =: M_T Y_T = 0\,. \tag{4.38}$$

where we stress that $G_4 - G_{\text{Tele},T} \neq 0$.

The advantage of the matrix representation of each individual sector is that in the end the final result will be a combination of each individual sector in a block diagonal form. In the same scope the total system can be written also in a block diagonal form as

$$\begin{pmatrix} M_s & 0 & 0 \\ 0 & M_V & 0 \\ 0 & 0 & M_T \end{pmatrix}\begin{pmatrix} Y_s \\ Y_V \\ Y_T \end{pmatrix} =: MY = 0\,, \tag{4.39}$$

where M_V and M_T are to be understood as $M_V \otimes I_2$ and $M_T \otimes I_2 = M_T I_2$, and I_2 being the identity matrix of dimension 2. This fix is needed in order to account for the fact that all indexed tensor fields carry 2 components, i.e, $\beta_j = (\beta_1, \beta_2)$ and $h_{ij} = (h_1, h_2)$.

At this stage physical information from the system can be extracted as to which dof are propagating. For example propagation of physical GW, mathematically corresponds to the tensor part of the perturbations and thus ensuring that they do propagate physics is consistent. The tensor perturbations dynamics is described in Eq. (4.38). In fact the tensor perturbations propagate at the speed of light and they are massless as indicated by their dispersion relation $\omega^2 - k^2 = 0$.

Regarding the whole system, the principal polynomial $\det M$ of the matrix M has to be calculated in order to determine if there is propagation or not. This principal polynomial in the theory of partial differential equations is called the principal polynomial [147, 148, 149] and it is denoted as the quantity $P(k)$. Denegeracy of the principal polynomial or demanding $P(k) = 0$ has roots, is exactly what generates the dispersion relations of a system. Hence the dispersion relation, if it exists, it is determined directly as the root of the equation $P(k) = 0$ which in principle is an equation relating the temporal Fourier variable ω along with the norm of the wave covector k as some functional form $f(\omega, k) = 0$. Hence, in order to calculate the principal polynomial, notice that $P(k) := \det(M) = \det(M_s)\det(M_V)\det(M_T)$ which is due to the block diagonal form of the total system as illustrated in Eq. (4.39). The principal polynomial is the product of each individual sector. Explicitly $P(k)$ attains the value

$$P(k) = -\tfrac{16384}{243}(G_4 - G_{\mathrm{Tele},T})^5 {G_{\mathrm{Tele},T_{\mathrm{vec}}}}^3 {G_{\mathrm{Tele},T_{\mathrm{ax}}}}^3 k^{12}(\omega^2 - k^2)^8\Big(\tilde{c}_1 + \tilde{c}_2(\omega^2 - k^2)\Big), \tag{4.40}$$

where the following quantities have been defined

$$\tilde{c}_1 := 2(G_{\mathrm{Tele},\phi\phi} + G_{2,\phi\phi})(2(G_4 - G_{\mathrm{Tele},T}) + 3G_{\mathrm{Tele},T_{\mathrm{vec}}}), \tag{4.41}$$

$$\tilde{c}_2 := -3(G_{\mathrm{Tele},I_2} - 2G_{4,\phi})^2 - 2(G_{\mathrm{Tele},X} + G_{2,X} - 2G_{3,\phi})(2(G_4 - G_{\mathrm{Tele},T}) + 3G_{\mathrm{Tele},T_{\mathrm{vec}}})\,. \tag{4.42}$$

it is also convenient to define the quantities

$$\tilde{c}_3 := -G_{\mathrm{Tele},\phi\phi} - G_{2,\phi\phi}\,, \tag{4.43}$$

$$\tilde{c}_4 := G_{\mathrm{Tele},X} + G_{2,X} - 2G_{3,\phi}\,, \tag{4.44}$$

and

$$Z_1 := -\frac{(G_4 - G_{\mathrm{Tele},T})G_{2,\phi\phi}}{-3{G_{4,\phi}}^2 + (G_4 - G_{\mathrm{Tele},T})(2G_{3,\phi} - G_{2,X})}\,, \tag{4.45}$$

$$Z_2 := \frac{(3(G_{\mathrm{Tele},I_2} - 2G_{4,\phi})^2 + 2(2(G_4 - G_{\mathrm{Tele},T}) + 3(G_{\mathrm{Tele},T_{\mathrm{vec}}}))(G_{\mathrm{Tele},X} + G_{2,X}))}{4(2(G_4 - G_{\mathrm{Tele},T}) + 3(G_{\mathrm{Tele},T_{\mathrm{vec}}}))}\,. \tag{4.46}$$

Judging from the form of the principal polynomial $P(k)$ in Eq. (4.40), it can be speculated that there might be branches of the BDLS theory in which there exist massive, massless modes or even combination of both. Of course a detailed analysis is needed in order to properly probe the exact content of the propagating modes. This analysis in order to be carried out in an exhaustive manner, all the cases where the principal polynomial Eq. (4.40) is non-degenerate i.e, $P(k) = 0$ for any k have to be singled out and individually studied.

One such example where $P(k) = 0$ arises, is by constraining the form of our Lagrangian as $G_{\mathrm{Tele},T_{\mathrm{vec}}} = 0$. The rest of these types of $P(k) = 0$ instances are methodically and exhaustively calculated. Along those instances, the corresponding dispersion relations are obtained. Then from each of these dispersion relations the speed of propagation and the mass(if it exists) of the GW are extracted.

In practice, once the dispersion relation is obtained, which is an equation of the form $f(\omega, k) = 0$ it is then solved for $\omega = \omega(k)$ and then substituted back into the system of Eq. (4.39) in order to arrive to the solution of the system. This solution will determine the number of dof for each corresponding case. It is expected that there is strong branching in the theory due to its high inclusiveness and hence the dof will depend from each case.

The method of solving the system of Eq. (4.39) for the case of GR will be illustrated. This case also includes the $f(T)$ gravity since they both are defined by $(G_4 - G_{\mathrm{Tele},T}) \neq 0$.

Reducing the system to just independent equations

$$\begin{pmatrix} M_{11} & 0 & 0 & 0 & 0 & 0 \\ M_{21} & -M_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{M_{11}}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{M_{11}}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{55} \end{pmatrix} \begin{pmatrix} \psi \\ \Phi \\ \beta_i \\ h_{ij} \end{pmatrix} = MY\,, \tag{4.47}$$

where the components M_i are expanded as

$$M_{11} = -4(G_4 - G_{\text{Tele},T})k^2\,, \quad M_{21} = 4(G_4 - G_{\text{Tele},T})(3\omega^2 - k^2)\,, \tag{4.48}$$

$$M_{55} = (G_4 - G_{\text{Tele},T})(-\omega^2 + k^2)\,. \tag{4.49}$$

Note that $(G_4 - G_{\text{Tele},T}) \neq 0$ in order for the tensor modes to be properly propagating. At this point considering the components, M_{ij} one could already deduce that only the tensor modes propagate as it is the case. Nevertheless, it will be proven by calculating in every possible detail since this exact sequence will be applied to any other case.

The principal polynomial for GR is found to be $P(k) = -16(G_4 - G_{\text{Tele},T})^6 k^8 (\omega^2 - k^2)^2$, which is non-degenerate since $(G_4 - G_{\text{Tele},T}) \neq 0$ due to demanding proper tensor perturbation propagation. The dispersion relation in this case is $P(k) = -16(G_4 - G_{\text{Tele},T})^6 k^8 (\omega^2 - k^2)^2 = 0$ which results in $\omega^2 - k^2 = 0$ that is substituted back in the system in Eq. (4.47). In this way, all the possible solutions are found and then the general solution is constructed as a linear combination of them. This in terms of linear algebra is translated as constructing the Null Space (the space of all solutions of the system) of the total matrix of the system M and then constructing the general solution which is just a linear combination of the elements of the Null space. The general solution for the dispersion relation $\omega = |k|$ is $Y_{|k|}$ is then

$$Y_{|k|} = (0, 0, 0, 0, A_1, A_2)^T \ . \tag{4.50}$$

where $A_i \in \mathbb{C}$ are the coefficients of the linear combination of the solution space of the system. From this solution only THE last two slots occupied by A_1, A_2 are non zero, which correspond to the tensor perturbations. Hence, the only propagating 2 dof are described by the tensorial modes. We note that the dispersion relation of the tensor modes is $\omega^2 - k^2 = 0$ which means that they are massless and travel at the speed of light. It should be noted that, in general, the maximum subscript appearing in the totallity of the coefficients A_i denotes the maximum number of propagating dof.

4.4 Solutions and Branching

The general solution of the system in Eq. (4.39) will be a linear combination of the elements of the null space of the matrix M. The coefficients of the linear combinations will in general be denoted as $A_i \in \mathbb{C}$ for the massless dispersion relation $\omega^2 - k^2 = 0$ and $B_i \in \mathbb{C}$ for the massive dispersion relation $\omega^2 - k^2 = m^2$. These coefficients also act as labels for the number of pdof. Hence each A_i or B_i denotes onepdof. If there are both massless and massive branches there is a solution space for each sector and thus the sum of their dimensions is the total number pdof. The solutions that correspond to the massless dispersion relation $\omega^2 - k^2 = 0$ will be denoted as $Y_{|k|}$ and the ones corresponding to a massive dispersion relation $\omega^2 - k^2 = m^2$ as $Y_{|m|}$. Finally, these solutions will be equal and compared with the general column vector Y defined in Eq. (4.39).

After calculating each solution $Y_{|k|}$ or $Y_{|m|}$, the computation of the electric components of the Riemann tensor will follow. These components are also split into massless and massive cases, denoted by $\mathring{R}_{0i0j}(Y_{|k|})$ and $\mathring{R}_{0i0j}(Y_{|m|})$. Finally, due the nature of the solutions only the most unique and interesting ones will be further discussed. Although the list is completely exhaustive, there are a few cases that are physically equivalent while others do not manifest any interesting physical significance.

Case 0: Horndeski $\left(G_{\mathrm{Tele},T_{\mathrm{vec}}} = 0,\ G_{\mathrm{Tele},T_{\mathrm{ax}}} = 0,\ G_{\mathrm{Tele},I_2} = 0,\ G_{\mathrm{Tele},X} = 0,\ G_{\mathrm{Tele},\phi\phi} = 0\right)$

The Horndeski case is described by the principal polynomial

$$\begin{aligned} P(k) &= -16(G_4 - G_{\mathrm{Tele},T})^5 k^8 (\omega^2 - k^2)^2 \\ &\quad \times \left((G_4 - G_{\mathrm{Tele},T})G_{2,\phi\phi} + (-3G_{4,\phi}{}^2 + (G_4 - G_{\mathrm{Tele},T})(2G_{3,\phi} - G_{2,X}))(\omega^2 - k^2)\right), \end{aligned} \tag{4.51}$$

which splits into two subcases:

Case 0.I ($G_{2,\phi\phi} \neq 0$ and $-3G_{4,\phi}{}^2 + (G_4 - G_{\mathrm{Tele},T})\left(2G_{3,\phi} - G_{2,X}\right) \neq 0$).

In this case the principal polynomial reads

$$\begin{aligned} P(k) &= -16(G_4 - G_{\mathrm{Tele},T})^5 k^8 (\omega^2 - k^2)^2 \\ &\quad \times \left((G_4 - G_{\mathrm{Tele},T})G_{2,\phi\phi} + (-3G_{4,\phi}{}^2 + (G_4 - G_{\mathrm{Tele},T})(2G_{3,\phi} - G_{2,X}))(\omega^2 - k^2)\right), \end{aligned} \tag{4.52}$$

from which it is evident that it is non-degenerate and there are two dispersion relations $\omega^2 - k^2 = 0$ massless speed of light propagation and

$$(G_4 - G_{\mathrm{Tele},T})G_{2,\phi\phi} + (-3G_{4,\phi}{}^2 + (G_4 - G_{\mathrm{Tele},T})(2G_{3,\phi} - G_{2,X}))(\omega^2 - k^2) = 0$$

through which an effective mass is defined as

$$m^2 = -\frac{(G_4 - G_{\text{Tele},T})G_{2,\phi\phi}}{-3G_{4,\phi}{}^2 + (G_4 - G_{\text{Tele},T})(2G_{3,\phi} - G_{2,X}))} > 0\,,$$

the corresponding solutions are then for the massless sector

$$Y_\omega = \left(\delta\phi, \psi, \Phi, \beta_i, h_{ij}\right)\,, \tag{4.53}$$

$$Y_{|k|} = \left(0, 0, 0, 0, 0, \frac{2A_1}{k^2}, \frac{2A_2}{k^2}\right)^T\,, \tag{4.54}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.55}$$

and for the massive sector

$$Y_{|m|} = \left(2(G_4 - G_{\text{Tele},T})B_1, -G_{4,\phi}B_1, G_{4,\phi}B_1, 0, 0, 0, 0\right)^T\,, \tag{4.56}$$

$$\mathring{R}_{0i0j}(Y_{|m|}) = G_{4,\phi}\begin{pmatrix} (m^2+k^2)B_1 & 0 & 0 \\ 0 & (m^2+k^2)B_1 & 0 \\ 0 & 0 & m^2 B_1 \end{pmatrix}. \tag{4.57}$$

The full Horndeski theory assumes 3 pdof, 2 of which are the tensor modes described by the parameters (A_1, A_2) in the massless sector Eq. (4.54) and the remaining dof is a scalar described by the parameter B_1 in the massive sector Eq. (4.56). It is evident that by setting $G_{4,\phi} = 0$ one can completely hide the massive scalar from the polarization detectors Eq. (4.57) but it will still propagate as can be seen from Eq. (4.56).

Case 0.II ($G_{2,\phi\phi} = 0$ and $-3G_{4,\phi}{}^2 + (G_4 - G_{\text{Tele},T})\left(2G_{3,\phi} - G_{2,X}\right) = 0$).

A solution of this system that covers the whole solution manifold is

$$G_{2,X} = -\frac{3G_{4,\phi}{}^2}{(G_4 - G_{\text{Tele},T})} + 2G_{3,\phi}\,,$$

and then the principal polynomial becomes

$$P(k) = -8(G_4 - G_{\text{Tele},T})^5 G_{4,\phi} k^8 (\omega^2 - k^2)^2\,, \tag{4.58}$$

which describes only massless propagation. Further assuming that $G_{4,\phi} \neq 0$ the solution is

$$Y_\omega = \left(\delta\phi, \psi, \beta_i, h_{ij}\right)\,, \tag{4.59}$$

$$Y_{|k|} = \left(0, 0, 0, 0, \frac{2A_1}{k^2}, \frac{2A_2}{k^2}\right)^T\,, \tag{4.60}$$

whereas for $G_{4,\phi} = 0$ the solution becomes

$$Y_\omega = \left(\psi, \beta_i, h_{ij}\right)\,, \tag{4.61}$$

$$Y_{|k|} = \left(0, 0, 0, \frac{2A_1}{k^2}, \frac{2A_2}{k^2}\right)^T\,, \tag{4.62}$$

nevertheless for both cases $G_{4,\phi} = 0$ and $G_{4,\phi} \neq 0$ the electric components of the Riemann tensor are identical

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.63}$$

Although the algebraic solutions are different as can be seen from Eq. (4.60) and Eq. (4.62) the final physical solution is the same, i.e only tensor modes. This is also reflected in the polarizations described by Eq. (4.63) since their value does not depend at all by the choice of $G_{4,\phi}$.

Case 1: Full BDLS theory ($G_{\text{Tele},T_{\text{vec}}} \neq 0$, $G_{\text{Tele},T_{\text{ax}}} \neq 0$, $\tilde{c}_1 \neq 0$, $\tilde{c}_2 \neq 0$)

In this most general case the principal polynomical reads as

$$P(k) = -\frac{16384}{243}(G_4 - G_{\text{Tele},T})^5 G_{\text{Tele},T_{\text{vec}}}{}^3 G_{\text{Tele},T_{\text{ax}}}{}^3 k^{12}(\omega^2 - k^2)^8\Big(\tilde{c}_1 + \tilde{c}_2(\omega^2 - k^2)\Big), \tag{4.64}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are defined in Eq. (4.41)-Eq. (4.42). It is evident that it is non-degenerate and there are two dispersion relations, $\omega^2 - k^2 = 0$ which describes massless speed of light propagation and $\tilde{c}_1 + \tilde{c}_2(\omega^2 - k^2) = 0$ which describes massive propagation with an effective mass $m^2 = -\frac{\tilde{c}_1}{\tilde{c}_2} > 0$. The solution form for this case is

$$Y_\omega = \Big(\delta\phi, \psi, \Phi, \chi, \sigma, \beta_j, \Sigma_j, \Lambda_j, h_{ij}\Big)\,, \tag{4.65}$$

which for the massless sector the solution is

$$Y_{|k|} = \Big(0, -\frac{A_1}{k^2}, -\frac{2A_1}{G_{\text{Tele},T_{\text{vec}}} k^2}(G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}}),$$

$$\frac{iA_1}{G_{\text{Tele},T_{\text{vec}}} |k|^3}(2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}}), A_2, -A_3,$$
$$-A_4, i|k|A_3, i|k|A_4, A_3, A_4, \frac{2A_5}{k^2}, \frac{2A_6}{k^2}\Big)^T, \tag{4.66}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 + A_5 & A_6 & 0 \\ A_6 & A_1 - A_5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.67}$$

whereas for the massive sector it becomes

$$Y_{|m|} = \Big(-2(2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}})B_1, -(G_{\text{Tele},I_2} - 2G_{4,\phi})B_1, (G_{\text{Tele},I_2} - 2G_{4,\phi})B_1,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big)^T, \tag{4.68}$$

$$\mathring{R}_{0i0j}(Y_{|m|}) = \Big(G_{\text{Tele},I_2} - 2G_{4,\phi}\Big) \begin{pmatrix} (m^2 + k^2)B_1 & 0 & 0 \\ 0 & (m^2 + k^2)B_1 & 0 \\ 0 & 0 & m^2 B_1 \end{pmatrix}. \tag{4.69}$$

The system in total assumes 7 pdof which are divided as 6 in the massless sector described by Eq. (4.66) and 1 in the massive sector described by Eq. (4.68). The massless sector is parametrized by $(A_1, .., A_6)$ which are packed as 2 scalars (A_1, A_2), one vector (A_3, A_4) and the tensor modes (A_5, A_6). Regarding the polarization content, the massless sector enjoys the usual tensor (A_5, A_6) polarizations along with the breathing (A_1) mode. The massive sector contains one massive scalar described by B_1. It is evident that by setting $\Big(G_{\text{Tele},I_2} - 2G_{4,\phi}\Big) \to 0$ in Eq. (4.69) the massive scalar mode becomes undetectable to the polarization detectors although it will still be propagating as can be seen from Eq. (4.68).

Case 2 ($G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} \neq 0,\ \tilde{c}_1 = 0,\ \tilde{c}_2 = 0$)

A pair of solutions of the system $\tilde{c}_1 = 0, \tilde{c}_2 = 0$ that covers the whole solution manifold are

$$G_{\text{Tele},T_{\text{vec}}} = -\tfrac{2}{3}(G_4 - G_{\text{Tele},T})\,, \tag{4.70}$$

$$G_{4,\phi} = \tfrac{1}{2}G_{\text{Tele},I_2}\,, \tag{4.71}$$

and

$$G_{3,\phi} = \frac{(3(G_{\text{Tele},I_2} - 2G_{4,\phi})^2 + 2(2(G_4 - G_{\text{Tele},T}) + 3(G_{\text{Tele},T_{\text{vec}}}))(G_{\text{Tele},X} + G_{2,X}))}{4(2(G_4 - G_{\text{Tele},T}) + 3(G_{\text{Tele},T_{\text{vec}}}))}\,, \tag{4.72}$$

$$G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi}\,. \tag{4.73}$$

which demands further subcases to be consistently probed:

Case 2.I ($G_{\text{Tele},T_{\text{vec}}} = -\tfrac{2}{3}(G_4 - G_{\text{Tele},T}),\ G_{4,\phi} = \tfrac{1}{2}G_{\text{Tele},I_2}$)

The principal polynomial of the system reads as

$$P(k) = \tfrac{131072}{19683}(G_4 - G_{\text{Tele},T})^8(G_{\text{Tele},T_{\text{ax}}})^3 k^{12}(\omega^2 - k^2)^7\Big(\tilde{c}_3 + \tilde{c}_4(\omega^2 - k^2)\Big)\,, \tag{4.74}$$

which in turn means that more subcases are needed:

Case 2.I.a ($G_{\text{Tele},T_{\text{ax}}} \neq 0,\ \tilde{c}_3 \neq 0, \tilde{c}_4 \neq 0$)

The corresponding principal polynomial is

$$P(k) = \tfrac{131072}{19683}(G_4 - G_{\mathrm{Tele},T})^8 G_{\mathrm{Tele},T_{\mathrm{ax}}}{}^3 k^{12}(\omega^2 - k^2)^7\Big(\tilde{c}_3 + \tilde{c}_4(\omega^2 - k^2)\Big). \tag{4.75}$$

The general form of the solution assumes the form

$$Y_\omega = \Big(\delta\phi, \Phi, \chi, \sigma, \beta_i, \Sigma_i, \Lambda_i, h_{ij}\Big)\ , \tag{4.76}$$

which for the massless sector becomes

$$Y_{|k|} = \left(0, 0, 0, A_1, -A_2, -A_3, i\,|k|\,A_2, i\,|k|\,A_3, A_2, A_3, \frac{2A_4}{k^2}, \frac{2A_5}{k^2}\right)^T\ , \tag{4.77}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_4 & A_5 & 0 \\ A_5 & -A_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.78}$$

whereas for the massive sector

$$Y_{|m|} = (B_1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T\ , \tag{4.79}$$

$$\mathring{R}_{0i0j}(Y_{|m|}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.80}$$

This is a peculiar case where only σ propagates from the scalars in the massless sector and $\delta\phi$ from the massive sector. However, none of them leave a polarization imprint. This is due to the fact that they are not coupled to any metric (Φ, χ) scalar dof and hence they cannot have a polarization imprint in neither the massless nor the massive sectors.

Case 2.I.b $(G_{\text{Tele},T_{\text{ax}}} \neq 0,\ \tilde{c}_3 = 0,\ \tilde{c}_4 = 0)$

The solution of $\tilde{c}_3 = 0, \tilde{c}_4 = 0$ is

$$G_{3,\phi} = \frac{(G_{\text{Tele},X} + G_{2,X})}{2}\,, \quad G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi}\,,$$

so the principal polynomial reads as

$$P(k) = \tfrac{131072}{19683}(G_4 - G_{\text{Tele},T})^8 G_{\text{Tele},T_{\text{ax}}}{}^3 k^{12} (\omega^2 - k^2)^7\,, \tag{4.81}$$

where it is evident that only massless modes propagatee. The general form of the solution assumes the form

$$Y_\omega = \left(\Phi, \chi, \sigma, \beta_i, \Sigma_i, \Lambda_i, h_{ij}\right)\,. \tag{4.82}$$

which reduces to

$$Y_{|k|} = \left(0, 0, A_1, -A_2, -A_3, i\,|k|\,A_2, i\,|k|\,A_3, A_2, A_3, , \frac{2A_4}{k^2}, \frac{2A_5}{k^2}\right)^T\,, \tag{4.83}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_4 & A_5 & 0 \\ A_5 & -A_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{4.84}$$

Although, a there are a few pdof, in the end only the tensor modes appear in the polarization signature.

Case 2.II ($G_{3,\phi} = Z_2,\ G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi}$)

In this case Z_2 is defined in (4.46) and taking into consideration $2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}} \neq 0$ and then we have:

$$P(k) = \tfrac{16384}{243}(G_4 - G_{\text{Tele},T})^5 G_{\text{Tele},T_{\text{vec}}}{}^3 G_{\text{Tele},T_{\text{ax}}}{}^3 (G_{\text{Tele},I_2} - 2G_{4,\phi}) k^{12} (\omega^2 - k^2)^8 \,. \tag{4.85}$$

this form of the principal polynomial induces further subsubcases:

Case 2.II.a ($G_{\text{Tele},I_2} - 2G_{4,\phi} \neq 0.$)

In this case the principal polynomial reads

$$P(k) = \tfrac{16384}{243}(G_4 - G_{\text{Tele},T})^5 (G_{\text{Tele},T_{\text{vec}}})^3 (G_{\text{Tele},T_{\text{ax}}})^3 (G_{\text{Tele},I_2} - 2G_{4,\phi}) k^{12} (\omega^2 - k^2)^8 \,, \tag{4.86}$$

which only includes a massless sector and leads to the general form of the solution

$$Y_\omega = \left(\delta\phi, \psi, \chi, \sigma, \beta_i, \Sigma_i, \Lambda_i, h_{ij}\right) \,, \tag{4.87}$$

that attains the values

$$Y_{|k|} = \left(-\frac{4(G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}})}{(G_{\text{Tele},I_2} - 2G_{4,\phi})k^2} A_1, -\frac{A_1}{k^2}, \frac{iA_1}{k^3}, A_2, -A_3, -A_4, ikA_3, ikA_4\right), \tag{4.88}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 + A_5 & A_6 & 0 \\ A_6 & A_1 - A_5 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.89}$$

Case 2.II.b ($G_{\text{Tele},I_2} - 2G_{4,\phi} = 0.$)

The principal polynomial of the system in this case is

$$P(k) = -\tfrac{16384}{243}(G_4 - G_{\mathrm{Tele},T})^4 G_{\mathrm{Tele},T_{\mathrm{vec}}}{}^3 (G_4 - G_{\mathrm{Tele},T} + G_{\mathrm{Tele},T_{\mathrm{vec}}}) G_{\mathrm{Tele},T_{\mathrm{ax}}}{}^3 k^{12} (\omega^2 - k^2)^7 \,. \tag{4.90}$$

Case 2.II.b.1 $(G_4 - G_{\mathrm{Tele},T} + G_{\mathrm{Tele},T_{\mathrm{vec}}} \neq 0)$

$$P(k) = -\tfrac{16384}{243}(G_4 - G_{\mathrm{Tele},T})^4 G_{\mathrm{Tele},T_{\mathrm{vec}}}{}^3 (G_4 - G_{\mathrm{Tele},T} + G_{\mathrm{Tele},T_{\mathrm{vec}}}) G_{\mathrm{Tele},T_{\mathrm{ax}}}{}^3 k^{12} (\omega^2 - k^2)^7 \,, \tag{4.91}$$

$$Y_\omega = \left(\Phi, \chi, \sigma, \beta_i, \Sigma_i, \Lambda_i, h_{ij}\right) \,, \tag{4.92}$$

$$Y_{|k|} = \left(0, 0, A_1, -A_2, -A_3, i\,|k|\,A_2, i\,|k|\,A_3, A_2, A_3, \frac{2A_4}{k^2}, \frac{2A_5}{k^2}\right)^T \,, \tag{4.93}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_4 & A_5 & 0 \\ A_5 & -A_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.94}$$

Case 2.II.b.2 $(G_4 - G_{\mathrm{Tele},T} + G_{\mathrm{Tele},T_{\mathrm{vec}}} = 0)$

$$P(k) = -\tfrac{4096}{243} i (G_4 - G_{\text{Tele},T})^7 (G_{\text{Tele},T_{\text{ax}}})^3 \omega k^8 (\omega^2 - k^2)^7 \,, \tag{4.95}$$

$$Y_\omega = \Big(\Phi, \sigma, \beta_i, \Sigma_i, \Lambda_i, h_{ij}\Big) \,, \tag{4.96}$$

$$Y_{|k|} = \left(0, A_1, -A_2, -A_3, i\,|k|\,A_2, i\,|k|\,A_3, A_2, A_3, \frac{2A_4}{k^2}, \frac{2A_5}{k^2}\right)^T \,, \tag{4.97}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_4 & A_5 & 0 \\ A_5 & -A_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.98}$$

<u>**Case 3**</u> ($G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ \tilde{c}_1 \neq 0,\ \tilde{c}_2 \neq 0$)

$$P(k) = -32 i (G_4 - G_{\text{Tele},T})^5 {G_{\text{Tele},T_{\text{vec}}}}^3 \omega^2 k^{10} (\omega^2 - k^2)^3 \Big(\tilde{c}_1 + \tilde{c}_2(\omega^2 - k^2)\Big) , \tag{4.99}$$

$$Y_\omega = \Big(\delta\phi, \psi, \Phi, \chi, \beta_i, \Sigma_i, h_{ij}\Big) \,, \tag{4.100}$$

$$Y_{|k|} = \Big(0, -\frac{A_1}{k^2}, -\frac{2(G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}})}{(G_{\text{Tele},T_{\text{vec}}})k^2}A_1, \frac{i(2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}})}{(G_{\text{Tele},T_{\text{vec}}})\,|k|^3}A_1, 0$$

$$, 0, 0, 0, \frac{2A_2}{k^2}, \frac{2A_3}{k^2}\Big)^T, \tag{4.101}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 + A_2 & A_3 & 0 \\ A_3 & A_1 - A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.102}$$

$$Y_{|m|} = \Big(-2(2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}})B_1, -(G_{\text{Tele},I_2} - 2G_{4,\phi})B_1, (G_{\text{Tele},I_2} - 2G_{4,\phi})B_1,^T$$

$$0, 0, 0, 0, 0, 0, 0\Big)^T, \tag{4.103}$$

$$\mathring{R}_{0i0j}(Y_{|m|}) = (G_{\text{Tele},I_2} - 2G_{4,\phi})\begin{pmatrix} (m^2 + k^2)B_1 & 0 & 0 \\ 0 & (m^2 + k^2)B_1 & 0 \\ 0 & 0 & m^2 B_1 \end{pmatrix}, \tag{4.104}$$

where we can hide the massive scalar from the polarizations by setting $(G_{\text{Tele},I_2} - 2G_{4,\phi}) = 0$.

<u>Case 4</u> ($G_{\text{Tele},T_{\text{vec}}} \neq 0$, $G_{\text{Tele},T_{\text{ax}}} = 0$, $\tilde{c}_1 = 0$, $\tilde{c}_2 = 0$)

A pair of solutions of the system $\tilde{c}_1 = 0, \tilde{c}_2 = 0$ that cover the whole solution manifold are

$$G_{\text{Tele},T_{\text{vec}}} = -\tfrac{2}{3}(G_4 - G_{\text{Tele},T}), \quad G_{4,\phi} = \tfrac{1}{2}G_{\text{Tele},I_2},$$

and

$$G_{3,\phi} = \frac{(3(G_{\mathrm{Tele},I_2} - 2G_{4,\phi})^2 + 2(2(G_4 - G_{\mathrm{Tele},T}) + 3G_{\mathrm{Tele},T_{\mathrm{vec}}})(G_{\mathrm{Tele},X} + G_{2,X}))}{4(2(G_4 - G_{\mathrm{Tele},T}) + 3G_{\mathrm{Tele},T_{\mathrm{vec}}})}, \quad (4.105)$$

$$G_{2,\phi\phi} = -G_{\mathrm{Tele},\phi\phi}\,. \quad (4.106)$$

which further indices the subcases:

Case 4.I $(G_{\mathrm{Tele},T_{\mathrm{vec}}} = -\frac{2}{3}(G_4 - G_{\mathrm{Tele},T}),\ G_{4,\phi} = \frac{1}{2}G_{\mathrm{Tele},I_2})$

By defining

$$\tilde{c}_3 = -G_{\mathrm{Tele},\phi\phi} - G_{2,\phi\phi}\,, \quad (4.107)$$

$$\tilde{c}_4 = G_{\mathrm{Tele},X} + G_{2,X} - 2G_{3,\phi}\,, \quad (4.108)$$

the principal polynomial of the system reads as

$$P(k) = \tfrac{256}{81} i (G_4 - G_{\mathrm{Tele},T})^8 \omega^2 k^{10} (\omega^2 - k^2)^2 \Big(\tilde{c}_3 + \tilde{c}_4(\omega^2 - k^2)\Big), \quad (4.109)$$

which in turn means that extra subcases are needed:

Case 4.I.a $(\tilde{c}_3 \neq 0,\ \tilde{c}_4 \neq 0)$

$$P(k) = \tfrac{256}{81} i (G_4 - G_{\mathrm{Tele},T})^8 \omega^2 k^{10} (\omega^2 - k^2)^2 \Big(\tilde{c}_3 + \tilde{c}_4(\omega^2 - k^2)\Big), \quad (4.110)$$

$$Y_\omega = \Big(\delta\phi, \psi, \chi, \beta_i, \Sigma_i, h_{ij}\Big)\,, \quad (4.111)$$

$$Y_{|k|} = \left(0,0,0,0,0,0,0,\frac{2}{k^2}A_1,\frac{2}{k^2}A_2\right)^T , \tag{4.112}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.113}$$

$$Y_{|m|} = (B_1,0,0,0,0,0,0,0,0)^T , \tag{4.114}$$

$$\mathring{R}_{0i0j}(Y_{|m|}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.115}$$

Case 4.I.b ($\tilde{c}_3 = 0,\ \tilde{c}_4 = 0$)

The solution of $\tilde{c}_3 = 0,\ \tilde{c}_4 = 0$ is

$$G_{3,\phi} = \tfrac{1}{2}(G_{\mathrm{Tele},X} + G_{2,X}) , \quad G_{2,\phi\phi} = -G_{\mathrm{Tele},\phi\phi} ,$$

and then we have:

$$P(k) = \tfrac{256}{81} i (G_4 - G_{\text{Tele},T})^8 \omega^2 k^{10} (\omega^2 - k^2)^2 \,, \tag{4.116}$$

$$Y_\omega = \left(\psi, \chi, \beta_i, \Sigma_i, h_{ij}\right) \,, \tag{4.117}$$

$$Y_{|k|} = \left(0, 0, 0, 0, 0, 0, \frac{2}{k^2} A_1, \frac{2}{k^2} A_2\right)^T \,, \tag{4.118}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.119}$$

<u>Case 4.II</u> ($G_{3,\phi} = Z_2,\ G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi},\ 2(G_4 - G_{\text{Tele},T}) + 3G_{\text{Tele},T_{\text{vec}}} \neq 0$)

In this case with Z_2 defined in (4.46) we have

$$P(k) = -32i (G_4 - G_{\text{Tele},T})^5 (G_{\text{Tele},T_{\text{vec}}})^3 (G_{\text{Tele},I_2} - 2G_{4,\phi}) \omega^2 k^{10} (\omega^2 - k^2)^3 \,, \tag{4.120}$$

and then more subcases appear.

<u>Case 4.II.a</u> ($G_{\text{Tele},I_2} - 2G_{4,\phi} \neq 0$)

$$P(k) = -32i(G_4 - G_{\text{Tele},T})^5 G_{\text{Tele},T_{\text{vec}}}{}^3 (G_{\text{Tele},I_2} - 2G_{4,\phi})\omega^2 k^{10}(\omega^2 - k^2)^3 \,, \tag{4.121}$$

$$Y_\omega = \left(\delta\phi, \psi, \chi, \beta_i, \Sigma_i, h_{ij}\right) \,, \tag{4.122}$$

$$Y_{|k|} = \left(-\frac{4(G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}})}{(G_{\text{Tele},I_2} - 2G_{4,\phi})k^2} A_1, -\frac{A_1}{k^2}, \frac{iA_1}{|k|^3}, 0, 0, 0, 0, \frac{2A_2}{k^2}, \frac{2A_3}{k^2}\right)^T \,, \tag{4.123}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 + A_2 & A_3 & 0 \\ A_3 & A_1 - A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.124}$$

Case 4.II.b ($G_{\text{Tele},I_2} - 2G_{4,\phi} = 0$)

The principal polynomial of the system in this case is

$$P(k) = 32i(G_4 - G_{\text{Tele},T})^4 G_{\text{Tele},T_{\text{vec}}}{}^3 (G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}})\omega^2 k^{10}(\omega^2 - k^2)^2 \,. \tag{4.125}$$

Case 4.II.b.1 ($G_4 - G_{\text{Tele},T} + G_{\text{Tele},T_{\text{vec}}} \neq 0$)

$$P(k) = 32i(G_4 - G_{\mathrm{Tele},T})^4 {G_{\mathrm{Tele},T_{\mathrm{vec}}}}^3 (G_4 - G_{\mathrm{Tele},T} + G_{\mathrm{Tele},T_{\mathrm{vec}}})\omega^2 k^{10}(\omega^2 - k^2)^2 \,, \tag{4.126}$$

$$Y_\omega = \left(\psi, \chi, \beta_i, \Sigma_i, h_{ij}\right) \,, \tag{4.127}$$

$$Y_{|k|} = \left(0, 0, 0, 0, 0, 0, \frac{2A_1}{k^2}, \frac{2A_2}{k^2}\right)^T \,, \tag{4.128}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.129}$$

<u>**Case 4.II.b.2**</u> ($G_4 - G_{\mathrm{Tele},T} + G_{\mathrm{Tele},T_{\mathrm{vec}}} = 0$)

$$P(k) = -8(G_4 - G_{\mathrm{Tele},T})^7 \omega^3 k^6 (\omega^2 - k^2)^2 \,, \tag{4.130}$$

$$Y_\omega = \left(\psi, \beta_i, \Sigma_i, h_{ij}\right) \,, \tag{4.131}$$

$$Y_{|k|} = \left(0, 0, 0, 0, 0, \frac{2A_1}{k^2}, \frac{2A_2}{k^2}\right)^T , \tag{4.132}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.133}$$

<u>Case 5</u> ($G_{\mathrm{Tele},T_{\mathrm{vec}}} = 0,\ G_{\mathrm{Tele},T_{\mathrm{ax}}} \neq 0,\ \tilde{c}_1 \neq 0,\ \tilde{c}_2 \neq 0$)

In this case we find the following quantities:

$$P(k) = -\tfrac{2048}{243} i(G_4 - G_{\mathrm{Tele},T})^5 {G_{\mathrm{Tele},T_{\mathrm{ax}}}}^3 \omega^2 k^{10} (\omega^2 - k^2)^3 \Big(\tilde{c}_1 + \tilde{c}_2(\omega^2 - k^2)\Big), \tag{4.134}$$

$$Y_\omega = \Big(\delta\phi, \psi, \Phi, \sigma, \beta_i, \Sigma_i, h_{ij}\Big) , \tag{4.135}$$

$$Y_{|k|} = \left(0, 0, 0, A_1, 0, 0, 0, 0, \frac{2A_2}{k^2}, \frac{2A_3}{k^2}\right)^T , \tag{4.136}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_2 & A_3 & 0 \\ A_3 & -A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{4.137}$$

$$Y_{|m|} = \Big(-4(G_4 - G_{\mathrm{Tele},T})B_1, -(G_{\mathrm{Tele},I_2} - 2G_{4,\phi})B_1, (G_{\mathrm{Tele},I_2} - 2G_{4,\phi})B_1, \tag{4.138}$$

$$0,0,0,0,0,0,0\Big)^T ,$$

$$\mathring{R}_{0i0j}(Y_{|m|}) = \Big(G_{\mathrm{Tele},I_2} - 2G_{4,\phi}\Big)\begin{pmatrix} (m^2+k^2)B_1 & 0 & 0 \\ 0 & (m^2+k^2)B_1 & 0 \\ 0 & 0 & m^2B_1 \end{pmatrix} . \tag{4.139}$$

Case 6: ($G_{\mathrm{Tele},T_{\mathrm{vec}}} = 0,\ G_{\mathrm{Tele},T_{\mathrm{ax}}} \neq 0,\ \tilde{c}_1 = 0,\ \tilde{c}_2 = 0$)

A solution set of the system that covers the whole solution manifold is

$$G_{3,\phi} = \frac{3(G_{\mathrm{Tele},I_2} - 2G_{4,\phi})^2}{8(G_4 - G_{\mathrm{Tele},T})} + \tfrac{1}{2}(G_{\mathrm{Tele},X} + G_{2,X}), G_{2,\phi\phi} = -G_{\mathrm{Tele},\phi\phi} , \tag{4.140}$$

which leads us to the principal polynomial

$$P(k) = -\tfrac{2048}{243} i(G_4 - G_{\mathrm{Tele},T})^5 G_{\mathrm{Tele},T_{\mathrm{ax}}}{}^3 (G_{\mathrm{Tele},I_2} - 2G_{4,\phi})\omega^2 k^{10}(\omega^2 - k^2)^3 , \tag{4.141}$$

which leads several subcases:

Case 6.I ($G_{\mathrm{Tele},I_2} - 2G_{4,\phi} \neq 0$)

$$P(k) = -\tfrac{2048}{243} i(G_4 - G_{\mathrm{Tele},T})^5 G_{\mathrm{Tele},T_{\mathrm{ax}}}{}^3 (G_{\mathrm{Tele},I_2} - 2G_{4,\phi})\omega^2 k^{10}(\omega^2 - k^2)^3 , \tag{4.142}$$

$$Y_\omega = \Big(\delta\phi, \psi, \sigma, \beta_i, \Sigma_i, h_{ij}\Big) , \tag{4.143}$$

$$Y_{|k|} = \left(0, 0, A_1, 0, 0, 0, 0, \frac{2A_2}{k^2}, \frac{2A_3}{k^2}\right)^T , \tag{4.144}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_2 & A_3 & 0 \\ A_3 & -A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.145}$$

<u>**Case 6.II**</u> ($G_{\text{Tele},I_2} - 2G_{4,\phi} = 0$)

$$P(k) = \tfrac{2048}{243} i (G_4 - G_{\text{Tele},T})^5 {G_{\text{Tele},T_{\text{ax}}}}^3 \omega^2 k^8 (\omega^2 - k^2)^3 , \tag{4.146}$$

$$Y_\omega = \left(\psi, \sigma, \beta_i, \Sigma_i, h_{ij}\right) , \tag{4.147}$$

$$Y_{|k|} = \left(0, A_1, 0, 0, 0, 0, \frac{2A_2}{k^2}, \frac{2A_3}{k^2}\right)^T , \tag{4.148}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_2 & A_3 & 0 \\ A_3 & -A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.149}$$

Case 7: $(G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ \tilde{c}_1 \neq 0,\ \tilde{c}_2 \neq 0)$

In this case we obtain:

$$P(k) = -4(G_4 - G_{\text{Tele},T})^5(\omega^2 - k^2)^2 k^8 \Big(\tilde{c}_1 + \tilde{c}_2(\omega^2 - k^2)\Big), \tag{4.150}$$

$$Y_\omega = \Big(\delta\phi, \psi, \Phi, \beta_i, h_{ij}\Big)\ , \tag{4.151}$$

$$Y_{|k|} = \left(0, 0, 0, 0, 0, \frac{2A_1}{k^2}, \frac{2A_2}{k^2}\right)^T\ , \tag{4.152}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.153}$$

$$Y_{|m|} = \Big(-4(G_4 - G_{\text{Tele},T})B_1, -(G_{\text{Tele},I_2} - 2G_{4,\phi})B_1, (G_{\text{Tele},I_2} - 2G_{4,\phi})B_1, 0, 0, 0, 0\Big)^T\ , \tag{4.154}$$

$$\mathring{R}_{0i0j}(Y_{|m|}) = (G_{\text{Tele},I_2} - 2G_{4,\phi}) \begin{pmatrix} (m^2+k^2)B_1 & 0 & 0 \\ 0 & (m^2+k^2)B_1 & 0 \\ 0 & 0 & m^2 B_1 \end{pmatrix} . \tag{4.155}$$

This case gives us exactly the full Horndeski Case 0i. in the limit $G_{\text{Tele},I_2} \to 0$. Thus the full Horndeski theory is a sub-branch of Case 7 and just like we discussed in 4.4 with $\left(G_{\text{Tele},I_2} - 2G_{4,\phi}\right) \to 0$ where it hides the massive scalar pdoffrom the polarization detectors.

<u>Case 8:</u> $(G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ \tilde{c}_1 = 0,\ \tilde{c}_2 = 0)$

A solution set of this system that covers the whole solution manifold is

$$G_{3,\phi} = \frac{3(G_{\text{Tele},I_2} - 2G_{4,\phi})^2}{8(G_4 - G_{\text{Tele},T})} + \tfrac{1}{2}(G_{\text{Tele},X} + G_{2,X}) , \quad G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi} , \tag{4.156}$$

which leads us to the principal polynomial

$$P(k) = 4(G_4 - G_{\text{Tele},T})^5 (G_{\text{Tele},I_2} - 2G_{4,\phi}) k^8 (\omega^2 - k^2)^2 . \tag{4.157}$$

The Horndeski Case 0ii. is a subcase for $G_{\text{Tele},I_2} \to 0$.

<u>Case 8.I</u> $(G_{\text{Tele},I_2} - 2G_{4,\phi} \neq 0)$

$$P(k) = 4(G_4 - G_{\text{Tele},T})^5 (G_{\text{Tele},I_2} - 2G_{4,\phi}) k^8 (\omega^2 - k^2)^2 , \tag{4.158}$$

$$Y_\omega = \left(\delta\phi, \psi, \beta_i, h_{ij}\right) , \tag{4.159}$$

$$Y_{|k|} = \left(0, 0, 0, 0, \frac{2A_1}{k^2}, \frac{2A_2}{k^2}\right)^T \, , \tag{4.160}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.161}$$

Case 8.II ($G_{\text{Tele},I_2} - 2G_{4,\phi} = 0$)

$$P(k) = -4(G_4 - G_{\text{Tele},T})^5 k^6 (\omega^2 - k^2)^2 \, , \tag{4.162}$$

$$Y_\omega = \left(\psi, \beta_i, h_{ij}\right) \, , \tag{4.163}$$

$$Y_{|k|} = \left(0, 0, 0, \frac{2A_1}{k^2}, \frac{2A_2}{k^2}\right)^T \, , \tag{4.164}$$

$$\mathring{R}_{0i0j}(Y_{|k|}) = \begin{pmatrix} A_1 & A_2 & 0 \\ A_2 & -A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.165}$$

4.5 Discussion of the Results

The results Sec. 4.4, from regarding the dynamical analysis of the BDLS theory, are illustrated in a compactified manner in Table 4.2. The classification of the solutions has been carried out by the non-degeneracy of the principal polynomial.

The next less trivial case of interest is Case 0 which is the standard Horndeski gravity, where the usual massive sector result is found exactly as in the standard Ref. [146, 150]. On top of that, a new massless sector is found where only tensor pertubations are propagating. This sector is explicitly distinct from GR. Thus, in this work the full dynamical analysis of Horndeski gravity in Minkowski is completed in the most exhaustive manner.

In Case 1, which is the most general case in BDLS theory, 7 propagating dof were found which can be though of as just two extra scalars and a vector in the massless sector compared to the standard Horndeski scenario (Case 0I). In turn, Case 2 explores the case where $\tilde{c}_1 = \tilde{c}_2 = 0$, that further leads to a rich set of sub-classes that all include at least one scalar and one vector set of propagating dof with the possibility of one massive scalar mode too.

Theory	**Case**	**pdof**	**Lagrangian Density** $\mathcal{L}_i \left(S_i = \frac{1}{2\kappa^2}\int d^4x\, e\, \mathcal{L}_i\right)$
GR or $f(T)$	-	2	$\mathring{R}$ or $f(T)$
Horndeski	0.I	3	Eqs. (3.8)-(3.10)
G_{Tele}	1	7	$G_{\text{Tele}}(\phi, X, T, T_{\text{ax}}, T_{\text{vec}}, I_2, J_1, J_3, J_5, J_6, J_8, J_{10})$
Generalized NGR	2.I.b	5	$f(T, T_{\text{ax}}, T_{\text{vec}})$
Generalized teleparallel dark energy	7	3	$-A(\phi)T - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$
Generalized Teleparallel Scalar Tensor	7	3	$F(\phi)T + P(\phi, X) - G_3(\phi, X)\Box\phi$
Tachyonic teleparallel gravity	7	3	$f(T, X, \phi)$

Table 4.1: These are some of the literature models shown against our analysis, as presented in Table. 4.2.

Cases	Conditions	Sectors					pdof
		Massless$\left(\omega^2 = k^2\right)$			Massive$\left(\omega^2 - k^2 = m^2\right)$		
		Scalar	Vector	Tensor	Scalar	m^2	
–	$G_4 - G_{\text{Tele},T} \neq 0$	-	-	1	-	-	2
0	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,$ $G_{\text{Tele},I_2} = 0, G_{\text{Tele},X} = 0, G_{\text{Tele},\phi\phi} = 0$						
0.I	$G_{2,\phi\phi} \neq 0$ and $-3{G_{4,\phi}}^2 + (G_4 - G_{\text{Tele},T})\left(2G_{3,\phi} - G_{2,X}\right) \neq 0$	-	-	1	1	Z_1	3
0.II	$G_{2,\phi\phi} = 0$ and $-3{G_{4,\phi}}^2 + (G_4 - G_{\text{Tele},T})\left(2G_{3,\phi} - G_{2,X}\right) = 0$	-	-	1	-	-	2
1	$G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} \neq 0$ $\tilde{c}_1 \neq 0, \tilde{c}_2 \neq 0$	2	1	1	1	$-\tilde{c}_1/\tilde{c}_2$	7
2	$G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} \neq 0, \tilde{c}_1 = 0, \tilde{c}_2 = 0$						
2.I	$G_{\text{Tele},T_{\text{ax}}} \neq 0, \tilde{c}_3 \neq 0, \tilde{c}_4 \neq 0$						
2.I.a	$G_{\text{Tele},T_{\text{ax}}} \neq 0, \tilde{c}_3 \neq 0, \tilde{c}_4 \neq 0$	1	1	1	1	$-\tilde{c}_3/\tilde{c}_4$	6
2.I.b	$G_{\text{Tele},T_{\text{ax}}} \neq 0, \tilde{c}_3 = 0, \tilde{c}_4 = 0$	1	1	1	-	-	5
2.II	$G_{3,\phi} = Z_2,\ G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi}$						
2.II.a	$G_{\text{Tele},I_2} - 2G_{4,\phi} \neq 0$	2	1	1	-	-	6
2.II.b	$G_{\text{Tele},I_2} - 2G_{4,\phi} = 0$	1	1	1	-	-	5
3	$G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} = 0, \tilde{c}_1 \neq 0, \tilde{c}_2 \neq 0$	1	-	1	1	$-\tilde{c}_1/\tilde{c}_2$	4
4	$G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} = 0, \tilde{c}_1 = 0, \tilde{c}_2 = 0$						
4.I	$G_{\text{Tele},T_{\text{vec}}} = -\frac{2}{3}(G_4 - G_{\text{Tele},T}),\ G_{4,\phi} = \frac{1}{2}G_{\text{Tele},I_2}$						
4.I.a	$\tilde{c}_3 \neq 0, \tilde{c}_4 \neq 0$	-	-	1	1	$-\tilde{c}_3/\tilde{c}_4$	3
4.I.b	$\tilde{c}_3 = 0, \tilde{c}_4 = 0$	-	-	1	-	-	2
4.II	$G_{3,\phi} = Z_2,\ G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi}$						
4.II.a	$G_{\text{Tele},I_2} - 2G_{4,\phi} \neq 0$	1	-	1	-	-	3
4.II.b	$G_{\text{Tele},I_2} - 2G_{4,\phi} = 0$	-	-	1	-	-	2
5	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} \neq 0, \tilde{c}_1 \neq 0, \tilde{c}_2 \neq 0$	1	-	1	1	$-\tilde{c}_1/\tilde{c}_2$	4
6	$G_{\text{Tele},T_{\text{vec}}} = 0, G_{\text{Tele},T_{\text{ax}}} \neq 0, \tilde{c}_1 = 0, \tilde{c}_2 = 0$	1	-	1	-	-	3
7	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ \tilde{c}_1 \neq 0, \tilde{c}_2 \neq 0$	-	-	1	1	$-\tilde{c}_1/\tilde{c}_2$	3
8	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} = 0, \tilde{c}_1 = 0, \tilde{c}_2 = 0$	-	-	1	-	-	2

Table 4.2: All Branches of the theory are represented with their respective pdof. To each of the scalar, vector and tensor components correspond 1,2 and 2 DoF respectively. The quantities $\tilde{c}_i$ are defined in (4.41), (4.42), (4.43) and (4.44) while Z_1 and Z_2 are defined in Eqs. (4.45)-(4.46)) and , respectively.

The rest of the Cases 3, 4, 5 and 7 are less general branches that include the rest of the combinations when $c_2 = c_3 = 0$, effectively rendering the vectorial dof non-dynamical. These branches include massless/massive scalar and tensorial dof.

In total the BDLS theory, contains a variety of branches which include all forms of massless SVT combinations and also massive scalars. Specifically, the massless sectors if applicable, they always include only one scalar pdof. This is a common feature shared amongst the most general scalar tensor type of theories, dubbed as Horndeski gravity. In order to gain more insight about how these pdof manifest in the real world, in the next section the polarization content of the BDLS theory will be probed.

4.6 Polarizations of gravitational waves

In metric theories of gravity there are only 6 GW polarizations allowed [151, 152]. These polarizations can be classified according to their helicity states as two tensor (helicity ± 2) modes plus (+) and cross ($\times$), two vector (helicity ± 1) modes called x and y and two scalar (helicity 0) modes named breathing and longitudinal modes. This is illustrated in Fig. 4.1. In general,the polarization content of GW can be probed by measuring the outputs of their relative amplitudes in the detectors [153, 154, 155, 156].

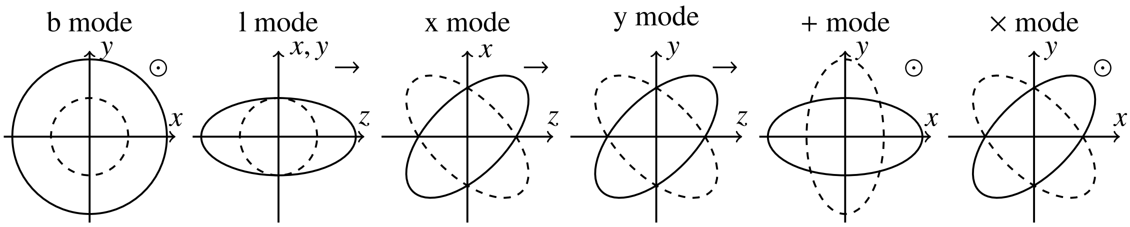

Figure 4.1: All possible polarizations of GW travelling in z-direction, starting with the scalar modes breathing, longitudinal, the vector modes x,y, and the tensor modes $+, \times$. The GW deforms a ring of freely falling test particles.

Currently, there is the possibility to distinguish among very specific subsets of all possible polarization combinations via a three-detector network. There are still degeneracies to be resolved which can be realized by introducing a five-detector network. In this way, the measurements will also be more accurate compared to the current three-detector scheme. In this regard, future measurements will shed more light on the status of the polarization content.

However, there have been reported in Ref. [35, 157, 158] some constraints on pure combinations of only tensor modes against only vector or only scalar modes, based on a quite simplified analysis relying on GR templates. In the analysis, using (GW170814, GW170817, and GW170818), it turned out that only the tensorial polarizations were not disfavoured. It should be clarified that the analysis suggests that a GW cannot only have scalar or only vector polarizations but is highly more likely to have only tensor polarizations. Nevertheless, this most certainly does not exclude any combination between different kinds of subsets such as tensor+scalar modes, tensor+vectors and other combinations as such.

The framework that mathematically describes the polarizations of GW is directly linked to the electric components of the Riemann tensor $\mathring{R}_{i0j0}$ [7]. This is due to the fact that the components $\mathring{R}_{i0j0}$ control the response of test particles that are freely falling in a gravitational field. More precisely, these components dictate the behaviour of the geodesic deviation equation Ref. [7, 159]

$$\ddot{x}_i = -\mathring{R}_{i0j0}x^j\,, \tag{4.166}$$

where dots represent coordinate time derivatives, $(t, x, y, z) = (0, 1, 2, 3)$, $i = \{1, 2, 3\}$ and $x^j = (x, y, z)$. There is a very useful tool that allows for a systematic study of GW polarizations called E(2) classification [142]. This tool is based on the Newmann-Penrose formalism [151] and allows us to categorise the polarizations of massless GW via the help of the representation of the little group , which is the two-dimensional Euclidean group

E(2). Using this tool one can parametrize the six independent components of $\mathring{R}_{i0j0}$ as

$$\mathring{R}_{0i0j} = \begin{pmatrix} \frac{1}{2}(\Re\Psi_4 + \Phi_{22}) & \frac{1}{2}\Im\Psi_4 & -2\Re\Psi_3 \\ \frac{1}{2}\Im\Psi_4 & -\frac{1}{2}(\Re\Psi_4 - \Phi_{22}) & 2\Im\Psi_3 \\ -2\Re\Psi_3 & 2\Im\Psi_3 & -6\Psi_2 \end{pmatrix}. \tag{4.167}$$

where $\Phi_{22}, \Psi_2, \Psi_3, \Psi_4$ are some of the Newmann-Penrose variables, $\Re$ represents the real part and $\Im$ the imaginary one. These variables can also be classified wrt their helicity states through

$$\begin{aligned} &\Psi_2 : s = 0\,, \quad \Phi_{22} : s = 0\,, \\ &\Psi_3 : s = -1\,, \overline{\Psi}_3 \ : s = 1\,, \\ &\Psi_4 : s = -2\,, \overline{\Psi}_4 \ : s = 2\,, \end{aligned} \tag{4.168}$$

where the overbar denotes complex conjugation. It can be directly deduced from Eq. (4.168) that Φ_{22}, Ψ_2 are related to scalar dof, Ψ_3 is related to vectorial dof and finally Ψ_4 is related to tensorial dof. A visualization of the parametrization of Eq. (4.167) can be found in Fig. (4.1).

For the needs of this analysis, the electric components of the Riemann tensor $\mathring{R}_{i0j0}$ will be split in an SVT manner and they will be expressed in terms of the gauge invariant variables (4.21)-(4.21) as

$$\mathring{R}_{0i0j} = \begin{pmatrix} \ddot{\psi} - \frac{1}{2}\ddot{h}_+ & -\frac{1}{2}\ddot{h}_\times & -\frac{1}{2}ik(\dot{\beta}_1 + \dot{\Lambda}_1) \\ -\frac{1}{2}\ddot{h}_\times & \ddot{\psi} + \frac{1}{2}\ddot{h}_+ & -\frac{1}{2}ik(\dot{\beta}_2 + \dot{\Lambda}_2) \\ -\frac{1}{2}ik(\dot{\beta}_1 + \dot{\Lambda}_1) & -\frac{1}{2}ik(\dot{\beta}_2 + \dot{\Lambda}_2) & \ddot{\psi} - k^2(\dot{\chi} + \Phi) \end{pmatrix}. \tag{4.169}$$

In contrast to the representation of Eq. (4.167), the SVT counterpart Eq. (4.169) is valid for both a massless and massive GW. As a matter of fact when the GW is massless then Eq. (4.169) and Eq. (4.167) coincide.

Combining the results of the analysis in Sec. 4.3 along with the electric components of the Riemann tensor defined in Eq. (4.169), all the available cases of the BDLS theory are exhaustively probed regarding their polarization content. The calculation process is straightforward since most of the information needed is already calculated in the form of solutions $Y_{|k|}$ and $Y_{|m|}$ in Sec. 4.4. These solutions just need to be replaced in Eq. (4.169). All of the resulting polarizations are presented into Table 4.3 which is very similar in structure to Table 4.2.

The first cases considered were also the most fundamental ones including GR and $f(T)$ gravity theories. It is already known [58, 160, 70] that these theories predict only tensor polarizations and thus the correct results were reproduced as validation for the analysis. The next step was to consider the standard Horndeski theory, Case 0 in Table 4.3, which is the most general scalar tensor theory based on curvature. For Case 0I, tensor polarizations were found for the massless sector along with a mix of breathing an longitudinal modes for the massive sector. On the contrary, for Case 0II which assumes only a massless sector with tensorial dof, just tensorial polarizations were found.

Regarding the TG framework, starting with the most general Case 1, the full BDLS theory, the breathing mode along with the tensor one were obtained for the massless sector. In the massive sector, on the other hand, both scalar polarizations were calculated which is an expected behaviour for the massless scalar sector since there were predicted two scalar dof as shown in Table 4.2. This behaviour is comparable to Case 0I and as such Case 1 can be thought of as Case 0I with just an extra breathing mode in the massless sector. This unique polarization imprint is only shared between Cases 1 and 3 although they differ in dof since the first case includes an additional massive scalar and one massless vector.

Between the rest of the less general Cases after Case 1, for the Cases 2.I.a, 2.I.b, 2.II.b, 5 and 6 although a massless scalar dof exists it does not leave any polarization imprint.

Cases		**Polarizations**							
	Conditions	Massless sector $\omega^2 = k^2$						Massive sector $\omega^2 - k^2 = m^2$	
		Scalar		Vector		Tensor		Scalar	
		b	l	x	y	+	×	b	l
-	$G_4 - G_{\text{Tele},T} \neq 0$	-	-	-	-	√	√	-	-
0	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ G_{\text{Tele},I_2} = 0\,,$ $G_{\text{Tele},X} = 0, G_{\text{Tele},\phi\phi} = 0$								
0.I	$G_{2,\phi\phi} \neq 0$ and $-3{G_{4,\phi}}^2 + (G_4 - G_{\text{Tele},T})\left(2G_{3,\phi} - G_{2,X}\right) \neq 0$	-	-	-	-	√	√	√	√
0.II	$G_{2,\phi\phi} = 0$ and $-3{G_{4,\phi}}^2 + (G_4 - G_{\text{Tele},T})\left(2G_{3,\phi} - G_{2,X}\right) = 0$	-	-	-	-	√	√	-	-
1	$G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} \neq 0,\ \tilde{c}_1 \neq 0,\ \tilde{c}_2 \neq 0$	√	-	-	-	√	√	√	√
2	$G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} \neq 0,\ \tilde{c}_1 = 0,\ \tilde{c}_2 = 0$								
2.I	$G_{\text{Tele},T_{\text{ax}}} \neq 0, \tilde{c}_3 \neq 0, \tilde{c}_4 \neq 0$								
2.I.a	$G_{\text{Tele},T_{\text{ax}}} \neq 0,\ \tilde{c}_3 \neq 0,\ \tilde{c}_4 \neq 0$	-	-	-	-	√	√	-	-
2.I.b	$c \neq 0,\ \tilde{c}_3 = 0,\ \tilde{c}_4 = 0$	-	-	-	-	√	√	-	-
2.II	$G_{3,\phi} = Z_2,\ G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi}$								
2.II.a	$G_{\text{Tele},I_2} - 2G_{4,\phi} \neq 0$	√	-	-	-	√	√	-	-
2.II.b	$G_{\text{Tele},I_2} - 2G_{4,\phi} = 0$	-	-	-	-	√	√	-	-
3	$G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ \tilde{c}_1 \neq 0,\ \tilde{c}_2 \neq 0$	√	-	-	-	√	√	√	√
4	$G_{\text{Tele},T_{\text{vec}}} \neq 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ \tilde{c}_1 = 0,\ \tilde{c}_2 = 0$								
4.I	$G_{\text{Tele},T_{\text{vec}}} = -\frac{2}{3}(G_4 - G_{\text{Tele},T}),\ G_{4,\phi} = \frac{1}{2}G_{\text{Tele},I_2}$								
4.I.a	$\tilde{c}_3 \neq 0, \tilde{c}_4 \neq 0$	-	-	-	-	√	√	-	-
4.I.b	$\tilde{c}_3 = 0, \tilde{c}_4 = 0$	-	-	-	-	√	√	-	-
4.II	$G_{3,\phi} = Z_2,\ G_{2,\phi\phi} = -G_{\text{Tele},\phi\phi}$								
4.II.a	$G_{\text{Tele},I_2} - 2G_{4,\phi} \neq 0$	√	-	-	-	√	√	-	-
4.II.b	$G_{\text{Tele},I_2} - 2G_{4,\phi} = 0$	-	-	-	-	√	√	-	-
5	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} \neq 0,\ \tilde{c}_1 \neq 0,\ \tilde{c}_2 \neq 0$	-	-	-	-	√	√	√	√
6	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} \neq 0,\ \tilde{c}_1 = 0,\ \tilde{c}_2 = 0$	-	-	-	-	√	√	-	-
7	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ \tilde{c}_1 \neq 0,\ \tilde{c}_2 \neq 0$	-	-	-	-	√	√	√	√
8	$G_{\text{Tele},T_{\text{vec}}} = 0,\ G_{\text{Tele},T_{\text{ax}}} = 0,\ \tilde{c}_1 = 0,\ \tilde{c}_2 = 0$	-	-	-	-	√	√	-	-

Table 4.3: All Branches of the BDLS theory with corresponding Polarizations. The quantity Z_2 is defined in the appendix (see Eqs. (4.45)-(4.46)) and the quantities $\tilde{c}_i$ are defined in (4.41), (4.42), (4.43) and (4.44), respectively.

The same exact pattern is shared for Cases 2.I.a and 4.I.a where there is instead a massive scalar that is invisible in the detectors. In addition, for the Cases 1, 2.I.a, 2.I.b, 2.II.a and 2.II.b which are the only ones that predict vectorial dof, this vectorial mode remains elusive to the polarization content. This peculiar situation of Cases that seem to predict specific dof which do not leave a polarization imprint can be attributed to two main reasons. The first one, is that those modes are not metrical dof and thus do not enter at all Eq. (4.169) which is directly responsible for the polarization response. The second reason, is that these fields although not metrical they still interact with the metrical fields but not strongly enough.

The simple Cases 2.I.b, 2.II.b, 4.I.b, 4.II.b, 6 and 8 were also found, in which there are only tensor polarizations. Obviously, they are trivial only up to the polarization content since some of them actually predict much more dof than just tensor modes. For example, Cases 4.I.b, 4.II.b and 8 predict only tensorial dof but the Cases 2.I.b, 2.II.b and 6 predict more dof as explained in Sec. 4.3.

In totality, the BDLS theory in contrast to its predicted dof which is quite rich in variety as illustrated in Sec. 4.3, assumes a similar behaviour also for its polarization content. All possibles types of polarizations were discovered and they correspond to 23 unique cases including massless and even massive sectors. Irrespective of the cases, in the end, there were no vector polarizations in any instance although there are predicted as dof.

4.7 Conclusion

In this section the number and nature of the dof of the BDLS theory in a Minkowski background was explored. The polarization content of the GW of the BDLS theory was probed, by using the results from the dynamical content of the theory found in Sec. 4.4. The BDLS theory is an analogue to the standard Horndeski theory that utilizes the TG framework. Instead of using the Levi-Civita connection which induces only curvature, the teleparallel connection is used which induces only torsion. The Lagrangian one can build, in general, depends on the underlying connection chosen for the geometry. In this instance, using the teleparallel connection allows for a much wider choice of potential scalar invariants compared to the usual curvature-based ones. This issue is covered in detail in Sec. 3.

In Sec. 4.2 an alternative method of performing tetrad perturbations was introduced by invoking the spacetime indexed tetrad labeled $\tau_{\mu\nu}$. This is just another way of performing tetrad perturbations and completely equivalent to the one already introduced in Sec.2.9.2. The difference in the two methods, relies solely in the calculational facilitation since it always easier and more natural to work with the same type of indices. Using this new representation of the tetrad perturbation and the scalar perturbation Eq. (4.2) the linearised field equations of the BLDS theory were calculated.

Introducing the SVT decomposition of $\tau_{\mu\nu}$ Eq. (4.19), in Sec. 4.3, the linear field equations (4.15)) – (4.16) were split into a 3+1 setting and then solved in order to determine the propagating dof. Then the number of propagating dof was calculated, whether they are massive or massless and the induced branching of the BDLS theory in Table 4.2 . All the relevant details of the tedious calculations need to generate Table 4.2 are included in Sec. 4.4.

A few of the highlights of the results are Cases 0.II and Case 1. Case 0.II is a never reported before sub-branch of the standard Horndeski theory which is not GR but it only entails 2 dof. In Case 1, which is the full BDLS theory, there are all SVT dof propagating which in total are 7 field, which 3 of them are 2 massless scalars and 1 massive scalar on top of vectorial and tensorial modes. The results of the dynamical analysis of Sec. 4.4 were compared against well known TG theories in Table 4.1. All these theories are subcases of the BDLS theory, their properties are known and they are also in agreement with the results of Table 4.2.

The results of Sec. 4.4 regarding the dof were then used in Sec. 4.6 in order to calculate the polarization content of the BLDS theory. All the relevant theory was introduced regarding the polarizations like the geodesic deviation Eq. (4.166) and the electric components of the Riemann tensor Eq. (4.169) which was also used in order to generate the Table (4.3). Specifically from these results, Case 0, among others, was found to be in complete agreement with the standard Ref. [146] and as a matter of fact it completes the analysis. Moreover, the BDLS theory includes a maximum of 4 polarizations for the massless sector and another 2 for the massive sector in general. This is distributed in various combinations amongst the branches of the theory. Nevertheless, there is no vector polarization in contrast to the existence of vectorial dof in some of the branches which may be interesting for future GW detections.

The polarization content of GW is one of the most fundamental aspects to be tested for a gravitational theory. It is rather like the next step after measuring the speed of GW since both are fundamental properties of waves. Polarization study is also becoming more important as new and more accurate measurements arise. TG theories have not been exhaustively tested yet but they show great promise in various areas of astrophysics [161, 30, 162, 163, 55, 164, 165, 48, 166]. BDLS theory encompasses the widest range of TG theories which are of scalar tensor type and produce second order field equations. Thus, by exhaustively studying the polarizations of this theory, a wide spectrum of TG theories can be confronted directly against experiments.

CHAPTER 5

COSMOLOGICAL PERTURBATIONS IN MODIFIED TELEPARALLEL GRAVITY MODELS: BOUNDARY TERM EXTENSION

In the last few decades it became clear that the background cosmology studies of a theory are not sufficient on their own [167]. This is linked to the behaviour of the dynamical dof under a specific choice of background solution. It may so happen that a specific class of background solutions, introduce instabilities in the theory. These singular cases can either be signaled by ill behaved linear perturbations or direct trivialization of constraints in the Hamiltonian analysis of the theory. On top of that, currently, there is no way that the background analysis is able to signal towards these problems [70]. The only way in order to check if there is a problem is by either performing a detailed Hamiltonian analysis or performing linear perturbations around the background solution in question [86]. Usually, linear perturbations are enough if there is a priory knowledge of the number of pdof of the theory, which is usually the case in cosmological perturbations.

From Minkowski perturbations it is also possible to obtain stability conditions which are imposed on the form of the theory [56, 168, 58]. These conditions properly constrain the theory or the background solutions in such a way that the pdof are well behaved or "healthy". In other words, the pdof should not attain anomalous propagation properties such as zero propagation speed, negative effective mass or anyway lead to negative unbounded energy.

Perturbations are also directly linked with observations. More specifically they are linked with the physical GW (tensor perturbations) and the formation of cosmic structures (scalar perturbations). In TG and more specifically in $f(T)$ theory there are a few works that deal with cosmological perturbations in a consistent way like in Ref. [44] which has been confirmed and widened in Refs.[65, 66, 67]. One of the earliest problems was the incorrect choice of perturbed tetrad which was missing 3 dof. However this did not affect most the analyses in $f(T)$ since these 3 dof completely drops off the field equations in flat FLRW backgrounds. Another prevalent problem quite related to the wrong perturbed tetrad was the over-fixing of gauge. However, this is not the case for more general theories like $f(T,B)$.

In this chapter, linear perturbations around a flat FLRW background for the $f(T,B)$ theory will be probed. First, a brief overview of the cosmology of $f(T,B)$ will be given in Sec. 5.1. Then the cosmological perturbations will be probed in Sec. 5.2, starting with the tensor perturbations Sec. 5.2.1 which lead directly to the GWPE. Subsequently, the vector perturbations will be studied in Sec. 5.2.2 which are followed by the scalar perturbations in Sec. 5.2.3. The scalar perturbations will then be used in order to calculate the matter density equation in Sec. 5.3 from which the effective gravitational constant will be derived in the sub-horizon approximation. The deflection parameter is also calculated along the effective gravitational constant throughout the analysis. Finally, in Sec. 5.4 an overview of the results will be given along with a discussion about their significance and comparison with the literature.

5.1 $f(T,B)$ gravity and Flat FLRW background

The boundary terms, in the action of a theory, being a total divergence term, do not contribute in the dynamical content of a theory. This is the case for TEGR as it is evident from Eq. (2.57), nevertheless the boundary term B becomes quite relevant in the modification $f(T,B)$. In this modification, B introduces higher order derivative terms in the field equations wrt the tetrad field. This is due to the fact that T is first order in the derivatives of the tetrad field whereas B is of second order. In this sense, the splitting of the Ricci scalar in Eq, (2.57) can be thought of as a separation between first order and second order derivatives.

In TG there is no direct equivalent of $f(\mathring{R})$ gravity but rather an analogue which is $f(T)$ since $f(\mathring{R}) = f(-T+B) \neq f(T)$. On the other hand, there is the superclass $f(T,B)$ that contains both $f(\mathring{R})$ and $f(T)$ [169, 170, 171, 160, 172, 172, 173]. This superclass, in all possible generality, includes arbitrary non-linear terms build from T and B. It is still a rather novel theory which may offer new insights regarding $f(T)$ and maybe $f(\mathring{R})$ gravity theories. The action of this theory is represented as

$$\mathcal{S}_{f(T,B)} = \frac{1}{2\kappa^2}\int d^4x\, e f(T,B) + \int d^4x\, e\mathcal{L}_m\,. \tag{5.1}$$

and its field equations are obtained through variation wrt the tetrad field as [169, 160]

$$\begin{aligned} 2\kappa^2\Theta^{\nu}{}_{\lambda} &= 2\delta^{\lambda}_{\nu}\mathring{\Box}f_B - 2\mathring{\nabla}^{\lambda}\mathring{\nabla}_{\nu}f_B + Bf_B\delta^{\lambda}_{\nu} + 4\Big[(\partial_\mu f_B) + (\partial_\mu f_T)\Big]S_{\nu}{}^{\mu\lambda} \\ &\quad + 4e^{-1}e^{A}{}_{\nu}\partial_\mu(eS_{A}{}^{\mu\lambda})f_T - 4f_T T^{\sigma}{}_{\mu\nu}S_{\sigma}{}^{\lambda\mu} - 2e^{A}{}_{\nu}f_T\omega^{B}{}_{A\mu}S_{B}{}^{\mu\lambda} - f\delta^{\lambda}_{\nu}\,, \end{aligned} \tag{5.2}$$

where $\Theta^{\nu}{}_{\lambda}$ is the energy-momentum tensor of $\mathcal{L}_m$ as defined in Eq. (2.69). The field equations of the spin connection are omitted since they are identical with the antisymmetrized form of Eq. (5.2). For simplicity, in what follows the Weitzenböck gauge ($\omega^{B}{}_{A\mu} \equiv 0$) is employed, hence the frame field is the sole variable.

The details of the background setup have already been described in Sec. 2.9. Furthermore, using the field Eqs. (5.2) together with the FLRW tetrad in Eq. (2.76) the Friedmann equations are obtained

$$3H(\dot{f}_B - 2Hf_T) + \frac{1}{2}(Bf_B - f) \quad = \quad \kappa^2\rho\,, \tag{5.3}$$

$$-\ddot{f}_B + 2f_T\dot{H} + 2H(3Hf_T + \dot{f}_T) + \frac{1}{2}(f - Bf_B) \quad = \quad \kappa^2 p\,, \tag{5.4}$$

where $\dot{f} := df/dt$ i.e, the overdots refer to derivatives with respect to cosmic time t, the energy density and pressure of matter are denoted as ρ and p respectively.

The Friedmann equations on top of their usefulness in background analyses [162] are also essential in applying perturbative schemes. It is worth mentioning that the effective fluid representation of the Eq. (5.3) which read as

$$3H^2 = \kappa^2\left(\rho + \rho_{\text{eff}}\right)\,, \tag{5.5}$$

$$3H^2 + 2\dot{H} = -\kappa^2\left(p + p_{\text{eff}}\right)\,, \tag{5.6}$$

where the fluid properties are defined as

$$\kappa^2\rho_{\text{eff}} := 3H^2\left(3F_B + 2F_T\right) - 3H\dot{F}_B + 3\dot{H}F_B + \frac{1}{2}F\,, \tag{5.7}$$

$$\kappa^2 p_{\text{eff}} := -\frac{1}{2}F - \left(3H^2 + \dot{H}\right)\left(3F_B + 2F_T\right) - 2H\dot{F}_T + \ddot{F}_B\,. \tag{5.8}$$

For this derivation the Lagrangian density $f(T,B)$ has been transformed into a TEGR plus modification form as indicated by the rule $f(T,B) \to -T + F(T,B)$. This effective fluid representation also includes the continuity equation [171]

$$\dot{\rho}_{\text{eff}} + 3H\left(\rho_{\text{eff}} + p_{\text{eff}}\right) = 0\,. \tag{5.9}$$

In this manner the Equation of State of the system is defined as

$$\omega_{\text{eff}} := \frac{p_{\text{eff}}}{\rho_{\text{eff}}} \tag{5.10}$$

$$= -1 + \frac{\ddot{F}_B - 3H\dot{F}_B - 2\dot{H}F_T - 2H\dot{F}_T}{3H^2\left(3F_B + 2F_T\right) - 3H\dot{F}_B + 3\dot{H}F_B - \frac{1}{2}F}\,. \tag{5.11}$$

In the ΛCDM limit ($F(T,B) = -2\Lambda$), this equation will approach an effective cosmological constant with $\omega_{\text{eff}} = -1$. It should be noted that the effective fluid representation is only valid at background level. The perturbations of a perfect fluid and those sourced from gravity are completely different structurally and thus one cannot reformulate one in terms of the other as with the background quantities.

5.2 Cosmological perturbations of $f(T,B)$ gravity

Before considering cosmological perturbations of the $f(T,B)$ theory it is worth mentioning that its spectrum in Minkowski spacetime has already been explored in Ref. [174]. It was shown that there is the usual massless graviton propagator with a $\sim -f_T$ modulation of the propagator plus an additional "scalaron" with a mass $\sim 1/\sqrt{-f_{BB}}$. Thus, in order to ensure stability, avoiding ghosts is ensured by imposing $f_T < 0$ and avoiding tachyons (see Sec. 2.8) is ensured likewise by $f_{BB} < 0$. This is also the situation for $f(\mathring{R})$ gravity, in Minkowski [27]. The fact that both $f(T,B)$ and $f(\mathring{R})$ predict the same polarizations [170, 160] might also be another indicator that these theories are more similar than they seem.

Using the perturbative framework introduced in Sec. 2.9.2 the cosmological perturbations of $f(T,B)$ theory in the flat FLRW will be probed. Although for the vector and tensor perturbations the full cases will be considered, for the scalar sector the sub-horizon limit will be used, for which a scale deep inside the Hubble radius $k >> aH$ is chosen. This is due to the fact that the scalar sector is highly involved and complex in analysing in arbitrary scales. On the other hand, deriving the matter density equation in the sub-horizon limit is quite important since identifying the modified gravitational Newton's constant G_{eff} is more straightforward.

Initiating the process [65, 67], a 3+1 split is employed by also using the SVT decomposition of the tetrad Eq. (2.85). This choice of perturbed tetrad is the most general and consistent since it includes 16 dof. The Weitzenböck gauge is also used which is valid at all perturbative orders. Hence, even at linear order only the tetrad will be considered. Utilizing this setup and the calculations from Sec. 2.9.4, the analysis of each sector is realized. First, the tensor sector will be probed, followed by the vector one and finally the scalar one.

5.2.1 Tensor Perturbations

Tensor perturbations are the most important sector of the perturbations since they are just the mathematical representation of the physical GW observed. In order to calculate their field equations, the field equations Eq. (5.2) are perturbed up to first order in the tetrad Eq. (2.85) and then the perturbation is restricted to just its tensor part as indicated in Eq. (3.47). For the case of $f(T,B)$ the GWPE is obtained as

$$\ddot{h}_{ij} + (3+\alpha_M)H\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 0\,, \tag{5.12}$$

for which the tensor excess speed α_T (2.120) is 0 meaning that there is light speed propagation [175] which is in agreement with recent observations as indicated in the multimessenger events of GW170817 [5] and GRB170817A [6]. On the other hand, the friction term α_M reads as

$$\alpha_M = \frac{1}{H}\frac{\dot{f}_T}{f_T}\,, \tag{5.13}$$

from which it is evident that the stability condition

$$f_T < 0, \tag{5.14}$$

since before simplifying the expression f_T multiplied the whole expression and its sign affected the sign of the speed of propagation. Hence, imposing $f_T < 0$, ghost instabilities are avoided in the tensor perturbations (see Sec. 2.8 for more details). In general, instabilities in the propagating dof signal to unphysical background solutions or unstable classses of models. If the instability can be remedied by restricting the class of possible models then the background solution is safe or the other way around. Moreover, this stability condition was already derived for the Minkowski background in Ref. [174], thus our result is in agreement with the literature.

The overall functional form of the GWPE Eq. (5.12), turns out to be identical to the $f(\mathring{R})$ [27] and also $f(T)$ [67] gravity theories, where for all of them the GW propagation speed is that of light.

5.2.2 Vector (and pseudovector) Perturbations

The vector perturbations are calculated by restricting the perturbed tetrad (2.85) to its vector components as

$$\left[\delta e^{A}{}_{\mu}\right] = \begin{bmatrix} 0 & a\beta_i \\ \delta^{I}{}_{i}b^{i} & a\delta^{Ii}\epsilon_{ijk}\sigma^{k} \end{bmatrix}, \tag{5.15}$$

where the gauge fixing $h_i \equiv 0$ has been already imposed. This specific gauge fixing is one of the available choices as indicated by the gauge transformation properties of the tetrad in Sec. 2.9.3. Linearizing the field equations wrt to the vector part of the tetrad (5.15), the field equations for the vector perturbations are obtained via the variables β_i and the σ_i

$$W_{[0i]}: \quad 0 \quad = \quad \sigma_i(\dot{f}_B + \dot{f}_T)\,, \tag{5.16}$$

$$W_{[ij]}(i \neq j): \quad 0 \quad = \quad \beta_i(\dot{f}_B + \dot{f}_T)\,, \tag{5.17}$$

which for $\dot{f}_B + \dot{f}_T \neq 0$ result in $\sigma_i = 0$ and $\beta_i = 0$. Hence only one equation is left that governs the evolution of b_i

$$W_{ij}(i \neq j): \quad 0 \quad = \quad \dot{b}_j + b_j H\,(2 + \alpha_M)\;. \tag{5.18}$$

This equation serves as a constraint equation since there are no second order time derivatives. As a result, b_i is not propagating and thus there are no pdof in the vector sector. It should be noted that the result of Eq. (5.18) has exactly the same functional form as the one reported in Ref. [67] for $f(T)$ gravity. This implies that adding B, as an argument in $f(T)$, does not introduce vectorial pdof in the flat FLRW background.

By imposing

$$\dot{f}_B + \dot{f}_T \equiv 0\,, \tag{5.19}$$

from Eq. (5.18), $f(T,B)$ reduces to $f(\mathring{R})$ gravity and all antisymmetric equations vanish trivially $W_{[\mu\nu]} \equiv 0$. Introducing further $Y_i := b_i - \beta_i$ then

$$W_{ij}(i \neq j): \quad 0 \quad = \quad \dot{Y}_j + Y_j\left(2H + \frac{\dot{f}_R}{f_R}\right)\,, \tag{5.20}$$

where

$$f_R = df/d\mathring{R}\,. \tag{5.21}$$

This equation describes the same physics as Eq. (5.18), thus no propagating vector perturbations exist. A very similar result holds true in $f(\mathring{R})$ theories [27]. Hence the vectorial part, at the level of the cosmological perturbations does not introduce any instabilities since there is no propagation.

5.2.3 Scalar Perturbations

The scalar sector of the perturbations, is always the most involved one due to the number of scalar dof that arise from the SVT decomposition of the tetrad or metric. Apart from their number, they also tend to couple to each other and any other additional external scalars such as matter fields. This is also the case for $f(T,B)$ gravity plus a perfect fluid. Using the same reasoning as in the other sectors, the tetrad perturbation (2.85) is restricted to its scalar part as

$$\left[\delta e^{A}{}_{\mu}\right] = \begin{bmatrix} \varphi & a\partial_i b \\ \delta^{I}{}_{i}\partial^{i} b & a\delta^{Ii}\left(-\psi\delta_{ij} + \epsilon_{ijk}\partial^{k}\sigma\right) \end{bmatrix}, \tag{5.22}$$

where the gauge has been fixed by setting $b \equiv \beta$ and $h \equiv 0$. Linearizing the field equations Eq. (5.2) and keeping only their scalar part, the following equations, that describe the dynamics of the scalar perturbations, are obtained

$$\begin{aligned} W_{00}: \quad \kappa^2\delta\rho &= 3H\delta\dot{f}_B + \Big(\frac{k^2}{a^2} + \frac{B}{2}\Big)\delta f_B - 6H^2\delta f_T - \frac{1}{2}f_T\delta T - \frac{2Hk^2 f_T}{a}b \\ &\quad +\dot{\psi}(12Hf_T - 3\dot{f}_B) + \frac{2k^2 f_T}{a^2}\psi + 6H\phi(2Hf_T - \dot{f}_B)\,, \end{aligned} \tag{5.23}$$

$$W_{ij}(i\neq j): \quad \psi - \phi = \frac{1}{f_T}(a(\dot{f}_T + \dot{f}_B)b - \delta f_B)\,, \tag{5.24}$$

$$\begin{aligned} W^{i}_{i}: \quad -\kappa^2\delta p &= \delta\ddot{f}_B + \delta f_B\left(\frac{2k^2}{3a^2} + \frac{B}{2}\right) - 2H\delta\dot{f}_T - 2(3H^2 + \dot{H})\delta f_T \\ &\quad -\frac{1}{2}f_T\delta T + 2f_T\ddot{\psi} + 2\dot{\psi}(6Hf_T + \dot{f}_T) + \frac{2k^2 f_T}{3a^2}\psi \\ &\quad +\dot{\phi}(2Hf_T - \dot{f}_B) - \frac{2k^2}{3a}(\dot{f}_B + 3Hf_T + \dot{f}_T)b \end{aligned}$$

$$+\phi\left(4f_T\left(-\frac{2k^2 f_T}{3a^2}+3H^2+\dot{H}-2\ddot{f}_B\right)+4H\dot{f}_T\right), \qquad (5.25)$$

where $\delta f_T = f_{TT}\delta T + f_{TB}\delta B$ and $\delta f_B = f_{BT}\delta T + f_{BB}\delta B$, while the antisymmetric contributions are

$$W_{0i}: \quad \kappa^2 av(p+\rho) = \delta\dot{f}_B - 3H\delta f_B + 2f_T\dot{\psi} - 2H\delta f_T + (2f_T H - \dot{f}_B)\phi\,, \qquad (5.26)$$

$$W_{i0}: \quad \kappa^2 av(p+\rho) = \delta\dot{f}_B - H\delta f_B + 2f_T\dot{\psi} + 2(\dot{f}_T + \dot{f}_B)\psi + (2f_T H - \dot{f}_B)\phi\,, \qquad (5.27)$$

$$W_{i0} - W_{0i}: \quad 0 = H(\delta f_T + \delta f_B) + \psi(\dot{f}_T + \dot{f}_B)\,, \qquad (5.28)$$

where the energy-momentum conservation in the case of dust is given by

$$\mathring{\nabla}_\mu \Theta_0{}^\mu: \quad \delta\dot{\rho} + 3H\delta\rho = \frac{\rho}{a}k^2 v + 3\dot{\psi}\rho\,, \qquad (5.29)$$

$$\mathring{\nabla}_\mu \Theta_i{}^\mu: \quad a\dot{v} + aHv = -\phi\,. \qquad (5.30)$$

As expected the scalar perturbations of the tetrad are the ones that couple with the scalar parts of the perturbed EMT components, the matter density $\delta\rho$ and δ. It is this relation that leads to the link between cosmological observables and the scalar perturbations. In the next section, the linearized field equations for the scalar sector (5.23)–(5.28) will be solved, in order to obtain the matter density equation that is closely related with the clustering of galaxies.

5.3 Matter perturbation equations in $f(T,B)$ gravity

In this section, the matter density equation of $f(T,B)$ gravity will be calculated. From this equation the induced modified Newton's constant G_{eff} will be extracted along with the deflection parameter Σ which is sensitive to weak lensing. The values of G_{eff} and

Σ will then be compared with known results in the literature and, of course, against the standard Newton's constant G_N.

The setup of calculating the matter density equation follows closely Refs. [27, 176]. In order to facilitate the calculations, the variable $V := av$ is introduced as a mere rescaling of the velocity scalar. The gauge invariant variable δ_{m}, called density contrast, is also introduced as

$$\delta_{\mathrm{m}} := \frac{\delta\rho}{\rho} + 3HV\,, \tag{5.31}$$

in order to allow for a gauge invariant representation of the matter density equation. Combining Eqs. (5.29)–(5.30) results in the relation

$$\delta\dot{\rho} + 3H\delta\rho = \frac{k^2\rho V}{a^2} + 3\rho\dot{\psi}\,, \tag{5.32}$$

which can be further used by taking its time derivative along with the scalar velocity V as

$$\dot{\delta}_{\mathrm{m}} = -\frac{\mathring{\nabla}^2 V}{a^2} + 3\dot{\psi} + 3\frac{d}{dt}(HV)\,, \tag{5.33a}$$

$$\dot{V} = -\varphi\,, \tag{5.33b}$$

From these equations, the matter density equation is derived as

$$\ddot{\delta}_{\mathrm{m}} + 2H\dot{\delta}_{\mathrm{m}} = \frac{\mathring{\nabla}^2\varphi}{a^2} + 3\ddot{\psi} + 3\frac{d^2}{dt^2}(HV) + 6H\dot{\psi} + 6H\frac{d}{dt}(HV)\,, \tag{5.34}$$

where the first term on the RHS is the typical Laplacian operator that implies the Poisson equation while the rest terms can be considered as corrections. At this stage Eq. (5.34) is valid for any regime but the analysis will proceed in the sub-horizon limit. This limit is defined as being deep inside the Hubble radius i.e, $k >> aH$, k being the norm of the wave covector. Thus, the dominant terms will be k and $\delta\rho$. Further implementing this approximating scheme

$$\left\{\frac{k^2}{a^2}|\phi|, \frac{k^2}{a^2}|\psi|, \frac{k^2}{a^2}|\beta|, \frac{k^2}{a^2}|\delta f_T|, \frac{k^2}{a^2}|\delta f_B|\right\} \gg \left\{H^2|\phi|, H^2|\psi|, H^2|\beta|, H^2|\delta f_T|, H^2|\delta f_B|\right\}, \tag{5.35}$$

and

$$|\dot{X}| \lesssim |HX| \quad \text{where} \quad X \in \left\{\phi, \psi, b, \delta f_T, \delta f_B, \dot{\phi}, \dot{\psi}, \dot{\beta}, \delta\dot{f}_T, \delta\dot{f}_B\right\}. \tag{5.36}$$

Using these inequalities, in Fourier space it follows that Eq. (5.34) becomes

$$\ddot{\delta}_{\rm m} + 2H\dot{\delta}_{\rm m} \simeq -\frac{k^2\phi}{a^2} = 4\pi\rho G_{\rm eff}\delta_{\rm m} = \frac{\kappa^2}{2}\rho G_{\rm eff}\delta_{\rm m}\,, \tag{5.37}$$

from which one can directly see that ϕ, specifically, relates $\delta_{\rm m}$ directly to the gravitational scalar dof. Henceforth, the linearized field equations need to be solved in order to determine what ϕ is in terms of $\delta\rho$. In this way, the explicit form of $G_{\rm eff}$ can be calculated is in terms of the background quantities of the theory.

Applying the Sub-horizon limit, in the Newtonian gauge for the scalar perturbations in Sec. 2.9.4, various needed perturbations of scalar combinations are obtained

$$\delta T \simeq -\frac{4H}{a}\left(k^2 b - 3aH(\psi + \phi)\right), \tag{5.38}$$

$$\delta B \simeq -\frac{2k^2}{a^2}(2abH - 2\psi + \phi)\,, \tag{5.39}$$

$$\delta f_T \simeq -\frac{2k^2}{a^2}\left(2abH\left(f_{TB} + f_{TT}\right) + f_{TB}(\phi - 2\psi)\right), \tag{5.40}$$

$$\delta f_B \simeq -\frac{2k^2}{a^2}\left(2abH(f_{BB} + f_{TB}) + f_{BB}(\phi - 2\psi)\right). \tag{5.41}$$

Thus the Eqs. (5.23)–(5.28) in the sub-horizon approximation are

$$W_{00}: \quad \kappa^2\delta\rho \simeq \left(\frac{2k^2 f_T}{a^2} - 3H\dot{f}_B\right)\psi + \left(\frac{k^2}{a^2} - 3\dot{H}\right)\delta f_B + 6H(Hf_T - \dot{f}_B)\phi$$

$$-6H^2\delta f_T\,, \tag{5.42}$$

$$W_{[0i]}:\quad 0 \simeq \psi(\dot{f}_B+\dot{f}_T)+H\delta f_B+H\delta f_T\,, \tag{5.43}$$

$$W_{ij}(i\neq j):\quad 0 = -ab(\dot{f}_B+\dot{f}_T)+\delta f_B+f_T\psi-f_T\phi\,, \tag{5.44}$$

$$\begin{aligned} W_i^i:\quad 0 \simeq\; & \delta f_B(18a^2\dot{H}-4k^2)+12a^2(4H^2+\dot{H})\delta f_T+4ak^2(\dot{f}_B+\dot{f}_T)b \\ & +\left(6a^2\left(H(\dot{f}_B-4\dot{f}_T)+2\ddot{f}_B\right)+4f_T(k^2-6a^2\dot{H})\right)\phi \\ & -4\psi\left(f_Tk^2+3a^2H\dot{f}_T\right)\,. \end{aligned} \tag{5.45}$$

For later convenience, the $W_{[0i]}$ component will be fully expanded

$$\begin{aligned} W_{[0i]}:\qquad 0 \simeq\; & -\frac{4H^2k^2(f_{BB}+2f_{TB}+f_{TT})}{a}b \\ & +\frac{\left(a^2((\dot{f}_B+\dot{f}_T)+12H^3(f_{TT}+f_{TB}))+4Hk^2(f_{BB}+f_{TB})\right)}{a^2}\psi \end{aligned} \tag{5.46}$$

$$-\frac{2H\left((f_{BB}+f_{TB})(k^2-6a^2\dot{H})-6a^2H^2(f_{TT}+f_{TB})\right)}{a^2}\phi\,. \tag{5.47}$$

due to its significance in the classification of the branching. In order to obtain a closed system which can be solved, $\left\{W_{00}, W_{[0i]}, W_{ij}\right\}$ have been chosen to be the constituents of the system. For the solution process it is also convenient to include the following variables

$$\Pi := f_B+f_T\,, \tag{5.48}$$

$$\Upsilon := f_{BB}+2f_{TB}+f_{TT} = \Pi_T+\Pi_B\,, \tag{5.49}$$

$$\Xi := f_{TB}^2-f_{TT}f_{BB} = -\Pi_T\Pi_B+f_{TB}\Upsilon\,. \tag{5.50}$$

The variable Π quantifies the deviation of $f(T,B)$ from $f(\mathring{R})$ theories of gravity where

$\Pi|_{f(\mathring{R})} \equiv 0$. These quantities will allow for a classification of the $f(T,B)$ models in three branches

1. $\{\Pi \neq \text{const}, \Upsilon \neq 0\}$
   Which can further be classified using $\Xi = -\Pi_T \Pi_B + f_{TB}\Upsilon$
   (a) $\{\Pi \neq \text{const}, \Upsilon \neq 0, \Xi \neq 0\}$ most general case of $f(T,B)$
   (b) $\{\Pi \neq \text{const}, \Upsilon \neq 0, \Xi = 0\}$ includes $f(T)$
2. $\{\Pi \neq \text{const}, \Upsilon \equiv 0\}$
   Which can further be classified using $\Upsilon \equiv 0 \Rightarrow \Pi_B \equiv -\Pi_T$ into Eq. (5.50) as $\Xi = \Pi_T^2 = \Pi_B^2$
   (a) $\{\Pi \neq \text{const}, \Upsilon = 0, \Xi \neq 0\}$
   (b) $\{\Pi = \text{const}, \Upsilon = 0, \Xi = 0\}$ the unique $f(\mathring{R})$ case

There is yet another important parameter called Σ_{def} that is sensitive to weak lensing, which is very important for fundamental physics in cosmology because it links the propagation of light with the effect of gravity. This parameter is defined as

$$\Sigma := \frac{1}{2}\frac{G_{\text{eff}}}{G}\left(1 + \frac{\psi}{\phi}\right) . \tag{5.51}$$

This parameter is a measure between the lensing potential $-(\phi + \psi)$ and the matter density contrast δ_{m}. Hence, in a way, Σ_{def} is an analog to G_{eff} but between the lensing potential and δ_{m}.

Before elaborating further on the branches, the defining conditions, $\Xi \equiv 0$ and $\Upsilon \equiv 0$ will be further discussed along with their solutions. Starting with with $\Xi \equiv 0$, it can be solved using separation of variables by assuming $f(T,B) = f_1(T)f_2(B)$ which yields

$$f(T,B) = f_0\Big((B + Bm - C_2)(T + mT - C_3 m)^m\Big)^{\frac{1}{m+1}} , \quad m \neq -1 , \tag{5.52}$$

$$f(T,B) = f_0 e^{C_1 T + C_2 B} , \quad m = -1 . \tag{5.53}$$

where f_0, C_1, C_2, C_3, m are constants. There is yet another class of solutions that attain single variable dependence as Φ, $f(T,B) = f(\Phi)$ where $\Phi = \Phi(T,B)$. The most popular models in this family are $\Phi \equiv \mathring{R} = -T + B$ and $\Phi \equiv T$ which represent $f(\mathring{R})$ and $f(T)$ theories of gravity. A few more examples include $f(TB) = c\sqrt{TB}$ which is actually the only acceptable model of the family $f(TB) = c\,(TB)^m$ and

$$f(T,B) = -T + F(B)\,. \tag{5.54}$$

As for the condition $\Upsilon \equiv 0$, it is solved by the family of solutions

$$f(T,B) = f_1(\mathring{R})X + f_2(\mathring{R})\,, \tag{5.55}$$

where $X = X(T,B)$ is any function such that $X_T + X_B \neq 0$ and $\Upsilon \equiv 0$. The condition $X_T + X_B \neq 0$ effectively means that $X \neq X(\mathring{R})$ so that the overall solution does not reduce to just $f(\mathring{R})$. The most straightforward form of these type of solutions is $X = (c_1 T^p + c_2 B^q + c_3\,(TB)^r)^m$ for which $X = c_1 T + c_2 B$ where $c_1, c_2 \in \mathbb{R}$ and $c_1 \neq -c_2$. It should be noted that $\Xi \equiv 0$ and $\Upsilon \equiv 0$ are solved trivially for $f(\mathring{R})$.

Before presenting each branch will all the relevant information, the table 5.1 regarding all the results is presented.

Class	Conditions	Models	G_{eff}	Σ
1	$\{\Pi \neq 0, \Upsilon \neq 0\}$			
1a	$\Xi \neq 0$	General $f(T,B)$	$-G\frac{4\Upsilon}{36H^2(f_{BB}f_{TT}+2\Xi)+3\Upsilon f_T}$	$-\frac{\Upsilon}{\Upsilon f_T+12H^2(f_{BB}f_{TT}+2\Xi)}$
1b	$\Xi = 0$	Includes $f(T)$	$G\frac{A_2}{A_6}$	$\frac{\Delta_4}{\Delta_{10}} = -\frac{A_2}{A_6}$
2	$\{\Pi \neq 0, \Upsilon \equiv 0\}$			
2a	$\Xi \neq 0$	Less general $f(T,B)$	$-G\frac{4}{3(f_T+12H^2 f_{TB})}$	$-\frac{1}{f_T+12H^2 f_{TB}}$
2b	$\Xi = 0$	Only $f(\mathring{R})$	$G\left(\frac{4}{3f_R}+\frac{1}{3(-f_R+3\frac{k^2}{a^2}f_{RR})}\right)$	$\frac{1}{f_R}$

Table 5.1: Summarizing the cases of all the subclasses of $f(T,B)$ in the sub-horizon approximation. For more details see [3].

In the next subsections, table 5.1 is explained in a branch by branch manner. For each individual branch, the linearized field equations for the scalar sector are solved and then G_{eff} along with Σ are calculated accordingly. Each branch is completely self-contained and complete, since also all technical details are also included.

5.3.1 Branch $\{\Pi \neq \textbf{const}, \Upsilon \neq 0, \Xi \neq 0\}$

In this branch, every argument attains a non-linear dependence and thus this is the most general case one can have. Solving Eq. (5.47) for b

$$b = \frac{(a^2(12H^3 f_{TT} + \dot{\Pi}) + 4Hk^2\Pi_B)}{4aH^2k^2\Upsilon}\psi + \frac{(6a^2(\Pi_B\dot{H} + H^2 f_{TT}) - k^2\Pi_B)}{2aHk^2\Upsilon}\phi\,, \tag{5.56}$$

which is then replaced into Eq. (5.44) and the ratio of the gravitational potential is found to be

$$\frac{\psi}{\phi} = \frac{2H\left(6a^4\dot{\Pi}(\Pi_B\dot{H} + H^2 f_{TT})\right)}{-a^4\dot{\Pi}(12H^3 f_{TT} + \dot{\Pi}) + 4a^2Hk^2(-2\Pi_B\dot{\Pi} + 12H^3 f_{BB}f_{TB} + H\Upsilon f_T) - 16H^2k^4\Xi} + \frac{2H\left(a^2k^2(-\Pi_B\dot{\Pi} - 24H^3 f_{BB}f_{TB} + 2H\Upsilon f_T + 24H\Xi\dot{H}) - 4Hk^4\Xi\right)}{-a^4\dot{\Pi}(12H^3 f_{TT} + \dot{\Pi}) + 4a^2Hk^2(-2\Pi_B\dot{\Pi} + 12H^3 f_{BB}f_{TB} + H\Upsilon f_T) - 16H^2k^4\Xi}, \tag{5.57}$$

then this equation is further substituted into Eq. (5.42) in order to obtain the general forms of the modified gravitational constant along with the deflection parameter Σ as

$$G_{\text{eff}} = G\frac{A_1k^2 + A_2k^4 + A_3k^6}{A_4 + A_5k^2 + A_6k^4 + A_7k^6}, \tag{5.58}$$

$$\Sigma = \frac{\Delta_1k^2 + \Delta_2k^4 + \Delta_3k^6 + \Delta_4k^8 + \Delta_5k^{10}}{\Delta_6 + \Delta_7k^2 + \Delta_8k^4 + \Delta_9k^6 + \Delta_{10}k^8 + \Delta_{11}k^{10}}, \tag{5.59}$$

where the A_i and Δ_i coefficients have been calculated as

$$A_1 = -a^4\Upsilon\dot{\Pi}(\dot{\Pi} + 12H^3 f_{TT}), \tag{5.60}$$

$$A_2 = -4a^2H\Upsilon(2\Pi_B\dot{\Pi} - 12H^3 f_{BB}f_{TB} - H\Upsilon f_T), \tag{5.61}$$

$$A_3 = -16H^2\Xi\Upsilon, \tag{5.62}$$

$$A_4 = -3a^6\dot{\Pi}(\Pi_B + \Pi_T)\Big[\tag{5.63}$$

$$\Pi'(6\dot{H}^2(\Pi_B - f_{TB}) + 6H^2\dot{H}(3f_{TB} - \Pi_B) + 18H^4(\Pi_T - f_{TB}) - H^2 f_T - H\dot{f}_T) \tag{5.64}$$

$$+ 6H^2\Big[\dot{H}(-24H^3(f_{TB} - \Pi_B)(f_{TB} - \Pi_T) + \Pi_B\dot{f}_T) + H\dot{\Pi}^2 \tag{5.65}$$

$$- 12H\dot{H}^2(f_{TB}(\Pi_B - f_{TB}) + \Xi) + H^2(2Hf_T + \dot{f}_T)(f_{TB} - \Pi_T)\Big]\Big] \tag{5.66}$$

$$A_7 = 12H^2\Xi(12H^2(f_{BB}f_{TT} + 2\Xi) + \Upsilon f_T)\,, \tag{5.67}$$

$$\Delta_1 = -\,a^4A_1\dot{\Pi}(12H\Pi_B\dot{H} - \dot{\Pi})\,, \tag{5.68}$$

$$\Delta_2 = a^2\left(-2H\Pi_B\dot{\Pi}(6a^2A_2\dot{H} - 5A_1) + a^2A_2\dot{\Pi}^2 + 8A_1H^2(-\Upsilon f_T - 6\Xi\dot{H})\right), \tag{5.69}$$

$$\Delta_3 = a^4A_3\dot{\Pi}^2 - 2a^2H\Pi_B\dot{\Pi}(6a^2A_3\dot{H} - 5A_2) + 8H^2\left(a^2A_2(-\Upsilon f_T - 6\Xi\dot{H}) + 3A_1\Xi\right), \tag{5.70}$$

$$\Delta_4 = 2H\left(4H\left(a^2A_3(-\Upsilon f_T - 6\Xi\dot{H}) + 3A_2\Xi\right) + 5a^2A_3\Pi_B\dot{\Pi}\right), \tag{5.71}$$

$$\Delta_5 = 24A_3H^2\Xi\,, \tag{5.72}$$

$$\Delta_6 = 2a^4A_4\dot{\Pi}(\dot{\Pi} + 12H^3f_{TT})\,, \tag{5.73}$$

$$\Delta_7 = 2a^2\left(\dot{\Pi}\left(a^2A_5(\dot{\Pi} + 12H^3f_{TT}) + 8A_4H\Pi_B\right)\right) \tag{5.74}$$

$$+\,96A_4H^4(f_{BB}f_{TT} + \Xi) + 48A_4H^4f_{TB}(f_{TT} - \Upsilon) - 4A_4H^2\Upsilon f_T\,, \tag{5.75}$$

$$\Delta_8 = -\,2a^4A_6\dot{\Pi}(-\dot{\Pi} - 12H^3f_{TT}) + 8a^2A_5H \tag{5.76}$$

$$+\left(-(H\Upsilon f_T - 2\Pi_B\dot{\Pi}) + 24H^3(f_{BB}f_{TT} + \Xi) + 12H^3f_{TB}(f_{TT} - \Upsilon)\right) + 32A_4H^2\Xi\,, \tag{5.77}$$

$$\Delta_9 = 2a^4A_7 - \dot{\Pi}(-\dot{\Pi} - 12H^3f_{TT}) + 32A_5H^2\Xi \tag{5.78}$$

$$+\,8a^2A_6H\left(-(H\Upsilon f_T - 2\Pi_B\dot{\Pi}) + 24H^3(f_{BB}f_{TT} + \Xi) + 12H^3f_{TB}(f_{TT} - \Upsilon)\right), \tag{5.79}$$

$$\Delta_{10} = 8H\left(a^2A_7\left(-(H\Upsilon f_T - 2\Pi_B\dot{\Pi}) + 24H^3(f_{BB}f_{TT} + \Xi) + 12H^3f_{TB}(f_{TT} - \Upsilon)\right) + 4A_6H\Xi\right), \tag{5.80}$$

$$\Delta_{11} = 32A_7H^2\Xi\,. \tag{5.81}$$

If one restricts to leading order then $A_3 \propto \Xi$, $A_7 \propto \Xi$ are the only coefficients, proportional to Ξ and the same happens with the coefficients $\Delta_5 \propto A_3$ and $\Delta_{11} \propto A_7$. Thus, the leading order parts turn out to be

$$G_{\text{eff}} = G\frac{A_3}{A_7} = -G\frac{4\Upsilon}{36H^2(f_{BB}f_{TT} + 2\Xi) + 3\Upsilon f_T}\,, \tag{5.82}$$

$$\Sigma \;=\; -\;\frac{\Upsilon}{\Upsilon f_T + 12H^2\left(f_{BB}f_{TT} + 2\Xi\right)}\,. \tag{5.83}$$

A prototype of models that represent this branch is for example $f(T,B) = f_1(T) + f_2(T)f_3(B) + f_4(B)$.

5.3.2 Branch $\{\Pi \neq \textbf{const}, \Upsilon \neq 0, \Xi = 0\}$

If $A_3 = A_7 \equiv 0$ where $\Xi \equiv 0$ then the leading terms are modified as

$$G_{\text{eff}} \;=\; G\frac{A_2}{A_6}\,, \tag{5.84}$$

$$\begin{aligned}\Sigma \;&=\; \frac{\Delta_4}{\Delta_{10}} = -\frac{A_2}{A_6} \\ &= -\frac{-A_3(4H\Upsilon f_T - 5\Pi_B\dot{\Pi})}{4A_7\left(-(H\Upsilon f_T - 2\Pi_B\dot{\Pi}) + 24H^3 f_{BB}f_{TT} + 12H^3 f_{TB}(f_{TT} - \Upsilon)\right)}\,.\end{aligned} \tag{5.85}$$

The coefficients A_2, A_6 turn out to be quite involved and hence G_{eff} like Σ have also very complicated forms. For this reason they are calculated explicitly for the models $f(T)$ and $f(T,B) \Rightarrow -T + f(B)$. For $f(T)$, expanding up to next to leading order Eq. (5.84)

$$G_{\text{eff}} \;=\; \frac{a^2\dot{f}_T(12H^3 f_{TT} + \dot{f}_T) - 4H^2k^2 f_T f_{TT}}{4H^2 f_T f_{TT}(6a^2H\dot{f}_T + k^2 f_T)}\,, \tag{5.86}$$

$$\Sigma \;=\; \frac{3a^2\dot{f}_T(8H^3 f_{TT} + \dot{f}_T) - 8H^2k^2 f_T f_{\rm TT}}{2f_T\Big(a^2\dot{f}_T(12H^3 f_{TT} - \dot{f}_T) + 4H^2k^2 f_T f_{TT}\Big)}\,, \tag{5.87}$$

which properly reproduces the standard (leading order) result $G_{\rm eff} = -G/f_T$ reported in Refs. [44, 47, 177].

Regarding the model $f(T,B) \to -T + f(B)$, Eq. (5.84) expanded up to next to leading order, the following forms are obtained

$$G_{\rm eff} \;=\; G\frac{4H(2\dot{f}_B + H)}{(\dot{f}_B + 2H)^2}\,, \tag{5.88}$$

$$\Sigma \;=\; \frac{a^2\dot{f}_B{}^2 + 4Hk^2 f_{BB}(2\dot{f}_B + H)}{3a^2 f_{BB}\dot{f}_B\Big(H^2(9\dot{f}_B + 14H) - 3\dot{H}(\dot{f}_B + 2H)\Big) + k^2 f_{BB}(\dot{f}_B + 2H)^2}\,. \tag{5.89}$$

It turns out that the resulting formulas for $f(T)$ and $f(T.B) \to -T + f(B)$ are simple enough compared to the general cases. This could lead to further observational investigations in the future.

5.3.3 Branch $\left\{\Pi \neq \text{const}, \Sigma = 0, \Xi = \Pi_T^2 = \Pi_B^2 \neq 0\right\}$

In this case it turns out that b completely drops out from Eq. (5.47), thus ψ can be solved as

$$\frac{\psi}{\phi} = \frac{2H\Pi_T(k^2 - 6a^2\dot{H})}{4Hk^2\Pi_T - a^2\dot{\Pi}}\,, \tag{5.90}$$

which is then replaced into Eq. (5.44) in order to solve for b as

$$b = \frac{f_T\left(2H\Pi_T(6a^2\dot{H} + k^2) - a^2\dot{\Pi}\right) + 2\dot{\Pi}(k^2 - 6a^2\dot{H})(f_{TB} + \Pi_T)}{(4Hk^2\Pi_T - a^2\dot{\Pi})^2}a\,\phi\,. \tag{5.91}$$

Substituting both in Eq. (5.42), $G_{\rm eff}$ and Σ are found to be

$$G_{\rm eff} = G\frac{Z_1k^2 + Z_2k^4 + Z_3k^6}{Z_4 + Z_5k^2 + Z_6k^4 + Z_7k^6}\,, \tag{5.92}$$

$$\Sigma = \frac{Y_1 k^2 + Y_2 k^4 + Y_3 k^6 + Y_4 k^8}{Y_5 + Y_6 k^2 + Y_7 k^4 + Y_8 k^6 + Y_9 k^8}\,, \tag{5.93}$$

where the Z_i and Y_i coefficients are calculated as

$$Z_1 = a^4 \dot{\Pi}^2\,, \tag{5.94}$$

$$Z_2 = -\,8a^2 H \dot{\Pi} \Pi_T\,, \tag{5.95}$$

$$Z_3 = 16H^2 \Xi\,, \tag{5.96}$$

$$Z_6 = -\,a^2 \dot{\Pi}^2 (f_{TB} + \Pi_T) \tag{5.97}$$

$$\begin{aligned} &12a^2 H^3 \Xi (5\dot{f}_B + 72H^3 f_{TT} + 60a^2 H f_{TB} \dot{H}) \\ &- 12a^2 H^3 \dot{\Pi} \left(f_{TB}(2f_{TT} - 7\Pi_T) + 2a^2 f_{TT} \Pi_T \right) \\ &+ 4a^2 H f_T \Pi_T \left(-6H^3 (f_{TT} + 2\Pi_T) + 9a^2 H \dot{H} \Pi_T + \dot{\Pi} \right)\,, \end{aligned} \tag{5.98}$$

$$Z_7 = -\,12H^2 \Xi (f_T + 12H^2 f_{TB})\,, \tag{5.99}$$

$$Y_1 = -\,a^2 A_1 (\dot{\Pi} + 12H\dot{H}\Pi_T)\,, \tag{5.100}$$

$$Y_2 = 6H\Pi_T (A_1 - 2a^2 A_2 \dot{H}) - a^2 A_2 \dot{\Pi}\,, \tag{5.101}$$

$$Y_3 = 6H\Pi_T (A_2 - 2a^2 A_3 \dot{H}) - a^2 A_3 \dot{\Pi}\,, \tag{5.102}$$

$$Y_4 = 6A_3 H \Pi_T\,, \tag{5.103}$$

$$Y_5 = -\,2a^2 A_4 \dot{\Pi}\,, \tag{5.104}$$

$$Y_6 = -\,2a^2 A_5 \dot{\Pi} + 8A_4 H \Pi_T\,, \tag{5.105}$$

$$Y_7 = -\,2a^2 A_6 \dot{\Pi} + 8A_5 H \Pi_T\,, \tag{5.106}$$

$$Y_8 = -\,2a^2 A_7 \dot{\Pi} + 8A_6 H \Pi_T\,, \tag{5.107}$$

$$Y_9 = 8A_7 H \Pi_T\,. \tag{5.108}$$

The leading order contributions in G_{eff} and Σ are then obtained as

$$G_{\text{eff}} = G\frac{Z_3}{Z_7} = -G\frac{4}{3(f_T + 12H^2 f_{TB})}, \tag{5.109}$$

$$\Sigma = \frac{Y_4}{Y_9} = -\frac{1}{f_T + 12H^2 f_{TB}}, \tag{5.110}$$

which are both much simpler than (5.84) and (5.85). This branch could be considered as a bit more general than just plain $f(\mathring{R})$.

5.3.4 Branch $\left\{\Pi = \text{const}, \Sigma = 0, \Xi = \Pi_T^2 = \Pi_B^2 \equiv 0\right\}$

This is a very unique and peculiar branch within the superclass of $f(T,B)$. In this branch, by default, all the antisymmetric field equations (5.47) vanish. This also holds without using any approximation. On top of that, once all quantities expanded, the scalar field b completely drops out from the field equations. Therefore only ψ and ϕ are left from the scalars. The condition $\Pi_T^2 = \Pi_B^2 \equiv 0$ means that $\Pi = f_T + f_B \equiv c$. This is just the defining condition of the $f(\mathring{R})$ theories.

Solving W_{ij} for ψ

$$\frac{\psi}{\phi} = \frac{a^2(12 f_{RR}\dot{H} + f_R) - 2k^2 f_{RR}}{a^2 f_R - 4k^2 f_{RR}}\,, \tag{5.111}$$

and substituting it in Eq. (5.42) we obtain

$$G_{\text{eff}} = \frac{8k^4 f_{RR} - 2a^2k^2 f_R}{-9a^4(f_R(4f_{RR}\dot{H}^2 + H\dot{f}_R) + 4Hf_{RR}\dot{f}_R\dot{H}) - 2a^2k^2(-15Hf_{RR}\dot{f}_R + 9f_Rf_{RR}\dot{H} + f_R^2) + 6k^4 f_R f_{RR}}, \tag{5.112}$$

$$\Sigma = \frac{6k^4 f_{RR} - 2a^2k^2(6f_{RR}\dot{H} + f_R)}{-9a^4(f_R(4f_{RR}\dot{H}^2 + H\dot{f}_R) + 4Hf_{RR}\dot{f}_R\dot{H}) - 2a^2k^2(-15Hf_{RR}\dot{f}_R + 9f_Rf_{RR}\dot{H} + f_R^2) + 6k^4 f_R f_{RR}}. \tag{5.113}$$

Employing further approximations of the form $|\dot{X}| \sim H|X|$ where X denotes background quantities, in conjunction with the matter dominated approximation $|f_R/(H^2 f_{RR})| \gg 0$ the standard results are obtained

$$G_{\text{eff}} \sim G\left(\frac{4}{3f_R} + \frac{1}{3(-f_R + 3\frac{k^2}{a^2}f_{RR})}\right), \tag{5.114}$$

$$\Sigma \sim \frac{1}{f_R}, \tag{5.115}$$

as also reported in the literature [27, 176]. Thus, apart from the self-consistency check for $f(T)$ gravity, standard results of $f(\mathring{R})$ gravity are properly reproduced from a more general point of view since $f(T,B)$ contains $f(\mathring{R})$.

5.4 Conclusion and discussion

Cosmological perturbations allow for a deeper understanding of the GW propagation and large scale structure to even late-time cosmology. In standard curvature based gravity one would need to perturb the metric tensor in order to study deviations from the background spacetime but in TG one needs to perturb the tetrad and then study the SVT parts since they decouple at first order. This is also followed by a choice of the spin connection and the Weitzenböck gauge is also employed for convenience.

The SVT sectors of the perturbations of the tetrad were calculated for the $f(T,B)$ theory. First, the tensor perturbations Eq. (3.47) were analysed for which the GWPE was found in Eq. (5.12). From this equation, the propagation speed of the tensor perturbations was obtained as the speed of light along with a zero effective mass both of which are in good agreement with recent multimessenger measurements. On top of that, the constraint $f_T < 0$ (5.14) must be imposed in order for the tensor perturbations to be stable. Comparing the GWPE Eq. (5.13) with its $f(T)$ and $f(\mathring{R})$ counterparts it was realized that they have exactly the same functional form which was expected since $f(T,B)$ is a superclass of both $f(T)$ and $f(\mathring{R})$.

Regarding the vector perturbations, they are following the behaviour of $f(T)$ and $f(\mathring{R})$, being non-dynamical. This is illustrated in Eq. (5.18) which is a constraint equation since no second order (or higher) time derivatives exist. The functional form of Eq. (5.18) is shared between $f(T)$ and $f(\mathring{R})$.

As for the scalar sector, since it is highly involved, it was probed in the sub-horizon limit, deep inside the Hubble radius where $k >> aH$ in order to derive the matter density Eq. (5.37). This equation encodes the growth of matter perturbations and also give us the the effective gravitational constant G_{eff} of $f(T,B)$ which is very important in studying the growth of structures in the Universe.

For the $f(T,B)$ theory it turns out that G_{eff} is expressed by three branches, that depend on the fundamental variables (5.48)–(5.50). These branches were defined in an exhaustive manner that includes every possible sub-class within the theory. Among them, $f(T)$ and $f(\mathring{R})$ are also included. Specifically, $f(T)$ falls in a much more general category compared than $f(\mathring{R})$ and this could also signal to difference in the pdof. In addition, models of the form $f(T,B) \to f(TB)$, have proven to be quite interesting for background cosmology [178, 171, 172, 162].

Finally, the values of G_{eff} found, include more than leading order terms, even for $f(T)$ as novel result in the literature, thus higher precision in numerics and data fittings will be possible.

Chapter 6

Generalised Proca theories in teleparallel gravity

Many of the proposals that go beyond the ΛCDM model have been focused on the framework of the Horndeski gravity which falls in the standard curvature based theories [13]. This is due to the fact that most of the sensible and natural theories of interest are already contained in some branch of the most general scalar tensor theory (Horndeski gravity Sec. 3.2). The term scalar tensor means, as it was explained in depth in Sec. 3.1, that on top of the metric tensor an extra scalar field is also considered along with its kinetic term and some generalised potential. Any field equations arising from the metric and the scalar field are second order equations [59, 60, 61] in order to avoid Ostrogradsky ghosts, as explained in Sec. 2.8. Surprisingly enough, the constraint coming from the multimessenger observation [179, 38] drastically limited the compliant models within the landscape of Horndeski theory.

Recently, a TG analog of Horndeski gravity was proposed in Ref. [64] and discussed in depth in Sec. 3.3. BDLS theory, in Sec. 3.5, was shown to produce a systematic way to revive previously ruled out models by producing a much more general gravitational wave propagation equation in Ref. [1]. The framework was also shown to be largely compatible with solar system tests through its parameterized post-Newtonian formalism in Ref. [180].

In addition, it is well known that the standard model of particle physics is described through several abelian and non-abelian vector fields as fundamental constituents that represent the gauge interactions [181]. Moreover, adding external fields was also motivated from Lovelock's theorem (Sec. 3.1), in order to generalize the GR paradigm. On top of that, one can always restrict any vector field into a scalar field by using the Stueckelberg method

$$A_\mu \to A_\mu + \partial_\mu \pi \,. \tag{6.1}$$

Hence, in this sense, there is strong motivation to generalize the scalar frameworks into vectorial ones, and probe the cosmic evolution through bosonic vector fields. These fields could also be candidates of explaining the nature or source of dark energy which drives the late time accelerated expansion [71, 72]. It may also be the case, that this use of vectors, impacts the relation of beyond ΛCDM theories to particle physics in a more straightforward manner. Another, potential outcome would be to shed further light into phenomena such as cosmic birefringence or cosmological principle tests [182]. The Proca theories, as direct generalizations of the Maxwell's theory of electromagnetism, where generalized [73, 183, 117, 184] into GP theory by including proper self-derivative interactions while preserving the propagation of only three vectorial dof from A_μ. These vector-tensor type of frameworks have been shown to support isotropic solutions along with screening mechanisms [73, 74].

TG, among other types of modified/alternative gravity theories, offers a unique perspective in formulating gravity theories beyond the ΛCDM by interchanging curvature with torsion in a geometrical manner (Sec. 2). Due to the nature of the coupling prescription [80] one can consider the standard Proca theory, which is just massive Maxwell's theory described by the action

$$\mathcal{S}_P = \int d^4x \sqrt{-g}\Big[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \Big] , \tag{6.2}$$

where $F^{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. This action describes the dynamics of a massive vector field assuming only three propagating dof, as part of the TG setting and also include more scalar invariants built from the torsion tensor and the vector field A_μ. Similarly built as the BDLS theory, this teleparallel Proca analogue was applied to a flat FLRW background in Sec. 6.3 in order to probe the evolution of the cosmological background. Finally, the key points of our construction along with our main results are summarized in Sec. 6.4.

6.1 Generalised Proca fields

6.1.1 Generalised Proca in flat space

The generalised Proca theories were first introduced in [71, 73, 74] as a mean of generalizing Eq. (6.2). In the procedure of generalizing, the propagating degrees of freedom of the vector field A_μ were demanded to be fixed to 3. The Lagrangian density of the GP theory

$$\mathcal{L}_{GP} = -\frac{1}{4}F^2 + \sum_{n=2}^{6} \alpha_n \mathcal{L}_n \,, \tag{6.3}$$

where $F^2 := F_{\mu\nu}F^{\mu\nu}$, α_n are arbitrary constants and $\mathcal{L}_n$ are the self-interacting contributions in our Lagrangian. The self-interaction terms are generated by the vector field α_n and its derivatives but their form and order is restricted by consistency arguments such as having fixed propagating dof of the vector field A_μ. This condition is enforced by trivializing the dynamics of the A_0 component which is realized by imposing that the Hessian matrix components $\mathcal{H}^{\mu\nu}_{\mathcal{L}_n}$ satisfy

$$\mathcal{H}^{\mu\nu}_{\mathcal{L}_n} = \frac{\partial^2 \mathcal{L}_n}{\partial \dot{A}_\mu \partial \dot{A}_\nu} \equiv 0, \quad \text{with} \quad \dot{A}_\mu = \partial_0 A_\mu \,, \tag{6.4}$$

for each individual component $\mathcal{L}_n$ vanishes. In terms of linear algebra, this is translated as nullifying the eigenvalue that corresponds to the A_0 component from the kinetic matrix $\mathcal{H}^{\mu\nu}_{\mathcal{L}_n}$. Technically, in order for this to be achieved, it is required that $\mathcal{H}^{00} = \mathcal{H}^{0i} = 0$ [74].

This guarantees that the determinant of the Hessian is degenerate. In this manner, the final form of the Lagrangian of GP in flat space is found to be

$$\mathcal{L}_2 = f_2(X, F, Y)\,, \tag{6.5}$$

$$\mathcal{L}_3 = f_3(X)\partial\cdot A\,, \tag{6.6}$$

$$\mathcal{L}_4 = f_4(X)[(\partial\cdot A)^2 - \partial_\mu A_\nu \partial^\nu A^\mu] + c_2\tilde{f}_4(X)F^2\,, \tag{6.7}$$

$$\mathcal{L}_5 = f_5(X)[(\partial\cdot A)^3 - 3(\partial\cdot A)\partial_\mu A_\nu\partial^\nu A^\mu + 2\partial_\rho A_\sigma \partial^\alpha A^\rho \partial^\sigma A_\alpha] + d_2\tilde{f}_5(X)\tilde{F}^{\mu\rho}\tilde{F}^\nu{}_\rho \partial_\mu A_\nu\,, \tag{6.8}$$

$$\mathcal{L}_6 = e_2 f_6(X)\tilde{F}^{\mu\nu}\tilde{F}^{\gamma\rho}\partial_\mu A_\gamma \partial_\nu A_\rho\,, \tag{6.9}$$

where $\tilde{F}_{\mu\nu} := \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $\partial\cdot A := \partial_\mu A^\mu$, $X := -A_\mu A^\mu/2$, $Y := A^\mu A^\nu F_\mu{}^\alpha F_{\nu\alpha}$, $f_{2,3,4,5,6}$ are arbitrary functions and $c_2,\ d_2,\ e_2$ are arbitrary constants. The arguments of the functions $f_{3,4,5,6}$ are fixed by demanding that they cannot contain derivatives of the vector field A_μ and that they should not interfere with integration by parts. The only exception is the f_2 function that contains the only possible combination of first order derivatives in the vector field. This function is also the simplest and most intuitive generalization that also includes first order self-derivatives along with a potential term $V(A^2)$.

For the other terms the non-constrained form of $\mathcal{L}_4$ will be used, in order to illustrate their origin, which is

$$\mathcal{L}_4 = f_4(X)[c_1(\partial\cdot A)^2 + c_2\partial_\mu A_\nu \partial^\mu A^\nu + c_3\partial_\mu A_\nu\partial^\nu A^\mu]\,, \tag{6.10}$$

where, $c_1,\ c_2,$ and c_3 are arbitrary constants. This expression describes the most general self-interactions of second order. The Hessian matrix for this term is

$$H^{\mu\nu}_{\mathcal{L}_4} = f_4\begin{pmatrix} 2(c_1+c_2+c_3) & 0 \\ 0 & -2c_2\delta_{ij}\end{pmatrix}. \tag{6.11}$$

It is obvious that one has seemingly two choices in order to nullify the dynamics of the A_0 term for this general $\mathcal{L}_4$ term. Since, $\mathcal{H}^{0i} = 0$ already, only $\mathcal{H}^{00} = 0 \Rightarrow 2(c_1 + c_2 + c_3) = 0$ needs to be imposed, which means that the c_i are linearly dependent. Normalizing $c_1 \equiv 1$, a solution of the system $(c_1 + c_2 + c_3) = 0$ is $c_3 = -(1 + c_2)$ which we use to re-write the $\mathcal{L}_4$ as

$$\begin{aligned}\mathcal{L}_4 &= f_4[(\partial \cdot A)^2 + c_2 \partial_\mu A_\nu \partial^\mu A^\nu - (1 + c_2)\partial_\mu A_\nu \partial^\nu A^\mu] \\ &= f_4[(\partial \cdot A)^2 - \partial_\mu A_\nu \partial^\nu A^\mu + c_2 F^2]\,, \end{aligned} \tag{6.12}$$

where, in the second line the derivatives have been simplified. Notice that the term proportional to c_2 could be absorbed in the function f_2 but the convention of Eq. (6.7) will be followed and will be left as it is. In general, this is the process followed in order to find the final form of the $\mathcal{L}_4$ terms that describe non-dynamical A_0 components. This process, in its totality, is iterative throughout all the pieces of the total Lagrangian and it ensures that A_0 does not propagate.

6.1.2 Generalised Proca in curved backgrounds

The Lagrangian described by the Eqs. (6.9) holds only for a flat background and a generalization into a non-trivial background is not a straightforward procedure. Non-trivial backgrounds in general assume dynamical metrics $g_{\mu\nu}$ and hence they introduce gravity. For proper Cauchy data problems, where the solutions exist and are uniquely determined, second order field equations need to be imposed for any field. This constraint affects also the vector field A_μ and extra caution is needed since self-interaction have been included, which could in principle produce higher order terms when coupled to $g_{\mu\nu}$ in non-trivial geometries [185]. One extra condition to preserve, is that A_0 should remain non-dynamical in any background. The above constraints make the convariantization, or coupling with a dynamical metric $g_{\mu\nu}$, of the Lagrangian described by the Eq. (6.9) a highly non-trivial task.

The way to proceed in this case is to use divergence-less tensors which are purely constructed by $g_{\mu\nu}$ and hence the proper covariantized Lagrangian density is given by [185]

$$\mathcal{L}_{\rm GP}^{\rm curved} = -\frac{1}{4}F^2 + \sum_{n=2}^{6}\beta_n\mathcal{L}_n\,, \tag{6.13}$$

with [185, 73]

$$\mathcal{L}_2 = G_2(X,F,Y)\,, \tag{6.14a}$$

$$\mathcal{L}_3 = G_3(X)\mathring{\nabla}\cdot A\,, \tag{6.14b}$$

$$\mathcal{L}_4 = G_4(X)\mathring{R} + G_{4,X}(X)[(\mathring{\nabla}\cdot A)^2 - \mathring{\nabla}_\mu A_\nu\mathring{\nabla}^\nu A^\mu]\,, \tag{6.14c}$$

$$\begin{aligned}\mathcal{L}_5 = {}& G_5(X)\mathring{G}_{\mu\nu}\mathring{\nabla}^\mu A^\nu - \frac{1}{6}G_{5,X}(X)[(\mathring{\nabla}\cdot A)^3 - 3(\mathring{\nabla}\cdot A)\mathring{\nabla}_\mu A_\nu\mathring{\nabla}^\nu A^\mu \\ &+2\mathring{\nabla}_\mu A_\nu\mathring{\nabla}^\alpha A^\mu\mathring{\nabla}^\nu A_\alpha] - \tilde{G}_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^\beta{}_\mu\mathring{\nabla}_\alpha A_\beta\,,\end{aligned} \tag{6.14d}$$

$$\mathcal{L}_6 = G_6(X)\mathring{L}^{\mu\nu\alpha\beta}\mathring{\nabla}_\mu A_\nu\mathring{\nabla}_\alpha A_\beta + \frac{1}{2}G_{6,X}(X)\tilde{F}^{\alpha\mu}\tilde{F}^{\beta\gamma}\mathring{\nabla}_\alpha A_\beta\mathring{\nabla}_\mu A_\gamma\,, \tag{6.14e}$$

where $\mathring{L}^{\mu\nu\alpha\beta}$ is the double dual Riemann tensor

$$\mathring{L}^{\mu\nu\alpha\beta} := \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\mathring{R}_{\rho\sigma\gamma\delta}\,, \tag{6.15}$$

which assumes the same symmetries as the Riemann tensor, i.e. $\mathring{L}^{\mu\nu\alpha\beta} = \mathring{L}^{\alpha\beta\mu\nu}$, $\mathring{L}^{\mu\nu\alpha\beta} = -\mathring{L}^{\nu\mu\alpha\beta}$ and $\mathring{L}^{\mu\nu\alpha\beta} = -\mathring{L}^{\mu\nu\beta\alpha}$ [74]. Note that for the last term in $\mathcal{L}_5$, $\tilde{G}_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^\beta{}_\mu\mathring{\nabla}_\alpha A_\beta$, and $\mathcal{L}_3$ no extra non-minimal coupling counter terms are needed since they are coupled linearly to the LC connection [74, 185].

On the whole, these are the steps towards setting up a proper GP gravity theory in arbitrary spacetimes. This exact procedure will be used in the next section where all these steps will be translated into the teleparallel analogue of GP theory.

6.2 Proca theories in teleparallel gravity

Applying the coupling prescription introduced in Sec. 2.5, which was also used in the BDLS theory in Sec. 3, the core GP theory as presented in Eqs. (6.14a)–(6.14e) is also preserved. Slightly re-writing them, by expanding the Ricci scalar in $\mathcal{L}_4$ as in Eq. (2.57), they read as

$$\mathcal{L}_2 = G_2(X, F, Y)\,, \tag{6.16a}$$

$$\mathcal{L}_3 = G_3(X)\mathring{\nabla} \cdot A\,, \tag{6.16b}$$

$$\mathcal{L}_4 = G_4(X)(-T + B) + G_{4,X}(X)[(\mathring{\nabla} \cdot A)^2 - \mathring{\nabla}_\mu A_\nu \mathring{\nabla}^\nu A^\mu]\,, \tag{6.16c}$$

$$\begin{aligned}\mathcal{L}_5 = {} & G_5(X)\mathring{G}_{\mu\nu}\mathring{\nabla}^\mu A^\nu - \frac{1}{6}G_{5,X}(X)[(\mathring{\nabla} \cdot A)^3 - 3(\mathring{\nabla} \cdot A)\mathring{\nabla}_\mu A_\nu \mathring{\nabla}^\nu A^\mu \\ & +2\mathring{\nabla}_\mu A_\nu \mathring{\nabla}^\alpha A^\mu \mathring{\nabla}^\nu A_\alpha] - \tilde{G}_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^\beta{}_\mu \mathring{\nabla}_\alpha A_\beta\,,\end{aligned} \tag{6.16d}$$

$$\mathcal{L}_6 = G_6(X)\mathring{L}^{\mu\nu\alpha\beta}\mathring{\nabla}_\mu A_\nu \mathring{\nabla}_\alpha A_\beta + \frac{1}{2}G_{6,X}(X)\tilde{F}^{\alpha\mu}\tilde{F}^{\beta\gamma}\mathring{\nabla}_\alpha A_\beta \mathring{\nabla}_\mu A_\gamma\,. \tag{6.16e}$$

In general, the Riemann tensor can be expressed in terms of the tetrad and the TG connection by using Eq. (2.3) as

$$\mathring{R}^\rho{}_{\sigma\mu\nu} = -\overset{\circ}{\nabla}_\mu K^\rho{}_{\sigma\nu} + \overset{\circ}{\nabla}_\nu K^\rho{}_{\sigma\mu} - K^\beta{}_{\sigma\nu}K^\rho{}_{\beta\mu} + K^\beta{}_{\sigma\mu}K^\rho{}_{\beta\nu}\,. \tag{6.17}$$

Although this description for the covariantized form is complete for the core GP theory, which is expressed through the metric, by including TG contributions the situation is not that straightforward. It is known that for the TG connection the Lovelock's theorem, discussed in Sec. 3.1 is weakened as reported in [186, 187] and this dramatically increases the available pool of scalar invariants that can be constructed via the tetrad or the torsion tensor. This pool of scalars is what will shape the TG contribution of the GP action, with appropriate filtering conditions. The Lagrangian density that corresponds to the TG contribution will be denoted as $\mathcal{L}_{TP}$, thus the total Lagrangian density of the theory is $\mathcal{L}_{GP}^{Tele}$ will be given by

$$\mathcal{L}_{\mathrm{GP}}^{\mathrm{Tele}} = -\frac{1}{4}F^2 + \sum_{n=2}^{6} \beta_n \mathcal{L}_n + \mathcal{L}_{TP} \,, \tag{6.18}$$

where $\mathcal{L}_{TP}$ extends the GP theory into the TG framework. The exact form of $\mathcal{L}_{TP}$ will be determined from the scalar invariants that will be generated by the torsion tensor and more specifically from the ones constructed by the irreducible components of the torsion tensor Eq. (2.35) along with vector field A_μ.

Constructing these scalars, a few limiting conditions are needed, just like the case of the BDLS theory in Sec. 3.3. For this case the following rules are imposed

1. The resulting field equations must, at most, be of second order in both $e^A{}_\mu$ and A_μ;
2. A_μ must have maximum 3 degrees of freedom, A_0 being not dynamical;
3. Cannot be parity violating;
4. Must be linear in the torsion tensor;
5. Up to fourth order derivatives on A_μ, i.e, $\partial A \partial A \partial A \partial A \sim (\partial A)^4$.

These rules serve as physical and mathematical consistency keepers mostly. Starting with the first condition, the avoidance of the Ostrogradsky ghosts [188, 189] is guaranteed by restricting the derivative order to two for field equations of both the tetrad and the vector field A_μ. This is a standard condition in order to be consistent with the Cauchy data as it was also imposed for the BDLS case. Surprisingly this is not enough to ensure that there are no ghosts since, in general, the A_0 can still be propagating but as a ghost. In conjunction with the massive spin-1 representation of the Lorentz group which only carries three dynamical dof it is rather necessary to impose explicitly that the A_0 will be non-dynamical. Hence, all of the above criteria are combined into condition no 2. Thus far, conditions 1 and 2 are purely of physical and mathematical nature, the 3rd and 4th conditions are just to limit the available scalar invariant candidates. The only way to construct a sensible number of scalar invariants is to impose that all of these scalars are constructed linearly wrt to the torsion tensor. If one demanded quadratic dependence in the torsion tensor then the number of available scalars would drastically increase to hundreds.

n	Vectorial (v)	Axial (a)	Purely tensorial (t)
0	vA	-	$tAAA$
1	$vAFA$	$\epsilon aAFA$	$tAFA$
2	$vAFF$, $vA\widetilde{F}\widetilde{F}$	$\epsilon aAFF$, $\epsilon aA\tilde{F}\tilde{F}$	$tAFF$, $tA\widetilde{F}\widetilde{F}$
3	$vAFFF$, $vA\widetilde{F}\widetilde{F}F$	$\epsilon aAFFF$, $\epsilon aA\tilde{F}\tilde{F}F$	$tAFFF$, $tA\widetilde{F}\widetilde{F}F$
4	$vAFFFF$, $vA\widetilde{F}\widetilde{F}\widetilde{F}\widetilde{F}$, $vA\widetilde{F}\widetilde{F}FF$	$\epsilon aAFFFF$, $\epsilon aA\tilde{F}\tilde{F}\tilde{F}\tilde{F}$, $\epsilon aA\tilde{F}\tilde{F}FF$	$tAFFFF$, $tA\widetilde{F}\widetilde{F}\widetilde{F}\widetilde{F}$, $tA\widetilde{F}\widetilde{F}FF$

Table 6.1: Generators of scalars – These are the independent components from which all the other terms can be obtained by permuting the indices.

Having as guide the afformentioned conditions, all possible scalars are illustrated schematically in Table[1] 6.1, where n denotes the index of the expansion of the product

$$\prod_n \mathring{\nabla}_{\mu_n} A_{\nu_n} = \mathring{\nabla}_{\mu_1} A_{\nu_1} \mathring{\nabla}_{\mu_2} A_{\nu_2}, .., \mathring{\nabla}_{\mu_n} A_{\nu_n}. \tag{6.19}$$

In this expansion, since there are no indices explicitly written, it is assumed that it holds for all possible index configurations, i.e, the ”generator” $\{vAF\}$ can be expanded along all possible index configurations as

$$I_2 \;=\; v_\beta A_\alpha F^{\alpha\beta}, \tag{6.20}$$

[1]Table 6.1 was partially generated with the help of the xAct packages [190, 191, 192, 193, 194, 195, 196].

$$I_3 = v_\beta A_\alpha F^{\beta\alpha}, \tag{6.21}$$

Along these lines the full set of scalars is calculated by expanding all the generators from Table 6.1 in all possible index configurations. The generator groups will be denoted with brackets like in the example after the Table 6.1, $\left\{vA\mathring{\nabla}A\right\}$. Note that the following sets of scalars are the full list of possible independent scalars.

Torsion vector component v_μ

$\{vA\}$

$$I_1 \coloneqq v_\mu A^\mu, \tag{6.22}$$

$\{vAF\}$

$$I_2 \coloneqq A_\alpha v_\beta F^{\alpha\beta}, \tag{6.23}$$

$$I_3 \coloneqq A_\alpha v_\beta F^{\beta\alpha}, \tag{6.24}$$

$\left\{vAFF, vA\widetilde{F}\widetilde{F}\right\}$

$$I_4 \coloneqq A_\alpha F^2 v^\alpha, \tag{6.25}$$

$$I_5 \coloneqq A_\alpha F^\alpha{}_\gamma F^{\beta\gamma} v_\beta, \tag{6.26}$$

$\left\{vAFFF, vA\widetilde{F}\widetilde{F}F\right\}$

$$I_6 \coloneqq A_\alpha F^\alpha{}_\mu F^\beta{}_\gamma F^{\mu\gamma} v_\beta, \tag{6.27}$$

$$I_7 \coloneqq A_\alpha F^{\alpha\beta} F^2 v_\beta, \tag{6.28}$$

$$\left\{vAFFFF, vA\widetilde{F}\widetilde{F}\widetilde{F}\widetilde{F}, vA\widetilde{F}\widetilde{F}FF\right\}$$

$$I_8 \coloneqq A_\alpha F_{\beta\nu} F_\gamma{}^\beta F^{\gamma\mu} F_\mu{}^\nu v^\alpha, \tag{6.29}$$

$$I_9 \coloneqq A_\alpha F^4 v^\alpha, \tag{6.30}$$

$$I_{10} \coloneqq A_\alpha F^\alpha{}_\mu F^\beta{}_\gamma F^{\gamma\nu} F^\mu{}_\nu v_\beta, \tag{6.31}$$

$$I_{11} \coloneqq A_\alpha F^\alpha{}_\gamma F^{\beta\gamma} F^2 v_\beta, \tag{6.32}$$

Torsion axial component a_μ

$$\{\epsilon a A F\}$$

$$I_{12} \coloneqq A^\alpha a^\beta \epsilon_{\alpha\beta\gamma\mu} F^{\mu\gamma}, \tag{6.33}$$

$$\left\{\epsilon aAFF, \epsilon aA\tilde{F}\tilde{F}\right\}$$

$$I_{13} \coloneqq A_\alpha a_\beta \epsilon^{\beta\mu\nu\gamma} F^\alpha{}_\mu F_{\nu\gamma}, \tag{6.34}$$

$$I_{14} \coloneqq A_\alpha a_\beta \epsilon^{\alpha\mu\nu\gamma} F^\beta{}_\mu F_{\nu\gamma}, \tag{6.35}$$

$$I_{15} \coloneqq A_\alpha a^\alpha \epsilon^{\gamma\mu\nu\beta} F_{\gamma\mu} F_{\nu\beta}, \tag{6.36}$$

$$\left\{\epsilon aAFFF, \epsilon aA\tilde{F}\tilde{F}F\right\}$$

$$I_{16} \coloneqq A_\alpha a_\beta \epsilon^{\alpha\beta\mu\nu} F_{\gamma\rho} F^{\gamma}{}_{\mu} F^{\rho}{}_{\nu}, \tag{6.37}$$

$$I_{17} \coloneqq A_\alpha a_\beta \epsilon^{\mu\rho\gamma\nu} F^{\alpha}{}_{\mu} F^{\beta}{}_{\rho} F_{\gamma\nu}, \tag{6.38}$$

$$I_{18} \coloneqq A_\alpha a_\beta \epsilon^{\beta\rho\gamma\nu} F^{\alpha}{}_{\mu} F_{\gamma\nu} F^{\mu}{}_{\rho}, \tag{6.39}$$

$$I_{19} \coloneqq A_\alpha a_\beta \epsilon^{\alpha\rho\gamma\nu} F^{\beta}{}_{\mu} F_{\gamma\nu} F^{\mu}{}_{\rho}, \tag{6.40}$$

$$I_{20} \coloneqq A_\alpha a_\beta \epsilon^{\nu\rho\gamma\mu} F^{\alpha\beta} F_{\gamma\mu} F_{\nu\rho}, \tag{6.41}$$

$$I_{21} \coloneqq A_\alpha a_\beta \epsilon^{\alpha\beta\nu\rho} F_{\nu\rho} F^2, \tag{6.42}$$

$$\left\{\epsilon aAFFFF, \epsilon aA\tilde{F}\tilde{F}\tilde{F}\tilde{F}, \epsilon aA\tilde{F}\tilde{F}FF\right\}$$

$$I_{22} \coloneqq A_\alpha a_\beta \epsilon^{\beta\mu\rho\sigma} F^{\alpha}{}_{\mu} F^{\gamma}{}_{\sigma} F_{\nu\gamma} F^{\nu}{}_{\rho}, \tag{6.43}$$

$$I_{23} \coloneqq A_\alpha a_\beta \epsilon^{\alpha\mu\rho\sigma} F^{\beta}{}_{\mu} F^{\gamma}{}_{\sigma} F_{\nu\gamma} F^{\nu}{}_{\rho}, \tag{6.44}$$

$$I_{24} \coloneqq A_\alpha a_\beta \epsilon^{\rho\gamma\nu\sigma} F^{\alpha}{}_{\mu} F^{\beta}{}_{\rho} F^{\mu}{}_{\gamma} F_{\nu\sigma}, \tag{6.45}$$

$$I_{25} \coloneqq A_\alpha a^\alpha \epsilon^{\mu\beta\nu\sigma} F_{\gamma\rho} F^{\gamma}{}_{\mu} F_{\nu\sigma} F^{\rho}{}_{\beta}, \tag{6.46}$$

$$I_{26} \coloneqq A_\alpha a_\beta \epsilon^{\mu\gamma\nu\sigma} F^{\alpha}{}_{\mu} F^{\beta}{}_{\rho} F_{\nu\sigma} F^{\rho}{}_{\gamma}, \tag{6.47}$$

$$I_{27} \coloneqq A_\alpha a_\beta \epsilon^{\beta\gamma\nu\sigma} F^{\alpha}{}_{\mu} F^{\mu}{}_{\rho} F_{\nu\sigma} F^{\rho}{}_{\gamma}, \tag{6.48}$$

$$I_{28} \coloneqq A_\alpha a_\beta \epsilon^{\alpha\gamma\nu\sigma} F^{\beta}{}_{\mu} F^{\mu}{}_{\rho} F_{\nu\sigma} F^{\rho}{}_{\gamma}, \tag{6.49}$$

$$I_{29} \coloneqq A_\alpha a_\beta \epsilon^{\sigma\gamma\nu\rho} F^{\alpha}{}_{\mu} F^{\beta\mu} F_{\nu\rho} F_{\sigma\gamma}, \tag{6.50}$$

$$I_{30} \coloneqq A_\alpha a^\alpha \epsilon^{\nu\rho\sigma\beta} F_{\nu\rho} F_{\sigma\beta} F^2, \tag{6.51}$$

$$I_{31} \coloneqq A_\alpha a_\beta \epsilon^{\beta\rho\sigma\gamma} F^\alpha{}_\rho F_{\sigma\gamma} F^2, \tag{6.52}$$

$$I_{32} \coloneqq A_\alpha a_\beta \epsilon^{\alpha\rho\sigma\gamma} F^\beta{}_\rho F_{\sigma\gamma} F^2, \tag{6.53}$$

Purely tensorial component $t_{\alpha\beta\gamma}$

$$\{tAAA\}$$

$$I_{33} \coloneqq A^\alpha A^\beta A^\gamma t_{\alpha\beta\gamma}, \tag{6.54}$$

$$\{tAF\}$$

$$I_{34} \coloneqq A^\alpha t_{\alpha\beta\gamma} F^{\gamma\beta}, \tag{6.55}$$

$$I_{35} \coloneqq A^\alpha t_{\alpha\gamma\beta} F^{\gamma\beta}, \tag{6.56}$$

$$I_{36} \coloneqq A^\alpha t_{\beta\gamma\alpha} F^{\gamma\beta}, \tag{6.57}$$

$$\left\{tAFF, tA\widetilde{F}\widetilde{F}\right\}$$

$$I_{37} \coloneqq A_\alpha F_{\beta\mu} F^\beta{}_\gamma t^{\alpha\mu\gamma}, \tag{6.58}$$

$$I_{38} \coloneqq A_\alpha F^\alpha{}_\beta F_{\gamma\mu} t^{\beta\gamma\mu}, \tag{6.59}$$

$$I_{39} \coloneqq A_\alpha F_{\beta\mu} F^\beta{}_\gamma t^{\gamma\mu\alpha}, \tag{6.60}$$

$$\left\{tAFFF, tA\widetilde{F}\widetilde{F}F\right\}$$

$$I_{40} := A_\alpha F_{\beta\mu} F^\beta{}_\gamma F^\mu{}_\nu t^{\alpha\gamma\nu}, \tag{6.61}$$

$$I_{41} := A_\alpha F_{\beta\gamma} F^2 t^{\alpha\beta\gamma}, \tag{6.62}$$

$$I_{42} := A_\alpha F^\alpha{}_\beta F_{\gamma\nu} F^\gamma{}_\mu t^{\beta\nu\mu}, \tag{6.63}$$

$$I_{43} := A_\alpha F^\alpha{}_\beta F^\beta{}_\gamma F_{\mu\nu} t^{\gamma\mu\nu}, \tag{6.64}$$

$$I_{44} := A_\alpha F^\alpha{}_\beta F_{\gamma\nu} F^\gamma{}_\mu t^{\mu\nu\beta}, \tag{6.65}$$

$$\left\{tAFFFF, tA\widetilde{F}\widetilde{F}\widetilde{F}\widetilde{F}, tA\widetilde{F}\widetilde{F}FF, tAFFFF\right\}$$

$$I_{45} := A_\alpha F_\beta{}^\mu F^\beta{}_\gamma F_{\mu\nu} F^\nu{}_\rho t^{\alpha\rho\gamma}, \tag{6.66}$$

$$I_{46} := A_\alpha F_{\beta\mu} F^\beta{}_\gamma F^2 t^{\alpha\mu\gamma}, \tag{6.67}$$

$$I_{47} := A_\alpha F^\alpha{}_\beta F_{\gamma\nu} F^\gamma{}_\mu F^\nu{}_\rho t^{\beta\mu\rho}, \tag{6.68}$$

$$I_{48} := A_\alpha F^\alpha{}_\beta F_{\gamma\mu} F^2 t^{\beta\gamma\mu}, \tag{6.69}$$

$$I_{49} := A_\alpha F^\alpha{}_\beta F^\beta{}_\gamma F_{\mu\rho} F^\mu{}_\nu t^{\gamma\rho\nu}, \tag{6.70}$$

$$I_{50} := A_\alpha F^\alpha{}_\beta F^\beta{}_\gamma F^\gamma{}_\mu F_{\nu\rho} t^{\mu\nu\rho}, \tag{6.71}$$

$$I_{51} := A_\alpha F_\beta{}^\mu F^\beta{}_\gamma F_{\mu\nu} F^\nu{}_\rho t^{\gamma\rho\alpha}, \tag{6.72}$$

$$I_{52} := A_\alpha F_{\beta\mu} F^\beta{}_\gamma F^2 t^{\gamma\mu\alpha}, \tag{6.73}$$

$$I_{53} := A_\alpha F^\alpha{}_\beta F^\beta{}_\gamma F_{\mu\rho} F^\mu{}_\nu t^{\nu\rho\gamma}. \tag{6.74}$$

It should be stressed that for $n > 1$ and onwards the derivatives $\nabla_\mu A_\nu$ are directly substituted by $F_{\mu\nu}$ in order to keep A_0 as non-dynamical. Another important point is that these teleparallel scalars will only have an effect on non-trivial geometries, where the tetrad and the torsion tensor are not trivial. Going back to flat spacetime the GP [73, 71] theory is recovered as presented by Eqs.(6.9).

At this stage, all the available scalars can be packed in $\mathcal{L}_{TP}$ as

$$\mathcal{L}_{TP} \coloneqq G_{TP}\left(X, F, Y, I_1, I_2, .., I_{54}\right) \tag{6.75}$$

where the I's are defined in the Sec. 6.2. Let us repeat that $\mathcal{L}_{TP}$, in flat spacetime, will be directly absorbed into $\mathcal{L}_2$ from Eq. (6.16a) due to the trivialization of all the I's. Having filtered out quite a lot the available scalar invariants there are still 54 new scalars which could be argued that they are still a lot. This actually depends on the background in question, since in highly symmetric backgrounds only very few of them will survive. This is actually the case for a spatially flat FLRW background as it is illustrated in Sec. 6.3 where only four of these I's survive.

6.3 Cosmological background in teleparallel Proca Theories

In this section, the teleparallel contribution will be probed in a spatially flat FLRW background and its interplay with the vector field A_μ. The conventions of Sec. 2 for the metric and the tetrad are used. The vector field is assumed to be homogeneous and time dependent, in order to comply with the FLRW symmetries and it reads as

$$A_\mu = (A(t), 0, 0, 0)\ , \tag{6.76}$$

where $A(t)$ is a time dependent function that with argument the cosmic time. The only teleparallel scalar that survives from Sec. 6.2 is

$$I_1 = 3AH\,, \tag{6.77}$$

on the other hand the purely GP scalars are given by

$$X = \frac{1}{2}A^2\,, \tag{6.78}$$

$$Y = 0 = F\,. \tag{6.79}$$

In terms of the teleparallel contribution in the Lagrangian this can be seen as $G_{TP} = G_{TP}(X, I_1, I_2, I_3, I_4)$. A Universe filled by a perfect fluid with energy density ρ and pressure p, is also considered for this analysis. Hence, the Friedmann equations turn out to be

$$\mathcal{A}_{\rm TP} + \Sigma_{i=2}^{5}\mathcal{A}_i = \rho\,, \tag{6.80}$$

$$\mathcal{B}_{\rm TP} + \Sigma_{i=2}^{5}\mathcal{B}_i = p\,, \tag{6.81}$$

where the first Friedmann equation reads as

$$\mathcal{A}_2 = G_2 - A^2 G_{2,\rm X}\,, \tag{6.82}$$

$$\mathcal{A}_3 = -3HA^3 G_{3,X}\,, \tag{6.83}$$

$$\mathcal{A}_4 = 6H^2 G_4 - 6\left(2G_{4,X} + G_{4,XX}A^2\right)H^2 A^2\,, \tag{6.84}$$

$$\mathcal{A}_5 = G_{5,XX}H^3 A^5 + 5G_{5,X}H^3 A^3\,, \tag{6.85}$$

$$\mathcal{A}_{\rm TP} = G_{TP} - A\left(AG_{\rm TP,X} + 6HG_{\rm TP,I_1}\right)\,, \tag{6.86}$$

while the second the second Friedmann equation reads as

$$\mathcal{B}_2 = G_2\,, \tag{6.87}$$

$$\mathcal{B}_3 = -A\dot{A}G_{3,\mathrm{X}}\,, \tag{6.88}$$

$$\mathcal{B}_4 = 2G_4\left(3H^2 + 2\dot{H}\right) - 2G_{4,\mathrm{X}}\left(3H^2A + 2H\dot{A} + 2\dot{H}A\right) - 4G_{4,\mathrm{XX}}HA^3\dot{A}\,, \tag{6.89}$$

$$\mathcal{B}_5 = G_{5,\mathrm{XX}}H^2A^4\dot{A} + G_{5,\mathrm{X}}HA^2\left(2\dot{H}A + 2H^2A + 3H\dot{A}\right)\,, \tag{6.90}$$

$$\mathcal{B}_{\mathrm{TP}} = G_{TP} - 3AG_{\mathrm{TP,I_1I_1}}\left(H\dot{A} + A\dot{H}\right) - G_{\mathrm{TP,I_1}}\left(\dot{A} + 3AH\right) - A^2\dot{A}G_{\mathrm{TP,I_1X}}\,. \tag{6.91}$$

On top of these equations, the vector conservation equation can be obtained as

$$\mathcal{P}_{\mathrm{TP}} + \Sigma_{i=2}^{5}\mathcal{P}_i = 0\,, \tag{6.92}$$

where

$$\mathcal{P}_2 = AG_{2,\mathrm{X}}\,, \tag{6.93}$$

$$\mathcal{P}_3 = 3A^2HG_{3,\mathrm{X}}\,, \tag{6.94}$$

$$\mathcal{P}_4 = 6AH^2G_{4,\mathrm{X}} + 6H^2A^3G_{4,\mathrm{XX}}\,, \tag{6.95}$$

$$\mathcal{P}_5 = -3H^3A^2G_{5,\mathrm{X}} - H^3A^4G_{5,\mathrm{XX}}\,, \tag{6.96}$$

$$\mathcal{P}_{\mathrm{TP}} = AG_{\mathrm{TP,X}} + 3HG_{\mathrm{TP,I_1}}\,. \tag{6.97}$$

The system of Eqs. (6.80),(6.81),(6.92) describes the background dynamics of the Teleparallel Proca Theory proposed in this section. As an internal check, by trivializing the terms generated from G_{TP} one directly recovers the set of Friedmann equations for the standard GP theory as presented in [73]. Although, only the scalar I_1 survives in this highly symmetric background, once more complicated ansatzes of the vector field A_μ are chosen or the Weitzenböck gauge is dropped the equations will get quite complicated. Nonetheless, the evolution is drastically enriched compared to the standard case.

6.4 Conclusion

The generalization of Maxwell theory, which inevitably introduces a massive vector field, was a very interesting idea for a paradigm shift, where the community took a first step at generalizing the scalar field theories to vector ones. On top of it, considering derivative self-interactions gave birth to the GP gravity theory, which could be non-trivially covariantized, as in Eqs. (6.5)–(6.9), into arbitrary backgrounds retaining second order field equations as illustrated in detail in Sec. 6.1.2.

A TG analog of GP theory was introduced by using the standard GR coupling prescription in order to obtain a consistent covariantization scheme. In this context, five guiding principles were introduced in order to reliably and consistently shape our teleparallel contributing Lagrangian terms. These five principles are: (i) second order equations of motion are produced, in order to avoid Ostrogradsky instabilities and Cauchy data problems; (ii) the vector field must have three dof, which is both a physical and mathematical requirement; (iii) does not violate parity, since there is no strong evidence otherwise; (iv) Lagrangian contributions must be, at most, linear in the torsion tensor (v) vector field derivatives must appear less than fourth order; providing a reasonable cutoff. These rules efficiently provide a well defined framework to construct a TG Proca Lagrangian term Eq. (6.75) on top of the GP core theory. Note that the construction of Eq. (6.75) is quite sensitive to the underlying rules one chooses since just demanding scalar invariants quadratic in torsion, the number of potential available scalars jumps to a few hundreds.

There are obviously a few ways towards further generalizations of our TG contribution but they need more solid physical filtering rules. In general, one may have potentially hundreds of terms to work with by loosening the linearity in torsion but then new rules will be needed for handling the amount of potential scalar invariants. Another route, is to follow the beyond GP idea where there are higher than second order field equations and as it turns out the three propagating dof for the vector field can be still preserved by appropriate constraints but then one needs to be very careful about ghosts and unbound Hamiltonians [182]. The idea of Eq. (6.75) is that is just one of the simplest and intuitive Lagrangian terms one can consider at first, amongst the myriads of potential combinations.

Putting the TG Lagrangian contribution of Eq. (6.75) into perspective, the respective Friedmann Eqs. (6.80),(6.81),(6.92) have been calculated in Sec. (6.3). These equations describe the background dynamics of the system in a homogeneous and isotropic Universe filled with an ideal fluid. Judging from the final form of the surviving scalar I_1, it is evident that the choice of teleparallel scalars is quite sensitive to the choice of specific highly symmetric backgrounds. This could also serve as a potential filter for specific applications, i.e, scalars fined tuned specifically for cosmology. Although, only this one scalar was non-trivial, the complexity of the final solution depends quite a lot on the choice of the Weitzenböck gauge.

On the whole, it would be quite interesting to further probe these types of teleparallel contributions on top of the standard big classes of theories like GP and even the standard Horndeski theory as illustrated in Sec. 3. It was evident that the background cosmology dynamics were drastically modified compared to the standard GP theory. A similar situation is expected in the linearized regime too, judging partially from the background dynamics but also from the actual number of available scalars.

Chapter 7

Conclusion and Discussion

7.1 Summary of results and Conclusions

In this dissertation, the class of teleparallel theories of gravity was introduced along with the motivation of its conception by Einstein (see Sec. 1). In TG, geometry is solely expressed through the torsion tensor and the role of the fundamental variable is assumed by the tetrad field and spin connection pairs. This change of fundamental variable introduces its own intricacies, like the fact that the tetrad field – spin connection is unique up to LLT while the metric is invariant under them [76]. Moreover the spin connection, in TG does not depend on the tetrad field and has to be considered as an extra variable in its own right. As a consequence the teleparallel spin connection attains each own set of field equations. Due to curvature being zero, there is always a special class of frame fields for which the spin connection is trivial. This is also consistent with the fact that the spin connection is also non-dynamical [23] since it represents the dof of the Lorentz group. This is quite different compared to GR where the metric is the sole variable since the LC connection depends on the metric itself. In this context, using the tetrad as a fundamental variable renders the metric a derived quantity as shown in Eq. (2.8).

There is a specific model within the TG framework called the TEGR which is dynamically equivalent to GR, as was shown in Sec. 2. This means that TEGR and GR can not be discriminated by classical experiments but they could differ at the quantum level, where the gravitational action plays a much bigger role in the dynamics of the theory. Nevertheless, modified teleparallel theories and curvature based ones are very distinct from each other. The most standard example includes the class of $f(T)$ against $f(\mathring{R})$ gravity theories. These two classes are very different due to Eq. (2.57), which is the fundamental equation that relates the teleparallel connection with the LC connection. Nonetheless, both of these classes are included in the most general class of $f(T,B)$ gravity. Any other type of modification or extension like Horndeski theory and BDLS theory (see Sec. 3) and even GP and its teleparallel extension are different theories (see Sec. 6). Analogous to modified theories of gravity based on GR, a plethora of theories can be constructed in TG which exhibit a distinct phenomenology compared to their curvature analogs.

It was shown that there is dynamical equivalence between TEGR and GR, hence the same exact gravitational system can be described by using only torsion instead of curvature. This is not exactly the case for the modifications therein. As a matter of fact, $f(\mathring{R})$ [27] gravity is not dynamically equivalent to $f(T)$ gravity as one can directly see from the Eq. (2.57). Taking this as a starting point we introduced and focused on two major modifications of TEGR i.e, $f(T,B)$ and BDLS gravity theories.

In Sec. 3, the construction of the BDLS theory was presented along with the GWPE on a flat FLRW background. A concise review of Lovelock's theorem and Horndeski gravity was also included for completeness and smoother transition to the BDLS theory. Lovelock's theorem states that in four dimensions the most general second order system of field equations wrt the metric, dynamically, is just GR plus a constant term. This theorem serves as a guide on how to generalize curvature based theories in a way that are not just GR. Including also a scalar field, in four dimensions, while demanding the most general scalar tensor theory, one ends up with the Horndeski theory. Although quite famous and well studied, it turned out that it got heavily constrained after the GW 170817 and GRB 170817A events [123]. One of the ways to avoid this is by extending it. That is how the

BDLS theory was motivated on top of incorporating the TG framework.

The BDLS theory was studied perturbatively both in Minkowski (Sec. 4) and flat FLRW (Sec. 3) spacetimes. Specifically, in flat FLRW only the tensor perturbations were studied in order to derive its GWPE which resulted in an equation of the form (2.119) where the tensor excess $\alpha_T \neq 0$ is calculated in (3.48) and the friction term $\alpha_M \neq 0$ is given in (2.121) (Sec. 3.4). In order to comply with the speed of light propagation, $\alpha_T = 0$ is required. In contrast to Horndeski theory, the BDLS theory is not severely constrained by such demand. In fact, in a way this allows for a revival of the class of Horndeski models that were previously discarded since $\alpha_T = 0$ lead to $G_4 = G_5 = 0$ (see Sec. 3.5). This can also be understood as extending G_4 and G_5 by a term proportional to J_5 which is of pure teleparallel origin. Nonetheless, this is a very general result that holds for any model within the BDLS class.

In the Minkowski background, the BDLS theory was also studied perturbatively in its entirety, in order to determine the full dynamical content and polarizations of the theory. First, the linearized field equations were calculated (4.15)–(4.16) and then they were split in a SVT manner (4.34)–(4.36)–(4.38). Solving each sector separately and then combining them in order to determine the total system resulted in an exhaustive branching of the BDLS theory as illustrated in Sec. 4.4.

The most important cases from the analysis are Cases 0.II and Case 1. In Case 0.II a novel sub-branch of the Horndeski theory was found, that assumes two tensorial pdof but is not GR. This branch was not reported before in the literature. Regarding Case 1, which corresponds to the full BDLS theory, it was found that there are seven pdof that correspond to two scalars, one vectorial, one tensorial pdof in the massless sector and one scalar in the massive sector. The overall results of the analysis for every sub-branch are presented in Table 4.2 and comparison with known theories from the literature are presented in Table 4.1.

The dynamical analysis of the BDLS theory described in Table 4.2 was used also in order to probe the polarization content of the theory. The calculation of the polarizations is determined by the electric components of the Riemann tensor (4.169) that dictate the behavior of the geodesic deviation equation (4.166). Replacing the solutions that generate Table 4.2 into the electric components of the Riemann tensor (4.169) the polarization content is calculated and presented compactly in Table (4.3).

From these results, Case 0.I was also in agreement with Ref. [146] describing tensor polarizations for the massless sector and both scalar polarizations for the massive sector. On top of that, the newly found sub-branch of Case 0.II is consistently described by just tensor polarizations. With regards to the total BDLS theory, described in Case 1, the seven pdof are imprinted as breathing scalar and tensor polarizations in the massless sector while in the massive sector there are both scalar polarizations.

Although, for Case 1 and any other case where vectorial pdof are calculated (see Table 4.2), they are not manifested in the polarization Table (4.3). The underlying reason for these elusive pdof in the polarization signature, is attributed to the fact that they do not correspond to or couple strongly with metrical pdof. In the end, the polarizations depend on the electric components of the Riemann tensor which are directly calculated from the metric.

In a similar way, using perturbations in a SVT form, in flat FLRW the class of $f(T,B)$ theories [169] was probed in Chapter. 5. First, the tensor perturbations were studied since they constitute the GWPE which represents the physically observed GW. The resulting equation is (5.12) and is by default in agreement with the multimessenger constraint [36, 37, 38, 39, 40] since the speed of propagation is one. In addition, from the friction term α_M in Eq. (5.13) the stability condition $f_T < 0$ is derived. This condition assumes global validity within the whole $f(T,B)$ class and once imposed only physically relevant models are singled out in the context of a flat FLRW.

The vector perturbations were calculated next, where it was found that they are not dynamical. This was evident from Eq. (5.18) which describes a constraint. The last and more involved sector that was calculated was the scalar one that resulted in the field equations Eqs. (5.23)--(5.28). In general, these equations are very cumbersome to be solved analytically, thus they were further probed in the sub-horizon limit in order to derive the matter density equation (5.37). This equation is fully determined once the effective gravitational constant G_{eff} is calculated. Since this parameter fully determines the matter perturbation equations it is thus quite important regarding the growth of structures in the Universe. As a matter of fact, there is the possibility of new physics such as non-spatially flat cosmology at super-horizon limits. Thus, analyses of the matter density equation at the sub-horizon limits is a first step towards fully understanding the matter density equation and its implications.

For $f(T,B)$, G_{eff} assumes a branch like form as thoroughly explored in Sec. 5.3. The first two branches, being the major ones, are restricted within the teleparallel framework, whereas the third branch is the unique case of $f(\mathring{R})$ gravity. The first branch describes the most general, fully non-linear $f(T,B)$ models whereas the rest branches describe more and more linear ones. On top of calculating the most general G_{eff} within $f(T,B)$, a more general expression of G_{eff} was derived for $f(T)$ that includes higher order corrections.

In a similar way as constructing the BDLS theory, the teleparallel GP was constructed in Sec. 6. GP theories effectively generalize Maxwell theory by introducing mass terms (general potential) along with all possible self-interactions of the vector field, while maintaining only three propagating pdof. Note that, this construction is valid only in four dimensions at its current form.

The teleparallel GP version effectively generalizes the GP theories by introducing the most general teleparallel term build by linear in torsion scalars (6.75). This is one of the crucial differences wrt BDLS theory where the extra teleparallel term is build from quadratic terms build from torsion (3.28). Hence, the extra teleparallel contribution (6.75) effectively affects only the gravitational pdof. All the steps involved in the construction of the scalars is presented in an algorithmic and exhaustive manner in Sec. 6.2.

Moreover, as a first application of the teleparallel GP construction, the Friedmann equations were calculated in Sec. 6.3. Due to the symmetry of the flat FLRW background and the ansatz chosen for the vector field, only the scalars $I_{1,}I_2$ and X are non-trivial. Nevertheless, the resulting Friedmann equations (6.80), (6.81), (6.92) are quite involved, thus enriching the standard GP cosmological evolution.

7.2 Wider impact of this work

In $f(T, B)$ theory the next most natural course of action is to calculate explicitly, without any approximation, the dynamics of the scalar sector in the spatially flat FLRW spacetime. In this way, conclusions for the full cosmological stability of the theory will be available since the full dynamics and number of pdof will be known. This will potentially impact the form of cosmologically accepted $f(T, B)$ models. Knowledge of the full spectrum of perturbations further enables tests in late time cosmology using GW data, the Hubble parameter $H(z)$ versus redshift z data points. In addition, $S8$ tension can also be tested in the context of late time cosmology in very technical way as pure data fit.

On the other hand, simulations can be performed such as MCMC and CLASS and then be compared with data by taking into consideration early time boundary conditions that describe the early universe. In this context, the H_0 and S_8 tensions can be probed along the age of the universe. This could be understood as an evolutionary type of simulation that uses the power spectrum of the CMB in contrast to the redshift z. For these types of tests, knowledge of the perturbations is essential in contrast to the late time cosmological one.

Another very important type of application is testing if there is any variation of fundamental constants such as the fine structure constant, the speed of light, the gravitational constant, the proton-to-electron mass ratio and the Cosmological constant. As a first example, determining the gravitational constant demands complete knowledge of the scalar sector of the perturbations. In a similar way, the determination of the speed of light through a theory is calculated from its tensor perturbations and more specifically from the form of the GWPE it predicts. Last but not least, the fine structure constant in GR turns out to be constant since the EMT tensor is conformaly invariant. This need not be the case for a modified theory of gravity in general.

In addition, since for both $f(T,B)$ and the BDLS theory their GWPE are known, the luminosity distance can be calculated from GW data along with the electromagnetic counterpart in order to realize the constraint Eq. (2.123). It should be noted that, data from the standard sirens will be needed, in order for this constraint to be calculated, which is part of the next generation of GW detectors. Along the same direction, Einstein telescope and LISA which belong in the next generation detectors will also provide us with an extra set of data in order to further check the value of the Hubble parameter.

Just like the $f(T,B)$, BDLS and teleparallel GP theories can be very similarly further probed. Due to the status of their current progress a full analysis of their cosmological perturbations would be one of the next steps. Further more since for both theories the Friedmann equations are already calculated, they can be further probed by late time data.

Knowledge of the Friedmann equations leads also to their dynamical analysis in the phase space. This is yet another popular way of constraining theories from a background level perspective.

References

[1] Bahamonde S, Dialektopoulos KF, Gakis V, Levi Said J. Reviving Horndeski theory using teleparallel gravity after GW170817. Phys Rev D. 2020;101(8):084060.

[2] Bahamonde S, Caruana M, Dialektopoulos KF, Gakis V, Hohmann M, Levi Said J, et al. Gravitational Wave Propagation and Polarizations in the Teleparallel analog of Horndeski Gravity. 2021 5;.

[3] Bahamonde S, Gakis V, Kiorpelidi S, Koivisto T, Levi Said J, Saridakis EN. Cosmological perturbations in modified teleparallel gravity models: Boundary term extension. Eur Phys J C. 2021;81(1):53.

[4] Nicosia GP, Levi Said J, Gakis V. Generalised Proca theories in teleparallel gravity. Eur Phys J Plus. 2021;136(2):191.

[5] Abbott BP, et al. GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral. Phys Rev Lett. 2017;119(16):161101.

[6] Goldstein A, et al. An Ordinary Short Gamma-Ray Burst with Extraordinary Implications: Fermi-GBM Detection of GRB 170817A. Astrophys J. 2017;848(2):L14.

[7] Misner CW, Thorne KS, Wheeler JA. Gravitation. No. pt. 3 in Gravitation. W. H. Freeman; 1973.

[8] Weinberg S. The Cosmological Constant Problem. Rev Mod Phys. 1989;61:1–23.

[9] Einstein A. Zur allgemeinen Relativitätstheorie. Sitzungsber preuss Akad Wiss. 1915;1:778–786.

[10] Weyl H. Symmetry. Princeton University Press; 2015.

[11] Klein O. Quantum Theory and Five-Dimensional Theory of Relativity. (In German and English). Z Phys. 1926;37:895–906.

[12] Aghanim N, et al. Planck 2018 results. VI. Cosmological parameters. Astron Astrophys. 2020;641:A6. [Erratum: Astron.Astrophys. 652, C4 (2021)].

[13] Clifton T, Ferreira PG, Padilla A, Skordis C. Modified Gravity and Cosmology. Phys Rept. 2012;513:1–189.

[14] Capozziello S, De Laurentis M. Extended Theories of Gravity. Phys Rept. 2011;509:167–321.

[15] Nojiri S, Odintsov SD, Oikonomou VK. Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution. Phys Rept. 2017;692:1–104.

[16] Goenner HFM. On the History of Unified Field Theories. Part II. (ca. 1930 - ca. 1965). Living Rev Rel. 2014;17:5.

[17] Moller C. Further remarks on the localization of the energy in the general theory of relativity. Annals Phys. 1961;12:118–133.

[18] Pellegrini C, Plebanski J. Tetrad Fields and Gravitational Fields. Kgl Danske Videnskab Selskab, Mat Fys Skrifter. 1963;2(4).

[19] Møller C. Further remarks on the localization of the energy in the general theory of relativity. Annals of Physics. 1961;12(1):118–133.

[20] Brown JD, York JW Jr. Quasilocal energy and conserved charges derived from the gravitational action. Phys Rev. 1993;D47:1407–1419.

[21] Hehl FW, von der Heyde P, Kerlick GD, Nester JM. General relativity with spin and torsion: Foundations and prospects. Rev Mod Phys. 1976 Jul;48:393–416.

[22] Hayashi K, Nakano T. Extended translation invariance and associated gauge fields. Prog Theor Phys. 1967;38:491–507.

[23] Aldrovandi R, Pereira JG. An Introduction to Geometrical Physics. World Scientific; 1995.

[24] Rubin VC, Ford J W Kent. Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions. Astrophys J. 1970;159:379–403.

[25] Riess AG, et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. AstronJ. 1998;116:1009–1038.

[26] Riess AG, et al. Type Ia supernova discoveries at z ¿ 1 from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution. Astrophys J. 2004;607:665–687.

[27] De Felice A, Tsujikawa S. f(R) theories. Living Rev Rel. 2010;13:3.

[28] Nojiri S, Odintsov SD. Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models. Phys Rept. 2011;505:59–144.

[29] Ferraro R, Fiorini F. Modified teleparallel gravity: Inflation without inflaton. Phys Rev. 2007;D75:084031.

[30] Linder EV. Einstein's Other Gravity and the Acceleration of the Universe. Phys Rev. 2010;D81:127301. [Erratum: Phys. Rev.D82,109902(2010)].

[31] Sotiriou TP, Faraoni V. f(R) Theories Of Gravity. Rev Mod Phys. 2010;82:451–497.

[32] Capozziello S, De Laurentis M, Faraoni V. A Bird's eye view of f(R)-gravity. Open Astron J. 2010;3:49.

[33] Bahamonde S, Dialektopoulos KF, Escamilla-Rivera C, Farrugia G, Gakis V, Hendry M, et al. Teleparallel Gravity: From Theory to Cosmology. 2021;.

[34] Abbott BP, et al. Observation of Gravitational Waves from a Binary Black Hole Merger. Phys Rev Lett. 2016;116(6):061102.

[35] Abbott BP, et al. GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence. Phys Rev Lett. 2017;119(14):141101.

[36] Lombriser L, Taylor A. Breaking a Dark Degeneracy with Gravitational Waves. JCAP. 2016;03:031.

[37] Lombriser L, Lima NA. Challenges to Self-Acceleration in Modified Gravity from Gravitational Waves and Large-Scale Structure. Phys Lett B. 2017;765:382–385.

[38] Sakstein J, Jain B. Implications of the Neutron Star Merger GW170817 for Cosmological Scalar-Tensor Theories. Phys Rev Lett. 2017;119(25):251303.

[39] Ezquiaga JM, Zumalacárregui M. Dark Energy After GW170817: Dead Ends and the Road Ahead. Phys Rev Lett. 2017;119(25):251304.

[40] Baker T, Bellini E, Ferreira PG, Lagos M, Noller J, Sawicki I. Strong constraints on cosmological gravity from GW170817 and GRB 170817A. Phys Rev Lett. 2017;119(25):251301.

[41] Bamba K, Capozziello S, De Laurentis M, Nojiri S, Sáez-Gómez D. No further gravitational wave modes in $F(T)$ gravity. Phys Lett. 2013;B727:194–198.

[42] Abedi H, Capozziello S. Gravitational waves in modified teleparallel theories of gravity. Eur Phys J C. 2017;78(6):474.

[43] Cai YF, Li C, Saridakis EN, Xue L. $f(T)$ gravity after GW170817 and GRB170817A. Phys Rev D. 2018;97(10):103513.

[44] Zheng R, Huang QG. Growth factor in $f(T)$ gravity. JCAP. 2011;03:002.

[45] Anagnostopoulos FK, Basilakos S, Saridakis EN. Bayesian analysis of $f(T)$ gravity using $f\sigma_8$ data. Phys Rev D. 2019;100(8):083517.

[46] Nunes RC, Pan S, Saridakis EN. New observational constraints on $f(T)$ gravity through gravitational-wave astronomy. Phys Rev. 2018;D98(10):104055.

[47] Nesseris S, Basilakos S, Saridakis EN, Perivolaropoulos L. Viable $f(T)$ models are practically indistinguishable from ΛCDM. Phys Rev. 2013;D88:103010.

[48] Farrugia G, Said JL. Stability of the flat FLRW metric in $f(T)$ gravity. Phys Rev. 2016;D94(12):124054.

[49] Verde L, Treu T, Riess AG. Tensions between the Early and the Late Universe. Nature Astron. 2019;3:891.

[50] Di Valentino E, et al. Snowmass2021 - Letter of interest cosmology intertwined II: The hubble constant tension. Astropart Phys. 2021;131:102605.

[51] Di Valentino E, et al. Cosmology intertwined III: f8 and S8. Astropart Phys. 2021;131:102604.

[52] Di Valentino E, Mena O, Pan S, Visinelli L, Yang W, Melchiorri A, et al. In the realm of the Hubble tension—a review of solutions. Class Quant Grav. 2021;38(15):153001.

[53] Perivolaropoulos L, Skara F. Challenges for ΛCDM: An update. 2021 5;.

[54] Escamilla-Rivera C. Precision cosmology in modified and extended theories of gravity: An insightful test. Astron Nachr. 2021;342(1-2):63–68.

[55] Briffa R, Capozziello S, Levi Said J, Mifsud J, Saridakis EN. Constraining teleparallel gravity through Gaussian processes. Class Quant Grav. 2020;38(5):055007.

[56] Bardeen JM. Gauge Invariant Cosmological Perturbations. Phys Rev D. 1980;22:1882–1905.

[57] Kodama H, Sasaki M. Cosmological Perturbation Theory. Prog Theor Phys Suppl. 1984;78:1–166.

[58] Mukhanov VF, Feldman HA, Brandenberger RH. Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions. Phys Rept. 1992;215:203–333.

[59] Kobayashi T, Yamaguchi M, Yokoyama J. Generalized G-inflation: Inflation with the most general second-order field equations. Prog Theor Phys. 2011;126:511–529.

[60] Gleyzes J, Langlois D, Piazza F, Vernizzi F. Essential Building Blocks of Dark Energy. JCAP. 2013;1308:025.

[61] Koyama K. Cosmological Tests of Modified Gravity. Rept Prog Phys. 2016;79(4):046902.

[62] Baumann D, McAllister L. Inflation and String Theory. Cambridge Monographs on Mathematical Physics. Cambridge University Press; 2015.

[63] Romania MG, Tsamis NC, Woodard RP. Quantum Gravity and Inflation. Lect Notes Phys. 2013;863:375–395.

[64] Bahamonde S, Dialektopoulos KF, Levi Said J. Can Horndeski Theory be recast using Teleparallel Gravity? Phys Rev D. 2019 Sep;100(6):064018.

[65] Izumi K, Ong YC. Cosmological Perturbation in f(T) Gravity Revisited. JCAP. 2013;1306:029.

[66] Wu YP, Geng CQ. Matter Density Perturbations in Modified Teleparallel Theories. JHEP. 2012;11:142.

[67] Golovnev A, Koivisto T. Cosmological perturbations in modified teleparallel gravity models. JCAP. 2018;11:012.

[68] Li M, Miao RX, Miao YG. Degrees of freedom of $f(T)$ gravity. JHEP. 2011;07:108.

[69] Ferraro R, Guzmán MJ. Hamiltonian formalism for f(T) gravity. Phys Rev D. 2018;97(10):104028.

[70] Blagojević M, Nester JM. Local symmetries and physical degrees of freedom in $f(T)$ gravity: a Dirac Hamiltonian constraint analysis. Phys Rev D. 2020;102(6):064025.

[71] Heisenberg L. Generalised Proca Theories. In: 52nd Rencontres de Moriond on Gravitation; 2017. p. 233–241.

[72] de Rham C, Pozsgay V. New class of Proca interactions. Phys Rev D. 2020;102(8):083508.

[73] Heisenberg L. Generalization of the Proca Action. JCAP. 2014;05:015.

[74] Allys E, Peter P, Rodriguez Y. Generalized Proca action for an Abelian vector field. JCAP. 2016;02:004.

[75] Hehl FW, McCrea JD, Mielke EW, Ne'eman Y. Metric affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance. Phys Rept. 1995;258:1–171.

[76] Ortín T. Gravity and Strings. Cambridge Monographs on Mathematical Physics. Cambridge University Press; 2004.

[77] Weitzenböock R. ‘Invariantentheorie’. Noordhoff, Gronningen; 1923.

[78] Beltrán Jiménez J, Heisenberg L, Koivisto TS. Teleparallel Palatini theories. JCAP. 2018;08:039.

[79] Cai YF, Capozziello S, De Laurentis M, Saridakis EN. f(T) teleparallel gravity and cosmology. Rept Prog Phys. 2016;79(10):106901.

[80] Aldrovandi R, Pereira JG. Teleparallel Gravity: An Introduction. vol. 173 of Fundamental Theories of Physics. Dordrecht: Springer; 2013.

[81] Li B, Sotiriou TP, Barrow JD. $f(T)$ gravity and local Lorentz invariance. Phys Rev. 2011;D83:064035.

[82] Ringström H. The Cauchy Problem in General Relativity. ESI lectures in mathematics and physics. European Mathematical Society; 2009.

[83] Motohashi H, Suyama T. Third order equations of motion and the Ostrogradsky instability. Phys Rev D. 2015;91(8):085009.

[84] Weinberg S. Cosmology. Cosmology. OUP Oxford; 2008.

[85] Wolf WJ, Lagos M. Cosmological Instabilities and the Role of Matter Interactions in Dynamical Dark Energy Models. Phys Rev D. 2019;100(8):084035.

[86] Baumann D, Senatore L, Zaldarriaga M. Scale-Invariance and the Strong Coupling Problem. JCAP. 2011;05:004.

[87] Peacock JA. Cosmological Physics. Cambridge Astrophysics. Cambridge University Press; 1999.

[88] Melchiorri A, et al. A measurement of omega from the North American test flight of BOOMERANG. Astrophys J Lett. 2000;536:L63–L66.

[89] Hinshaw G, et al. Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results. Astrophys J Suppl. 2013;208:19.

[90] Aiola S, et al. The Atacama Cosmology Telescope: DR4 Maps and Cosmological Parameters. JCAP. 2020;12:047.

[91] Balkenhol L, et al. Constraints on ΛCDM extensions from the SPT-3G 2018 EE and TE power spectra. Phys Rev D. 2021;104(8):083509.

[92] Dodelson S. Modern Cosmology. Academic Press; 2003.

[93] Aghanim N, et al. Planck 2018 results. VI. Cosmological parameters. Astron Astrophys. 2020;641:A6. [Erratum: Astron.Astrophys. 652, C4 (2021)].

[94] Di Valentino E, Melchiorri A, Silk J. Planck evidence for a closed Universe and a possible crisis for cosmology. Nature Astron. 2019;4(2):196–203.

[95] Handley W. Curvature tension: evidence for a closed universe. Phys Rev D. 2021;103(4):L041301.

[96] Di Valentino E, Melchiorri A, Silk J. Investigating Cosmic Discordance. Astrophys J Lett. 2021;908(1):L9.

[97] Clarkson C, Maartens R. Inhomogeneity and the foundations of concordance cosmology. Class Quant Grav. 2010;27:124008.

[98] Ellis GFR, van Elst H. Cosmological models: Cargese lectures 1998. NATO Sci Ser C. 1999;541:1–116.

[99] Ellis GFR. The Bianchi models: Then and now. Gen Rel Grav. 2006;38:1003–1015.

[100] Malik KA, Wands D. Cosmological perturbations. Phys Rept. 2009;475:1–51.

[101] Malik KA, Matravers DR. A Concise Introduction to Perturbation Theory in Cosmology. Class Quant Grav. 2008;25:193001.

[102] Saltas ID, Sawicki I, Amendola L, Kunz M. Anisotropic Stress as a Signature of Nonstandard Propagation of Gravitational Waves. Phys Rev Lett. 2014;113(19):191101.

[103] Riazuelo A, Uzan JP. Quintessence and gravitational waves. Phys Rev. 2000;D62:083506.

[104] Ezquiaga JM, Zumalacárregui M. Dark Energy in light of Multi-Messenger Gravitational-Wave astronomy. Front Astron Space Sci. 2018;5:44.

[105] Nishizawa A. Generalized framework for testing gravity with gravitational-wave propagation. I. Formulation. Phys Rev D. 2018 May;97:104037.

[106] Belgacem E, et al. Testing modified gravity at cosmological distances with LISA standard sirens. JCAP. 2019;1907(07):024.

[107] Abbott BP, et al. Gravitational Waves and Gamma-rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A. Astrophys J Lett. 2017;848(2):L13.

[108] de Rham C, Deskins JT, Tolley AJ, Zhou SY. Graviton Mass Bounds. Rev Mod Phys. 2017;89(2):025004.

[109] Abbott TMC, et al. First Cosmology Results using Type Ia Supernovae from the Dark Energy Survey: Constraints on Cosmological Parameters. Astrophys J. 2019;872(2):L30.

[110] Horndeski GW. Second-order scalar-tensor field equations in a four-dimensional space. Int J Theor Phys. 1974 Sep;10:363–384.

[111] Kobayashi T. Horndeski theory and beyond: a review. Rept Prog Phys. 2019;82(8):086901.

[112] Will CM. Theory and Experiment in Gravitational Physics. 2nd ed. Cambridge University Press; 2018.

[113] Lasenby AN, Jones AW, Dabrowski Y. Observations of the cosmic microwave background and implications for cosmology and large scale structure. Phil Trans Roy Soc Lond A. 1999;357:35.

[114] Corbelli E, Salucci P. The Extended Rotation Curve and the Dark Matter Halo of M33. Mon Not Roy Astron Soc. 2000;311:441–447.

[115] Lovelock D. The Einstein tensor and its generalizations. J Math Phys. 1971;12:498–501.

[116] Lovelock D. The four-dimensionality of space and the einstein tensor. J Math Phys. 1972;13:874–876.

[117] De Felice A, Heisenberg L, Kase R, Mukohyama S, Tsujikawa S, Zhang Yl. Effective gravitational couplings for cosmological perturbations in generalized Proca theories. Phys Rev D. 2016;94(4):044024.

[118] Armendariz-Picon C, Damour T, Mukhanov VF. k - inflation. Phys Lett B. 1999;458:209–218.

[119] Chiba T, Okabe T, Yamaguchi M. Kinetically driven quintessence. Phys Rev D. 2000;62:023511.

[120] Armendariz-Picon C, Mukhanov VF, Steinhardt PJ. A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration. Phys Rev Lett. 2000;85:4438–4441.

[121] Deffayet C, Pujolas O, Sawicki I, Vikman A. Imperfect Dark Energy from Kinetic Gravity Braiding. JCAP. 2010;1010:026.

[122] Germani C, Kehagias A. New Model of Inflation with Non-minimal Derivative Coupling of Standard Model Higgs Boson to Gravity. Phys Rev Lett. 2010;105:011302.

[123] Abbott BP, et al. Gravitational Waves and Gamma-rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A. Astrophys J Lett. 2017;848(2):L13.

[124] Deffayet C, Gao X, Steer DA, Zahariade G. From k-essence to generalised Galileons. Phys Rev. 2011;D84:064039.

[125] Capozziello S, Dialektopoulos KF, Sushkov SV. Classification of the Horndeski cosmologies via Noether Symmetries. Eur Phys J. 2018;C78(6):447.

[126] Nicolis A, Rattazzi R, Trincherini E. The Galileon as a local modification of gravity. Phys Rev D. 2009;79:064036.

[127] Deffayet C, Esposito-Farese G, Vikman A. Covariant Galileon. Phys Rev D. 2009;79:084003.

[128] Martin-Moruno P, Nunes NJ, Lobo FSN. Horndeski theories self-tuning to a de Sitter vacuum. Phys Rev. 2015;D91(8):084029.

[129] Charmousis C, Copeland EJ, Padilla A, Saffin PM. General second order scalar-tensor theory, self tuning, and the Fab Four. Phys Rev Lett. 2012;108:051101.

[130] Gubitosi G, Linder EV. Purely Kinetic Coupled Gravity. Phys Lett. 2011;B703:113–118.

[131] Gannouji R, Sami M. Galileon gravity and its relevance to late time cosmic acceleration. Phys Rev. 2010;D82:024011.

[132] Li B, Barreira A, Baugh CM, Hellwing WA, Koyama K, Pascoli S, et al. Simulating the quartic Galileon gravity model on adaptively refined meshes. JCAP. 2013;1311:012.

[133] Elvang H, Hadjiantonis M, Jones CRT, Paranjape S. On the Supersymmetrization of Galileon Theories in Four Dimensions. Phys Lett. 2018;B781:656–663.

[134] Sushkov SV. Exact cosmological solutions with nonminimal derivative coupling. Phys Rev. 2009;D80:103505.

[135] Starobinsky AA, Sushkov SV, Volkov MS. The screening Horndeski cosmologies. JCAP. 2016;1606(06):007.

[136] Matsumoto J, Sushkov SV. General dynamical properties of cosmological models with nonminimal kinetic coupling. JCAP. 2018;1801(01):040.

[137] Sushkov SV. Realistic cosmological scenario with non-minimal kinetic coupling. Phys Rev. 2012;D85:123520.

[138] Amendola L. Cosmology with nonminimal derivative couplings. Phys Lett. 1993;B301:175–182.

[139] Saridakis EN, Sushkov SV. Quintessence and phantom cosmology with non-minimal derivative coupling. Phys Rev. 2010;D81:083510.

[140] Ghosh A, Johnson-Mcdaniel NK, Ghosh A, Mishra CK, Ajith P, Del Pozzo W, et al. Testing general relativity using gravitational wave signals from the inspiral, merger and ringdown of binary black holes. Class Quant Grav. 2018;35(1):014002.

[141] Barack L, et al. Black holes, gravitational waves and fundamental physics: a roadmap. Class Quant Grav. 2019;36(14):143001.

[142] Will CM. The Confrontation between general relativity and experiment. Living Rev Rel. 2006;9:3.

[143] Liang D, Gong Y, Hou S, Liu Y. Polarizations of gravitational waves in $f(R)$ gravity. Phys Rev. 2017;D95(10):104034.

[144] Gong Y, Hou S. Gravitational Wave Polarizations in $f(R)$ Gravity and Scalar-Tensor Theory. EPJ Web Conf. 2018;168:01003.

[145] Maggiore M, et al. Science Case for the Einstein Telescope. JCAP. 2020;03:050.

[146] Hou S, Gong Y, Liu Y. Polarizations of Gravitational Waves in Horndeski Theory. Eur Phys J C. 2018;78(5):378.

[147] Hohmann M, Krˇ sšák M, Pfeifer C, Ualikhanova U. Propagation of gravitational waves in teleparallel gravity theories. Phys Rev D. 2018;98(12):124004.

[148] Hörmander L. The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis. Classics in Mathematics. Springer Berlin Heidelberg; 2015.

[149] Hörmander L. The Analysis of Linear Partial Differential Operators II. Classics in Mathematics. Springer Berlin Heidelberg; 2015.

[150] Gong Y, Hou S. The Polarizations of Gravitational Waves. Universe. 2018;4(8):85.

[151] Eardley DM, Lee DL, Lightman AP, Wagoner RV, Will CM. Gravitational-wave observations as a tool for testing relativistic gravity. Phys Rev Lett. 1973;30:884–886.

[152] Eardley DM, Lee DL, Lightman AP. Gravitational-wave observations as a tool for testing relativistic gravity. Phys Rev D. 1973;8:3308–3321.

[153] Will CM. The Confrontation between General Relativity and Experiment. Living Rev Rel. 2014;17:4.

[154] Chatziioannou K, Yunes N, Cornish N. Model-Independent Test of General Relativity: An Extended post-Einsteinian Framework with Complete Polarization Content. Phys Rev D. 2012;86:022004. [Erratum: Phys.Rev.D 95, 129901 (2017)].

[155] Isi M, Pitkin M, Weinstein AJ. Probing Dynamical Gravity with the Polarization of Continuous Gravitational Waves. Phys Rev D. 2017;96(4):042001.

[156] Callister T, Biscoveanu AS, Christensen N, Isi M, Matas A, Minazzoli O, et al. Polarization-based Tests of Gravity with the Stochastic Gravitational-Wave Background. Phys Rev X. 2017;7(4):041058.

[157] Abbott BP, et al. Tests of General Relativity with GW170817. Phys Rev Lett. 2019;123(1):011102.

[158] Abbott BP, et al. Tests of General Relativity with the Binary Black Hole Signals from the LIGO-Virgo Catalog GWTC-1. Phys Rev D. 2019;100(10):104036.

[159] Carroll SM. Spacetime and Geometry. Cambridge University Press; 2019.

[160] Farrugia G, Said JL, Gakis V, Saridakis EN. Gravitational Waves in Modified Teleparallel Theories. Phys Rev. 2018;D97(12):124064.

[161] Bamba K, Capozziello S, Nojiri S, Odintsov SD. Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. Astrophys Space Sci. 2012;342:155–228.

[162] Escamilla-Rivera C, Levi Said J. Cosmological viable models in $f(T,B)$ gravity as solutions to the H_0 tension. Class Quant Grav. 2020;37(16):165002.

[163] Finch A, Said JL. Galactic Rotation Dynamics in f(T) gravity. Eur Phys J C. 2018;78(7):560.

[164] Levi Said J, Mifsud J, Parkinson D, Saridakis EN, Sultana J, Adami KZ. Testing the violation of the equivalence principle in the electromagnetic sector and its consequences in $f(T)$ gravity. JCAP. 2020;11:047.

[165] Farrugia G, Said JL, Ruggiero ML. Solar System tests in f(T) gravity. Phys Rev. 2016;D93(10):104034.

[166] Bahamonde S, Levi Said J, Zubair M. Solar system tests in modified teleparallel gravity. JCAP. 2020;10:024.

[167] Joyce A, Khoury J. Strong Coupling Problem with Time-Varying Sound Speed. Phys Rev D. 2011;84:083514.

[168] Bardeen JM. Gauge-invariant cosmological perturbations. Physical Review D. 1980;22(8):1882.

[169] Bahamonde S, Böhmer CG, Wright M. Modified teleparallel theories of gravity. Phys Rev D. 2015;92(10):104042.

[170] Capozziello S, Capriolo M, Transirico M. The gravitational energy-momentum pseudotensor: the cases of $f(R)$ and $f(T)$ gravity. Int J Geom Meth Mod Phys. 2018;15:1850164.

[171] Bahamonde S, Capozziello S. Noether Symmetry Approach in $f(T,B)$ teleparallel cosmology. Eur Phys J. 2017;C77(2):107.

[172] Bahamonde S, Zubair M, Abbas G. Thermodynamics and cosmological reconstruction in $f(T,B)$ gravity. Phys Dark Univ. 2018;19:78–90.

[173] Wright M. Conformal transformations in modified teleparallel theories of gravity revisited. Phys Rev. 2016;D93(10):103002.

[174] Koivisto T, Tsimperis G. The Spectrum of Teleparallel Gravity. Universe. 2019 Mar;5(3):80.

[175] Copeland EJ, Kopp M, Padilla A, Saffin PM, Skordis C. Dark energy after GW170817 revisited. Phys Rev Lett. 2019 Feb;122(6):061301.

[176] Tsujikawa S. Matter density perturbations and effective gravitational constant in modified gravity models of dark energy. Phys Rev D. 2007;76:023514.

[177] Nunes RC, Bonilla A, Pan S, Saridakis EN. Observational Constraints on $f(T)$ gravity from varying fundamental constants. Eur Phys J. 2017;C77(4):230.

[178] Franco GAR, Escamilla-Rivera C, Levi Said J. Stability analysis for cosmological models in $f(T,B)$ gravity. Eur Phys J C. 2020;80(7):677.

[179] Creminelli P, Vernizzi F. Dark Energy after GW170817 and GRB170817A. Phys Rev Lett. 2017;119(25):251302.

[180] Bahamonde S, Dialektopoulos KF, Hohmann M, Levi Said J. Post-Newtonian limit of Teleparallel Horndeski gravity. Class Quant Grav. 2020;38(2):025006.

[181] Weinberg S. Frontmatter. vol. 1. Cambridge University Press; 1995.

[182] Heisenberg L. A systematic approach to generalisations of General Relativity and their cosmological implications. Phys Rept. 2019;796:1–113.

[183] De Felice A, Heisenberg L, Kase R, Mukohyama S, Tsujikawa S, Zhang Yl. Cosmology in generalized Proca theories. JCAP. 2016;06:048.

[184] Emami R, Mukohyama S, Namba R, Zhang Yl. Stable solutions of inflation driven by vector fields. JCAP. 2017;03:058.

[185] Beltran Jimenez J, Heisenberg L. Derivative self-interactions for a massive vector field. Phys Lett B. 2016;757:405–411.

[186] Gonzalez PA, Vasquez Y. Teleparallel Equivalent of Lovelock Gravity. Phys Rev D. 2015;92(12):124023.

[187] González PA, Reyes S, Vásquez Y. Teleparallel Equivalent of Lovelock Gravity, Generalizations and Cosmological Applications. JCAP. 2019;07:040.

[188] Ostrogradsky M. Mémoires sur les équations différentielles, relatives au problème des isopérimètres. Mem Acad St Petersbourg. 1850;6(4):385–517.

[189] Woodard RP. Avoiding dark energy with 1/r modifications of gravity. Lect Notes Phys. 2007;720:403–433.

[190] Martin-Garcia JM, Portugal R, Manssur LRU. The Invar Tensor Package. Comput Phys Commun. 2007;177:640–648.

[191] Martin-Garcia JM, Yllanes D, Portugal R. The Invar tensor package: Differential invariants of Riemann. Comput Phys Commun. 2008;179:586–590.

[192] Martín-García JM. xPerm: fast index canonicalization for tensor computer algebra. CoRR. 2008;abs/0803.0862.

[193] Brizuela D, Martin-Garcia JM, Mena Marugan GA. xPert: Computer algebra for metric perturbation theory. Gen Rel Grav. 2009;41:2415–2431.

[194] Gomez-Lobo AGP, Martin-Garcia JM. Spinors: a Mathematica package for doing spinor calculus in General Relativity. Comput Phys Commun. 2012;183:2214–2225.

[195] Pitrou C, Roy X, Umeh O. xPand: An algorithm for perturbing homogeneous cosmologies. Class Quant Grav. 2013;30:165002.

[196] Nutma T. xTras : A field-theory inspired xAct package for mathematica. Comput Phys Commun. 2014;185:1719–1738.