L-Università ta' Malta
Institute of Space
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The 3+1 Formalism in Teleparallel and Symmetric Teleparallel Gravity

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Abstract

In this dissertation, both a tetrad and a metric 3+1 formulation for a general affine connection while also assuming metricity is developed. By employing a space and time split of the usual space time manifold, a spatial version of the fundamental variables is obtained. Finding the Gauss-like equations for any tensor through which gravity is expressed, a general foundation for the two formalisms is set up. Using this foundation the general form of the evolution equations of the 3-tetrad and 3-metric as they are dragged along the normal vector to the spatial foliations are derived. Finally through the choice of two different connections assuming metricity, and another case assuming the coincident gauge with non-metricity, the relevant 3+1 formulations for General Relativity, the Teleparallel Equivalent of General Relativity and the Symmetric Teleparallel Equivalent of General Relativity are respectively derived up to the latest state of the research. By obtaining the 3+1 formalisms with respect to each of these three different geometric interpretations of gravity we achieve what is called the 3+1 formalism in the geometric trinity of gravity. Building on the fully consistent system of equations obtained in the Symmetric Teleparallel Equivalent of General Relativity a more stable structure for this system is derived in the form of a BSSN-like formalism.

1 Introduction

Gravitational waves first appeared as a result of the theoretical predictions of general relativity (GR). Observed to be travelling at the speed of light, gravitational waves of significant magnitudes manifest as ripples on the space-time fabric. Such waves are generated when any mass moves through space however the largest waves are generated by massive astronomical events such as supernovae, compact star mergers and black hole mergers. Multiple research initiatives have been undertaken to search for these elusive vibrations throughout the past hundred years with a substantial increase of effort in the past few decades. While gravitational waves are thought to dissipate at a rate of $\frac{1}{r}$, given the huge distances between stellar objects and the even greater distances between events of significant magnitudes, these ripples are so small by the time they reach Earth that they are of the order of 10^{-21} [2]. In fact, even Einstein himself doubted that they will ever be detected.

In the past years groundbreaking work has been carried out in order to detect and observe gravitational waves through The Laser Interferometer Gravitational-Wave Observatory (LIGO). In 2015 the first confirmed detection was announced [3] and a number of others have since been confirmed [4]. One of the most significant events was the first detection of a binary neutron star collision which was announced to have been detected both through gravitational waves and also through electromagnetic

radiation [5, 6]. These detections are leading us to a new era of astrophysics with detectors like The Laser Interferometer Space Antenna (LISA) [7]. The final aim of this project is to pave the way for the study of gravitational radiation on the space-time manifold as generated in a Universe governed by alternative theories of gravity. Specifically, gravitational radiation produced through the interaction of black holes and compact objects.

Before going into alternative theories however, it is important to briefly cover how such simulations are obtained in GR along with their use. This process starts with a branch of physics called Numerical relativity. Here, numerical methods are used in order to solve and study different gravitationally bound systems starting from a particular system of equations. Assuming one has a consistent set of evolution equations which are well posed and hyperbolic, things that will all be defined and elaborated further later on in this work, one can start to produce simulations. In Gr the base form of these equations is known as the ADM formalism with a more numerically stable reformulation being the BSSN formalism, among others. Once such equations are available various numerical methods can be used depending on the physical system being considered. The simulations can be dynamical, stationary or static and the gravitational interactions can be specified to be taking place in vacuum and/or in some form of fluid matter. For the sake of brevity and concision we will be sticking to dynamical vacuum simulations for the time being as these are the kind used to produce compact object mergers that generate gravitational waves [8, 9].

The main aim of generating such gravitational simulations is the production of gravitational wave profiles, known as templates [10, 11], from likely mergers. Specific mergers are considered likely if they result from compact objects that are theoretically abundant. Obtaining such profiles is important in two ways. Firstly, they indicate what profiles should be expected by Gravitational wave observatories like Ligo, Virgo, etc. This information helps with planning both the construction of new facilities as well as the calibration of the current ones. Secondly, such simulations help with the identification of gravitational wave detections once they are recorded. Through the knowledge of which compact objects and mergers produce which profiles through simulations, actual detections can be classified. That being said, there is an obvious downside to this method. The profiles generated assume Gr is correct and so any classified detections are model dependant.

In the last hundred years, GR has proven to be an invaluable theory for explaining gravity in our Universe. Unfortunately cases where GR, in tandem with only baryonic matter, fails to agree with observation do exist. Some of the more noticeable ones are galactic rotation curves and the expansion of the Universe. With the addition of dark matter and dark energy, what is called the ΛCDM model, has even overcome most of GR's shortcomings including the ones mentioned above. That being said, a major problem still remains. Both dark matter and dark energy have never been directly observed and are open to different manifestations within the theory. In the past few years a further issue has emerged surrounding the Hubble constant. When comparing the value of the Hubble constant calculated through the type Ia supernovae (SNe Ia), the Cepheids in NGC 4258, the Milky Way and the Large Magellanic Cloud [12, 13] as well as gravitationally lensed quasars [14] with the the value obtained through Planck measurements, the values differ from 4 to 6σ. It is thus important to consider other explanations for GR's departure from observational data. Alternative theories of gravity consider the scenario where our standard picture of gravity could be flawed. In this project it is considered that our current understanding of the way energy densities interact with the space-time fabric may be better served through the teleparallel approach. This means that the energy densities themselves would not be considered the problem but instead our geometric interpretation of gravity may be the issue.

In this work the group of alternative theories of gravity that fall under the name of teleparallel theories of gravity will be the main ones considered. While GR treats gravity through the lens of curvature, teleparallel theories of gravity treat it either as torsion or non-metricity on the space-time fabric. Thus two teleparallel theories are defined, Torsional Gravity (TG) and Symmetric Teleparallel Gravity (STG). In particular, TG has a further distinction from the GR and STG theories. This distinction is that instead of using the metric as its fundamental observable it uses what is called the tetrad. The use of the tetrad is beneficial since it creates a direct relationship between inertial and non-inertial frames of reference. On the other hand, STG, uses the metric as its fundamental variable same as GR. The teleparallel theories of gravity have been found to reproduce results frequently identical to GR once specific Lagrangians are chosen. In these cases the names of these two theories become the Teleparallel Equivalent of General Relativity (TEGR) and the Symmetric Teleparallel Equivalent of General Relativity (STEGR). Unfortunately, these theories naturally include most of GR's shortcomings. That being said, modified models of teleparallel gravity, $f(T)$ gravity and $f(Q)$ gravity, have recently proven to be good alternatives [15, 16, 17]. While still based on torsion, $f(T)$ gravity is to TEGR what $f(R)$ gravity is to GR. Similarly $f(Q)$ gravity is still based on non-metricity as is STEGR. While these modified theories of gravity will not be considered in this report, the following work sets the necessary underlying formulations which can later be extended to such theories helping to determine their validity in this area.

Recently some work has been done on gravitational wave theory in $f(T)$ and $f(Q)$ gravity such as in Ref.[18, 19, 20, 21] however these works are related to polarization and cannot give the insight that can be retrieved through the numerical relativistic approaches considered here. While further work needs to be done on the analytical side of gravitational waves in teleparallel theories of gravity, no work has yet been done in order to obtain the necessary formulations of teleparallel theories in order to test and produce simulations of gravitational wave profiles from events such as star collapses, binary black hole and binary neutron star mergers, to name a few. In this work a similar analysis is to be done to that carried out for standard gravity in Refs.[8, 9, 22] for the two teleparallel theories of gravity discussed above.

Once finalized this work will pave the way to a further analysis of $f(T)$ and $f(Q)$ gravity as is done in Ref.[23] for $f(R)$ gravity where the merger of a binary black hole system was studied. The conclusion of this study was that the distinction between $f(R)$ gravity and GR is large enough that LISA and other future detectors could potentially be capable of accurate enough measurements to differentiate between the two theories. Such simulations in teleparallel theories of gravity would potentially allow us to narrow down the classes of possible theories in this regime thereby getting us ever closer to a possible better understanding of gravity as a whole and the ways through which it can be interpreted.

In order to obtain gravitational wave simulations in TEGR and STEGR it is first necessary to develop what is called an ADM formalism for them. The ADM formalism was first introduced by Richard Arnowitt, Stanley Deser and Charles W. Misner [24]. It was developed as a new way to approach the formulation of the field equations in GR so that they may be evolved in time. The basis for this ADM is what is know as the 3+1 formalism where four dimensional space time is sliced into three dimensional surfaces called hypersurfaces with each slice occupying a particular time instance [22]. The formulation consists of four main equations. Two are evolution equations along a temporal vector, one for the spatial fundamental variable and one for the evolution of that fundamental variable. The second two are constraint equations which relate the energy and the momentum of the considered system to the fundamental variable and its evolution. This formulation is beneficial as it is very suitable for generating numerical solutions from the field equations. This can be used for various endeavors. Among the use cases, it is often used in order to simulate gravitational waves from various sources [8, 9]. As of yet no ADM or 3+1 formalism has been fully developed for teleparallel theories of gravity. Once this is done it will serve as the much needed back bone for numerical calculations within such theories making it an important contribution to the overall area.

In order to set up the underlying mathematics needed for this, in Chapter 2 an overview of teleparallel theories of gravity is given where the two theories' origins and development are discussed. The necessity and benefit for a tetrad formulation in such theories that are built around torsion is explored with regards to TEGR. The idea of a local spin connection as a separate second fundamental variable from the view point of such theories is also addressed. With regards to STEGR and non-metricity theories, the equation simplifying benefit of taking the metric as the fundamental variable while also assuming the coincident gauge is explored. In both TEGR and STEGR the field equations are considered and both their differences to and their relationship with the GR field equations are discussed. Their equivalence at the level of equations to the GR Field equations is also highlighted.

While the main focus of this work is to build the necessary framework for future gravitational wave simulations in teleparallel theories of gravity, the more general aim of the project is to open up the possibility for simulations to be carried out in as large a number of extended theories of gravity as possible. As such the basis for a 3+1 formulation for a general affine linear connection while assuming non-metricity is derived in Chapter 3. Keeping the connection general with non-metricty allows for all curvature, torsion and non-metricity terms to survive. Another generalization in this chapter is that both a metric and a tetrad 3+1 formulation are derived. By the end, all possible definitions, relations and evolution equations (both tetrad and metric) are obtained up to the point where field equations are necessary and a connection needs to be chosen in order to move forward and obtain usable equations.

Such a fully generalized 3+1 formulation is beneficial as depending on the theory of choice, one may choose whichever affine linear connection one needs and the equations will simplify accordingly. This also allows for a consistency check by taking the connection to be the Levi-Civita connection and confirming that the 3+1 metric formalism simplifies to the well known GR 3+1 and ADM formalism derived in Refs.[8, 9, 22, 24]. This in fact is what is carried out in the first section of Chapter 4. In the second section of this chapter the Weitzenböck connection is considered while assuming metricity and a tetrad 3+1 formulation is derived for teleparallel theories while keeping spin zero. Finally both torsion terms and curvature terms are taken to vanish while keeping non-metricity and assuming the coincident gauge. This produces the STEGR evolution equations along with its correspoing constraint equations. While the evolution equations in the GR and STEGR theories are derived to their final form an issue with the finalization of the TEGR tetrad 3+1 formulation is discovered and discussed.

In Chapter 5 the final STEGR ADM system of equations are considered and tested. Their relationship and separation from the GR ADM equations are studied in more depth even up to the most basic partial derivative versions of the evolution equations. Known spatial solution to the GR evolution equations are then tested in order to further confirm the viability of the derived STEGR system of equations. Here these equations are tested using four standard spatial metrics. The Schwarzchild, Isotropic, Painlev´e-Gullstrand and Kerr-Schild metrics are the ones considered. Due to the fact that the equations being considered are non-hyperbolic, the BSSN formulation for the STEGR ADM is derived eliminating all second order mixed derivatives and restructuring the equations to a hyperbolic and well-posed form.

In the final chapter an overview of the results obtained is presented and a plan for the future of this work is discussed. It is described how such a formalism can be used in order to obtain gravitational wave simulations in a number of different simulation programs some of which will have to be partly redeveloped in order to be used for teleparallel theories of gravity. Among such programs will be the Cactus Computational Toolkit. Cactus is an open source program for generating simulations in areas varying form numerical relativity to fluid dynamics to quantum gravity. An example of its use in generating simulations for gravitational waves can be found in Ref.[25]. Here gravitational radiation simulations were carried out stemming from the collapse of neutron stars and rotating black holes. This toolkit has already been built on the cluster, Dante, in the Institute of Space Sciences and Astronomy at the University of Malta (ISSA) and a number of simulations, including a binary black hole merger simulation has been carried out in GR. The results of this simulation are also presented in Chapter 6.

Thus by combining theory and simulation it is the intention of this work to lead to answers for the questions; Is there an alternative 3+1 formulation which is more or just as beneficial to produce gravitational wave simulations as GR?; Do particular alternative theories of gravity predict the correct wave forms and mechanics of gravitational waves induced by dense stellar mergers?; Do they predict the same as GR?; If not, then what is causing the variation and does it coincide with actual data?

Due to recent advances in technology such as the LIGO experiment and the proposed LISA experiment, this is a very exiting time to be studying gravitational waves. As time goes by, the use of gravitational waves will be at the forefront of astrophysical event detection systems and observation projects. As such, it is very important that they are studied holistically. The lack of evidence for exotic matter and energy make alternative theories of gravity a strong contender in our quest for understanding the Universe and we would be remiss if the resulting gravitational radiation aspects go untested.

2 Teleparallel Theories of Gravity

In this chapter the two teleparallel theories of gravity within the geometric trinity of gravity [26] are set up. Specifically the torsion- and non-metricity-based formulations of gravity are considered. Through these formulations field equations can be derived that are mathematically equivalent to the classical curvature-based field equations one finds in General Relativity (GR). These two theories are the Teleparallel Equivalent of General Relativity (TEGR) and Symmetric Teleparallel Equivalent of General Relativity (STGR). The bases of these theories leading to the field equations as well as any underlying nuances and equations are set up in order to prepare for their eventual 3+1 decomposition in the following chapters.

2.1 Building the Geometric Structure for our Theories

While gravity is expressed through a different geometric construct in such theories most of the fundamental elements that they are built on remain the same. Before delving further into these differences it is important to define the properties of a number of geometric entities such as the space-time manifold and the objects through which it is expressed. These properties will also be useful later on in the following chapters while developing the 3+1 formalism mentioned in the previous

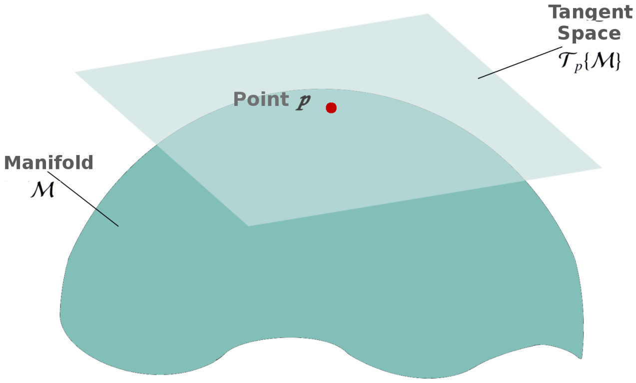

Figure 2.1: This diagram represents a two dimensional representation of spacetime as a manifold with the plane at the red dot representing the point's tangent space $\mathcal{T}_p\{\mathcal{M}\}$

chapter.

A manifold, $\mathcal{M}$, can be defined as a four dimensional, smooth and infinitely differentiable surface on which all events take place and all mathematical entities discussed in this work are bound to or inhabit. At each point p on the manifold a tangent space can be defined composed of all vectors located at that point. This is denoted by $\mathcal{T}_p\{\mathcal{M}\}$ and is illustrated graphically in Fig.(2.1). It should be noted that the graphic is simply a example of this notion where a two dimensional manifold is viewed from a three dimensional space. The set of all tangent spaces from all points constructs the tangent bundle for that manifold, $\mathcal{T}\{\mathcal{M}\}$. For each tangent space $\mathcal{T}_p\{\mathcal{M}\}$, a dual vector space exists, $\mathcal{T}_p^*\{\mathcal{M}\}$. This is composed of all linear maps from the tangent space to the real number set or, more mathematically, it is composed of all vectors such that for each vector V^μ in $\mathcal{T}_p\{\mathcal{M}\}$ there exists a dual vector w_μ in $\mathcal{T}_p^*\{\mathcal{M}\}$ that maps $w_\mu(V^\mu) \rightarrow \mathbb{R}$. The vector spaces are each defined through their own set of basis vectors that will be the backbone of all the measurements and calculations carried out on the vector space themselves as well as the manifold [27].

The basis vectors for a vector space are the smallest set of independent vectors through which all vectors in the vector space can be built. While in general any minimal set of vectors with the above properties can be used as a basis, here it is convenient to link the basis to the coordinate system being used for the sake of physical interpretation. The basis for the tangent space can thus be defined as $\partial_\mu = \partial/\partial x^\mu$ and those for the dual or co-tangent space can be defined as dx^μ. Here the relationship between the basis vectors and the coordinate system is evident as they are operators with respect to the coordinate system coordinates x^μ themselves. These basis are intentionally designed to be orthogonal to each other and together produce the delta tensor $dx^\mu(\partial_\nu) = \delta^\mu_\nu$.

Finally it is necessary to define an entity which is built to measure the distances between objects on the manifold. This entity is of course the metric tensor tied to this manifold and is denoted by $g^{\mu\nu}$. The metric tensor is used to contract vectors and co-vectors together while also having properties such as $g^{\mu\nu}g_{\mu\chi} = \delta^\nu_\chi$ and $g^{\mu\nu}g_{\mu\nu} = 4$ when constructed on 4 dimensional space times [27, 28].

GR was originally formulated in order to accommodate for the lack of accelerating frames in special relativity and by extension, through the strong equivalence principal, the gravitational effects [29]. One did not replace the other but simply matures it to incorporate a broader spectrum of events. In one way or another this must also be true for our alternative theories. Vectors, dual vectors and events inhabiting and/or taking place on our manifold must interact with the gravitational fields in a global sense while at the same time originating from a local, inertial, special relativistic space. The local space is in essence a tangent space of the manifold in its own right. The local vector spaces will play an integral part in this work especially when considering torsional gravity and tetrad formulations.

It is now necessary to find a way to distinguish between local and global vectors and tensors. From this point on-words Greek indices will be used to denote global entities and capital Latin indices will be used to denote local entities. These local spaces must also have basis vectors of their own which are defined as $\partial_A = \partial/\partial x^A$ and dx^A, mirroring the global basis vectors discussed earlier these are called non-coordinate basis [30]. These basis vectors are also orthogonal in their own right. The metric that will be used for these local spaces needs to be a flat/inertial metric. While any inertial metric can be chosen, for the sake of simplicity the Minkowski metric, η^{AB}, will be used as the local metric throughout this work.

Having defined all the necessary basis vectors of the individual local (inertial) and global (non-inertial) frames, the tensor that maps vectors and tensors to and from each kind of frame can be defined. This mapping tensor is known as the tetrad. It can also be referred to as a vierbein (four-legged) if a four dimensional space is being considered or vielbine (manny-legged) if it is many dimensional [30]. This tensor is denoted by $e^A{}_\mu$, where the first index is always a local one and the second is always a global one. In reality we need to define two tensors, one that maps the tangent vector fields and one that maps the cotangent vector fields to their inertial counterparts. Fortunately, these turn out to be each other's inverses in that their contraction results in the delta function [28]

$$e^A{}_\mu e_A{}^\nu = \delta^\nu_\mu, \tag{2.1}$$

$$e^A{}_\mu e_B{}^\mu = \delta^A_B.$$

The tetrad allows us to obtain the local basis from the global ones, that is, $dx^A = e^A{}_\mu dx^\mu$ and $\partial_A = e_A{}^\mu \partial_\mu$, and vice versa. These tetrads are considered fundamental variables and in fact can be used to build the global metric from the local inertial Minkowski metric. This is done by contracting each of the two free local indices of the Minkowski metric with a separate tetrad as depected in Fig.(2.2)

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu. \tag{2.2}$$

Physically the tetrads represent the observer [31] but unfortunately these conditions

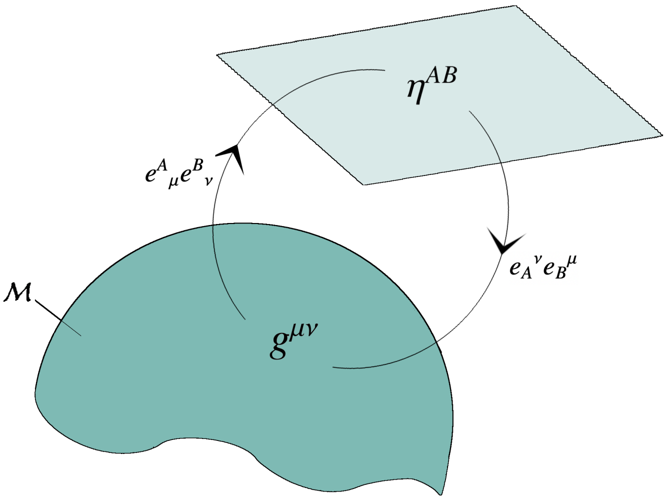

Figure 2.2: This diagram represents the mapping of the global metric on the manifold to the local Minkowski metric on the local frame through tetrad contraction

are not enough to define a unique form of the tetrad. As such there is an element of choice when it comes to choosing a tetrad that produces a particular metric. In fact, there are an infinite number of tetrads that can build the same global metric from the Minkowski metric.

An important operator to define in any theory is the derivative related to the manifold. In the case of scalar tensors the partial derivative is enough but when considering vectors or tensors this is not the case. Due to the fact that non-inertial and non-flat space-times are being considered, it is not possible to simply use the partial derivative. This derivative would thus also need to account for coordinate systems that are built around flat, non-constant metrics. For example, spherical coordinates in flat space. A new derivative called the covariant derivative is thus constructed that is capable of accounting for the departure from the inertial frame, flatness and the Minkowski metric. An integral part of the covariant derivative is the connection $\Gamma^{\nu}_{\lambda\mu}$. This is a general connection without any ties to any theory. The only assumption made here is that it is an affine connection, that is, it relates vectors as they are parallel transported over the various charts making up the manifold. The connection is tied directly to the geometric method being used to define variations on the manifold and the metric and so its definition is dependant on which theory is being considered. Throughout this work however, we adopt the following general convention when taking the covariant derivative of any vector, dual vector and tensor so long as the connection is affine [28]

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\lambda\,\mu} V^\lambda, \tag{2.3}$$

$$\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^\lambda_{\nu\,\mu} w_\lambda. \tag{2.4}$$

Since the connection is not generally symmetric on the bottom two indices, the choice of placing the derivative index as the final, bottom index of the connection is significant and should be considered a convention throughout this work. In the case of GR this is irrelevant due to the Levi-Civita being symmetric however this is not generally so and is certainly not so in the case of Teleparallel gravity.

When considering tetrad formulations another necessary ingredient is the spin connection, $\omega^B_{A\mu}$ [28, 32]. In the general case, and even in most specific theories, the tetrad/spin pair cannot be fully determined through the metric and the global connection. Similarly to the global connection it is not a tensor and its litteral definition depends on the theory being considered. The spin connection mainly arises as a result of the covariant derivative being applied to vectors/tensors that have local indices. In general for a tensor $A^A{}_\nu$

$$\nabla_\mu A^A{}_\nu = \partial_\mu A^A{}_\nu - \Gamma^\lambda_{\nu\,\mu} A^A{}_\lambda + \omega^A_{B\,\mu} A^B{}_\nu. \tag{2.5}$$

In the case of teleparallel theories the connection is fully inertial and is defined in such a way that it accounts for the local Laurence invariance of a theory built on tetrads as the fundamental variable. As seen in Ref.[15], in such theories the metric is invariant when the composing tetrad is transformed by a local laurentz matrix

$\Lambda^A{}_B$ that satisfies $\eta_{AB}\Lambda^A{}_C\Lambda^A{}_D = \eta_{AB}$, that is, it preserves flatness. It should be noted that the global connection is not invariant under such a tetrad transformation unless the spin connection is also transformed as

$$\omega^A{}_{B\mu} \rightarrow \omega`^A{}_{B\mu} = \Lambda^A{}_C(\Lambda^{-1})^D{}_B\,\omega^C{}_{D\mu} + \Lambda^A{}_C\partial_\mu(\Lambda^{-1})^C{}_B. \tag{2.6}$$

This indicates that the tetrad spin pair are unique up to a laurentz transformation.

The reason we have built this general geometric structure for our theories both on the metric and the tetrad is that they can both be used as fundamental variables. The choice of which variable to move forward with depends on which theory one is interestied in developing, that is, what connection is chosen. In the case of curvature and non-metricity based theories such as GR and STGR the metric tensor is usually taken to be the fundamental variable. In the case of torsional gravity theories like TEGR there is no choice in the matter. The TEGR connection, the Teleparallel connection and, by extension, the torsion tensor cannot be expressed purely in terms of the metric and so torsional theories have to be built around the tetrad as their fundamental variable. Independent of which fundamental variable is chosen or needed these theories each lead to field equations which can be used in order to solve for the fundamental tensor's components. These equations are fundamental for building the final 3+1 system of equations and will be expanded on in the following sections of this chapter.

2.2 Teleparallel Equivalent of General Relativity

In 1915 Albert Einstein propelled our understanding of gravity forward through his theory of General Relativity. After this, one of his major endeavours was to unite gravitation and electromagnetism. Unfortunately, he did not manage to fulfill this work during his life time. One method that he attempted introduced torsional terms to which he tried to relate the electromagnetic tensor. While this was not successful, the torsion tensor was eventually used to build a separate theory of gravity. TEGR, or the teleparallel equivalent of GR, expresses gravity on a four dimensional space-time manifold through torsion rather than curvature as is done in GR. This can be seen visually in Fig.(2.3) [15]. In this theory the torsion tensor is used in order to build field equations that relate masses, energy densities and the space-time geometry in a way that is equivalent to GR resulting in the same exact results [31, 28, 33].

Similar to how GR is built on the Levi-Civita connection, torsional theories of gravity are built on the Teleparallel connection, $\hat{\Gamma}^{\lambda}_{\mu\nu}$. This new connection is torsion full and curvature free, that is, all curvature based tensors identically vanish when written through this connection

$$\hat{R}^{\rho}{}_{\sigma\mu\nu} \equiv 0\,. \tag{2.7}$$

The connection itself is built on two fundamental dynamical variables, that is, the tetrad and the spin connection and can be defined as

$$\hat{\Gamma}^{\lambda}_{\mu\nu} := e_A{}^{\lambda}\partial_\nu e^A{}_\mu + e_A{}^{\lambda} e^B{}_\mu \omega^A_{B\nu}\,. \tag{2.8}$$

While in the general case the two fundamental variables are independent of each other, in teleparallel theories the two form a fundamental pare. In such cases the spin connection is tied to the tetrad if the theory is to be Lorentz invariant. Unfortunately there is no way that is currently known to express the spin connection purely in terms of the tetrad itself. This introduces the possibility of obtaining two sets field equations by varying the Lagrangian both with respect to the tetrad and with respect to the spin. It turns out that in the case of torsional theories the spin field equations are identical to the antisymmetric part of the tetrad field equations, further emphasizing the link between the variables. The choice of spin connection is thus treated as a gauge choice determined by the choice of tetrad. Two types of

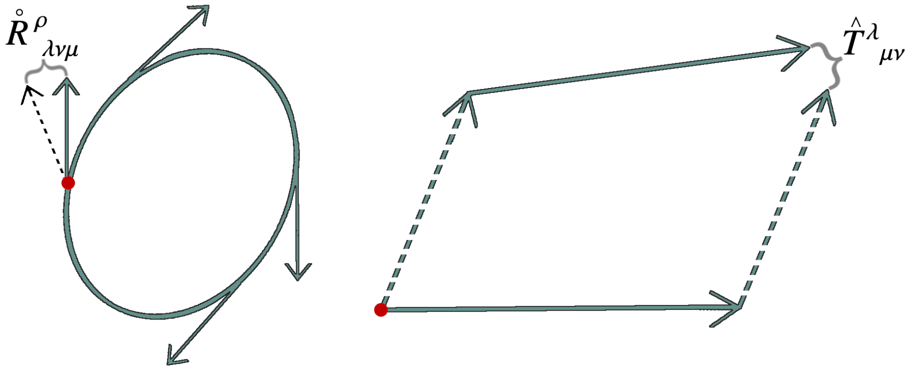

Figure 2.3: This diagram represents the different deformation of space time as produced through theories built around curvature and torsion.

tetrads are thus defined, proper tetrads, whose associated spin connection vanishes, and improper tetrads whose spin connection is non-vanishing and contributes to the overall theory [32].

Both in torsional and curvature based theories the covariant derivative of the metric is considered to be zero. This property is called metricity and it ensures that the definition of the metric is invariant across the considered space-time. This is also the property through which the Levi-Civita is defined in GR. Similarly, the Teleparallel connection in TEGR is defined by setting the covariant derivative of the tetrad equal to zero while setting the vanishing curvature constraint. That being said, assuming a vanishing $\hat{\nabla}_{\lambda} e^{a}{}_{\mu}$ does not necessarily imply metricity. Expanding the covariant derivative of the metric while taking the local metric to be the minkowski metric

$$\begin{aligned}
\hat{\nabla}_\lambda g_{\mu\nu} &= \hat{\nabla}_\lambda \eta_{AB} e^A{}_\mu e^B{}_\nu \\
&= e^A{}_\mu e^B{}_\nu \hat{\nabla}_\lambda \eta_{AB} + \eta_{AB} e^B{}_\nu \hat{\nabla}_\lambda e^A{}_\mu + \eta_{AB} e^A{}_\mu \hat{\nabla}_\lambda e^B{}_\nu \\
&= e^A{}_\mu e^B{}_\nu \hat{\nabla}_\lambda \eta_{AB} + 0 + 0 \\
&= e^A{}_\mu e^B{}_\nu (\partial_\lambda \eta_{AB} - \omega^C_{A\ \lambda} \eta_{CB} - \omega^C_{B\ \lambda} \eta_{AC}).
\end{aligned} \tag{2.9}$$

it is noted that if metricity is required then the spin connection needs to necessarily be anti-symmetric on the first two indices, $\omega_{BA\lambda} = -\omega_{AB\lambda}$. A specific instance that also satisfies metricity is choosing proper tetrads which results in a vanishing spin connection. For the sake of simplifying the calculations, throughout this work only proper tatrads are considered in relation to torsional gravity, resulting in zero spin connection and metricity.

The next step in building our theory is to introduce the gravitating tensors responsible for relating gravity to geometry. Taking a general covariant derivative, ∇, and a general connection, $\Gamma^\lambda_{\mu\nu}$, the general form of the gravitating tensors for GR and TEGR can be defined. In the case of GR this tensor is know as the Riemann tensor, $R^\mu{}_{\nu\sigma\rho}$, and in TEGR the gravitating tensor is the torsion tensor $T^\lambda{}_{\mu\nu}$. Starting from the commutator of such a general covariant derivative one obtains [31, 27]

$$\nabla_{[\nu}\nabla_{\mu]} V^\lambda = T^\sigma{}_{\mu\nu} \nabla_\sigma V^\lambda + V^\sigma R^\lambda{}_{\sigma\nu\mu} \,, \tag{2.10}$$

$$\nabla_{[\nu}\nabla_{\mu]} V_\lambda = T^\sigma{}_{\mu\nu} \nabla_\sigma V_\lambda + V_\sigma R^\sigma{}_{\lambda\mu\nu} \,, \tag{2.11}$$

where V^ν and w_ν are a vector and a dual vector respectively. From this, the following general definitions follow [28]

$$R^{\rho}{}_{\lambda\nu\mu} = \partial_\nu \Gamma^{\rho}_{\lambda\mu} - \partial_\mu \Gamma^{\rho}_{\lambda\nu} + \Gamma^{\rho}_{\alpha\nu}\Gamma^{\alpha}_{\lambda\mu} - \Gamma^{\rho}_{\alpha\mu}\Gamma^{\alpha}_{\lambda\nu}\,, \tag{2.12}$$

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\mu\nu}\,. \tag{2.13}$$

In GR the Levi-Civita is chosen to be the connection leading to a curvature full and torsion free theory, due to it being symmetric in the bottom two indices

$$\overset{\circ}{R}{}^{\rho}{}_{\lambda\nu\mu} = \partial_\nu \overset{\circ}{\Gamma}{}^{\rho}_{\lambda\mu} - \partial_\mu \overset{\circ}{\Gamma}{}^{\rho}_{\lambda\nu} + \overset{\circ}{\Gamma}{}^{\rho}_{\alpha\nu}\overset{\circ}{\Gamma}{}^{\alpha}_{\lambda\mu} - \overset{\circ}{\Gamma}{}^{\rho}_{\alpha\mu}\overset{\circ}{\Gamma}{}^{\alpha}_{\lambda\nu}\,, \tag{2.14}$$

$$\overset{\circ}{T}{}^{\lambda}{}_{\mu\nu} = 0. \tag{2.15}$$

In TEGR the Teleparallel connection is chosen producing a torsion full and curvatureless theory

$$\hat{R}^{\mu}{}_{\nu\sigma\rho} = 0, \tag{2.16}$$

$$\hat{T}^{\lambda}{}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\nu\mu} - \hat{\Gamma}^{\lambda}_{\mu\nu}. \tag{2.17}$$

The Torsion tensor is thus always antisymmetric in the bottom two indices [31]. Finally we define what is called the tortion vector. This vector is the result of contracting the first index of the torsion tensor with its last,

$$\hat{T}_{\mu} = \hat{T}^{\lambda}{}_{\mu\lambda} = \delta^{\nu}_{\lambda}\hat{T}^{\lambda}{}_{\mu\nu}. \tag{2.18}$$

As discussed in the previous section, TEGR is equivalent to GR at the level of equations but it is not geometrically equivalent in terms of the ingredients that lead up to the field equations. It also paints a completely different picture of the gravitational phenomena it treats.

A pivotal term in any theory is the Lagrangian on which the action is applied. In the case of torsional gravity, and more particularly TEGR, this term is the torsion scalar. In order to derive this scalar, the departure of the Teleparallel connection from the Levi-Civita connection is first considered. The resulting tensor is called the contortion tensor and can also be written purely in terms of the torsion tensor

$$\hat{K}^{\sigma}{}_{\mu\nu} := \hat{\Gamma}^{\sigma}_{\mu\nu} - \mathring{\Gamma}^{\sigma}_{\mu\nu} \tag{2.19}$$

$$= \frac{1}{2}\left(\hat{T}_{\mu}{}^{\sigma}{}_{\nu} + \hat{T}_{\nu}{}^{\sigma}{}_{\mu} - \hat{T}^{\sigma}{}_{\mu\nu}\right) . \tag{2.20}$$

Finally the superpotential tensor is defined as

$$\hat{S}_{A}{}^{\mu\nu} := \hat{K}^{\mu\nu}{}_{A} - e_{A}{}^{\nu}\hat{T}^{\alpha\mu}{}_{\alpha} + e_{A}{}^{\mu}\hat{T}^{\alpha\nu}{}_{\alpha} . \tag{2.21}$$

This tensor is related to the guage current for teleparallel gravity $J_{A}{}^{\nu} = -\frac{1}{e}\frac{\partial\mathcal{L}}{\partial e^{A}{}_{\nu}}$ through the field equations discussed below. This particular form of the superpotential is usefull as contracting it fully with the torsion tensor gives the torsion scalar discussed above

$$\hat{T} := \frac{1}{2}\hat{S}_{A}{}^{\mu\nu}\hat{T}^{A}{}_{\mu\nu} , \tag{2.22}$$

$$= \frac{1}{4}\hat{T}^{a}{}_{\mu\nu}\hat{T}_{a}{}^{\mu\nu} + \frac{1}{2}\hat{T}^{a}{}_{\mu\nu}\hat{T}^{\nu\mu}{}_{a} - \hat{T}^{a}{}_{\mu a}\hat{T}^{\nu\mu}{}_{\nu}. \tag{2.23}$$

It should be noted that changing the coefficients in front of the quadratic torsion tensor terms gives a modified teleparallel theory of gravity called New General Relativity [34]. In this theory the coefficients are usually kept as unknown constants and fit according to real-world data in an attempt to produce a theory which is better suited to explain certain phenomena that GR, and by extension TEGR, fail to predict. Another way to modify TEGR is through taking functions of the torsion scalar as described in Eq.(2.22). This theory is called $f(T)$ gravity [16, 35, 36].

Having obtained the relevant Lagrangian for TEGR, the essence of the difference between GR and TEGR, and the reason for the equivalence despite of this difference, can be discussed. Starting from the relationship between the Ricci scalar (the GR Lagrangian), R, and the torsion scalar, T, the equivalence will be traced down to the field equations themselves.

Basing the action on the torsion scalar gives

$$\mathcal{S}_{\mathrm{TEGR}} = -\frac{1}{2\kappa^2}\int d^4x\, e\hat{T} + \int d^4x\, e\mathcal{L}_{\mathrm{m}}\,, \tag{2.24}$$

where $\kappa^2 = 8\pi G$, $\mathcal{L}_{\mathrm{m}}$ is the matter Lagrangian and $e = \det e^A{}_\mu = \sqrt{-g}$. Unlike in GR where the Lagrangian is varied with respect to the metric, in TEGR it is varied with respect to the tetrad. This is done in order to produce the tetrad field equations from this action. As discussed above one may also carry out the variation with respect to the spin connection but since it only results in the antisymetric part of the tetrad field equations this is redundant.

The most common representation of the field equation in TEGR is given by [37, 31, 32]

$$\begin{aligned}\mathring{G}_{\mu\nu} \equiv \hat{G}_{\mu\nu} \quad &:= e^{-1}e^A{}_\mu g_{\nu\rho}\partial_\sigma(e\hat{S}_a{}^{\rho\sigma}) - \hat{S}_B{}^\sigma{}_\nu \hat{T}^B{}_{\sigma\mu} \\ &+\tfrac{1}{4}\hat{T}g_{\mu\nu} - e^A{}_\mu \omega^B{}_{A\sigma}\hat{S}_{B\nu}{}^\sigma = \kappa^2\Theta_{\mu\nu}\,,\end{aligned} \tag{2.25}$$

where $\Theta_{\alpha\sigma} = 8\pi\mathcal{T}_{\mu\nu}$, such that $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor. Of course at this point the spin connection still has not been set to vanish and so appears in the field equations.

Setting the spin to zero, which is also know as taking the Weitzenböck gauge, and taking into consideration the anti-symmetric properties of the torsion tensor, the contortion tensor and the Superpotential, the field equations above can be expanded to a more convenient form. The new form for the field equations given below is usefull when considering the equivalence between GR and TEGR

$$\frac{1}{2}g_{\alpha\sigma}\hat{T} - \hat{S}^{\lambda\beta}{}_{\sigma}\hat{K}_{\beta\lambda\alpha} - \mathring{\nabla}^{\lambda}\hat{S}_{\alpha\lambda\sigma} = \Theta_{\alpha\sigma}. \tag{2.26}$$

It is now noted that contracting the first and last indices of the Superpotential gives

$$\hat{S}^{\sigma}{}_{\mu\sigma} = 2\hat{K}^{\sigma}{}_{\mu\sigma} = 2\hat{T}^{\sigma}{}_{\mu\sigma}. \tag{2.27}$$

Using this observation and by re-writing the Riemann tensor in terms of the contortion tensor one can re-formulate the field equations as

$$\mathring{R}_{\sigma\alpha} = \frac{1}{2}g_{\alpha\sigma}\left(\hat{T} + \mathring{R}\right) - \hat{S}^{\lambda\beta}{}_{\sigma}\hat{K}_{\beta\lambda\alpha} - \mathring{\nabla}^{\lambda}\hat{S}_{\alpha\lambda\sigma}. \tag{2.28}$$

This explicitly shows the equivalence between the GR and TEGR field equations [37].

Through the knowledge that these two Lagrangians produce equivalent equations [31] one can easily determine that they are related through a boundary term [35]

$$\mathring{R}(e) = -\hat{T} + B, \tag{2.29}$$

where $B = \frac{2}{e}\partial_{\mu}\left(e\hat{T}^{\lambda}{}_{\lambda\mu}\right) = 2\mathring{\nabla}_{\mu}\hat{T}^{\lambda}{}_{\lambda\mu}$ is the boundary term. This implies that if one takes $-\hat{T} + B$ as the Lagrangian, then the variation will produce the exact field equations produced through the Einstein Hilbert action that uses the Ricci scalar.

It turns out that the form of the field equations givien in Eq.(2.26) is very convenient when it comes to deriving the 3+1 for TEGR and so will be the one considered in the following chapters.

As was hinted at before, TEGR can be extended to other theories of gravity by taking functions of the torsion tensor as the Lagrangian. Namely this is known $f(\hat{T})$ gravity and is an analogue to the $f(\mathring{R})$ class of theories [38, 39] that extend the Einstein Hilbert action. When considering $f(\hat{T})$ gravity, the equivalence relation that was discussed with respect to GR and TEGR is lost. That is, a particular $f(\hat{T})$ function does not necessarily produce equivalent field equations to those produced by the same $f(\mathring{R})$ function. This results from the fact that not all functions are distributive over addition [16], so $f(\mathring{R}) = f(-\hat{T} + B)$ is not necessarily equal to $f(-\hat{T}) + f(B)$. The reason why $f(\hat{T})$ theories are of particular interest is that out of the three possible extended Lagrangians, namely those of $\mathring{R}$, $\hat{T}$ and B, $f(\hat{T})$ is the only one that produces second-order field equations [16, 35, 36].

In this case, the action for an arbitrary function of the torsion scalar, $f(T)$, is given by

$$S = \frac{1}{4\kappa}\int d^4x\, e\, f(\hat{T}), \tag{2.30}$$

where $\kappa = \frac{1}{4\pi G}$ and $e = \det e^A{}_\mu = \sqrt{-g}$. Here $-g$ is the determinant of the global metric tensor. When varied with respect to the tetrad the following field equations for $f(\hat{T})$ gravity result[31]

$$\begin{aligned} E_A{}^\mu \equiv e^{-1} f_{\hat{T}} \partial_\nu \left(e\hat{S}_A{}^{\mu\nu} \right) + f_{\hat{T}\hat{T}} S_A{}^{\mu\nu} \partial_\nu \hat{T} \\ - f_{\hat{T}} \hat{T}^B{}_{\nu A} \hat{S}_B{}^{\nu\mu} + \frac{1}{2} f(\hat{T}) e_A{}^\mu = \tilde{\kappa} \Theta_A{}^\mu, \end{aligned} \tag{2.31}$$

where $\Theta_A{}^\mu \equiv 8\pi\hat{\mathcal{T}}_A{}^\nu$ and $f_{\hat{T}}$ and $f_{\hat{T}\hat{T}}$ denote the first and second derivatives of $f(\hat{T})$ with respect to $\hat{T}$.

Taking $f(\hat{T}) = -\hat{T}$ reduces the field equations, Eq.(2.31), to the TEGR ones discussed above. Similarly to $f(\mathring{R})$, a lot of investigations have been carried out considering various functions of the torsion scalar. While the $f(\hat{T})$ field equations are generally difficult to solve analytically for the fundamental variable being considered, due to being second order they are easier to solve numerically when compared to their forth-order counterparts.

While $f(T)$ gravity is not considered explicitly in this work, it is the scope of this project to build a basis for deriving various 3+1 formalisms in extended theories of gravity that in the long run might lead to significant numerical results that will shape our understanding of the fabric of space time and gravity as a whole.

2.3 Symmetric teleparallel equivalent of general relativity

Having thoroughly described gravitation through curvature and torsional geometries, all that is left is to consider the third and final geometric deformation entity, non-metricity. In this case gravity is purely expressed through the non-metricity tensor while all torsional and curvature terms vanish. While in all other theories the covariant derivative of the metric is taken to be zero, this is not the case in STEGR. In this theory the length of a vector that is being parallel transported along the manifold tied to the metric is non-uniform as seen in Fig.(2.4) [26, 15]. This non-metricity tensor is thus defined through the covariant derivative of the inverse metric

$$\overset{\diamond}{Q}_{\lambda\mu\nu} := \overset{\diamond}{\nabla}_{\lambda} g_{\mu\nu}\,. \tag{2.32}$$

This tensor can be related to the covariant derivative of the metric itself through

$$\overset{\diamond}{Q}_{\lambda}{}^{\mu\nu} = -\overset{\diamond}{\nabla}_{\lambda} g^{\mu\nu}\,, \tag{2.33}$$

and hence $\overset{\diamond}{Q}{}^{\alpha\mu\nu} = -g^{\alpha\beta}\overset{\diamond}{\nabla}_{\beta} g^{\mu\nu}$. Through this we identify the metric as the fundamental variable in this theory. This provides us with a theory that is less of a conceptual leap form GR than TEGR was.

Having defined all three geometric entities that can individually be used to express the extent of space time deformation, that is, gravity, it is important to study what links them. The trinity of gravity can be characterized through the definition of a

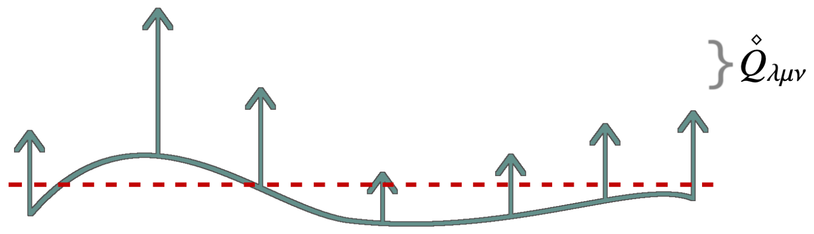

Figure 2.4: This diagram represents the deformation of space time as produced through theories built around non-metricity.

general linear affine connection that is given by [40, 41]

$$\Gamma^{\alpha}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu} + \hat{K}^{\alpha}{}_{\mu\nu} + \overset{\diamond}{L}{}^{\alpha}{}_{\mu\nu}\,, \tag{2.34}$$

where $\overset{\diamond}{L}{}^{\alpha}{}_{\mu\nu}$ represents the disformation tensor which embodies the contribution of the non-metricity tensor [42, 43] in the connection. This tensor is defined as

$$\overset{\diamond}{L}{}^{\alpha}{}_{\mu\nu} := \frac{1}{2}g^{\lambda\alpha}\left(Q_{\lambda\mu\nu} - Q_{\mu\lambda\nu} - Q_{\nu\lambda\mu}\right)\,. \tag{2.35}$$

The disformation tensor shares a number of characteristics with GR's Levi-Civita connection including the symmetry in the final two indices. The link between these two will be made clearer later on in this section.

While, in general, STG has all the three gravitating tensors, curvature, torsion and non-metricity, the class of STG being considered in this work is one with vanishing curvature

$$\overset{\diamond}{R}{}^{\rho}{}_{\sigma\mu\nu} \equiv 0\,, \tag{2.36}$$

and vanishing torsion

$$\mathring{T}^{\rho}{}_{\mu\nu} \equiv 0\,. \tag{2.37}$$

In order to satisfy this condition a specific class of connections are considered, of which the most general symmetric teleparallel connection is

$$\mathring{\Gamma}^{\alpha}{}_{\mu\nu} := \frac{\partial x^{\alpha}}{\partial \xi^{\sigma}} \frac{\partial^2 \xi^{\sigma}}{\partial x^{\mu} \partial x^{\nu}}\,, \tag{2.38}$$

where $\xi^{\sigma} = \xi^{\sigma}(x)$ is an arbitrary function of spacetime position. One specific case within this class is derived from vanishing connection components through the coordinate transformation

$$x^{\mu} \to \xi^{\mu}(x^{\nu})\,. \tag{2.39}$$

In this particular case, the connection (2.38) can be shown to be purely a gauge connection. This implies that it is always possible to find a coordinate transformation where the connection components vanish. This specific case is called the coincident gauge [43] and will be the gauge considered during this work. It is chosen as it is useful in simplifying all equations involved in the $3+1$ derivation carried out in the following chapters.

Having established the necessary building blocks of this geometric method it is possible for theories to be constructed based on the disformation tensor alone such that the gravitational effect is communicated through the non-metricity tensor rather than torsion or curvature. One specific theory that will be considered in this work is STGR, the Symmetric Teleparallel Equivalent of General Relativity. As the name suggests, this is equivalent to GR at the level of equations analogously to how TEGR was also equivalent. In this theory the Lagrangian is taken to be

$$\mathring{\mathcal{L}}_{\text{STEGR}} = \frac{\sqrt{-g}}{16\pi G} \mathring{Q}\,, \tag{2.40}$$

where the non-metricity scalar $\overset{\diamond}{Q}$ is defined as

$$\overset{\diamond}{Q} = g^{\mu\nu}\left(\overset{\diamond}{L}^{\alpha}{}_{\beta\mu}\overset{\diamond}{L}^{\beta}{}_{\nu\alpha} - \overset{\diamond}{L}^{\alpha}{}_{\beta\alpha}\overset{\diamond}{L}^{\beta}{}_{\mu\nu}\right). \tag{2.41}$$

In order to explicitly see the equivalence between STGR and GR the Einstein-Hilbert Lagrangian can be rewritten in terms of the Levi-Civita connection as [44]

$$\overset{\circ}{\mathcal{L}}_{EH} = \frac{\sqrt{-g}}{16\pi G}\overset{\circ}{R} = \overset{\circ}{\mathcal{L}}_{E} + \overset{\circ}{B}, \tag{2.42}$$

where $\mathcal{L}_E$ is constructed by the Levi-Civita connection and represents the Einstein Lagrangian contribution [45] and is defined as

$$\overset{\circ}{\mathcal{L}}_{E} := \frac{\sqrt{-g}}{16\pi G}\, g^{\mu\nu}\left(\overset{\circ}{\Gamma}^{\alpha}{}_{\beta\mu}\overset{\circ}{\Gamma}^{\beta}{}_{\nu\alpha} - \overset{\circ}{\Gamma}^{\alpha}{}_{\beta\alpha}\overset{\circ}{\Gamma}^{\beta}{}_{\mu\nu}\right). \tag{2.43}$$

Here the boundary term is defined by

$$\overset{\circ}{B} = \frac{\sqrt{-g}}{16\pi G}\left(g^{\alpha\mu}\overset{\circ}{\nabla}_{\alpha}\overset{\circ}{\Gamma}^{\nu}{}_{\mu\nu} - g^{\mu\nu}\overset{\circ}{\nabla}_{\alpha}\overset{\circ}{\Gamma}^{\alpha}{}_{\mu\nu}\right), \tag{2.44}$$

which is a total divergence term and therefore vanishes when deriving the field equations using this Lagrangian. The significance of the Einstein-Hilbert Lagrangian is that it completes the Einstein Lagrangian by including the boundary term $\overset{\circ}{B}$ which renders the theory covariant [40].

Assuming the coincident gauge such that the connection vanishes ($\overset{\diamond}{\Gamma}^{\alpha}{}_{\mu\nu} \equiv 0$) has a number of consequences on the theory. As was previously mentiond all curvatrue and torsion terms identically vanish, however, another significant consequence is that the covariant derivative becomes identical to the ordinary partial derivative

$$\overset{\diamond}{\nabla}_{\mu} \rightarrow \partial_{\mu}. \tag{2.45}$$

From this it follows that the non-metricity tensor is simply the partial of the metric leading to the disformation tensor being simply the negative of the Christoffel symbol

$$\mathring{L}^{\alpha}{}_{\mu\nu} = -\mathring{\Gamma}^{\alpha}{}_{\mu\nu} \,. \tag{2.46}$$

Through this we have proven that the field equations of the STEGR Lagrangian must produce equivalent field equations as GR even though it uses the non-metricity tensor as the geometric entity to describe the deformation of space time [46]. Hence, GR and STEGR turn out to be dynamically equivalent, as was the case with TEGR and GR.

The Einstein Lagrangian Eq.(2.43) needs the boundary term defined in Eq.(2.44) in order to preserve its invariance under diffeomorphism. STGR on the other hand stays diffeomorphically invariant even when undergoing an arbitrary coordinate transformation of the Lagrangian Eq. (2.40), without the need for a boundary term. This is a general property of the theory and is not dependant on the coincident gauge being chosen [47, 48]. The STGR action can thus be though of as another method to covariantize the Einstein action, similar to how the Einstein-Hilbert action makes it covariant. The difference is that the Einstein-Hilbert action accomplishes this by adding a boundary term and STEGR achieves the same thing by making the shift to a symmetric teleparallel action based on the non-metricity tensor.

Having reviewed the Lagrangian for this theory extensively the STGR action is thus given by [49, 50]

$$S_G = \int d^4x \, [\frac{\sqrt{-g}}{2\kappa^2} \, \mathring{Q} + \sqrt{-g}\mathcal{L}_m] \,, \tag{2.47}$$

which naturally leads to the conjugate to the STEGR Lagrangian

$$\begin{aligned}\mathring{P}^{\alpha}{}_{\mu\nu} &:= \frac{1}{2\sqrt{-g}}\frac{\partial\left(\sqrt{-g}\mathring{Q}\right)}{\partial\mathring{Q}_{\alpha}{}^{\mu\nu}}\\ &= \frac{1}{4}\mathring{Q}^{\alpha}{}_{\mu\nu} - \frac{1}{4}\mathring{Q}_{(\mu}{}^{\alpha}{}_{\nu)} - \frac{1}{4}g_{\mu\nu}\mathring{Q}^{\alpha\beta}{}_{\beta} + \frac{1}{4}\left[\mathring{Q}_{\beta}{}^{\beta\alpha}g_{\mu\nu} + \frac{1}{2}\delta^{\alpha}{}_{(\mu}\mathring{Q}_{\nu)}{}^{\beta}{}_{\beta}\right].\end{aligned} \tag{2.48}$$

This tensor provides us with an alternative way to describe the non-metricity scalar as $\mathring{Q} = -\mathring{Q}_{\alpha\mu\nu}\mathring{P}^{\alpha\mu\nu}$ [51]. Taking the variation of this action with respect to the fundamental variable, in this case the metric tensor, the field equations for this theory are finally derived and are given by [49, 52]

$$2\mathring{\nabla}_{\alpha}\left(\sqrt{-g}\mathring{P}^{\alpha}{}_{\mu\nu}\right) - q_{\mu\nu} - \frac{\sqrt{-g}\,\mathring{Q}}{2}g_{\mu\nu} = \kappa^2\sqrt{-g}\Theta_{\mu\nu}\,, \tag{2.49}$$

where

$$\begin{aligned}\frac{1}{\sqrt{-g}}q_{\mu\nu} =& \frac{1}{4}\left(2\mathring{Q}_{\alpha\beta\mu}\mathring{Q}^{\alpha\beta}{}_{\nu} - \mathring{Q}_{\mu\alpha\beta}\mathring{Q}_{\nu}{}^{\alpha\beta}\right) - \frac{1}{2}\mathring{Q}_{\alpha\beta\mu}\mathring{Q}^{\beta\alpha}{}_{\nu}\\ &- \frac{1}{4}\left(2\mathring{Q}_{\alpha}{}^{\beta}{}_{\beta}\mathring{Q}^{\alpha}{}_{\mu\nu} - \mathring{Q}_{\mu}{}^{\beta}{}_{\beta}\mathring{Q}_{\nu}{}^{\beta}{}_{\beta}\right) + \frac{1}{2}\mathring{Q}_{\beta}{}^{\beta}{}_{\alpha}\mathring{Q}^{\alpha}{}_{\mu\nu}\,.\end{aligned} \tag{2.50}$$

In general STG the connection is independent of the metric and thus the Lagrangian also needs to be varied with respect to the connection itself producing independent connection field equations. These equations are related to the hypermomentum tensor in much the same way as the metric field equations are related to the energy momentum tensor. Further to this similarity, the hypermomentum tensor is a product of varying the matter Lagrangian with respect to the connection. That being said, choosing the coincident gauge trivially satisfies the connection field equation and the assumption of a vanishing hypermomentum [50] leading to

$$\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\left(\sqrt{-g}\mathring{P}^{\mu\nu}{}_{\alpha}\right) = 0\,. \tag{2.51}$$

Together, the metric field Eqs.(2.49) and connection field Eqs.(2.51) represent the total DoFs of the dynamics in STEGR. Through these field equations the final evolution equations and constraint equations of the 3+1 formalism in STGR will be derived in the following chapters.

3 A 3+1 Formulation for a General Linear Affine Connection Without Metricity

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In this chapter a rigorous analysis of the 3+1 formalism is carried out. Two new formalisms are constructed, one tetrad based and one metric based, each of which is built around a general affine connection without assuming metricity. All equations considered in this chapter that are based on this general connection will have curvature, torsion, spin and non-metricity terms present. Due to the fact that no specific connection is chosen no geometric entity vanishes, this is left for Chapter 4 where specific connections like the Levi-Civita connection and the Teleparallel connection are considered.

Through the further development the 3+1 formalism of the two fundamental variables, it is shown that that the metric and tetrad variants are consistent and one can successfully derive the metric formulation through the tetrad one. Before starting with the derivations some further clarification of notation is required. Primarily terms such as 3-vector and 3-tensor, and equivalently spatial vector and spatial tensor, refer to 4-dimensional entities mapped onto a 3-dimensional plane or foliation. They should not be confused with the spatial part of a tensor where the temporal columns and rows are omitted. From this point forward 3-tensors that have the same symbol as their space-time counterparts will be denoted with a (3) superscript or subscript, $e^{A(3)}_{\nu}$. Finally, purely local tensors will be denoted with a ‘ ˜ ’ annotation, for example $\tilde{n}^A$.

3.1 Basic Definitions

In order to build the foundation for the formalisms that are going to be developed later on in this chapter, a number of definitions are presented in this section. The main interactions between the fundamental elements of the formalisms are also derived.

To start a 4-dimensional global space time manifold M is defined. A metric $g_{\mu\nu}$ associated with this manifold, built from a tetrad $e^a{}_\nu$ is also defined such that

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu. \tag{3.1}$$

It is assumed that the manifold from our manifold-metric pair, (M,$g_{\mu\nu}$), can be foliated into non-intersecting spacelike 3-surfaces denoted by Σ as can be seen in Fig.(3.1). Each of these hypersurfaces are thus a level spatial surface at an instant of some scalar function which we call “t” out of convenience. This scalar function is taken to be the global time function and thus the foliations become vector spaces containing all spatial vectors and co-vectors at an instant in time.

Having defined the spatial entities on which our formulation will be defined, it is equally as important to develop their temporal counterpart. Given a singular temporal dimension, the one form Ω_ν is defined as the covariant derivative of the time function defined above, $\nabla_\nu t$. Out of convenience the vector is then normalised

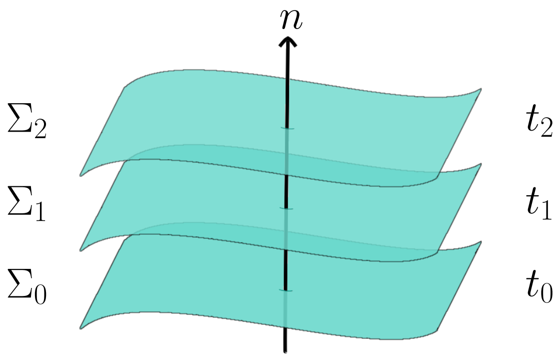

Figure 3.1: This diagram represents the foliation of the manifold into non-intersecting spacelike 3-surfaces. The arrow is pointing towards the direction of increasing time.

by finding its norm through

$$\begin{aligned} |\Omega|^2 &= g^{\mu\nu}\nabla_\mu t \nabla_\nu t \\ &= g^{00} \\ &= -\frac{1}{\alpha^2}, \end{aligned} \tag{3.2}$$

where α is the lapse function and ∇_μ is the general covariant derivative. The lapse function can be interpreted as the temporal separation between one foliation and the next. It is also the main term in the temporal coefficient of the metric tensor, g^{00}. Due to the sign of the one form's magnitude, the lapse function is taken to be positive in order to preserve the negative of $\mathbf{\Omega}_\nu$ resulting in it being timelike and all hypersurfaces, $\mathbf{\Sigma}$, everywhere spacelike.

The general covariant derivative used to define the above one form is associated with the general affine connection, $\Gamma^{\nu}_{\rho\lambda}$, defined in chapter 2. It is recalled that this derivative also adheres to non-metricity [52, 49]

$$\nabla_{\lambda} g_{\mu\nu} = Q_{\lambda\mu\nu} \,, \tag{3.3}$$

$$\nabla_{\lambda} g^{\mu\nu} = -\, Q_{\lambda}{}^{\mu\nu} \,, \tag{3.4}$$

where $Q_{\lambda\mu\nu}$ is the non-metricity tensor.

When considering a tensor with mixed indices, say $A^{a\nu}{}_{b\mu}$, the following covariant derivative convention will be adhered to throughout this work

$$\begin{aligned} \nabla_{\lambda} A^{A\nu}{}_{B\mu} = A^{C\rho}{}_{B\mu} \omega^{A}{}_{C\lambda} + A^{A\rho}{}_{B\mu} \Gamma^{\nu}_{\rho\lambda} \\ - A^{A\rho}{}_{C\mu} \omega^{C}{}_{B\lambda} - A^{A\nu}{}_{B\rho} \Gamma^{\rho}_{\mu\lambda} \,. \end{aligned} \tag{3.5}$$

Here $\omega^{A}{}_{C\lambda}$ is the spin connection tied to the local frame.

Dividing the one form $\mathbf{\Omega}$ by its norm, the unit normal vector to the foliations as well as its inverse/co-vector are defined as

$$n_{\nu} := -\alpha \Omega_{\nu}, \tag{3.6}$$

$$\begin{aligned} n^{\nu} &:= -g^{\mu\nu}(\alpha \Omega_{\mu}) \\ &= -g^{\mu\nu}(\alpha \nabla_{\mu}(t)) \,, \end{aligned} \tag{3.7}$$

such that

$$n^{\nu} n_{\nu} = -1 \,. \tag{3.8}$$

The normal vector and co-vector are defined as can be seen in Eqs. (3.6, 3.7) so that their contraction produces a negative sign. This sign aligns the normal towards the direction of increasing time [22].

The normal vector and co-vector can be expressed at the level of components in terms of the lapse function α and the shift vector β as in standard gravity [8, 22],

$$n_\mu = (-\alpha, 0, 0, 0) \ , \tag{3.9}$$

$$n^\mu = \left(\frac{1}{\alpha}, -\frac{1}{\alpha}\beta^i\right) . \tag{3.10}$$

In order to fully split the manifold into its purely spatial and temporal parts, the normal vector can be used to define the spatial metric, or 3-metric, on the global foliation Σ as

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu \ , \tag{3.11}$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \ . \tag{3.12}$$

A consequence of this metric – inverse metric pair is the spatial mapping tensor that maps tensors on the manifold $(M, g_{\mu\nu})$ onto the spatial plane. This tensor can be derived through contracting the spatial metric with the inverse metric $g_{\mu\nu}\gamma^{\lambda\mu}$ resulting in the following relation

$$\gamma^\mu{}_\nu = \delta^\mu_\nu + n^\mu n_\nu \ . \tag{3.13}$$

Analysing the terms of this tensor one may be tempted to think of it as the spatial delta or 3-delta. This notion is correct up to a point. This tensor does exhibit the usual properties of a delta tensor when contracted with spatial tensors and vectors. That being said, this stems from its decomposition terms rather than it necessarily being an identity matrix. Expanding such a contraction, $\gamma^\mu{}_\nu V^\nu_{(3)} = \delta^\mu_\nu V^\nu_{(3)} + n^\mu n_\nu V^\nu_{(3)}$, one observes that delta-like properties stem from the orthogonality of spatial tensors and vectors with the normal vector. As is demonstrated below, once Eq.(3.13) is observe at the level of component it is shown that it has off diagonal temporal components, similar to the inverse spatial metric.

$$\gamma^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}, \tag{3.14}$$

$$\gamma_{\mu\nu} = \begin{pmatrix} \beta^k\beta_k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}, \tag{3.15}$$

$$\gamma^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \beta^i \\ 0 & \gamma^i{}_j \end{pmatrix}. \tag{3.16}$$

Using the the above matrices and Eqs. (3.11 – 3.13) it is possible to derive the standard space-time metric and inverse metric components

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{1}{\alpha^2}\beta^i \\ \frac{1}{\alpha^2}\beta^i & \gamma^{ij} - \frac{1}{\alpha^2}\beta^i\beta^j \end{pmatrix}, \tag{3.17}$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^k\beta_k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}. \tag{3.18}$$

The following relations are some useful consequences of the previous definitions that will be utilized throughout the following derivations

$$\gamma^{\mu\lambda}\gamma_{\lambda\nu} = \gamma^{\mu}{}_{\nu}\,, \tag{3.19}$$

$$\gamma^{\mu}{}_{\nu}V^{(3)}_{\mu} = V^{(3)}_{\nu}\,, \tag{3.20}$$

$$n^{\mu}V^{(3)}_{\mu} = 0\,, \tag{3.21}$$

$$n^{\mu}\gamma_{\mu\nu} = 0\,. \tag{3.22}$$

Now that the global manifold, all its components and fundamental variables have been defined, it is important to consider an inertial/local manifold which will be denoted by $\tilde{\boldsymbol{M}}$. Along with its associated metric, which we choose to be the Minkowski metric η_{ab}, the non-intersecting spacelike local 3-surfaces $\tilde{\Sigma}$ are defined. Similar to before, the normal vector to the local foliations $\tilde{n}^a$ and the the local spatial metric $\tilde{\gamma}_{ab}$ can also be defined.

Since non-metricity is being assumed, the covariant derivative of the Minkowski metric is non-zero in this case. This results in the following relationship

$$\nabla_\lambda \eta_{AB} = \partial_\lambda \eta_{AB} - \eta_{CB}\omega^C{}_{A\lambda} - \eta_{AC}\omega^C{}_{B\lambda} \tag{3.23}$$

$$= -\eta_{CB}\omega^C{}_{A\lambda} - \eta_{AC}\omega^C{}_{B\lambda} \,.$$

In theories with metricity, the above relationship would result in the antisymmetry of the spin connection in its first two indices.

Through the choice of the Minkowski metric as the local/inertial metric, it is possible to show that the local normal and its inverse are the constant vectors

$$\tilde{n}_A = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.24}$$

$$\tilde{n}^A = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

This comes about through the fact that for an inertial frame the lapse function is one and the shift vector is a zero vector. That being said, if a different local coordinate system were to be chosen, such as a spherical coordinates, then the normal vector $\tilde{n}^A$ would again be dependant on the local spatial shift vector $\tilde{\beta}^{\tilde{i}}$, just like the standard global one [8].

Having defined the necessary global and local frames as well as their 3+1 spatial and temporal splits, it is possible to reintroduce the tetrad and attempt to define an equivalent form to Eqs. (3.17, 3.18) for it.

Using Eq. (3.1) as a starting point we get that

$$g_{\mu\nu} = \begin{pmatrix} e^{\tilde{0}}{}_0 e^{\tilde{0}}{}_0 \eta_{\tilde{0}\tilde{0}} + e^{\tilde{i}}{}_0 e^{\tilde{j}}{}_0 \eta_{\tilde{i}\tilde{j}} & e^{\tilde{0}}{}_i e^{\tilde{0}}{}_0 \eta_{\tilde{0}\tilde{0}} + e^{\tilde{i}}{}_i e^{\tilde{j}}{}_0 \eta_{\tilde{i}\tilde{j}} \\ e^{\tilde{0}}{}_0 e^{\tilde{0}}{}_j \eta_{\tilde{0}\tilde{0}} + e^{\tilde{i}}{}_0 e^{\tilde{j}}{}_j \eta_{\tilde{i}\tilde{j}} & e^{\tilde{0}}{}_i e^{\tilde{0}}{}_j \eta_{\tilde{0}\tilde{0}} + e^{\tilde{i}}{}_i e^{\tilde{j}}{}_j \eta_{\tilde{i}\tilde{j}} \end{pmatrix} . \tag{3.26}$$

Comparing this to Eq. (3.18), we get that

$$e^{\tilde{0}}{}_i e^{\tilde{0}}{}_j \eta_{\tilde{0}\tilde{0}} + e^{\tilde{i}}{}_i e^{\tilde{j}}{}_j \eta_{\tilde{i}\tilde{j}} = \gamma_{ij} \, . \tag{3.27}$$

If one were working in a 3-dimensional setting Eq. (3.1) would reduce to $e^{\tilde{i}}{}_i e^{\tilde{j}}{}_j \eta_{\tilde{i}\tilde{j}} = \gamma_{ij}$. This is still the case if we simply restrict the current 4-dimensional tensors to their spatial indices. Noting this and that $\eta_{\tilde{0}\tilde{0}} = -1$, it can be concluded that $e^{\tilde{0}}{}_i = 0$.

Using these results the shift vector is found to be $\beta_j = e^{\tilde{i}}{}_i e^{\tilde{j}}{}_0 \eta_{\tilde{i}\tilde{j}}$, from which it is concluded that $e^{\tilde{i}}{}_0 = e^{\tilde{i}}{}_m \gamma^{mn} \beta_n$. Substituting this in the second term of the metric temporal component g_{00} as given in Eq. 3.26, $e^{\tilde{i}}{}_0 e^{\tilde{j}}{}_0 \eta_{\tilde{i}\tilde{j}} = \beta^k \beta_k$ is obtained which implies that $e^{\tilde{0}}{}_0 = \pm\alpha$. Noticing that $e_A{}^\nu = \eta_{AB} g^{\mu\nu} e^B{}_\mu$, while using the above result, the following form of the space time tetrad is achieved

$$e^A{}_\nu = \begin{pmatrix} \pm\alpha & e^{\tilde{i}}{}_k \beta^k \\ 0 & e^{\tilde{i}}{}_j \end{pmatrix}, \tag{3.28}$$

$$e_A{}^\nu = \begin{pmatrix} \pm\frac{1}{\alpha} & 0 \\ \mp\frac{1}{\alpha}\beta^j & e_{\tilde{i}}{}^j \end{pmatrix} . \tag{3.29}$$

Before continuing with this tetrad decomposition it is important to consider the two defined normal vectors, the local one and the global one. It should be noted that these are not necessarily simply the same vector mapped onto one another by a tetrad

$$\tilde{n}_A = e_A{}^\nu n_\nu \, . \tag{3.30}$$

Expanding both sides separately one gets

$$\begin{aligned}\tilde{n}_A &= -\tilde{\alpha}\tilde{\nabla}_A(\tilde{t}) \\ &= -\tilde{\alpha}\tilde{\partial}_A\tilde{t} \\ &= (-1, 0, 0, 0),\end{aligned} \tag{3.31}$$

$$\begin{aligned}e_A{}^{\nu} n_{\nu} &= -e_A{}^{\nu}\alpha\nabla_{\nu}(t) \\ &= -\alpha e_A{}^{0}\partial_0 t \\ &= -\alpha e_A{}^{0} \\ &= (\mp 1, -\alpha e_{\tilde{i}}{}^{0}).\end{aligned} \tag{3.32}$$

This shows that Eq.(3.30) is only the case if $e_{\tilde{i}}^{0} = 0$ and the unknown signs of Eq.(3.28) are such that $e_0{}^0$ is negative. Fortunately, in our case, $e_{\tilde{i}}^{0}$ is in fact 0 and we are free to choose the sign of $e_0{}^0$ due to our choice of the global metric, Eq. (3.17) and local metric, the Minkowski metric. Through this relation and its variants, it is thus possible to refine the tetrad components found in Eq. (3.28) by choosing a consistent sign

$$e^A{}_{\nu} = \begin{pmatrix} \alpha & e^{\tilde{i}}{}_k\beta^k \\ 0 & e^{\tilde{i}}{}_j \end{pmatrix}, \tag{3.33}$$

$$e_A{}^{\nu} = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ -\frac{1}{\alpha}\beta^j & e_{\tilde{i}}{}^j \end{pmatrix}. \tag{3.34}$$

Here it is worth noting that these results are in agreement with the results obtained in Refs. [53, 54], at least up to the global index of the tetrad.

Having derived a full equivalent to Eq. (3.17) for the tetrad, the next step is to find an equivalent to the spatial metric $\gamma_{\mu\nu}$, say $\theta^{A}{}_{\mu}$, such that

$$\theta^{A}{}_{\mu} = e^{A}{}_{\mu} + U^{A}{}_{\mu}\,, \tag{3.35}$$

where $U^{A}{}_{\mu}$ is some tensor that will embody the temporal part of the tetrad.

In order to determine $\theta^{A}{}_{\mu}$ and $U^{A}{}_{\mu}$ at the component level it is required to consider what properties are necessary for these tensors to have. The only property that is required from $\theta^{A}{}_{\mu}$ is that it should be orthogonal to the normal vectors, that is, that it inhabits the spatial foliations. Due to having two unknowns in Eq. (3.35) one is free to choose any definition for $U^{A}{}_{\mu}$ so long as it conserves the orthogonality of $\theta^{A}{}_{\mu}$. For convenience this tensor is thus tentatively defined as

$$U^{A}{}_{\mu} = \tilde{n}^{A} n_{\mu}\,. \tag{3.36}$$

Below, it is shown that this definition indeed preserves the required orthogonality of $\theta^{A}{}_{\mu}$ with both the global and the local normal vectors assuming that the metrics chosen support Eq. (3.30).

$$\begin{aligned} n^{\mu}\theta^{A}{}_{\mu} &= n^{\mu}e^{A}{}_{\mu} + n^{\mu}n_{\mu}\tilde{n}^{A} \\ &= \tilde{n}^{A} - \tilde{n}^{A} \\ &= 0\,, \end{aligned} \tag{3.37}$$

$$\begin{aligned} n_{A}\theta^{A}{}_{\mu} &= \tilde{n}_{A}e^{A}{}_{\mu} + n^{\mu}\tilde{n}_{A}\tilde{n}^{A} \\ &= n^{\mu} - n^{\mu} \\ &= 0\,. \end{aligned} \tag{3.38}$$

Similarly, the inverse tetrad can be successfully split which produces the final two fundamental variable decompositions

$$\theta^A{}_\mu = e^A{}_\mu + \tilde{n}^A n_\mu \,, \tag{3.39}$$

$$\theta_A{}^\mu = e_A{}^\mu + \tilde{n}_A n^\mu \,. \tag{3.40}$$

Through the above equations the components of the θ tensors can finally be determined as

$$\theta^A{}_\nu = \begin{pmatrix} 0 & e^{\tilde{i}}{}_k \beta^k \\ 0 & e^{\tilde{i}}{}_j \end{pmatrix} , \tag{3.41}$$

$$\theta_A{}^\nu = \begin{pmatrix} 0 & 0 \\ 0 & e_{\tilde{i}}{}^j \end{pmatrix} . \tag{3.42}$$

These forms of $\theta^A{}_\mu$ and $U^A{}_\mu$ produces some relations that will prove convenient while developing the rest of the $3 + 1$ formalisms. Some of these relations are

$$\theta^A{}_\mu \theta_A{}^\nu = \gamma^\nu{}_\mu \,, \tag{3.43}$$

$$\theta^A{}_\mu \theta_B{}^\mu = \tilde{\gamma}^A{}_B \,, \tag{3.44}$$

$$\tilde{\gamma}_{AB} \theta^A{}_\mu \theta^B{}_\nu = \gamma_{\mu\nu} \,, \tag{3.45}$$

$$\tilde{\gamma}^{AB} \theta_A{}^\mu \theta_B{}^\nu = \gamma^{\mu\nu} \,. \tag{3.46}$$

By using these relations, a number of equations can be derived including the verification of the relationship between the 3-metrics and the 4-metrics at the level of tetrads.

$$\begin{aligned}\gamma_{\nu\mu} &= \theta^{A}{}_{\nu}\theta^{B}{}_{\mu}\tilde{\gamma}_{AB} \\ &= (e^{A}{}_{\nu} + \tilde{n}^{A}n_{\nu})\theta^{B}{}_{\mu}\tilde{\gamma}_{AB} \\ &= e^{A}{}_{\nu}(e^{B}{}_{\mu} + \tilde{n}^{B}n_{\mu})\tilde{\gamma}_{AB} \\ &= e^{A}{}_{\nu}e^{B}{}_{\mu}(\eta_{AB} + \tilde{n}_{A}\tilde{n}_{B}) \\ &= g_{\nu\mu} + n_{\nu}n_{\mu}.\end{aligned} \tag{3.47}$$

Once again it is noted that similar to the $\gamma^{\mu}{}_{\nu}$, the $\theta^{A}{}_{\mu}$ tensor should not be confused with the purely spatial part of the space-time tetrad $e^{\tilde{i}}{}_{j}$. That being said, for ease of reference this tensor will be re-labeled as $e^{A(3)}_{\mu}$ since it embodies the role of the spatial tetrad.

Having defined all of the necessary fundamental variable decompositions, their properties and their components, the next step is to define the spatial covariant derivative of a general spatial tensor, say $A^{A}{}_{\mu}$.

$$D_{\nu}A^{A}{}_{\mu} = \gamma^{\lambda}{}_{\mu}\tilde{\gamma}^{A}{}_{B}\gamma^{\rho}{}_{\nu}\nabla_{\rho}A^{B}{}_{\lambda} \tag{3.48}$$

It can be shown that this derivative follows the Leibniz rule only when applied to spatial tensors and that it produces additional terms when applied to spacetime tensors or vectors [22]. Some noteworthy results when applying this derivative to the normal vectors being considered are

$$D_\nu n^\mu = \partial^{(3)}_\nu n^\mu + n^\lambda \Gamma^{\mu(3)}_{\lambda\nu} \tag{3.49}$$
$$= \partial^{(3)}_\nu n^\mu \,,$$

$$D_\nu \tilde{n}^A = \partial^{(3)}_\nu \tilde{n}^A + \tilde{n}^B \omega^{A(3)}_{B\nu} \tag{3.50}$$
$$= \partial^{(3)}_\nu \tilde{n}^A$$
$$= 0 \,.$$

It can also be shown that this spatial covariant derivative inherits the non-metricity property from its spacetime counterpart when applied to the spatial metric, so long as this is also the case for the general space-time covariant derivative applied to the space-time metric. Similarly it will inherit metricity if that is not the case.

Further to the definitions given in Eqs. (3.3,3.5), and as a consequence of the general space-time covaraint derivative, the following relations will also prove useful in the development of the $3+1$ formallisms proposed. These relations are

$$n^\lambda \nabla_\rho n_\lambda = \frac{1}{2} n^\lambda n^\nu Q_{\rho\lambda\nu} \,, \tag{3.51}$$

$$n_\lambda \nabla_\rho n^\lambda = -\frac{1}{2} n^\lambda n^\nu Q_{\rho\lambda\nu} \,. \tag{3.52}$$

Defining the acceleration vector a_λ as $n^\sigma \nabla_\sigma n_\lambda$, one then obtains

$$a^\epsilon = n^\sigma \nabla_\sigma n^\epsilon + n^\sigma n_\lambda Q_\sigma{}^{\lambda\epsilon} \,, \tag{3.53}$$

$$n^\lambda a_\lambda = \frac{1}{2} n^\sigma n^\lambda n^\epsilon Q_{\sigma\lambda\epsilon} \,. \tag{3.54}$$

It is now possible to start delving into the underlying geometric tensors that make

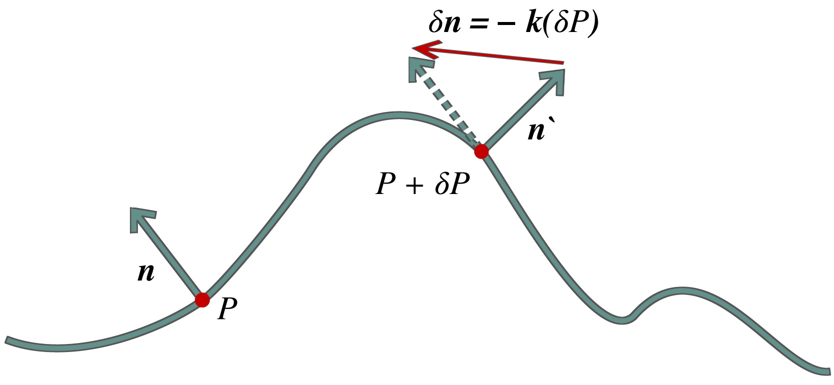

Figure 3.2: This diagram represents the change in pointing direction of the normal vector as it is moved from one point to another along the surface of a foliation.

up at least some of the $3 + 1$ formalisms that will be considered in this work. The extrinsic curvature is one such tensor and is defined as [22]

$$k_{\alpha\beta} := -\gamma^{\nu}{}_{\alpha}\gamma^{\mu}{}_{\beta}\nabla_{\nu}n_{\mu}\,, \tag{3.55}$$

here the negative sign is applied to enforce the convention of forward moving time. From this definition it is possible to conclude that this tensor is a purely spatial tensor and can be thought of as the measure of change of direction of the normal vector n as it moves along the surface of some foliation Σ [8]. This is interpreted graphically in Fig.(3.2) [9]. In GR it can also be interpreted as the rate of change of the spatial metric as it is dragged along the normal vector field in a way that is independent of coordinates. In essence this would be the lie derivative of the spatial metric along the normal vector [8, 27]. That being said, this secondary definition is only true in GR as in the general case as well as in other theories the extrinsic curvature is not the only term that results when taking this derivative. As such a more appropriate definition would be that it is the curvature contribution to the rate of change of the spatial metric as it is dragged along the normal vector field independently from the coordinates.

An alternative definition of the extrinsic curvature can also be easily derived through the expansion of the spatial deltas at the front of the definition. This gives

$$k_{\nu\mu} = -\nabla_\nu n_\mu - n_\nu a_\mu - \frac{1}{2} n^\rho n^\beta n_\mu \gamma^\alpha{}_\nu Q_{\alpha\rho\beta} \,. \tag{3.56}$$

It should be noted that the extrinsic curvature with raised indices is defined as

$$k^{\alpha\beta} = -\gamma^{\nu\alpha}\gamma^{\mu\beta}\nabla_\nu n_\mu \,,$$

and that, due to non-metricity the full mapping of the covariant derivative of the normal covector is given by

$$\gamma^{\nu\alpha}\gamma^\beta{}_\mu \nabla_\nu n^\mu = -n_\lambda \gamma^{\alpha\nu}\gamma^\beta{}_\mu Q_\nu{}^{\lambda\mu} - k^{\alpha\beta} \,. \tag{3.57}$$

Having amassed all this background for a generalized $3+1$ formalism, the terms which are used in order to characterize gravitation in various theories can now be considered. For convenience, the definitions of the Riemann and torsion tensors, written in terms of a general affine connection $\Gamma^\rho_{\lambda\mu}$ [28], as well as the non-metricity tensor are provided here once again

$$R^\rho{}_{\lambda\nu\mu} = \partial_\nu \Gamma^\rho_{\lambda\mu} - \partial_\mu \Gamma^\rho_{\lambda\nu} + \Gamma^\rho_{\alpha\nu}\Gamma^\alpha_{\lambda\mu} - \Gamma^\rho_{\alpha\mu}\Gamma^\alpha_{\lambda\nu} \,, \tag{3.58}$$

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} \,, \tag{3.59}$$

$$Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu} \,. \tag{3.60}$$

Here we note that, for a general affine connection, the Riemann tensor only possesses a single antisymmetry in its final two indices $R^\chi{}_{\lambda[\rho\pi]}$, the rest of the symmetries of this tensor as used in GR are a consequence of the Levi-Civita connection.

Given these definitions, it is possible to assign the antisymmetric part of the extrinsic curvature directly to that of the torsion tensor though

$$k_{[\alpha\beta]} = \gamma^{\nu}{}_{\alpha}\gamma^{\mu}{}_{\beta}n_{\sigma}T^{\sigma}{}_{\nu\mu}\,. \tag{3.61}$$

At this point the relationship between the gravitating tensors and their purely spatial counterparts is considered. Essentially the basis of what we will call the Gauss-like equations for these tensors is derived. The name Gauss-like is derived from the equation in standard gravity relating the space time Riemann to the purely spatial Riemann tensor. The commutator of the second order covariant derivative of a general vector, V^{λ}, and covector, V_{λ}, is once again considered and given here for ease of reference [27]

$$\nabla_{[\nu}\nabla_{\mu]}V^{\lambda} = T^{\sigma}{}_{\mu\nu}\nabla_{\sigma}V^{\lambda} + V^{\sigma}R^{\lambda}{}_{\sigma\nu\mu}\,, \tag{3.62}$$

$$\nabla_{[\nu}\nabla_{\mu]}V_{\lambda} = T^{\sigma}{}_{\mu\nu}\nabla_{\sigma}V_{\lambda} + V_{\sigma}R^{\sigma}{}_{\lambda\mu\nu}\,. \tag{3.63}$$

An important property to note is that this commutator is independent of nonmetricy so the Gauss-like equation for this tensor will be obtained through other means.

Considering the commutator of the purely spatial covariant derivative defined in Eq. (3.48) when applied to a purly spatial vector $V^{\beta}_{(3)}$ and spatial co-vector $V^{(3)}_{\beta}$, one obtains the following equations

$$\begin{aligned} D_{[\sigma}D_{\alpha]}V^{\beta} &= T^{\lambda(3)}_{\ \alpha\sigma}D_{\lambda}V^{\beta}_{(3)} + V^{\lambda}_{(3)}R^{\beta(3)}_{\ \lambda\sigma\alpha} \\ &= \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\beta}{}_{\chi}\left(T^{\lambda(4)}_{\ \pi\rho}D_{\lambda}V^{\chi}_{(3)} + V^{\lambda}_{(3)}R^{\chi(4)}_{\ \lambda\rho\pi}\right) \\ &- V^{\lambda}_{(3)}k_{[\alpha|\lambda}k_{|\sigma]}{}^{\beta} - \gamma^{\rho}{}_{[\sigma|}\gamma^{\beta}{}_{\chi}V^{\lambda}_{(3)}k_{|\alpha]\lambda}n_{\mu}Q_{\rho}{}^{\mu\chi}\,, \end{aligned} \tag{3.64}$$

and

$$D_{[\sigma}D_{\alpha]}V_\beta = T^{\lambda(3)}_{\alpha\sigma}D_\lambda V^{(3)}_\beta + V^{(3)}_\lambda R^{\lambda(3)}_{\beta\alpha\sigma} \tag{3.65}$$

$$= \gamma^\rho{}_\sigma\gamma^\pi{}_\alpha\gamma^\chi{}_\beta\left(T^{\lambda(4)}_{\pi\rho}D_\lambda V^{(3)}_\chi + V^{(3)}_\lambda R^{\lambda(4)}_{\chi\pi\rho}\right)$$

$$- V^\lambda_{(3)}k_{[\alpha|\lambda}k_{|\sigma]\beta} - \gamma^\pi{}_{[\alpha|}n_\lambda V^{(3)}_\nu Q_\pi{}^{\lambda\nu}k_{|\sigma]\beta}\,.$$

As they are presented here these equations provide little insight apart from the reliance of the deformation of the foliations being solely dependant on curvature and torsional terms. The lack of non-metricity terms is a result of that specific gravitating entity acting on the non uniformity of the metric as it appears at different points on the foliations rather than the deformation of the foliations themselves. That being said, once one does away with the general connection being considered here and chooses a specific connection, Eqs.(3.64) provides the ideal relations necessary to derive relationships between the Riemann tensor and the torsion tensor and their purely spatial counterparts. In essence their Gauss and Gauss-like equations respectively.

When considering the non-metricity term $Q_{\alpha\mu\nu}$, however, it is not necessary to choose a specific connection or theory in order to derive its Gauss-like equation. Even using the general affine connection considered in this chapter one can derive its relationship to its purely spatial counterpart through applying the spatial covariant derivative to the space-time metric

$$D_\rho g_{\mu\nu} = \gamma^\lambda{}_\rho\gamma^\beta{}_\mu\gamma^\alpha{}_\nu\nabla_\lambda g_{\beta\alpha} = \gamma^\lambda{}_\rho\gamma^\beta{}_\mu\gamma^\alpha{}_\nu Q_{\lambda\beta\alpha} \tag{3.66}$$

$$= \gamma^\lambda{}_\rho\gamma^\beta{}_\mu\gamma^\alpha{}_\nu\nabla_\lambda\left(\gamma_{\beta\alpha} - n_\beta n_\alpha\right)$$

$$= D_\rho\gamma_{\mu\nu} = Q^{(3)}_{\rho\mu\nu}\,.$$

From this, we determine that, independent of the theory being considered, the Gauss equation for the non-metricity term is

$$\gamma^{\lambda}{}_{\rho}\gamma^{\beta}{}_{\mu}\gamma^{\alpha}{}_{\nu}Q_{\lambda\beta\alpha} = Q^{(3)}_{\rho\mu\nu} . \quad (3.67)$$

With this, the underlying definitions and exploration of geometric entities has been exhausted and enough background has been built to start considering a preliminary formulation of evolution equations. Once a theory is chosen and such equations are finalized they will be what allows one to carry out numerical simulation in the various theories.

3.2 Lie derivatives and generalized evolution equations

In this section, the focus shifts to obtaining the evolution equations for the $3 + 1$ formalisms that are to be derived in this chapter. Lie derivatives are the operators used in order to obtain these evolution equations. The reason why Lie derivatives are considered instead of any other type of derivative is that they can show the evolution of a particular vector or tensor as they are dragged along any other vector in a way that is independent of the coordinate system being used [8, 9]. In Fig.(3.3) the comparison between infinitesimally dragging a vector (or tensor) V along a vector χ from a point P to a point U and mapping vector V at P to U through an infinitesimal coordinate transformation can be seen [9, 8, 27]. Taking the limit of transformation to zero one can derive the coordinate independant lie derivative for a general tensor as given in Eq.(3.68). This is particularly useful when one is trying to observe a fundamental variable's evolution along the positive time direction, that is, the normal vector to the foliations.

This derivative can be shown to be linear and also folows the Leibniz rule. The definition of the Lie derivative throughout this work, when applied to a general

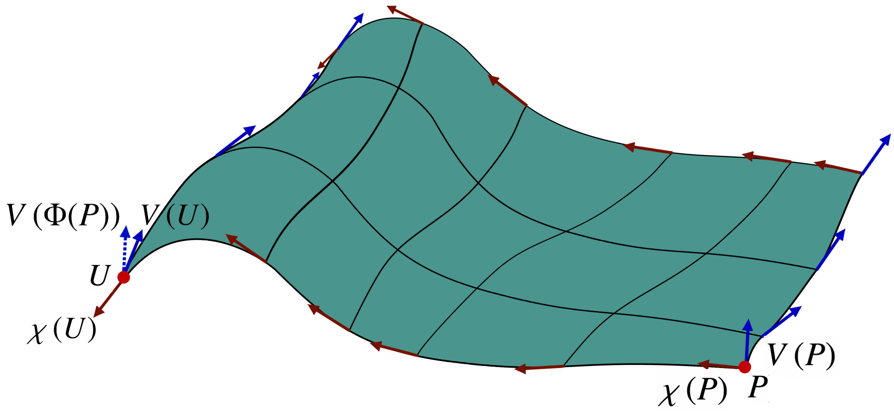

Figure 3.3: In this diagram the solid blue arrow depicts a vector V at point P on the manifold being dragged along a vector χ, depicted by the red arrows, to point U. The dashed blue arrow is the same vector V under an infinitesimal coordinate transformation Φ that maps P onto U.

global 4-vector, V^ν, and 4-covector, V_ν is [55, 56, 8]

$$\mathcal{L}_\chi V^\nu = \chi^\mu \partial_\mu V^\nu - V^\mu \partial_\mu \chi^\nu \tag{3.68}$$

$$= \chi^\mu \nabla_\mu V^\nu - V^\mu \nabla_\mu \chi^\nu + V^\mu \chi^\lambda T^\nu{}_{\lambda\mu} \,,$$

$$\mathcal{L}_\chi V_\nu = \chi^\mu \partial_\mu V_\nu + V_\mu \partial_\nu \chi^\mu \tag{3.69}$$

$$= \chi^\mu \nabla_\mu V_\nu + V_\mu \nabla_\nu \chi^\mu + V_\lambda \chi^\mu T^\lambda{}_{\mu\nu} \,.$$

Taking into consideration the Lie derivative of a tensor with mixed global and local indices along some vector it is important to consider what contributions the local indices make, if any. It turns out that in such a case the lie derivative only acts on the global indices due to the fact that gravitational effects can not be felt locally and as such there is nothing to evolve of the Minkowski metric. From a more general or geometric point of view, taking Eq. (3.1) into consideration, the evolution of the global metric does not necessarily imply the evolution of the local metric so long as the transformation or mapping tensors evolve alongside the global metric. In this case, this transformation tensor is the tetrad. The tetrad evolves with the global metric while keeping the secondary local Minkowski metric invariant. Given this, all lie derivatives of tensors with mixed indices will be considered to follow this general format

$$\mathcal{L}_{\chi} V^{\nu}{}_{A} = \chi^{\mu}\partial_{\mu}V^{\nu}{}_{A} - V^{\mu}{}_{A}\partial_{\mu}\chi^{\nu}\,. \tag{3.70}$$

As previously indicated, the Lie derivatives composing the evolution equations will be taken with respect to the global normal vector n_{μ} to the foliations. Specifically taking such a Lie derivative of a general 3-vector and 3-covector and testing for conservation of spatiality one gets

$$\begin{aligned} n^{\nu}\mathcal{L}_{n} V^{(3)}_{\nu} &= n^{\nu}n^{\mu}\partial_{\mu}V^{(3)}_{\nu} + n^{\nu}V^{(3)}_{\mu}\partial_{\nu}n^{\mu} \\ &= -V^{(3)}_{\nu}n^{\mu}\partial_{\mu}n^{\nu} + n^{\nu}V^{(3)}_{\mu}\partial_{\nu}n^{\mu} \\ &= -V^{(3)}_{\mu}n^{\nu}\partial_{\nu}n^{\mu} + n^{\nu}V^{(3)}_{\mu}\partial_{\nu}n^{\mu} \\ &= 0. \end{aligned} \tag{3.71}$$

$$
\begin{aligned}
n_\nu \mathcal{L}_n V^\nu_{(3)} &= n_\nu n^\mu \partial_\mu V^\nu_{(3)} - n_\nu V^\mu_{(3)} \partial_\mu n^\nu \\
&= -V^\nu_{(3)} n^\mu \partial_\mu n_\nu + V^\mu_{(3)} n^\nu \partial_\mu n_\nu \\
&= \partial_\mu n_\nu \left(-V^\nu_{(3)} n^\mu + V^\mu_{(3)} n^\nu\right) \\
&= -\partial_\mu (\alpha \partial_\nu t) \left(-V^\nu_{(3)} n^\mu + V^\mu_{(3)} n^\nu\right) \\
&= \frac{1}{\alpha} \partial_\mu (\alpha) n_\nu \left(-V^\nu_{(3)} n^\mu + V^\mu_{(3)} n^\nu\right) \\
&= -\frac{1}{\alpha} \partial_\mu (\alpha) V^\mu_{(3)} = \partial_\mu (Ln(\alpha^{-1})) V^\mu_{(3)},
\end{aligned}
\tag{3.72}
$$

Here, in the case of the covector, properties exhibited by spatial vectors and normal vectors first considered in Eqs. (3.8,3.19,3.6) were used. In the case of the vector only the orthogonality of spatial vectors with the normal vector, Eq. (3.19), was used. From Eqs. (3.71,3.72) it can be concluded that the Lie derivative with respect to the normal vector conserves the spatiality of the 3-covector but not of the 3-vector. This observation extends to any global upper and lower indices of any tensor.

The final consideration before the Lie derivative is applied to our formulation is reserved for a tensor with mixed indices. Specifically a spatial tensor with a local upper index and lower global index

$$
\begin{aligned}
\bar{n}_A \mathcal{L}_\chi U^{A(3)}_{\ \ \nu} =& \bar{n}_A \chi^\mu \partial_\mu U^{A(3)}_{\ \ \nu} + \bar{n}_A U^{A(3)}_{\ \ \mu} \partial_\nu \chi^\mu \\
=& - U^{A(3)}_{\ \ \nu} \chi^\mu \partial_\mu \bar{n}_A \\
=& 0\,,
\end{aligned}
\tag{3.73}
$$

where $\bar{n}_A$ represents the local variant of the normal vector. Through this we conclude that the Lie derivative of such a tensor preserves its speciality on the local index. That being said, this occurrence is dependant on our choice of the local metric, the

Minkowski metric.

Considering that the Lie derivative and its double action are to be applied to the inverse spatial metric and the spatial tetrad, it can be concluded that these operations will conserve the spatiality of these fundamental variables. These first and second order applications of the Lie derivative are what constitute the main evolution equations of each of the two $3+1$ formalisms being considered. In this chapter the final form of these equations will be limited by our choice of general affine connection. Further manipulation of the evolution equations in order to get them to a form that can be used to carry out numerical simulations requires a choice of theory, and by extension a specific connection. This provides field equations to be substituted into the evolution equations.

Having thoroughly defined the Lie derivative and its actions on the relevant tensor types, the initial forms of the metric and tetrad evolution equations can be derived.

Starting with the first lie derivative of the spatial metric along the global normal vector for a general affine connection with non-metricity one gets

$$\begin{aligned}\mathcal{L}_n\gamma_{\mu\nu} &= n^\lambda\partial_\lambda\gamma_{\mu\nu} + \gamma_{\lambda\mu}\partial_\nu n^\lambda + \gamma_{\lambda\nu}\partial_\mu n^\lambda \\ &= -2\gamma^\alpha{}_\nu\gamma^\beta{}_\mu n_\lambda\mathring{\Gamma}^\lambda_{\alpha\beta} + \gamma^\alpha{}_\nu\gamma^\beta{}_\mu\left(\nabla_{(\alpha}n_{\beta)} + n_\lambda\Gamma^\lambda_{(\alpha\beta)}\right) \\ &= -2\gamma^\alpha{}_\nu\gamma^\beta{}_\mu n_\lambda\mathring{\Gamma}^\lambda_{\alpha\beta} + k_{(\mu\nu)} + \gamma^\alpha{}_\nu\gamma^\beta{}_\mu n_\lambda\left(2\Gamma^\lambda_{\alpha\beta} + T^\lambda{}_{\alpha\alpha}\right) \\ &= \gamma_{\sigma(\nu|}\gamma^\beta{}_{|\mu).}n^\lambda T^\sigma{}_{\lambda\beta} + 2\gamma^\alpha{}_\nu\gamma^\beta{}_\mu n_\lambda L^\lambda{}_{\alpha\beta} - k_{(\mu\nu)} \\ &= A_{(\mu\nu)} + B_{(\mu\nu)} - k_{(\mu\nu)}\,.\end{aligned} \tag{3.74}$$

Here, in the second line, the spatial metric is expanded to produce two terms with the first term consisting of only the space time metric terms that can be reduced to the Levi Civita connection. with regards to the second term, an extrinsic curvature term is noted and the symmetry of the general connection is opened as a torsion tensor and a general connection term. Using the definition of the contortion tensor Eq.(2.19), the torsion term is expanded into a contortion term and a symmetric torsion term. Combining the Levi Civita term, the general connection and the contortion tensor as in Eq.(2.34) one obtains a disinformation tensor as seen in the line before last. In this line it is noted that, conveniently, each term represents one aspect of the three geometric deformations possible. The tensor $A_{(\mu\nu)}$ is built on the torsion tensor, the tensor $B_{(\mu\nu)}$ contains the disformation tensor that is defined using non-metricity tensors and $k_{(\mu\nu)}$ is the extrinsic curvature.

Taking the Lie derivative of each of these terms individually one gets

$$\begin{aligned}\mathcal{L}_n A_{\mu\nu} = &-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\nabla_{\lambda}T_{\alpha}{}^{\lambda}{}_{\beta} + D_{\lambda}T_{\nu}{}^{\lambda}{}_{\mu} \\ &+\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}a_{\sigma}T_{\alpha}{}^{\sigma}{}_{\beta} + \gamma^{\beta}{}_{\mu}n_{\sigma}n^{\epsilon}T_{\epsilon}{}^{\sigma}{}_{\beta}\gamma^{\alpha}{}_{\nu}a_{\alpha} \\ &-A_{\lambda\mu}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\alpha}{}_{\nu}n_{\epsilon}Q_{\alpha}{}^{\epsilon\sigma} + k_{\nu}{}^{\lambda}\right) \\ &-A_{\nu\lambda}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\beta}{}_{\mu}n_{\epsilon}Q_{\beta}{}^{\epsilon\sigma} + k_{\mu}{}^{\lambda}\right) \\ &+A_{\sigma\mu}A^{\sigma}{}_{\nu} + A_{\nu\sigma}A^{\sigma}{}_{\mu}\,,\end{aligned} \tag{3.75}$$

$$\begin{aligned}\mathcal{L}_n B_{\mu\nu} = &-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\nabla_{\lambda}L^{\lambda}{}_{\alpha\beta} + D_{\lambda}L^{\lambda}{}_{\mu\nu} \\ &+\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(a_{\sigma}L^{\sigma}{}_{\alpha\beta} + a_{(\beta|}L^{\sigma}{}_{\epsilon|\alpha)}n_{\sigma}n^{\epsilon}\right) \\ &-\gamma^{\epsilon}{}_{(\nu|}\gamma^{\chi}{}_{|\mu)}B_{\alpha\chi}\left(n_{\sigma}Q_{\epsilon}{}^{\sigma\alpha} + k_{\epsilon}{}^{\alpha}\right) \\ &+B_{\sigma(\mu|}A^{\sigma}{}_{|\nu)}\,,\end{aligned} \tag{3.76}$$

$$\begin{aligned}
\mathcal{L}_n k_{\mu\nu} =& \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu} n^{\lambda}\nabla_{\lambda}\nabla_{\alpha} n_{\beta} \\
&+ \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(\frac{1}{2}n^{\chi}n^{\epsilon}Q_{\alpha\chi\epsilon}a_{\beta} + a_{\beta}a_{\alpha}\right) \\
&- k_{\lambda\mu}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\alpha}{}_{\nu}n_{\epsilon}Q_{\alpha}{}^{\epsilon\sigma} + k_{\nu}{}^{\lambda}\right) \\
&- k_{\nu\lambda}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\beta}{}_{\mu}n_{\epsilon}Q_{\beta}{}^{\epsilon\sigma} + k_{\mu}{}^{\lambda}\right) \\
&+ k_{\sigma\mu}A^{\sigma}{}_{\nu} + k_{\nu\sigma}A^{\sigma}{}_{\mu}\,.
\end{aligned} \tag{3.77}$$

For a more detailed derivation refer to Appendix A .

Following the same procedure for the first Lie derivative of the spatial tetrad the following relation is obtained

$$\begin{aligned}
\mathcal{L}_n e^{A(3)}_{\ \nu} =& n^{\mu}\partial_{\mu}e^{A(3)}_{\ \nu} + e^{A(3)}_{\ \mu}\partial_{\nu}n^{\mu} \\
=& n^{\mu}\nabla_{\mu}e^{A(3)}_{\ \nu} + e^{A(3)}_{\ \mu}\nabla_{\nu}n^{\mu} \\
&+ n^{\mu}e^{A(3)}_{\ \lambda}\left(\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\mu\nu}\right) - n^{\mu}e^{B(3)}_{\ \nu}\omega^{A}{}_{B\mu} \\
=& \tilde{\gamma}^{A}{}_{C}\gamma^{\alpha}{}_{\nu}n^{\mu}\nabla_{\mu}e^{C}{}_{\alpha} - e^{A(3)}_{\ \mu}n_{\epsilon}\gamma^{\mu}{}_{\sigma}\gamma^{\alpha}Q_{\alpha}{}^{\epsilon\sigma} \\
&- e^{A(3)}_{\ \mu}\gamma_{\alpha\nu}k^{\alpha\mu} + e^{A(3)}_{\ \lambda}\gamma^{\lambda}{}_{\epsilon}\gamma^{\alpha}{}_{\nu}n^{\mu}T^{\epsilon}{}_{\mu\alpha} \\
&- e^{F(3)}_{\ \nu}\tilde{\gamma}^{B}{}_{F}\tilde{\gamma}^{A}{}_{C}n^{\mu}\omega^{C}{}_{B\mu}, \\
=& \mathcal{Q}^{A}{}_{\nu} - \mathcal{B}^{A}{}_{\nu} - k_{\nu}{}^{A} + A^{A}{}_{\nu} - C^{A}{}_{\nu}\,,
\end{aligned} \tag{3.78}$$

It is noted that along with the mixed index equivalents of the extrinsic curvature and the torsion term another three terms appear. Rather than having a single non-metricity entry represented by the disformation tensor, non-metricity is represented in two parts. The first term $\mathcal{Q}^{A}{}_{\nu}$ embodies the equivalent of the non-metricity property of the tetrad and the second, $\mathcal{B}^{A}{}_{\nu}$ is built around a single non-metricity tensor. The final term, $\mathcal{C}^{A}{}_{\nu}$ is a result of a non-vanishing spin connection.

Similar to the double action of the Lie derivative taken when considering the metric, a general form of the Lie derivative of all these terms is taken as well.

$$\begin{aligned}\mathcal{L}_n U^{A}{}_{\nu} =& \tilde{\gamma}^{A}{}_{C}\gamma^{\alpha}{}_{\nu}n^{\lambda}\left(\nabla_{\lambda}U^{C}{}_{\alpha} - \nabla_{\alpha}U^{C}{}_{\lambda}\right) \\ & - U^{B}{}_{\nu}\mathcal{C}^{A}{}_{B} + U^{A}{}_{\lambda}A^{\lambda}{}_{\nu}\,,\end{aligned} \tag{3.79}$$

where $U^{A}{}_{\nu}$ is considered to be a general spatial tensor. This general form can be substituted by each of the terms in the first Lie derivative of the tetrad to obtain a final form for the second Lie derivative while taking the general connection with non-metricity.

Having derived the necessary lie derivatives of the spatial metric and the spatial tetrad separately, it is possible to check for consistancy by making sure that the resulting equations, Eq. (3.74, 3.78), adhere to the initial spatial metric and spatial tetrad relation $\gamma_{\mu\nu} = e^{A(3)}_{\ \mu} e^{B(3)}_{\ \nu}\tilde{\gamma}_{AB}$. Expanding the lie derivative of the left hand side gives

$$\mathcal{L}_n(e^{A(3)}_{\ \mu} e^{B(3)}_{\ \nu}\tilde{\gamma}_{AB}) = e^{A(3)}_{\ \mu} e^{B(3)}_{\ \nu}\mathcal{L}_n(\tilde{\gamma}_{AB}) + e^{B(3)}_{\ \nu}\tilde{\gamma}_{AB}\mathcal{L}_n(e^{A(3)}_{\ \mu}) + e^{A(3)}_{\ \mu}\tilde{\gamma}_{AB}\mathcal{L}_n(e^{B(3)}_{\ \nu}). \tag{3.80}$$

Taking the first term into consideration one gets

$$\begin{aligned}\mathcal{L}_n(\tilde{\gamma}_{AB}) =& n^{\lambda}\partial_{\lambda}\tilde{\gamma}_{AB} \\ =& n^{\lambda}\partial_{\lambda}\left(\eta_{AB} + \tilde{n}_A\tilde{n}_B\right) \\ =& 0,\end{aligned} \tag{3.81}$$

where the final result stems directly from the choice of the local metric. Taking the second and third terms of Eq. (3.80) and expanding according to Eq. (3.78), the extrinsic curvature term, $-k_{(\mu\nu)}$, and the torsional term $A_{(\mu\nu)}$ come out naturally once they are combined again.

$$e^{B(3)}_{\ \nu}\tilde{\gamma}_{AB}\mathcal{L}_n(e^{A(3)}_{\ \mu}) = e^{A(3)}_{\ \nu}\tilde{\gamma}_{AC}\gamma^{\alpha}{}_{\mu}n^{\lambda}\nabla_{\lambda}e^{C}{}_{\alpha} - \gamma_{\sigma\nu}\gamma^{\alpha}{}_{\mu}n_{\epsilon}Q_{\alpha}{}^{\epsilon\sigma} - k_{\mu\nu} - \gamma_{\epsilon\nu}\gamma^{\alpha}{}_{\mu}n^{\sigma}T^{\epsilon}{}_{\sigma\alpha} - e^{A}{}_{\nu}\tilde{\gamma}_{CA}e^{F(3)}_{\ \mu}\tilde{\gamma}^{D}{}_{F}n^{\lambda}w^{C}{}_{D\lambda} \tag{3.82}$$

$$e^{A(3)}_{\ \mu}\tilde{\gamma}_{AB}\mathcal{L}_n(e^{B(3)}_{\ \nu}) = e^{B(3)}_{\ \mu}\tilde{\gamma}_{BC}\gamma^{\alpha}{}_{\nu}n^{\lambda}\nabla_{\lambda}e^{C}{}_{\alpha} - \gamma_{\sigma\mu}\gamma^{\alpha}{}_{\nu}n_{\epsilon}Q_{\alpha}{}^{\epsilon\sigma} - k_{\nu\mu} - \gamma_{\epsilon\mu}\gamma^{\alpha}{}_{\nu}n^{\sigma}T^{\epsilon}{}_{\sigma\alpha} - e^{B}{}_{\mu}\tilde{\gamma}_{CB}e^{F(3)}_{\ \nu}\tilde{\gamma}^{D}{}_{F}n^{\lambda}w^{C}{}_{D\lambda} \tag{3.83}$$

Considering the non-metricity terms and substituting Eq. (2.35) into the expression one gets

$$-\gamma^{\sigma}{}_{\mu}\gamma^{\alpha}{}_{\nu}n^{\epsilon}Q_{(\alpha|\epsilon|\sigma)} = -\gamma^{\sigma}{}_{\mu}\gamma^{\alpha}{}_{\nu}n^{\epsilon}\left(Q_{\epsilon\alpha\sigma} - 2L_{\epsilon\alpha\sigma}\right), \tag{3.84}$$

where the second term in the above equation gives the expected $B_{(\mu\nu)}$ term that features in the first lie of the spatial metric. Opening up the first term of Eq. (3.84) it can be shown to cut out exactly with the spin terms and the terms that contain the covariant derivative of the tetrad. With that, it has been confirmed that Eq. (3.80) does in fact reproduce the first lie derivative of the metric making the two formalisms consistent. An interesting aside to this is that once metricity is assumed, this relation is an alternative way to show the antisymmetry in the first two indices of the spin connection.

This chapter is concluded by presenting the action of a Lie derivative of a general spatial tensor along αn^{μ}. This is considered as it will be significant later on in the development of the $3+1$ formulaitons. It can be shown that even for our general case such a Lie derivative is itself spatial. Expanding this case according to the general definition of the lie derivative Eqs.(3.68-3.69) one gets

$$\begin{aligned}\mathcal{L}_{\alpha n} X^{\epsilon_1 \dots \epsilon_i}{}_{\sigma_1 \dots \sigma_j} =& \alpha n^{\lambda} \partial_{\lambda} X^{\epsilon_1 \dots \epsilon_i}{}_{\sigma_1 \dots \sigma_j} \\ & - \sum_{p=1}^{i} X^{\epsilon_1 .. \lambda_p .. \epsilon_i}{}_{\sigma_1 \dots \sigma_j} \partial_{\lambda_p} (\alpha n^{\epsilon_p}) \\ & + \sum_{q=1}^{j} X^{\epsilon_1 .. \epsilon_i}{}_{\sigma_1 .. \lambda_q .. \sigma_j} \partial_{\sigma_q} \left(\alpha n^{\lambda_q}\right)\end{aligned} \tag{3.85}$$

where where i and j are integers.

Taking $n^{\sigma_k} \mathcal{L}_n X^{\epsilon_1 \dots \epsilon_i}{}_{\sigma_1 \dots \sigma_j}$ for $1 \leq k \leq j$ results in the second term of Eq. (3.85) and all of the third term except for when $k = q$ to vanish. What remains of Eq. (3.85) is

$$\begin{aligned}n^{\sigma_k} \mathcal{L}_{\alpha n} X^{\epsilon_1 \dots \epsilon_i}{}_{\sigma_1 \dots \sigma_j} =& \alpha n^{\sigma_k} n^{\lambda} \partial_{\lambda} X^{\epsilon_1 \dots \epsilon_i}{}_{\sigma_1 \dots \sigma_j} \\ & + n^{\sigma_k} X^{\epsilon_1 .. \epsilon_i}{}_{\sigma_1 .. \lambda .. \sigma_j} \partial_{\sigma_k} \left(\alpha n^{\lambda}\right) \\ =& \alpha n^{\sigma_k} n^{\lambda} \Big(\partial_{\lambda} X^{\epsilon_1 \dots \epsilon_i}{}_{\sigma_1 .. \sigma_k .. \sigma_j} \\ & - \partial_{\sigma_k} X^{\epsilon_1 .. \epsilon_i}{}_{\sigma_1 .. \lambda .. \sigma_j}\Big) \\ =& 0 \,.\end{aligned} \tag{3.86}$$

Taking $n_{\epsilon_k} \mathcal{L}_n X^{\epsilon_1 \dots \epsilon_i}{}_{\sigma_1 \dots \sigma_j}$ for $1 \leq k \leq i$, results in the third term of Eq. (3.85) and all of the second term except for when $k = p$ to become zero. what remains of Eq. (3.85) is

$$
\begin{aligned}
n_{\epsilon_k}\mathcal{L}_{\alpha n}X^{\epsilon_1\ldots\epsilon_i}{}_{\sigma_1\ldots\sigma_j} =&\alpha n_{\epsilon_k}n^{\lambda}\partial_{\lambda}X^{\epsilon_1\ldots\epsilon_i}{}_{\sigma_1\ldots\sigma_j} \\
&- n_{\epsilon_k}X^{\epsilon_1..\lambda..\epsilon_i}{}_{\sigma_1\ldots\sigma_j}\partial_{\lambda}\left(\alpha n^{\epsilon_k}\right) \\
=&n_{\epsilon_k}\partial_{\lambda}\left(\alpha\right)\Big(n^{\epsilon_k}X^{\epsilon_1..\lambda..\epsilon_i}{}_{\sigma_1\ldots\sigma_j} \\
&-n^{\lambda}X^{\epsilon_1\ldots\epsilon_i}{}_{\sigma_1\ldots\sigma_j}\Big) \\
&+ X^{\epsilon_1..\lambda..\epsilon_i}{}_{\sigma_1\ldots\sigma_j}\partial_{\lambda}\left(\alpha\right) \\
=&0\,,
\end{aligned}
\tag{3.87}
$$

where

$$
\begin{aligned}
\alpha\partial_{\lambda}\left(n_{\epsilon_k}\right) =& -\alpha\partial_{\lambda}\left(\alpha\partial_{\epsilon_k}\{t\}\right) \\
=&n_{\epsilon_k}\partial_{\lambda}\left(\alpha\right)\,,
\end{aligned}
\tag{3.88}
$$

was used.

In this chapter the metric and tetrad $3+1$ formalisms have been developed for a general affine connection assuming non-metricity, and have been shown to be consistant with each other. At this point they can be developed no further unless a specific theory is chosen. The next step in the derivation would be the substitution of field equations into the evolution equations. This cannot be done in a theory independent way. In Chapter 4 a number of theories are considered in an attempt at finalizing the formalisms.

4 Gauss, Constraint and Evolution Equations in Specific Theories

In this chapter the generalized formalisms derived in Chapter 3 will be applied to three different specific theories each built around one of the three previously defined geometric deformation tensors. The first of these theories will be GR, a theory based on curvature, the second will be TEGR which is based on torsion and finally STEGR with the coincident gauge which is built around non-metricity. In each case the Gauss, constraint and evolution equations are derived and any issues that arise are analysed.

4.1 Gauss, Constraint and Evolution Equations in General Relativity

GR is built around the concept that gravity can be expressed geometrically through curvature. This can be characterized mathematically through the Levi-Civita connection that is defined purely in terms of the space-time metric as [27]

$$\mathring{\Gamma}^{\sigma}_{\mu\nu} := \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) . \tag{4.1}$$

Through this definition it is easily deduced that the connection is symmetric in its bottom two indices. It is also noted that this is not a tensor, that is, it does not Lorentz transform like a regular tensor [27].

Throughout this section all derivatives, such as covariant derivatives and Lie derivatives, as well as all tensors that characterize the geometry in some way are defined through this connection and are labeled by a "$\circ$" annotation.

When working with the Levi-Civita connection one should note the properties that its symmetry induces onto the tensors that are built on it. These properties have implications both on the theory in general as well as the resulting $3 + 1$ formalism that stems out of it.

Starting from the Riemann tensor as defined in Eq. (2.12). Through the symmetry of $\mathring{\Gamma}^{\sigma}_{\mu\nu}$, this tensor acquires two additional symmetries beyond the antisymmetry in its final two indices. Here all its symmetries are presented for ease of reference [27, 57]

$$\mathring{R}_{\rho\sigma\mu\nu} = -\mathring{R}_{\sigma\rho\mu\nu} , \tag{4.2}$$

$$\mathring{R}_{\rho\sigma\mu\nu} = -\mathring{R}_{\rho\sigma\nu\mu} , \tag{4.3}$$

$$\mathring{R}_{\rho\sigma\mu\nu} = \mathring{R}_{\mu\nu\rho\sigma} . \tag{4.4}$$

The Riemann tensor also acquires the following cyclic property

$$\mathring{R}_{\rho\sigma\mu\nu} + \mathring{R}_{\rho\mu\nu\sigma} + \mathring{R}_{\rho\nu\sigma\mu} = 0 . \tag{4.5}$$

When choosing to work with the Levi-Civita connection the torsion tensor vanishes. This is to be expected since this tensor is defined to be the antisymmetric part of the connection as seen in Eq. (2.22). By extension all tensors that are purely defined through the torsion tensor also vanish leading to a torsion free theory.

Another consequence of building a theory around the Levi-Civita connection is that it induces metricity. Taking the covariant derivative of the metric, expanding and substituting Eq. 4.1 one gets

$$\begin{aligned}\mathring{\nabla}_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - g_{\sigma\nu}\mathring{\Gamma}^{\sigma}_{\mu\lambda} - g_{\mu\sigma}\mathring{\Gamma}^{\sigma}_{\nu\lambda} \\ &= \partial_\lambda g_{\mu\nu} - \frac{1}{2}\left(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda}\right) \\ &\quad - \frac{1}{2}\left(\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda}\right) \\ &= 0\,,\end{aligned} \tag{4.6}$$

resulting in all tensors that are built around the non-metricity tensor, and by extension the disformation tensor, to vanish.

One of the most important operators in the formalisms being developed in this dissertation is Lie derivative. Choosing this connection makes it possible for the definition of the Lie derivative to be written in terms of partial derivatives and equivalently in terms of covariant derivatives. This holds so long as the Lie derivative is being operated on purely global tensors. The reason for this equivalence can easily be seen through Eqs. (3.68, 3.69) where it is shown that the difference between the partial derivative and the covariant derivative expressions is a torsion tensor term. Thus, in this theory when taking the Lie of any global tensor along any vector, the following are equivalent

$$\mathcal{L}_\chi V^\nu{}_{a\mu} = \chi^\sigma \partial_\sigma V^\nu{}_{a\mu} - V^\sigma{}_{a\mu} \partial_\sigma \chi^\nu + V^\nu{}_{a\sigma} \partial_\mu \chi^\sigma \tag{4.7}$$

$$= \chi^\sigma \mathring{\nabla}_\sigma V^\nu{}_{a\mu} - V^\sigma{}_{a\mu} \mathring{\nabla}_\sigma \chi^\nu + V^\nu{}_{a\sigma} \mathring{\nabla}_\mu \chi^\sigma \,.$$

That being said, when the Lie derivative is applied to a tensor with mixed indices, that is, both global and local, or a tensor with purely local indices, this does not apply. In this case the partial derivative expression would have an extra spin connection contribution for each of the local indices.

Due to the vanishing torsion tensor it is noted that the antisymmetric part of the extrinsic curvature also vanishes. This can easily be determined through Eq. (3.61). As a consequence, the extrinsic curvature in GR is symmetric on its two indices. For similar reasons the commutator of the covariant derivative of the one-from $\boldsymbol{\Omega}_\nu$, $\mathring{\nabla}_{[\mu} \boldsymbol{\Omega}_{\nu]} = \mathring{\nabla}_{[\mu} \mathring{\nabla}_{\nu]} t$, also reduces to zero. This can also be expanded to a single term containing the torsion tensor [8].

Having determined the basic consequences of choosing the Levi-Civita as the connection for GR, the next step is to determine the Gauss, Codazzi, constraint and evolution equations from the generalized $3+1$ formalisms derived in the previous chapter.

One of the main steps in building the $3+1$ formalism in any theory is the derivation of the relationship between the relevant gravitating spacetime tensor and its corresponding purely spatial counterpart. This is known as the Gauss equation. Since GR is curvature based the tensor which characterizes the geometric deformation due to gravity is the Riemann tensor. In Chapter 3 a general form of the commutator of the double covariant derivative of a general 3-vector V^λ, Eq. (3.64), was derived. Taking into consideration the implications of choosing the Levi-Civita as the connection in GR, the Gauss equation for the Riemann tensor turns out to be

[8]

$$\mathring{R}^{\beta(3)}{}_{\lambda\sigma\alpha} = \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\nu}{}_{\lambda}\gamma^{\beta}{}_{\chi}\mathring{R}^{\chi}{}_{\nu\rho\pi} - \mathring{k}_{[\alpha|\lambda}\mathring{k}_{|\sigma]}{}^{\beta} \,. \tag{4.8}$$

As required, this equation relates the 3-Riemann and the 4-Riemann. Conveniently the remaining term of the relation is made up of the extrinsic curvature, which is purely spatial in its own right. Contracting the Gauss equation once, the equivalent for the Ricci tensor is obtained and contracting again gives the Ricci scalar Gauss equation

$$\mathring{R}^{(3)}_{\lambda\alpha} = \gamma^{\pi}{}_{\alpha}\gamma^{\nu}{}_{\lambda}\gamma^{\rho}{}_{\chi}\mathring{R}^{\chi}{}_{\nu\rho\pi} - \mathring{k}_{\alpha\lambda}\mathring{k} + \mathring{k}_{\beta\lambda}\mathring{k}_{\alpha}{}^{\beta} \,, \tag{4.9}$$

$$\mathring{R}^{(3)} = \gamma^{\pi\nu}\gamma^{\rho}{}_{\chi}\mathring{R}^{\chi}{}_{\nu\rho\pi} - \mathring{k}^{2} + \mathring{k}_{\beta\lambda}\mathring{k}^{\lambda\beta} \,. \tag{4.10}$$

Taking the antisymmetry on the first two indices of the spatial derivative of the extrinsic curvature one gets the Codazzi equation

$$\mathring{D}_{[\mu}\mathring{k}_{\nu]\lambda} = \gamma^{\rho}{}_{\nu}\gamma^{\pi}{}_{\mu}\gamma^{\beta}{}_{\lambda}n^{\sigma}\mathring{R}_{\rho\pi\beta\sigma} \,. \tag{4.11}$$

While this equation seems random it will prove useful when simplifying the evolution equations for this theory. It is noted that up to this point all equations derived from the generalized form acquired in Chapter 3 agree with those of the standard GR $3+1$. [8, 22].

Taking into consideration the tetrad and local indices in general, another cosequence of choosing the Levi-Civita connection is that the spin connection can be written purely in terms of tetrads and first order partial derivatives of tetrads. By applying the covariant derivative to the tetrad, using the equivalent of the metricity property for tetrads and substituting Eq. (4.1) one can show that this relation is

$$\omega^{A}{}_{C\nu} = \frac{1}{2}\Big(\bar{e}^{A\alpha} e^{B}{}_{\nu} e_{C}{}^{\beta}\left(\partial_{\beta}\bar{e}_{B\alpha} - \partial_{\alpha}\bar{e}_{B\beta}\right)$$

$$+ e_{C}{}^{\alpha}\left(\partial_{\alpha}e^{A}{}_{\nu} - \partial_{\nu}e^{A}{}_{\alpha}\right) + \bar{e}^{A\alpha}\left(\partial_{\nu}\bar{e}_{C\alpha} - \partial_{\alpha}\bar{e}_{C\nu}\right)\Big)\,, \tag{4.12}$$

where, for the sake of simplicity, terms like $\eta^{AB}e_{B}{}^{\alpha}$ are written as $\bar{e}^{A\alpha}$. Through this equation the antisymmetry of the spin connection that was shown in Chapter 3 for theories that abide by metricity can easily be seen. While it is not part of this research project, this definition would be useful if a tetrad $3+1$ is considered for GR.

Finally the evolution equations are considered. Starting with the first Lie derivative of the spatial metric, Eq. (3.74) becomes

$$\mathcal{L}_{n}\gamma_{\mu\nu} = -2\mathring{k}_{\mu\nu}\,, \tag{4.13}$$

At this point what is known as the Ricci equation is derived. This equation is the second order Lie derivative of the spatial metric or the first Lie of Eq. (4.13). This equation is necessary due to the first Lie of the spatial metric being the extrinsic curvature. While $\mathring{k}_{\mu\nu}$ is itself fully spatial it is not possible to write it in terms purely spatial tensors. As such, we need the Lie of the extrinsic curvature in order to have a fully consistent set of equations. Effectively, the concept of requiring two equations for two unknowns where in our case the two unknowns are the spatial metric and the extrinsic curvature pair $(\gamma_{\mu\nu}, \mathring{k}_{\mu\nu})$. The two evolution equations for GR are thus

$$\mathcal{L}_n \gamma_{\mu\nu} = -2\mathring{k}_{\mu\nu}\,, \tag{4.14}$$

$$\begin{aligned}\mathcal{L}_n \mathring{k}_{\mu\nu} &= -\frac{1}{2}\mathcal{L}_n\mathcal{L}_n\gamma_{\mu\nu} \\ &= n^\rho \gamma^\alpha{}_\nu \gamma^\beta{}_\mu n_\chi \mathring{R}^\chi{}_{\alpha\rho\beta} - \mathring{D}_\mu \mathring{a}_\nu - \mathring{a}_\mu \mathring{a}_\nu - \mathring{k}_\mu{}^\rho \mathring{k}_{\nu\rho} \\ &= n^\rho \gamma^\alpha{}_\nu \gamma^\beta{}_\mu n_\chi \mathring{R}^\chi{}_{\alpha\rho\beta} - \frac{1}{\alpha}\mathring{D}_\mu \mathring{D}_\nu \alpha - \mathring{k}_\mu{}^\rho \mathring{k}_{\nu\rho}\,,\end{aligned} \tag{4.15}$$

where $\mathring{D}_\mu \mathring{a}_\nu + \mathring{a}_\mu \mathring{a}_\nu$ can be shown to be equivalent to $\frac{1}{\alpha}\mathring{D}_\mu \mathring{D}_\nu \alpha$ provided that the torsion tensor is zero.

In order to proceed with this calculation it is necessary to introduce the field equations. In the case of GR the field equations are as follows

$$\mathring{G}_{\mu\nu} \equiv \mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R} g_{\mu\nu} = 8\pi \mathcal{T}_{\mu\nu}\,, \tag{4.16}$$

where $\mathring{G}_{\mu\nu}$ is the Einstein Tensor and $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor. Noting that $n^\rho n_\chi = \gamma^\rho{}_\chi - \delta^\rho{}_\chi$, Eq.(4.15) becomes

$$\mathcal{L}_n \mathring{k}_{\mu\nu} = \gamma^\rho{}_\chi \gamma^\alpha{}_\nu \gamma^\beta{}_\mu \mathring{R}^\chi{}_{\alpha\rho\beta} - \gamma^\alpha{}_\nu \gamma^\beta{}_\mu \mathring{R}{}_{\alpha\beta} - \frac{1}{\alpha}\mathring{D}_\mu \mathring{D}_\nu \alpha - \mathring{k}_\mu{}^\rho \mathring{k}_{\nu\rho}\,, \tag{4.17}$$

where the first term on the left hand side is replaced using the Riemann Gauss equation, Eq.(4.8). In order to replace the second term with purely spatial tensors the GR field equations are contracted to give

$$\mathring{G} \equiv \mathring{R} - \frac{4}{2}\mathring{R} = 8\pi\mathcal{T}\,, \tag{4.18}$$

$$\mathring{R} = -8\pi\mathcal{T}\,, \tag{4.19}$$

where $\mathcal{T} = g^{\mu\nu}\mathcal{T}_{\mu\nu}$. The second term in Eq.(4.17) can be substituted by a full spatial mapping of the evolution equations after substituting for the Ricci scalar with Eq.(4.19)

$$\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\mathring{R}_{\alpha\beta} = \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\mathcal{T}_{\alpha\beta} + \frac{1}{2}\mathring{R}\gamma_{\mu\nu} \tag{4.20}$$

$$= 8\pi\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\mathcal{T}_{\alpha\beta} - 4\pi\mathcal{T}\gamma_{\mu\nu}\,,$$

which results in the following form of the Ricci equation

$$\mathcal{L}_n\mathring{k}_{\mu\nu} = \mathring{R}^{(3)}_{\nu\mu} + \mathring{k}_{\mu\nu}\mathring{k} - 2\mathring{k}_{\lambda\nu}\mathring{k}_{\mu}{}^{\lambda} - 8\pi\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}(\mathcal{T}_{\alpha\beta} - \frac{1}{2}\mathcal{T}g_{\alpha\beta}) - \frac{1}{\alpha}\mathring{D}_{\mu}\mathring{D}_{\nu}\alpha\,. \tag{4.21}$$

Defining the spatial Stress as

$$S_{\mu\nu} \equiv \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\mathcal{T}_{\alpha\beta}\,, \tag{4.22}$$

and noting that

$$\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}g_{\alpha\beta}g^{\lambda\chi}\mathcal{T}_{\lambda\chi} = \gamma_{\mu\nu}\left(\gamma^{\lambda\chi} - n^{\lambda}n^{\chi}\right)\mathcal{T}_{\lambda\chi}$$

$$= \gamma_{\mu\nu}\left(S - \rho\right)\,, \tag{4.23}$$

the final form of the second evolution equation for a metric 3+1 formalism in GR is obtained [8]

$$\mathcal{L}_n\mathring{k}_{\mu\nu} = \mathring{R}^{(3)}_{\nu\mu} + \mathring{k}_{\mu\nu}\mathring{k} - 2\mathring{k}_{\lambda\nu}\mathring{k}_{\mu}{}^{\lambda} - 8\pi\left(S_{\alpha\beta} - \frac{1}{2}\gamma_{\mu\nu}\left(S - \rho\right)\right) - \frac{1}{\alpha}\mathring{D}_{\mu}\mathring{D}_{\nu}\alpha\,. \tag{4.24}$$

The final terms that need to be written purely in terms of spatial tensors are the ones related to the energy momentum tensor. These can be substituted through the momentum and Hamiltonian constraints that are given below

$$\mathring{D}_\lambda \mathring{k} - \mathring{D}_\chi \mathring{k}_\lambda{}^\chi = 8\pi S_\lambda \,, \tag{4.25}$$

$$\mathring{R}^{(3)} + \mathring{k}^2 - \mathring{k}_{\beta\lambda}\mathring{k}^{\lambda\beta} = 16\pi\rho \,. \tag{4.26}$$

Here the momentum density S_λ is defined as $-\gamma^\epsilon{}_\lambda n^\nu \mathcal{T}_{\epsilon\nu}$. While all of the above is known and has been historically derived in standard gravity it was important to show that the same exact results can be achieved starting from the generalized $3+1$ formalism developed in Chapter 3. Comparing the above with literature it is found that the results here agree perfectly [8, 22].

4.2 Gauss Constraint and Evolution Equations in the Teleparallel Equivalent of General Relativity

In this section TG is considered, specifically a 3 + 1 decomposition is carried out on TEGR with the Weitzenböck gauge. As discussed in Section 2.1 all curvature terms are identically zero in this theory. Metricity is also assumed, this results in all non-metricity and disformation tensors vanishing.

As was done in the case of GR, the first step in obtaining the specific formalism is considering the connection in order to choose the fundamental variable. In this case the connection is the teleparallel connection that was defined in Eq. (2.8). This is given here again for ease of reference

$$\hat{\Gamma}^\lambda_{\mu\nu} := e_A{}^\lambda \partial_\nu e^A{}_\mu + e_A{}^\lambda e^B{}_\mu \omega^a_{B\nu} \,. \tag{4.27}$$

It can be observed that in general this is dependant on the 4-tetrad and the spin connection. All covariant and Lie derivatives as well as tensors built through the teleparallel connection will be denoted by a hat accent, $\hat{A}$. This is also the case for all purely spatial tensors built through the spatial Weitzenböck connection.

Unfortunately, unlike in GR, it is not possible to express the TG spin connection purely in terms of a single fundamental variable, say the tetrad. This is due to the fact that the spin connection represents a separate set of degrees of freedom in this theory. Given this, only pure tetrads are considered. Another way of putting this is that the Weitzenböck gauge will be considered throughout this section.

Due to the use of the Weitzenböck gauge the general teleparallel connection becomes the Weitzenböck connection variant. As a result the spin connection vanishes and so the connection becomes solely tetrad dependant. This also has the added benefit of simplifying most of the equations considered.

The main consequence of a vanishing spin connection on the development of the $3+1$ formulation is that the 4-covariant derivative of the normal vector can be shown to be zero

$$\hat{\nabla}_\alpha n^\sigma = e_A{}^\sigma \hat{\nabla}_\alpha \tilde{n}^A = e_A{}^\sigma \left(\partial_\alpha \tilde{n}^A + \tilde{n}^B \omega^B_{A\alpha}\right) , \tag{4.28}$$

where $\partial_\alpha \tilde{n}^A = 0$ through Eq.(3.24). There are a number of significant consequences of this among which are that the covariant derivative of the spatial metric is zero

$$\begin{aligned} \hat{\nabla}_\alpha \gamma^{\rho\sigma} &= \hat{\nabla}_\alpha \left(g^{\rho\sigma} + n^\rho n^\sigma\right) \\ &= n^\sigma \hat{\nabla}_\alpha n^\rho + n^\rho \hat{\nabla}_\alpha n^\sigma \\ &= 0 \,, \end{aligned} \tag{4.29}$$

the covariant of the 3-tetrad is also zero for the same reasons, the extrinsic curvature is zero

$$\begin{aligned}\hat{k}_{\mu\nu} &= -\gamma^{\lambda}{}_{\mu}\gamma^{\sigma}{}_{\nu}\hat{\nabla}_{\lambda}n_{\sigma} \\ &= -\gamma^{\lambda}{}_{\mu}\gamma^{\sigma}{}_{\nu}\hat{\nabla}_{\lambda}\left(e^{A}_{\sigma}\tilde{n}_{A}\right) \\ &= -\gamma^{\lambda}{}_{\mu}\gamma^{\sigma}{}_{\nu}e^{A}{}_{\sigma}\left(\partial_{\lambda}\tilde{n}_{A} - \tilde{n}_{B}\hat{w}^{B}_{A\lambda}\right) \\ &= 0\,,\end{aligned} \tag{4.30}$$

and the acceleration vector is also zero

$$\begin{aligned}\hat{a}_{\mu} &= n^{\lambda}\hat{\nabla}_{\lambda}n_{\mu} \\ &= 0\,.\end{aligned} \tag{4.31}$$

Having gone through the main consequences of choosing the Weitzenböck connection it is possible to start considering the tensor responsible for the geometric distortion of the space time fabric. This is of course the torsion tensor which is defined as in Eq. (2.22). The next step towards deriving the 3+1 formulation of teleparallel gravity is defining a Gauss-like equation for the torsion tensor. Starting from Eq. (3.64) and applying all of the above results the following simplified expression is obtained

$$\hat{D}_{\lambda}V^{\beta}_{(3)}\left(\hat{T}^{\lambda(3)}_{\alpha\sigma} - \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\pi\rho}\right) = 0\,, \tag{4.32}$$

where the vector $V^{\beta}_{(3)}$ is a general spatial vector. From this equation it can be concluded that there are three ways for this expression to be equal to zero. The first is for $\hat{D}_{\lambda}V^{\beta}_{(3)}$ to be always zero, this is of course impossible due to the generality of the spatial vector. The second is that $\left(\hat{T}^{\lambda(3)}_{\alpha\sigma} - \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\pi\rho}\right)$ is orthogonal to $\hat{D}_{\lambda}V^{\beta}_{(3)}$ for all spatial vectors. At this point a number of things are noted. $\hat{D}_{\lambda}V^{\beta}_{(3)}$ is purely spatial by nature of the spatial derivative definition. This spatiality is also independent of the spatial vector. It is also noted that $\left(\hat{T}^{\lambda(3)}_{\ \alpha\sigma} - \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\ \pi\rho}\right)$ is independent of the vector. The only way this is the reason the expression goes to zero is if $\left(\hat{T}^{\lambda(3)}_{\ \alpha\sigma} - \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\ \pi\rho}\right)$ is purely temporal. This is of course a contradiction given that it is made up of two spatial terms and can be tested by contracting a normal vector with each of the three free indices separately

$$n_{\lambda}\left(\hat{T}^{\lambda(3)}_{\ \alpha\sigma} - \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\ \pi\rho}\right) = 0\,, \tag{4.33}$$

$$n^{\alpha}\left(\hat{T}^{\lambda(3)}_{\ \alpha\sigma} - \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\ \pi\rho}\right) = 0\,,$$

$$n^{\sigma}\left(\hat{T}^{\lambda(3)}_{\ \alpha\sigma} - \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\ \pi\rho}\right) = 0\,,$$

implying that the bracket term is purely spatial. The remaining possibility is that $\left(\hat{T}^{\lambda(3)}_{\ \alpha\sigma} - \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\ \pi\rho}\right)$ is zero itself producing the Gauss equation for the torsion tensor

$$\hat{T}^{\lambda(3)}_{\ \alpha\sigma} = \gamma^{\rho}{}_{\sigma}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\ \pi\rho}\,. \tag{4.34}$$

Using this equation a number of variants can be obtained from it producing the Gauss-like equations for the torsion vector and scalar, the derivation of which can be seen below

$$\begin{aligned}\hat{T}^{(3)}_{\alpha} &= \gamma^{\rho}{}_{\lambda}\gamma^{\pi}{}_{\alpha}\gamma^{\lambda}{}_{\nu}\hat{T}^{\nu(4)}_{\ \pi\rho} \\ &= \hat{T}^{(4)}_{\alpha} + n_{\alpha}n^{\pi}\hat{T}^{(4)}_{\pi} + n_{\nu}n^{\rho}\hat{T}^{\nu(4)}_{\ \alpha\rho} \\ &= \gamma^{\lambda}{}_{\alpha}\hat{T}^{(4)}_{\lambda} + n_{\nu}n^{\rho}\hat{T}^{\nu(4)}_{\ \alpha\rho}\,,\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\hat{T}^{(3)} &= \frac{1}{4}\hat{T}^{(3)\mu\nu}_{\lambda}\hat{T}^{\lambda(3)}_{\mu\nu} + \frac{1}{2}\hat{T}^{\lambda(3)}_{\mu\nu}\hat{T}^{\nu\mu(3)}_{\lambda} - \hat{T}^{\lambda(3)}_{\mu\lambda}\hat{T}^{\nu\mu(3)}_{\nu} \\
&= \hat{T}^{(4)} + 2n^{\lambda}n^{\mu}\hat{T}^{\alpha}{}_{\nu\alpha}\hat{T}_{\lambda\mu}{}^{\nu} + \frac{1}{4}n^{\lambda}n^{\mu}\hat{T}_{\lambda}{}^{\nu\alpha}\hat{T}_{\mu\nu\alpha} \\
&\quad + n^{\lambda}n^{\mu}\hat{T}_{\lambda}{}^{\nu\alpha}\hat{T}_{\nu\mu\alpha} + \frac{1}{2}n^{\lambda}n^{\mu}\hat{T}_{\alpha\mu\nu}\hat{T}^{\nu}{}_{\lambda}{}^{\alpha} \\
&\quad + \frac{1}{2}n^{\lambda}n^{\mu}\hat{T}_{\nu\mu\alpha}\hat{T}^{\nu}{}_{\lambda}{}^{\alpha} - n^{\lambda}n^{\mu}\hat{T}^{\alpha}{}_{\mu\alpha}\hat{T}^{\nu}{}_{\lambda\nu} \\
&= \hat{T}^{(4)} + \frac{2}{\alpha}\hat{T}^{\nu(3)}\partial^{(3)}_{\nu}(\alpha) + \frac{1}{2}\hat{A}^{\nu}_{\lambda}\left(\hat{A}^{\lambda}_{\nu} + \hat{A}^{\lambda}_{\nu}\right) - \hat{A}^{2}\,,
\end{aligned} \tag{4.36}$$

where $\hat{A}^{B}{}_{\nu}$ is the first order Lie derivative of the spatial tetrad along the normal vector to the foliations n and $\hat{A}$ is its fully contracted scalar.

Having defined all the required Gauss-like equations it is possible to move on to deriving the necessary evolution equations and constraint equations for the completion of the torsional $3+1$ formalism. The first step in obtaining these equations is to consider the theory's field equations. In the case of TEGR the field equations being considered are as given in Refs.[37, 32] and provided in Eq. (2.25). Taking into consideration the Weitzenböck gauge and after some minor restructuring of the equations, they can be written as

$$\begin{aligned}
&\hat{S}_{\alpha\sigma}{}^{\rho}\hat{T}^{\lambda}{}_{\rho\lambda} - \hat{S}^{\rho\lambda}{}_{\sigma}\hat{T}_{\rho\lambda\alpha} - \frac{1}{2}\hat{S}_{\alpha}{}^{\rho\lambda}\hat{T}_{\sigma\lambda\rho} \\
&\qquad + \frac{1}{2}g_{\alpha\sigma}\hat{T} - \hat{\nabla}^{\lambda}\hat{S}_{\alpha\lambda\sigma} = \Theta_{\alpha\sigma}\,.
\end{aligned} \tag{4.37}$$

Contracting these equations, it is also possible to obtain an alternative expression for the torsion scalar in terms of the energy-momentum scalar

$$\hat{T} = \Theta + 2\hat{T}^{\alpha}{}_{\rho\alpha}\hat{T}^{\lambda}{}_{\lambda}{}^{\rho} + 2\hat{\nabla}_{\rho}\hat{T}^{\lambda}{}_{\lambda}{}^{\rho}\,. \tag{4.38}$$

With these field equations in mind it is possible to delve into the evolution equations of our torsional formalism. Since the connection, and by extension the torsion tensor, can only be written in terms of the tetrad and not in terms of the metric, the evolution of the spatial tetrad is considered. Starting with the first Lie derivative of the tetrad and applying all the consequences of our choice of connection and gauge Eq. (3.78) reduces to

$$\begin{aligned}\mathcal{L}_n e^{A(3)}_{\nu} &= n^{\lambda} e^{A(3)}_{\rho} \gamma^{\sigma}{}_{\nu} \hat{T}^{\rho}{}_{\lambda\sigma} \\ &= \hat{A}^{A}{}_{\nu}\,.\end{aligned} \tag{4.39}$$

Here it is observed that all terms apart from the torsional term have vanished. Similarly to what was done in the case of GR, the second order Lie derivative of the fundamental variable, in this case the tetrad, is considered according to Eq. (3.79) and the Weitzenböck gauge

$$\begin{aligned}\mathcal{L}_n \hat{A}^{A}{}_{\nu} =& \tilde{\gamma}^{A}{}_{C} \gamma^{\alpha}{}_{\nu} n^{\lambda} \left(\hat{\nabla}_{\lambda} \hat{A}^{C}{}_{\alpha} - \hat{\nabla}_{\alpha} \hat{A}^{C}{}_{\lambda} \right) + \hat{A}^{A}{}_{\lambda} \hat{A}^{\lambda}{}_{\nu} \\ =& \tilde{\gamma}^{A}{}_{C} \gamma^{\alpha}{}_{\nu} n^{\lambda} \hat{\nabla}_{\lambda} \hat{A}^{C}{}_{\alpha} + \tilde{\gamma}^{A}{}_{C} \gamma^{\alpha}{}_{\nu} \hat{A}^{C}{}_{\lambda} \hat{\nabla}_{\alpha} n^{\lambda} + \hat{A}^{A}{}_{\lambda} \hat{A}^{\lambda}{}_{\nu} \\ =& \tilde{\gamma}^{A}{}_{C} \gamma^{\alpha}{}_{\nu} n^{\lambda} \hat{\nabla}_{\lambda} \hat{A}^{C}{}_{\alpha} + \hat{A}^{A}{}_{\lambda} \hat{A}^{\lambda}{}_{\nu} \\ =& \tilde{\gamma}^{A}{}_{C} \gamma^{\alpha}{}_{\nu} n^{\lambda} \hat{\nabla}_{\lambda} \left(\tilde{\gamma}^{C}{}_{B} \gamma^{\epsilon}{}_{\alpha} n^{\sigma} \hat{T}^{B}{}_{\sigma\epsilon} \right) + \hat{A}^{A}{}_{\lambda} \hat{A}^{\lambda}{}_{\nu} \\ =& \hat{D}_{\rho} \hat{T}^{A\rho}_{(3)\nu} + \hat{A}^{A}{}_{\rho} \hat{A}^{\rho}{}_{\nu} + e^{A(3)}_{\sigma} \gamma^{\sigma}{}_{\lambda} \gamma^{\alpha}{}_{\nu} \hat{\nabla}_{\rho} \hat{T}^{\lambda}{}_{\alpha}{}^{\rho}\,.\end{aligned} \tag{4.40}$$

Here, in the second line, Eq.(4.28) is used to eliminate the second term. In the fourth line the same equation and the Leibniz rule were used to reduce $\hat{\nabla}_{\lambda} \left(\tilde{\gamma}^{C}{}_{B} \gamma^{\epsilon}{}_{\alpha} n^{\sigma} \hat{T}^{B}{}_{\sigma\epsilon} \right)$ to $\tilde{\gamma}^{C}{}_{B} \gamma^{\epsilon}{}_{\alpha} n^{\sigma} \hat{\nabla}_{\lambda} \hat{T}^{B}{}_{\sigma\epsilon}$. The new first term is then expanded using $n^{\lambda} n^{\sigma} = \gamma^{\lambda\sigma} - g^{\lambda\sigma}$ resulting in the final form of this equation. At this point it is not possible to proceed any further without the field equations. Substituting Eqs. (4.37, 4.38) into the evolution equation through the term $e^{A(3)}_{\sigma} \gamma^{\sigma}{}_{\lambda} \gamma^{\alpha}{}_{\nu} \hat{\nabla}_{\rho} \hat{T}^{\lambda}{}_{\alpha}{}^{\rho}$ and converting all space time terms in terms of purely spatial tensors the final form of the second evolution equation is achieved

$$\mathcal{L}_n e^{A(3)}_{\nu} = n^{\lambda} e^{A(3)}_{\rho} \gamma^{\sigma}{}_{\nu} \hat{T}^{\rho}{}_{\lambda\sigma} \tag{4.41}$$
$$= \hat{A}^{A}{}_{\nu},$$

$$\mathcal{L}_n \hat{A}^{a}{}_{\nu} = \hat{D}_{\rho} \hat{T}^{A\rho}_{(3)\nu} + \hat{A}^{A}{}_{\rho} \hat{A}^{\rho}{}_{\nu} + e^{A(3)}_{\sigma} \Big[\hat{A}_{\nu}{}^{\rho} \hat{A}^{\sigma}{}_{\rho} \tag{4.42}$$
$$- \hat{A}^{\rho}{}_{\nu} \hat{A}_{\rho}{}^{\sigma} - \frac{1}{2} \hat{T}^{(3)\rho\chi}_{\nu} \hat{T}^{\sigma(3)}_{\rho\chi} + \hat{T}^{\rho(3)\chi}_{\nu} \left(\hat{T}_{\rho}{}^{\sigma(3)}{}_{\chi} + \hat{T}_{\chi}{}^{\sigma(3)}{}_{\rho} \right)$$
$$+ \frac{2}{\alpha^2} \partial^{(3)}_{\nu}(\alpha) \partial^{\sigma}_{(3)}(\alpha) - \hat{T}^{\chi(3)}_{\rho\chi} \left(\hat{T}^{(3)\sigma\rho}_{\nu} + \hat{T}^{\sigma(3)\rho}_{\nu} \right)$$
$$- \frac{1}{\alpha} \partial^{(3)}_{\rho}(\alpha) \left(\hat{T}^{(3)\sigma\rho}_{\nu} + \hat{T}^{\sigma(3)\rho}_{\nu} \right) - \hat{A} \left(\hat{A}_{\nu}{}^{\sigma} + \hat{A}^{\sigma}{}_{\nu} \right)$$
$$+ \hat{D}_{\nu} \hat{T}^{\rho\sigma}_{(3)\rho} + \hat{D}^{\sigma} \hat{T}^{\rho(3)}_{\nu\rho} + \hat{D}_{\nu} \left(\frac{1}{\alpha} \partial^{\sigma}_{(3)}\{\alpha\} \right)$$
$$+ \hat{D}^{\sigma} \left(\frac{1}{\alpha} \partial^{(3)}_{\nu}\{\alpha\} \right) + 8\pi G \left(2S_{\nu}{}^{\sigma} - \gamma^{\sigma}{}_{\nu} \{S - \rho\} \right) \Big]$$
$$- e^{A(3)}_{\sigma} \gamma^{\sigma}{}_{\lambda} \gamma^{\alpha}{}_{\nu} \hat{\nabla}_{\rho} \hat{T}_{\alpha}{}^{\lambda\rho},$$

where $S_{\nu}{}^{\sigma}$ and S are the spatial stress and scalar stress while ρ is the density. For a more in depth derivation of the second evolution equation please refer to Appendix B.1.

From this equation one may note an important issue with this formulation of TEGR. As was previously stated, it is necessary for the right hand side of the second evolution equation to be written purely in terms of spatial tensors. This will allow all the terms to be written in terms of the spatial tetrad and its spatial derivatives. Unfortunately, one term does not conform to this requirement. The term $e^{A(3)}_{\sigma} \gamma^{\sigma}{}_{\lambda} \gamma^{\alpha}{}_{\nu} \hat{\nabla}_{\rho} \hat{T}_{\alpha}{}^{\lambda\rho}$ stems from the field equations themselves. It is actually the symmetry of the original term that is substituted for in the evolution equations $e^{A(3)}_{\sigma}\gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\hat{\nabla}_{\rho}\hat{T}^{\lambda}{}_{\alpha}{}^{\rho}$. Usually terms like this vanish thorough the field equation substitution but due to the evolution equations being asymmetric and the field equations being symmetric this extra term persists. One can trace the source of this issue to the asymmetry of the tetrad itself. If symmetric tetrads are considered exclusively the formulation would be complete and consistent, however, while such tetrads may be plausible in certain situations they are by no means general. The only way to truly consider symmetric tetrads as the general case would be to re-introduce a non-zero spin connection. This would open a can of worms in its own right due to the spin connection's lack of tetrad based definition and by extension our inability to produce a Gauss equation for it. The last option considered would be to take the spatial metric as the fundamental variable and generate the evolution equations accordingly. In that case it can be shown that the evolution equations are indeed consistent and the issue term vanishes, however, there is a catch. The tensors the evolution equations are built around cannot be expressed purely in terms of the metric instead of the tetrad, making it impossible to solve for the metric numerically.

While this outcome may seem inconclusive, there is a hope that some constraint equation exists that would fix this issue. Such an equation is either not yet known or somehow mistakenly not considered during this work. Another possibility is that this issue is indicating that one cannot omit the spin connection in such a formulation and in order to proceed one must first successfully develop a more robust analytical relationship between the spin connection and the tetrad. While all this is certainly an issue, it is still important that the formalism is still finalized to the latest extent of this research.

Through the first evolution equation of the spatial tetrad one can extract an interesting relationship that mirrors the relationship between GR and TEGR discussed in Chapter 3. Starting by taking the symmetry of the resulting tensor that is a product of this evolution equation $\hat{A}^{A}{}_{\nu}$, one can derive the following

$$\begin{aligned}
\hat{A}^{A}{}_{\nu} + \hat{A}_{\nu}{}^{A} &= n^{\lambda} e^{A(3)}_{\rho} \gamma^{\sigma}{}_{\nu} \left(\hat{T}^{\rho}{}_{\lambda\sigma} + \hat{T}_{\sigma\lambda}{}^{\rho} \right) \\
&= n_{\lambda} e^{A(3)}_{\rho} \gamma^{\sigma}{}_{\nu} \gamma^{\rho\epsilon} \left(2\hat{K}^{\lambda}{}_{\epsilon\sigma} + \hat{T}^{\lambda}{}_{\epsilon\sigma} \right) \\
&= n_{\lambda} e^{A(3)}_{\rho} \gamma^{\sigma}{}_{\nu} \gamma^{\rho\epsilon} 2 \left(\hat{\Gamma}^{\lambda}{}_{\epsilon\sigma} - \mathring{\Gamma}^{\lambda}{}_{\epsilon\sigma} \right) \\
&= e^{A(3)}_{\rho} \gamma^{\sigma}{}_{\nu} \gamma^{\rho\epsilon} 2 \left(\partial_{\sigma} n_{\epsilon} - \partial_{\sigma} n_{\epsilon} + \mathring{\nabla}_{\sigma} n_{\epsilon} \right) \\
&= -2\mathring{k}_{\nu}{}^{a} \,. \qquad (4.43)
\end{aligned}$$

This derivation is thus another way of showing the direct equivalence at the level of equations that GR, based on the Levi-Civita, has with TEGR, with the Weitzenböck gauge.

The remaining equations that are necessary in order to finalize the 3 + 1 system of equations for this theory are the constraint equations, namely the momentum constraint and the Hamiltonian constraint. These can be obtained by contracting the field equations, Eq. (4.37), with $\gamma^{\alpha}{}_{\chi} n^{\sigma}$ and $n^{\alpha} n^{\sigma}$ respectively. The space time terms of the resulting equations are then all converted in terms of purely spatial tensors resulting in their final form

$$\begin{aligned}
\hat{T}^{(3)}_{\rho} \left(\hat{A}^{\rho}{}_{\chi} + \hat{A}_{\chi}{}^{\rho} \right) - \hat{A}^{\lambda\rho} \left(\hat{T}^{(3)}_{\lambda\chi\rho} + \hat{T}^{(3)}_{\rho\chi\lambda} \right) \\
- 2\hat{D}_{\chi} \hat{A} - \hat{D}_{\lambda} \hat{A}^{\lambda}{}_{\chi} = 16\pi G S_{\lambda} \,, \qquad (4.44)
\end{aligned}$$

$$\begin{aligned}
2\hat{T}^{\lambda}_{(3)} \hat{T}^{(3)}_{\lambda} + \frac{1}{2} \hat{A}^{\rho\sigma} \left(\hat{A}_{\rho\sigma} + \hat{A}_{\sigma\rho} \right) - \hat{A}^{2} + 2\hat{D}_{\lambda} \hat{T}^{\lambda}_{(3)} \\
+ 4\hat{D}_{\rho} \left(\frac{1}{\alpha} \partial^{\rho}_{(3)}(\alpha) \right) + \hat{T}^{(3)} = -16\pi G\rho \,. \qquad (4.45)
\end{aligned}$$

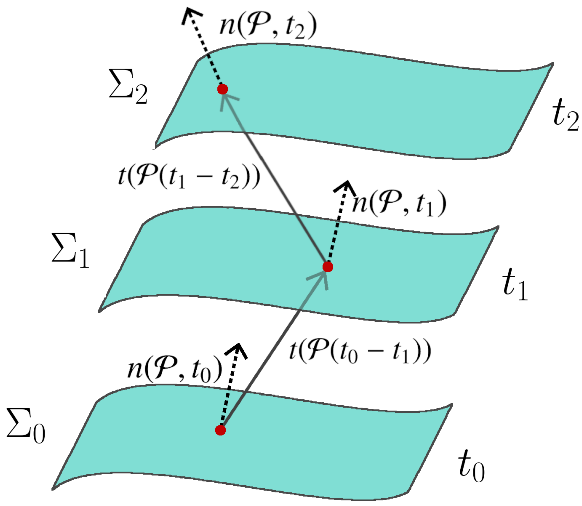

Figure 4.1: This diagram represents the difference between the normal vector and the time vector at a point $\mathcal{P}$ between foliations.

Further details on the derivation of these constraints can be found in Appendix B.2. Finally the expansion of the Lie derivatives themselves are considered in order to obtain a true evolution of the fundamental variable, the tetrad, in time. The time vector t^λ can now be considered and expressed as [8]

$$t^\lambda = \alpha n^\lambda + \beta^\lambda . \tag{4.46}$$

As such the Lie derivative of $\hat{A}^A{}_\nu$ along the time vector can be written as

$$\mathcal{L}_t \hat{A}^A{}_\nu = \alpha \mathcal{L}_n \hat{A}^A{}_\nu + \mathcal{L}_\beta \hat{A}^A{}_\nu , \tag{4.47}$$

where β^λ is the shift vector [58, 8]. The reason one needs to switch between the Lie with respect to the normal vector and the time vector is that the first of the two is not a natural time derivative. The reason for this is that the normal vector is not dual to the 1-form $\mathbf{\Omega}_\nu$ defined in Chapter 3, while the time vector is. This means that in the case of the normal vector their dot product is not unity but α^{-1}. As seen in Fig.(4.1) the time vector connects the points with the same spatial coordinates between time slices or foliations and as such is the ideal vector to use in order to observe evolution [8].

Treating the two resulting terms of Eq.(4.47) separately, the first term is composed of the laps function and the Lie derivative of $\hat{A}^A{}_\nu$ along the normal vector which we both know. The second term is yet unknown, however, expanding it one obtains

$$\mathcal{L}_\beta \hat{A}^A{}_\nu = \beta^\lambda \partial_\lambda \left(\hat{A}^A{}_\nu\right) + \hat{A}^A{}_\lambda \partial_\nu \left(\beta^\lambda\right) . \tag{4.48}$$

Expanding the spatial tetrad evolution equation in a similar way one gets

$$\mathcal{L}_t e^{A(3)}_{\ \nu} = \alpha \hat{A}^A{}_\nu + \beta^\lambda \left(\partial_\lambda \left\{e^{A(3)}_{\ \nu}\right\} - \partial_\nu \left\{e^{A(3)}_{\ \lambda}\right\}\right) . \tag{4.49}$$

With this the $3+1$ formalism in TEGR is concluded. A consistent set of equations have been derived that, at least for a symmetric tetrad, can potentially be used in order to produce numerical simulations of gravitational phenomena. As was stated before there are a number of avenues one may take in order to generalize this further including a more in depth analysis of the spin connection and potentially the use of some elusive third constraint equation that would solve the symmetry issue.

4.3 Gauss Constraint and Evolution Equations in Symmetric Teleparallel Gravity With the Coincident Gauge

In this section the decomposition of STG with the coincident gauge is considered, leading to a full $3+1$ formulation for the theory. As described in Sec. 2.3 this theory is composed of two field equations, one set is obtained through the variation of the chosen Lagrangian with respect to the metric tensor Eq. (2.49) and the other through the variation of the same Lagrangian with respect to the connection Eq. (2.51). Since in this work STEGR is considered with the coincident gauge, the second set of field equations, those obtained through the connection, are innately satisfied and do not need to surface in the $3+1$ derivation. The reason for this is that through the coincident gauge the connection itself vanishes. This has a number of consequences among which is that the covariant derivative and the partial derivative become one and the same when considering purely global indices. Other consequences are that all curvature and torsion tensors as well as any terms built around them vanish as well. Considering the extrinsic curvature one gets

$$\begin{aligned}\overset{\diamond}{k}_{\mu\nu} &= -\gamma^{\lambda}{}_{\mu}\gamma^{\sigma}{}_{\nu}\overset{\diamond}{\nabla}_{\lambda}n_{\sigma} \\ &= \gamma^{\lambda}{}_{\mu}\gamma^{\sigma}{}_{\nu}\partial_{\lambda}\left[\alpha\partial_{\sigma}(t)\right] \\ &= \gamma^{\lambda}{}_{\mu}\gamma^{\sigma}{}_{\nu}\partial_{\sigma}(t)\partial_{\lambda}(\alpha) \\ &= -\gamma^{\lambda}{}_{\mu}\gamma^{\sigma}{}_{\nu}\frac{1}{\alpha}n_{\sigma}\partial_{\lambda}(\alpha) \\ &= 0\,,\end{aligned} \tag{4.50}$$

so the extrinsic curvature also vanishes along with any spatial map of the covariant derivative of the normal vector so long as the map is acted on the normal vector's index. Using this, the acceleration vector reduces to

$$\begin{aligned}a_{\mu} &= n^{\lambda}\overset{\diamond}{\nabla}_{\lambda}n_{\mu} \\ &= n^{\lambda}\partial_{\lambda}n_{\mu} \\ &= n^{\lambda}n_{\mu}\frac{1}{\alpha}\partial_{\lambda}\alpha\,,\end{aligned} \tag{4.51}$$

implying that it is purely temporal.

As with the other theories the Gauss-like equation is considered first. In this theory the gravitating tensor, or the geometric deformation tensor, is the non-metricity tensor that is defined as the covariant derivative of the inverse metric tensor as seen in both Chapter 2 and Chapter 3. The Gauss-like equation for this tensor was derived in Chapter 3 as it is independant of the theory being considered so long as non-metricity is assumed. This is given here once again for ease of reference along with the Gauss-like equation of the disformation tensor that will prove useful in the subsequent calculations

$$\gamma^{\lambda}{}_{\rho}\gamma^{\beta}{}_{\mu}\gamma^{\alpha}{}_{\nu}\overset{\diamond}{Q}_{\lambda\beta\alpha} = \overset{\diamond}{Q}{}^{(3)}_{\rho\mu\nu}\,, \tag{4.52}$$

$$\gamma^{\lambda}{}_{\rho}\gamma^{\beta}{}_{\mu}\gamma^{\alpha}{}_{\nu}\overset{\diamond}{L}_{\lambda\beta\alpha} = \overset{\diamond}{L}{}^{(3)}_{\rho\mu\nu}\,. \tag{4.53}$$

Here we note that in both cases the spatial variant of each tensor is simply the direct mapping of their space time counterparts.

Having obtained the necessary Gauss-like equations the next step is to consider the evolution equations for STEGR. Since this theory is built around the metric tensor as its fundamental variable, the metric field equations as well as the first and second order Lie derivatives of the spatial metric are considered. Starting with the STEGR Field equations Eq. (2.49), these equations are rewritten in such a way that they are mainly based on the disformation tensor. This is done for convenience as this way they are easier to substitute into the second order Lie derivative of the spatial metric later on. Starting from Eq.(2.48) and Eq.(2.50), expanding them and manipulating some of the terms one gets

$$\overset{\diamond}{P}{}^{\alpha}{}_{\mu\nu} = \frac{1}{2}L^{\alpha}{}_{\mu\nu} + \frac{1}{4}g_{\mu\nu}\left(\overset{\diamond}{L}_{\epsilon}{}^{\epsilon\alpha} - \overset{\diamond}{L}{}^{\alpha\epsilon}{}_{\epsilon}\right) - \frac{1}{4}\delta^{\alpha}_{(\mu|}L^{\epsilon}{}_{\epsilon|\nu)}, \tag{4.54}$$

$$\frac{1}{\sqrt{-g}}q_{\mu\nu} = \overset{\diamond}{Q}{}^{\alpha\beta}{}_{\nu}\overset{\diamond}{L}_{\alpha\beta\mu} - \frac{1}{2}\overset{\diamond}{Q}_{\mu\alpha\beta}\overset{\diamond}{L}_{\nu}{}^{\alpha\beta} + \frac{1}{2}\overset{\diamond}{Q}_{\alpha\mu\nu}\left(\overset{\diamond}{L}_{\sigma}{}^{\sigma\alpha} - \overset{\diamond}{L}{}^{\alpha\sigma}{}_{\sigma}\right) + \overset{\diamond}{L}_{\sigma}{}^{\sigma}{}_{\mu}\overset{\diamond}{L}_{\epsilon}{}^{\epsilon}{}_{\nu}. \tag{4.55}$$

Noting that $\hat{\nabla}_\alpha Log\left(\sqrt{-g}\right) = \frac{1}{\sqrt{-g}}\hat{\nabla}_\alpha \sqrt{-g}$ and thus $\hat{\nabla}_\alpha Log\left(\sqrt{-g}\right) = \frac{1}{2}\mathring{Q}_\alpha$,

$$\frac{2}{\sqrt{-g}}\mathring{\nabla}_\alpha\left(\sqrt{-g}\mathring{P}^\alpha{}_{\mu\nu}\right) = -2\mathring{L}_\sigma{}^\sigma{}_\alpha\mathring{P}^\alpha{}_{\mu\nu} + 2\hat{\nabla}_\alpha\mathring{P}^\alpha{}_{\mu\nu}. \tag{4.56}$$

Substituting all of the above into Eq. (2.49) one gets

$$\begin{aligned}\partial_\alpha\mathring{L}^\alpha{}_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Theta + \mathring{L}_\sigma{}^\sigma{}_\alpha\mathring{L}^\alpha{}_{\mu\nu}\\ + \frac{1}{2}\partial_{(\mu|}\mathring{L}^\epsilon{}_{\epsilon|\nu)} + \mathring{Q}^{\sigma\epsilon}{}_\nu\mathring{L}_{\sigma\epsilon\mu} - \frac{1}{2}\mathring{Q}_{\mu\nu\epsilon}\mathring{L}_\nu{}^{\sigma\epsilon}.\end{aligned} \tag{4.57}$$

Another modification that should be noted here is the replacement of the scalar metricity tensor in terms of fully contracted disformation tensors and the energy momentum scalar

$$\mathring{Q} = \mathring{L}_\lambda{}^\lambda{}_\alpha\left(\mathring{L}^{\alpha\epsilon}{}_\epsilon - \mathring{L}_\epsilon{}^{\epsilon\alpha}\right) + \partial_\alpha\left(\mathring{L}_\epsilon{}^{\epsilon\alpha} - \mathring{L}^{\alpha\epsilon}{}_\epsilon\right) - \Theta\,. \tag{4.58}$$

This relation can be obtained through a combination of the contraction of the original field equations and the definition of the scalar non-metricity itself, $\mathring{Q} = -\mathring{Q}_{\alpha\mu\nu}\mathring{P}^{\alpha\mu\nu}$ [51].

The metric evolution equations in STEGR assuming the coincident gauge are now considered. Taking the generalized evolution equations Eqs. (3.74, 3.76) and considering all the consequences of the coincident gauge, the two evolution equations reduce to

$$\begin{aligned}
\mathcal{L}_n \gamma_{\mu\nu} &= \overset{\diamond}{B}_{(\mu\nu)} \\
&= 2\overset{\diamond}{B}_{\mu\nu}\,, \qquad (4.59)\\
\mathcal{L}_n \overset{\diamond}{B}_{\mu\nu} &= -\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\partial_{\lambda}\overset{\diamond}{L}^{\lambda}{}_{\alpha\beta} + \partial^{(3)}_{\lambda}\overset{\diamond}{L}^{\lambda}{}_{\mu\nu} \\
&\quad + \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\overset{\diamond}{a}_{\sigma}\overset{\diamond}{L}^{\sigma}{}_{\alpha\beta} - \gamma^{\epsilon}{}_{(\nu|}\gamma^{\chi}{}_{|\mu)}\overset{\diamond}{B}_{\alpha\chi}n_{\sigma}\overset{\diamond}{Q}_{\epsilon}{}^{\sigma\alpha}\,. \qquad (4.60)
\end{aligned}$$

At this point it is noted that the first term of the second evolution equation can be substituted with the field equations, Eq. (4.57). Substituting the field equations one gets

$$\begin{aligned}
\mathcal{L}_n \overset{\diamond}{B}_{\mu\nu} = -\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\Big(&\Theta_{\beta\alpha} - \frac{1}{2}g_{\beta\alpha}\Theta + \overset{\diamond}{L}_{\sigma}{}^{\sigma}{}_{\lambda}\overset{\diamond}{L}^{\lambda}{}_{\beta\alpha} + \frac{1}{2}\partial_{(\beta|}\overset{\diamond}{L}^{\epsilon}{}_{\epsilon|\alpha)} + \overset{\diamond}{Q}^{\lambda\sigma}{}_{\alpha}\overset{\diamond}{L}_{\lambda\sigma\beta} \qquad (4.61)\\
&- \frac{1}{2}\overset{\diamond}{Q}_{\beta\lambda\sigma}\overset{\diamond}{L}_{\alpha}{}^{\lambda\sigma}\Big) + \partial^{(3)}_{\lambda}\overset{\diamond}{L}^{\lambda}{}_{\beta\alpha} + \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\overset{\diamond}{a}_{\sigma}\overset{\diamond}{L}^{\sigma}{}_{\alpha\beta} - \gamma^{\epsilon}{}_{(\nu|}\gamma^{\chi}{}_{|\mu)}\overset{\diamond}{B}_{\alpha\chi}n_{\sigma}\overset{\diamond}{Q}_{\epsilon}{}^{\sigma\alpha}\,.
\end{aligned}$$

Writing all spacetime terms in terms of purely spatial tensors and expanding the energy momentum tensor in terms of the spatial stress $S_{\mu\nu}$, its trace S and the density ρ, the final form of the second evolution equation in STEGR is obtained

$$\begin{aligned}
\mathcal{L}_n \overset{\diamond}{B}_{\mu\nu} =& -8\pi G\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S - \rho\right)\right] \\
&+ \frac{1}{2}\overset{\diamond}{Q}_{\epsilon}{}^{\sigma(3)}{}_{\sigma}\overset{\diamond}{L}{}^{\epsilon(3)}_{\mu\nu} + \frac{1}{\alpha}\partial^{(3)}_{\epsilon}\left(\alpha\right)\overset{\diamond}{L}{}^{\epsilon(3)}_{\mu\nu} + \partial^{(3)}_{\sigma}\overset{\diamond}{L}{}^{\sigma(3)}_{\mu\nu} \\
&- \overset{\diamond}{B}_{\mu\nu}\overset{\diamond}{B} - \frac{1}{2}\partial^{(3)}_{(\nu|}\overset{\diamond}{L}{}^{\sigma(3)}_{\sigma|\mu)} + \frac{1}{2}\partial^{(3)}_{(\nu|}\left(\frac{1}{\alpha}\partial^{(3)}_{|\mu)}\alpha\right) \\
&- \frac{1}{2}\overset{\diamond}{Q}{}^{\lambda\sigma(3)}{}_{\nu}\overset{\diamond}{Q}{}^{(3)}_{[\lambda\sigma]\mu} + \frac{1}{4}\overset{\diamond}{Q}{}^{(3)}_{\mu\lambda\sigma}\overset{\diamond}{Q}{}^{(3)\lambda\sigma}_{\nu} + \overset{\diamond}{B}{}^{\sigma}{}_{(\nu|}\overset{\diamond}{B}_{\sigma|\mu)} \\
&+ \frac{1}{\alpha^{2}}\partial^{(3)}_{\nu}\alpha\,\partial^{(3)}_{\mu}\alpha\,. \qquad (4.62)
\end{aligned}$$

For a more extensive derivation please refer to Appendix C.1. The remaining equations that need to be considered in order to produce a fully consistant system of equations are the Hamiltonian and momentum constraints. These can be obtained by contracting the modified version of the field equations Eq. (4.57) in the standard way as was done in the previous sections and then once again converting any spacetime terms in terms of purely spatial tensors

$$\begin{aligned}16\pi G\rho =& \frac{1}{2}\overset{\diamond}{Q}_{\epsilon}{}^{\sigma(3)}{}_{\sigma}\overset{\diamond}{L}^{\epsilon}{}_{\nu(3)}{}^{\nu} + \gamma^{\mu\nu}\partial^{(3)}_{\epsilon}\overset{\diamond}{L}^{\epsilon(3)}_{\mu\nu} - \overset{\diamond}{B}^{2} - \partial^{\nu}_{(3)}\overset{\diamond}{L}^{\epsilon(3)}_{\epsilon\nu} \\ & - \frac{1}{2}\overset{\diamond}{Q}^{\lambda\sigma\nu}_{(3)}\overset{\diamond}{Q}^{(3)}_{[\lambda\sigma]\nu} + \frac{1}{4}\overset{\diamond}{Q}^{\lambda\sigma\nu}_{(3)}\overset{\diamond}{Q}^{(3)}_{\lambda\sigma\nu} + \overset{\diamond}{B}^{\sigma\nu}\overset{\diamond}{B}_{\sigma\nu}\,,\end{aligned} \tag{4.63}$$

$$8\pi G S_{\beta} = \frac{1}{2}\overset{\diamond}{B}^{\epsilon\lambda}\overset{\diamond}{Q}^{(3)}_{\beta\epsilon\lambda} + \overset{\diamond}{L}^{\theta\rho(3)}_{\rho}\overset{\diamond}{B}_{\beta\theta} - \gamma^{\sigma\rho}\partial^{(3)}_{\beta}\left(\overset{\diamond}{B}_{\rho\sigma}\right) + \gamma^{\sigma\rho}\partial^{(3)}_{\rho}\left(\overset{\diamond}{B}_{\beta\sigma}\right)\,. \tag{4.64}$$

For a more extensive derivation of these constraints please refer to Appendix C.2. Finally, similarly to what was done in the torsional gravity section, the Lie derivative of the second evolution equation is broken down in order to get the evolution of the $B_{\mu\nu}$ in time

$$\mathcal{L}_{t}\overset{\diamond}{B}_{\mu\nu} = \alpha\mathcal{L}_{n}\overset{\diamond}{B}_{\mu\nu} + \mathcal{L}_{\beta}\overset{\diamond}{B}_{\mu\nu}\,, \tag{4.65}$$

where general Lie derivative properties have been used [58, 8] and the vector β_{μ} is the shift vector. The first term here consists of the laps fuction and Eq. (4.62), which are both known. Expanding the second term of Eq. (4.65), the following relation is obtained

$$\mathcal{L}_{\beta}\overset{\diamond}{B}_{\mu\nu} = \beta^{\lambda}\partial^{(3)}_{\lambda}\left(\overset{\diamond}{B}_{\mu\nu}\right) + \overset{\diamond}{B}_{\mu\lambda}\partial^{(3)}_{\nu}\left(\beta^{\lambda}\right) + \overset{\diamond}{B}_{\lambda\nu}\partial^{(3)}_{\mu}\left(\beta^{\lambda}\right)\,. \tag{4.66}$$

The same expansion is carried out on the first evolution equation, the first Lie derivative of the spatial metric

$$\mathcal{L}_{t}\gamma_{\mu\nu} = \alpha 2\overset{\diamond}{B}_{\mu\nu} + \beta^{\lambda}\partial^{(3)}_{\lambda}\left\{\gamma_{\mu\nu}\right\} + \gamma_{\mu\lambda}\partial^{(3)}_{\nu}\left\{\beta^{\lambda}\right\} + \gamma_{\lambda\nu}\partial^{(3)}_{\mu}\left\{\beta^{\lambda}\right\}\,. \tag{4.67}$$

With this, the full 3 + 1 decomposition of STEGR with the coincident gauge is complete. A fully consistent set of equations have been derived. In this case all of the final equations are written fully in terms of purely spatial tensors and as such the issues that were present with the TEGR formulation are not present. The equations are thus in par with those of the GR 3 + 1 formalism

In this Chapter a number of Important results have been presented that shed light into how each of the three theories based on the three separate geometric deformation types behave when considered through the lens of a 3+1 formalism. Starting from the 3+1 decomposition for a general affine connection assuming non-metricity, Eqs.(3.74 - 3.78), three separate decompositions were derived.

In the case of Gr, and thus when all geometric deformation terms vanish apart from curvature ones, it was confirmed that the aforementioned general equations do indeed reduce to the standard ADM evolution equations as well as the expected Hamiltonian and momentum constraint equations. This helped to solidify the validity of the general equations and justifying pushing forward with the two remaining theories.

In the case of TEGR, based on torsion, the tetrad formulation was considered while choosing to work in the spin zero gauge. While the constraint equations were successfully derived and the evolution equations were written purely in terms of spatial terms on the right hand side, a problem arose while linking to the field equations. The lack of symmetry of the evolution equations, a direct consequence of asymmetric nature of the tetrad, meant that one covaraint derivative term was not substituted out through the field equations. This led to an incomplete formulation, apart from the case of a symmetric tetrad with spin zero.

Finally STEGR was considered while taking the coincident guage. In this case the metric formulation was considered and both the constraint and evolution equations were fully and successfully derived. For this reason, the STEGR 3+1 formalism will be the one considered in the next chapter were a validation analysis is carried out and a modified formulation of these equations is derived in order to pave the way for future stable simulations.

5 Application of 3+1 formalism in STGR

In this chapter the final STEGR system of equations are considered and tested in order to further show their usefulness and consistency with their GR counterparts. In the first Section, a number of known spatial metric solutions to the GR 3+1 system of equations are tested. If they are indeed solutions to the STEGR equations, such metrics would be the building blocks of the spatial initial conditions once simulations are considered [8]. In the second section an equivalent of the BSSN (Baumgarte, Shapiro, Shibata and Nakamura) formalism in GR is derived from the current STEGR 3+1 system of equations. This formalism was originally built to address the long term numerical instability of the GR ADM equations and is thus carried out with the intention that the STEGR equations become hyperbolic and well-posed leading to potentially more stable simulations [8, 9].

5.1 Testing Known Solutions

GR and STEGR are mathematically bound to produce the same solutions. The reason for this is that at the level of the Lagrangian they are only separated by a boundary term. This boundary term disappears when considering the action, that is, when integrating over the manifold. Varying the Lagrangians with respect to the respective fundamental variables produces different filed equations, but ones that are mathematically bound to have the same solution, assuming Stokes theorem.

The explicit similarities and differences between the two 3+1 formalisms are now highlighted. The relationship between extrinsic curvature in GR and the $B_{\mu\nu}$ tensor in STEGR, is considered first. It is noted that the tensor $B_{\mu\nu}$ will be called the extrinsic metricity from this point on. Starting from the definition of the extrinsic curvature one can derive

$$
\begin{aligned}
\mathring{k}_{\alpha\beta} &:= -\gamma^{\nu}{}_{\alpha}\gamma^{\mu}{}_{\beta}\mathring{\nabla}_{\nu}n_{\mu} \\
&= -\gamma^{\nu}{}_{\alpha}\gamma^{\mu}{}_{\beta}(\partial_{\nu}n_{\mu} - n_{\rho}\mathring{\Gamma}^{\rho}{}_{\mu\nu}) \\
&= -\gamma^{\nu}{}_{\alpha}\gamma^{\mu}{}_{\beta}\left[-\partial_{\nu}\left(\alpha\partial_{\mu}t\right) - n_{\rho}\frac{1}{2}g^{\lambda\rho}\left(-\partial_{\lambda}g_{\mu\nu} + \partial_{\mu}g_{\lambda\nu} + \partial_{\nu}g_{\lambda\mu}\right)\right] \\
&= -\gamma^{\nu}{}_{\alpha}\gamma^{\mu}{}_{\beta}\frac{1}{2}n^{\lambda}\left(\overset{\diamond}{Q}_{\lambda\mu\nu} - \overset{\diamond}{Q}_{\mu\lambda\nu} - \overset{\diamond}{Q}_{\nu\lambda\mu}\right) \\
&= -\gamma^{\nu}{}_{\alpha}\gamma^{\mu}{}_{\beta}n^{\lambda}\overset{\diamond}{L}_{\lambda\mu\nu} \\
&= -\overset{\diamond}{B}_{\mu\nu},
\end{aligned}
\tag{5.1}
$$

where in the third line the first term vanishes since $-\partial_{\nu}\left(\alpha\partial_{\mu}t\right) = \frac{1}{\alpha}n_{\mu}\partial_{\nu}\alpha$. It has thus been shown that the extrinsic curvature and the extrinsic metricity are one and the same up to a sign that is the result of a chosen convention. This implies that the first lie of the metric of STEGR is trivially the same as the that of GR up to any use of the covariant derivative. Having defined this equivalence thoroughly it is important to also highlight the difference between these theories, that is, apart from their geometric origins. As they are a direct consequence of the field equations themselves, the second evolution equations of GR and STEGR are now considered. Here by second evolution equation one understands Eq.(4.24) with its left hand side expanded into a partial with respect to time as in [8] for GR and Eq.(4.65) with Eqs.(4.62,4.66) substituted into it for STEGR. It should be noted that since we will be concentrating on vacuum solutions all tensors derived from the energy momentum tensors have been set to zero. These terms would not contribute to the equations any differently as the energy momentum tensor was treated the same way in both theories. Expanding the equations up to their most basic constituents, the partial derivatives of the metric, lapse function and the shift vector, one gets

$$
\begin{aligned}
\partial_t \mathring{k}_{\mu\nu} = {} & \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\overset{(3)}{\partial}_{\mu}\beta_{\nu}}{2\alpha} + \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\overset{(3)}{\partial}_{\nu}\beta_{\mu}}{2\alpha} - \frac{1}{2}\overset{(3)}{\partial}_{\epsilon}\alpha\partial^{\epsilon}_{(3)}\gamma_{\mu\nu} - \frac{\overset{(3)}{\partial}_{\epsilon}\beta_{\mu}\partial^{\epsilon}_{(3)}\beta_{\nu}}{2\alpha} - \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\overset{(3)}{\partial}_{\zeta}\overset{(3)}{\partial}_{\epsilon}\gamma_{\mu\nu} + \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\overset{(3)}{\partial}_{\zeta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\epsilon} \\
& - \frac{\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\zeta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\epsilon}}{2\alpha} + \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\overset{(3)}{\partial}_{\zeta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\epsilon} - \frac{\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\zeta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\epsilon}}{2\alpha} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta_{\zeta}\overset{(3)}{\partial}_{\epsilon}\gamma_{\nu\Theta}\partial^{\zeta}_{(3)}\gamma_{\mu\eta}}{2\alpha} + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta_{\zeta}\overset{(3)}{\partial}_{\epsilon}\gamma_{\eta\Theta}\partial^{\zeta}_{(3)}\gamma_{\mu\nu}}{4\alpha} \\
& - \frac{\beta^{\epsilon}\beta_{\zeta}\overset{(3)}{\partial}_{\epsilon}\alpha\partial^{\zeta}_{(3)}\gamma_{\mu\nu}}{2\alpha^{2}} + \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\beta_{\zeta}\partial^{\zeta}_{(3)}\gamma_{\mu\nu}}{2\alpha} - \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\gamma_{\nu\zeta}\partial^{\zeta}_{(3)}\beta_{\mu}}{2\alpha} + \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\zeta}\gamma_{\nu\epsilon}\partial^{\zeta}_{(3)}\beta_{\mu}}{2\alpha} - \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\gamma_{\mu\zeta}\partial^{\zeta}_{(3)}\beta_{\nu}}{2\alpha} \\
& + \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\zeta}\gamma_{\mu\epsilon}\partial^{\zeta}_{(3)}\beta_{\nu}}{2\alpha} + \frac{\beta^{\epsilon}\beta_{\zeta}\partial^{\zeta}_{(3)}\overset{(3)}{\partial}_{\epsilon}\gamma_{\mu\nu}}{2\alpha} + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\zeta}\gamma_{\nu\Theta}\overset{(3)}{\partial}_{\eta}\gamma_{\mu\epsilon}}{2\alpha} + \frac{\gamma^{\zeta\eta}\beta_{\epsilon}\partial^{\epsilon}_{(3)}\gamma_{\mu\nu}\overset{(3)}{\partial}_{\eta}\beta_{\zeta}}{2\alpha} - \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\overset{(3)}{\partial}_{\zeta}\gamma_{\nu\eta}\partial^{\eta}_{(3)}\gamma_{\mu\epsilon} \\
& + \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\overset{(3)}{\partial}_{\eta}\gamma_{\nu\zeta}\partial^{\eta}_{(3)}\gamma_{\mu\epsilon} + \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\overset{(3)}{\partial}_{\zeta}\gamma_{\epsilon\eta}\partial^{\eta}_{(3)}\gamma_{\mu\nu} - \frac{\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\zeta}\gamma_{\epsilon\eta}\partial^{\eta}_{(3)}\gamma_{\mu\nu}}{2\alpha} - \frac{1}{4}\gamma^{\epsilon\zeta}\alpha\overset{(3)}{\partial}_{\eta}\gamma_{\epsilon\zeta}\partial^{\eta}_{(3)}\gamma_{\mu\nu} \\
& + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta_{\zeta}\partial^{\zeta}_{(3)}\gamma_{\mu\eta}\overset{(3)}{\partial}_{\Theta}\gamma_{\nu\epsilon}}{2\alpha} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\eta}\gamma_{\mu\epsilon}\overset{(3)}{\partial}_{\Theta}\gamma_{\nu\zeta}}{2\alpha} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta_{\zeta}\partial^{\zeta}_{(3)}\gamma_{\mu\nu}\overset{(3)}{\partial}_{\Theta}\gamma_{\epsilon\eta}}{2\alpha} + \frac{\beta^{\epsilon}\partial^{\zeta}_{(3)}\beta_{\nu}\overset{(3)}{\partial}_{\mu}\gamma_{\epsilon\zeta}}{2\alpha} \\
& + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta_{\zeta}\partial^{\zeta}_{(3)}\gamma_{\nu\eta}\overset{(3)}{\partial}_{\mu}\gamma_{\epsilon\Theta}}{2\alpha} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\eta}\gamma_{\nu\epsilon}\overset{(3)}{\partial}_{\mu}\gamma_{\zeta\Theta}}{2\alpha} + \frac{1}{2}\partial^{\epsilon}_{(3)}\alpha\overset{(3)}{\partial}_{\mu}\gamma_{\nu\epsilon} + \frac{1}{4}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\overset{(3)}{\partial}_{\zeta}\gamma_{\eta\Theta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\epsilon} \\
& - \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\eta}\beta_{\zeta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\epsilon}}{2\alpha} - \frac{1}{2}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\overset{(3)}{\partial}_{\Theta}\gamma_{\zeta\eta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\epsilon} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\epsilon}\gamma_{\eta\Theta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\zeta}}{4\alpha} + \frac{\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\epsilon}\alpha\overset{(3)}{\partial}_{\mu}\gamma_{\nu\zeta}}{2\alpha^{2}} \\
& + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\Theta}\gamma_{\epsilon\eta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\zeta}}{2\alpha} - \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\beta_{\zeta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\eta}}{2\alpha} + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\zeta}\gamma_{\epsilon\Theta}\overset{(3)}{\partial}_{\mu}\gamma_{\nu\eta}}{2\alpha} - \frac{\partial^{\epsilon}_{(3)}\beta_{\nu}\overset{(3)}{\partial}_{\mu}\beta_{\epsilon}}{2\alpha} \\
& + \frac{\overset{(3)}{\partial}_{\epsilon}\beta_{\nu}\overset{(3)}{\partial}_{\mu}\beta^{\epsilon}}{2\alpha} - \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\gamma_{\nu\eta}\overset{(3)}{\partial}_{\mu}\beta_{\zeta}}{2\alpha} + \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\eta}\gamma_{\nu\epsilon}\overset{(3)}{\partial}_{\mu}\beta_{\zeta}}{2\alpha} + \frac{\beta_{\epsilon}\partial^{\epsilon}_{(3)}\gamma_{\nu\zeta}\overset{(3)}{\partial}_{\mu}\beta^{\zeta}}{2\alpha} - \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\zeta}\gamma_{\nu\epsilon}\overset{(3)}{\partial}_{\mu}\beta^{\zeta}}{2\alpha} \\
& + \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\gamma_{\zeta\eta}\overset{(3)}{\partial}_{\mu}\beta_{\nu}}{4\alpha} - \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\alpha\overset{(3)}{\partial}_{\mu}\beta_{\nu}}{2\alpha^{2}} + \frac{\gamma^{\epsilon\zeta}\overset{(3)}{\partial}_{\zeta}\beta_{\epsilon}\overset{(3)}{\partial}_{\mu}\beta_{\nu}}{2\alpha} - \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\eta}\gamma_{\epsilon\zeta}\overset{(3)}{\partial}_{\mu}\beta_{\nu}}{2\alpha} + \frac{\beta^{\epsilon}\partial^{\zeta}_{(3)}\beta_{\mu}\overset{(3)}{\partial}_{\nu}\gamma_{\epsilon\zeta}}{2\alpha} \\
& - \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\mu}\beta^{\zeta}\overset{(3)}{\partial}_{\nu}\gamma_{\epsilon\zeta}}{2\alpha} + \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\mu}\beta_{\zeta}\overset{(3)}{\partial}_{\nu}\gamma_{\epsilon\eta}}{2\alpha} + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta_{\zeta}\partial^{\zeta}_{(3)}\gamma_{\mu\eta}\overset{(3)}{\partial}_{\nu}\gamma_{\epsilon\Theta}}{2\alpha} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\eta}\gamma_{\mu\epsilon}\overset{(3)}{\partial}_{\nu}\gamma_{\zeta\Theta}}{2\alpha} \\
& + \frac{1}{4}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\overset{(3)}{\partial}_{\mu}\gamma_{\epsilon\eta}\overset{(3)}{\partial}_{\nu}\gamma_{\zeta\Theta} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\mu}\gamma_{\epsilon\eta}\overset{(3)}{\partial}_{\nu}\gamma_{\zeta\Theta}}{2\alpha} + \frac{1}{2}\partial^{\epsilon}_{(3)}\alpha\overset{(3)}{\partial}_{\nu}\gamma_{\mu\epsilon} + \frac{1}{4}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\overset{(3)}{\partial}_{\zeta}\gamma_{\eta\Theta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\epsilon} \\
& - \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\eta}\beta_{\zeta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\epsilon}}{2\alpha} - \frac{1}{2}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\overset{(3)}{\partial}_{\Theta}\gamma_{\zeta\eta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\epsilon} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\epsilon}\gamma_{\eta\Theta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\zeta}}{4\alpha} + \frac{\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\epsilon}\alpha\overset{(3)}{\partial}_{\nu}\gamma_{\mu\zeta}}{2\alpha^{2}} \\
& + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\Theta}\gamma_{\epsilon\eta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\zeta}}{2\alpha} - \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\beta_{\zeta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\eta}}{2\alpha} + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\overset{(3)}{\partial}_{\zeta}\gamma_{\epsilon\Theta}\overset{(3)}{\partial}_{\nu}\gamma_{\mu\eta}}{2\alpha} - \frac{\partial^{\epsilon}_{(3)}\beta_{\mu}\overset{(3)}{\partial}_{\nu}\beta_{\epsilon}}{2\alpha} + \frac{\overset{(3)}{\partial}_{\mu}\beta^{\epsilon}\overset{(3)}{\partial}_{\nu}\beta_{\epsilon}}{2\alpha} \\
& + \frac{\overset{(3)}{\partial}_{\epsilon}\beta_{\mu}\overset{(3)}{\partial}_{\nu}\beta^{\epsilon}}{2\alpha} + \frac{\overset{(3)}{\partial}_{\mu}\beta_{\epsilon}\overset{(3)}{\partial}_{\nu}\beta^{\epsilon}}{2\alpha} - \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\gamma_{\mu\eta}\overset{(3)}{\partial}_{\nu}\beta_{\zeta}}{2\alpha} + \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\eta}\gamma_{\mu\epsilon}\overset{(3)}{\partial}_{\nu}\beta_{\zeta}}{2\alpha} + \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\mu}\gamma_{\epsilon\eta}\overset{(3)}{\partial}_{\nu}\beta_{\zeta}}{2\alpha} \\
& - \frac{\gamma^{\epsilon\zeta}\overset{(3)}{\partial}_{\mu}\beta_{\epsilon}\overset{(3)}{\partial}_{\nu}\beta_{\zeta}}{2\alpha} + \frac{\beta_{\epsilon}\partial^{\epsilon}_{(3)}\gamma_{\mu\zeta}\overset{(3)}{\partial}_{\nu}\beta^{\zeta}}{2\alpha} - \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\zeta}\gamma_{\mu\epsilon}\overset{(3)}{\partial}_{\nu}\beta^{\zeta}}{2\alpha} - \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\mu}\gamma_{\epsilon\zeta}\overset{(3)}{\partial}_{\nu}\beta^{\zeta}}{2\alpha} + \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\gamma_{\zeta\eta}\overset{(3)}{\partial}_{\nu}\beta_{\mu}}{4\alpha} \\
& - \frac{\beta^{\epsilon}\overset{(3)}{\partial}_{\epsilon}\alpha\overset{(3)}{\partial}_{\nu}\beta_{\mu}}{2\alpha^{2}} + \frac{\gamma^{\epsilon\zeta}\overset{(3)}{\partial}_{\zeta}\beta_{\epsilon}\overset{(3)}{\partial}_{\nu}\beta_{\mu}}{2\alpha} - \frac{\gamma^{\zeta\eta}\beta^{\epsilon}\overset{(3)}{\partial}_{\eta}\gamma_{\epsilon\zeta}\overset{(3)}{\partial}_{\nu}\beta_{\mu}}{2\alpha} - \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\overset{(3)}{\partial}_{\nu}\overset{(3)}{\partial}_{\mu}\gamma_{\epsilon\zeta} - \overset{(3)}{\partial}_{\nu}\overset{(3)}{\partial}_{\mu}\alpha
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
\partial_t \overset{\diamond}{B}_{\mu\nu} = &-\frac{\gamma_{\nu\zeta}\beta^{\epsilon}\partial^{(3)}_{\epsilon}\partial^{(3)}_{\mu}\beta^{\zeta}}{2\alpha} - \frac{\gamma_{\mu\zeta}\beta^{\epsilon}\partial^{(3)}_{\epsilon}\partial^{(3)}_{\nu}\beta^{\zeta}}{2\alpha} + \frac{1}{2}\partial^{(3)}_{\epsilon}\alpha\partial^{\epsilon}_{(3)}\gamma_{\mu\nu} - \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\partial^{(3)}_{\epsilon}\gamma_{\mu\nu}\partial^{(3)}_{\zeta}\gamma_{\eta\Theta}}{4\alpha} + \frac{\beta^{\epsilon}\beta^{\zeta}\partial^{(3)}_{\epsilon}\alpha\partial^{(3)}_{\zeta}\gamma_{\mu\nu}}{2\alpha^2} \\
&- \frac{\beta^{\epsilon}\partial^{(3)}_{\epsilon}\beta^{\zeta}\partial^{(3)}_{\zeta}\gamma_{\mu\nu}}{2\alpha} + \frac{\gamma^{\eta\Theta}\beta^{\epsilon}\beta^{\zeta}\partial^{(3)}_{\epsilon}\gamma_{\mu\eta}\partial^{(3)}_{\zeta}\gamma_{\nu\Theta}}{2\alpha} - \frac{\beta^{\epsilon}\partial^{(3)}_{\epsilon}\gamma_{\mu\nu}\partial^{(3)}_{\zeta}\beta^{\zeta}}{2\alpha} + \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\partial^{(3)}_{\zeta}\partial^{(3)}_{\epsilon}\gamma_{\mu\nu} \\
&- \frac{\beta^{\epsilon}\beta^{\zeta}\partial^{(3)}_{\zeta}\partial^{(3)}_{\epsilon}\gamma_{\mu\nu}}{2\alpha} - \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\partial^{(3)}_{\zeta}\partial^{(3)}_{\mu}\gamma_{\nu\epsilon} - \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\partial^{(3)}_{\zeta}\partial^{(3)}_{\nu}\gamma_{\mu\epsilon} + \frac{\gamma^{\eta\Theta}\gamma_{\nu\zeta}\beta^{\epsilon}\partial^{(3)}_{\epsilon}\gamma_{\mu\Theta}\partial^{(3)}_{\eta}\beta^{\zeta}}{2\alpha} \\
&+ \frac{\gamma^{\eta\Theta}\gamma_{\mu\zeta}\beta^{\epsilon}\partial^{(3)}_{\epsilon}\gamma_{\nu\Theta}\partial^{(3)}_{\eta}\beta^{\zeta}}{2\alpha} + \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\partial^{(3)}_{\zeta}\gamma_{\nu\eta}\partial^{\eta}_{(3)}\gamma_{\mu\epsilon} - \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\partial^{(3)}_{\eta}\gamma_{\nu\zeta}\partial^{\eta}_{(3)}\gamma_{\mu\epsilon} - \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\partial^{(3)}_{\zeta}\gamma_{\epsilon\eta}\partial^{\eta}_{(3)}\gamma_{\mu\nu} \\
&+ \frac{1}{4}\gamma^{\epsilon\zeta}\alpha\partial^{(3)}_{\eta}\gamma_{\epsilon\zeta}\partial^{\eta}_{(3)}\gamma_{\mu\nu} + \frac{\gamma^{\zeta\Theta}\gamma_{\mu\epsilon}\gamma_{\nu\eta}\partial^{(3)}_{\zeta}\beta^{\epsilon}\partial^{(3)}_{\Theta}\beta^{\eta}}{2\alpha} - \frac{1}{2}\partial^{\epsilon}_{(3)}\alpha\partial^{(3)}_{\mu}\gamma_{\nu\epsilon} - \frac{1}{4}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\partial^{(3)}_{\zeta}\gamma_{\eta\Theta}\partial^{(3)}_{\mu}\gamma_{\nu\epsilon} \\
&+ \frac{1}{2}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\partial^{(3)}_{\Theta}\gamma_{\zeta\eta}\partial^{(3)}_{\mu}\gamma_{\nu\epsilon} - \frac{\gamma_{\nu\epsilon}\partial^{(3)}_{\zeta}\beta^{\zeta}\partial^{(3)}_{\mu}\beta^{\epsilon}}{2\alpha} - \frac{\gamma^{\eta\Theta}\gamma_{\nu\zeta}\beta^{\epsilon}\partial^{(3)}_{\epsilon}\gamma_{\eta\Theta}\partial^{(3)}_{\mu}\beta^{\zeta}}{4\alpha} - \frac{\beta^{\epsilon}\partial^{(3)}_{\epsilon}\gamma_{\nu\zeta}\partial^{(3)}_{\mu}\beta^{\zeta}}{2\alpha} \\
&+ \frac{\gamma_{\nu\zeta}\beta^{\epsilon}\partial^{(3)}_{\epsilon}\alpha\partial^{(3)}_{\mu}\beta^{\zeta}}{2\alpha^2} - \frac{1}{4}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\partial^{(3)}_{\mu}\gamma_{\epsilon\eta}\partial^{(3)}_{\nu}\gamma_{\zeta\Theta} - \frac{1}{2}\partial^{\epsilon}_{(3)}\alpha\partial^{(3)}_{\nu}\gamma_{\mu\epsilon} - \frac{1}{4}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\partial^{(3)}_{\zeta}\gamma_{\eta\Theta}\partial^{(3)}_{\nu}\gamma_{\mu\epsilon} \\
&+ \frac{1}{2}\gamma^{\epsilon\zeta}\gamma^{\eta\Theta}\alpha\partial^{(3)}_{\Theta}\gamma_{\zeta\eta}\partial^{(3)}_{\nu}\gamma_{\mu\epsilon} - \frac{\gamma_{\mu\epsilon}\partial^{(3)}_{\zeta}\beta^{\zeta}\partial^{(3)}_{\nu}\beta^{\epsilon}}{2\alpha} - \frac{\gamma^{\eta\Theta}\gamma_{\mu\zeta}\beta^{\epsilon}\partial^{(3)}_{\epsilon}\gamma_{\eta\Theta}\partial^{(3)}_{\nu}\beta^{\zeta}}{4\alpha} \\
&- \frac{\beta^{\epsilon}\partial^{(3)}_{\epsilon}\gamma_{\mu\zeta}\partial^{(3)}_{\nu}\beta^{\zeta}}{2\alpha} + \frac{\gamma_{\mu\zeta}\beta^{\epsilon}\partial^{(3)}_{\epsilon}\alpha\partial^{(3)}_{\nu}\beta^{\zeta}}{2\alpha^2} - \frac{\gamma_{\epsilon\zeta}\partial^{(3)}_{\mu}\beta^{\epsilon}\partial^{(3)}_{\nu}\beta^{\zeta}}{2\alpha} + \frac{1}{2}\gamma^{\epsilon\zeta}\alpha\partial^{(3)}_{\nu}\partial^{(3)}_{\mu}\gamma_{\epsilon\zeta} + \partial^{(3)}_{\nu}\partial^{(3)}_{\mu}\alpha.
\end{aligned}
\tag{5.3}
$$

While presenting these equations here instead of in an appendix might seem odd, there is a good reason. Through this, it can be clearly visually seen that the STEGR evolution equation is extensively shorter than its GR counterpart. In fact the GR equation consists of 83 terms while the STEGR evolution equation consists of 37, this is considerably less than half its size while supposedly producing the same solutions. This of course has to be tested analytically.

Between the equivalence of the first lie and that of the field equations, at least at the level of solutions, both the first and second evolution equations for GR and STEGR should result in the same solutions if the starting point is the same spatial metric. Before continuing with this test however, a note is presented about the indices. Up to this point all Greek indices covered 4 dimensions, $(0, 1, 2, 3)$. Here it is noted that for the equations being considered this is no longer necessary and and all Greek indices from this point will be replaced by lower case Latin indices such as i, j, k, l, m, n representing purely spatial values $(1, 2, 3)$ with any time index denoted by t. The (3) annotation will also be removed from partial derivatives as well as tensors as no (4) tensors are considered beyond this point. All tensors and partial derivatives are the spatial variants unless they have a t index and this only ever occurs with partial derivatives.

Having a look at the equations above, all dummy indices that involve a metric $\gamma^{\eta\Theta}$ are automatically zero on the 0th index. All dummy indices involving a spatial partial derivative are also temporally zero. This leaves us with the case of free indices, In this case it is simply chosen to iterate only over the spatial indices [8, 9].

Following the procedure as set by reference [8], four metrics are chosen to be tested, the Schwarzschild, isotropic, Painlev´e-Gullstrand and the Kerr-Schild metric. Table 5.1 shows the findings for each metric. This was obtained by taking each evolution equation derived in the previous chapter pertaining to STEGR and expanding them to the simplest of partial derivative form possible as done in Eq.(5.3). The same was then done for the constraint equations just as an extra check that these metrics are actual solutions for these equations as well. The original field equations were not explicitly tested as the evolution equations and the constraint equations are a reformulation of these same equations. The mathematical equivalence of the original STEGR field equations with the GR ones has also been already shown, making the fact that these metrics are a solution to the GR field equations equivalent to them being solutions to the STEGR ones. In Appendix D.3 one may find the code utilized in order to obtain these results. In the first subsection of Appendix D.3 the necessary xAct packages are imported and all necessary geometric entities, tensors and scalars are defined. Here the metric is not singled out as a special tensor so that no extra simplification is done by xAct automatically. In the second section a general form of the components of the tensors that are to be the building blocks of the rest of the tensors are defined. These tensors are the shift vector, the metric and inverse metric and the spatial mapping tensor. In the third section the Evolution and Constraint equations for TEGR are defined and finally the Four Metrics are specified. The code

	Schwarzchild	Isotropic	Painlev´e-Gullstrand	Kerr-Schild
α	$\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}$	$\frac{1-\frac{M}{2r}}{1+\frac{M}{2r}}$	1	$\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$
β^i	0	0	$\left(\frac{2M}{r}\right)^{\frac{1}{2}} l^i$	$\frac{2M}{r}\alpha^2 l^i$
γ_{ij}	$\text{diag}\left\{\alpha^{-2}, r^2, r^2 sin^2(\theta)\right\}$	$1+\frac{M}{2r}^4\bar{\eta}_{ij}$	$\bar{\eta}_{ij}$	$\bar{\eta}_{ij}+\frac{2M}{r}l_i l_j$
$\overset{\diamond}{B}_{ij}$	0	0	$-\left(\frac{2M}{r^3}\right)^{\frac{1}{2}}\left(\bar{\eta}_{ij}-\frac{3}{2}l_i l_j\right)$	$-\frac{2M\alpha}{r^2}\left\{\bar{\eta}_{ij}-\left(2+\frac{M}{r}\right)l_i l_j\right\}$
$\overset{\diamond}{B}$	0	0	$-\frac{3}{2}\left(\frac{2M}{r^3}\right)^{\frac{1}{2}}$	$-\frac{2M\alpha^3}{r^2}\left(1+\frac{3M}{r}\right)$
$\partial_t\overset{\diamond}{B}_{ij}$	0	0	0	0
Hc	0	0	0	0
Mc_β	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

Table 5.1: A table presenting the values of the extrinsic metricity, its scalar, its evolution equation as well as the Hamiltonian Constraint and the Momentum Constraint against four chosen metrics. Here $\bar{\eta}_{ij} = \text{diag}\left\{1, r^2, r^2 sin^2(\theta)\right\}$ and $l^i = l_i = (1,0,0)$.

works by running these sections as cells within Mathematica. The first two sections are run first, then a metric is chosen and the corresponding section is run, finally the Evolution and Constraint equations code is run to see the output. The output of each case can be seen in Table 5.1. As one would expect, since in each case the lapse function, the shift vector and the metric were independent of time, the second evolution equation always resulted in a zero matrix. It should be noted that in the last row $\partial_t\overset{\diamond}{B}_{ij}$ was calculated through the right hand side of Eq. (5.3) and not by taking the partial derivative with respect to time of the extrinsic metricity in row 4. Comparing Table 5.1 with that in reference [8] it is noted that all values agree apart from a change in sign. This was expected through Eq. (5.1).

In this section the equivalence of the STEGR equations derived in Section 4.3 to the ones in GR has been put to the test. In each case this equivalence was confirmed while noting that there is significant difference in the underlying equations leading up to the same solutions. The fact that the second evolution equation is a much simpler set of differential equations bodes well for numerically solving this set of equations.

5.2 BSSN Formalism in STEGR

In the previous section it was determined that the STEGR 3+1 system of equations are viable and consistent with those obtained through GR. Unfortunately, they also suffer from similar issues that will cause numerical instability. This numerical instability is also an issue for the ADM equations in GR. Mainly, the issue is that equations are second order in space and contain mixed derivatives of the spatial metric. In GR's ADM formalism, even if the equations are cast into a first order system of equations they are still weekly hyperbolic and thus not well posed [8, 9]. Given that the equivalent STEGR equations have been shown to have the same general shape and contain the same types of terms it can be concluded that they suffer from the same issues. This also indicates that such issues can potentially be solved in a similar manner.

While obtaining a closed system of equations as described the the previous Chapter is essential for obtaining any numerical result, these will not be the equations that are to be placed inside code and evolved. At least it is highly unlikely. In order for a general differential equation for some general function ϕ

$$A\partial_x^2\phi + 2B\partial_x\partial_y\phi + C\partial_y^2\phi = \bar{\rho} \tag{5.4}$$

to be hyperbolic, the coefficients A, B and C must be real and differentiable and must adhere to $AC - B < 0$. If this is the case then it is possible for the differential equation above to be written as a wave equation returning real values, rather than complex ones, for the unknown and needed results [8]. Now assuming that our equations are indeed hyperbolic and thus can eventually be written in "wave equation form" the next step is to see if such a wave equation will be well-posed. Consider a general vector wave equation for an n dimensional vector $\boldsymbol{V}$

$$\partial_t \boldsymbol{V} + \boldsymbol{A}^i \partial_i \boldsymbol{V} = 0, \tag{5.5}$$

where A^i will be an $n \times n$ matrix. For the system to be well posed then the solution must not increase more rapidly than exponentially [8], that is, it is possible to define some norm $\| \;\; \|$ such that

$$\|V(t, x^i)\| \leq k e^{\alpha t} \|V(0, x^i)\|, \tag{5.6}$$

where k and α are constants independent of the initial conditions. Unfortunately it is well understood that such a feat is not trivial. That being said, due to the similarity of the STEGR ADM 3+1 equations to those of GR it is possible to follow a particular well known and standard procedure to achieve such a formulation. The method that will be considered here in order to convert the system of equations into a strongly hyperbolic and well posed form is the one used to express the Gr ADM equations as the BSSN formalism [59, 8, 9].

The first step is to define a new scalar field ϕ and conformaly transform the metric through it as seen here.

$$\bar{\gamma}_{ij} = e^{-4\phi} \gamma_{ij}, \tag{5.7}$$

$$\bar{\gamma}^{ij} = e^{4\phi} \gamma^{ij}. \tag{5.8}$$

The scalar field ϕ is chosen so that the determinant of the conformal metric $\bar{\gamma}$ is equal to that of the flat metric η, that is, equal to 1. This results in the following relation between the scalar field and the determinant of the original spatial metric

$$\det\left(\bar{\gamma}_{ij}\right) = 1 \tag{5.9}$$
$$= \det\left(e^{-4\phi}\gamma_{ij}\right)$$
$$= e^{-12\phi}\gamma,$$
$$\mathrm{Ln}\left(e^{-12\phi}\gamma\right) = \mathrm{Ln}\left(1\right) \tag{5.10}$$
$$= 0,$$
$$\phi = \frac{\mathrm{Ln}\left(\gamma\right)}{12}. \tag{5.11}$$

In the ADM like system of equations there are two main variables that need to be numerically solved. The first is the metric and the second is the extrinsic metricity. Given this, the next step is to consider the best way to treat the extrinsic metricity tensor and parametrize it in a way that is consistent with the new conformal spatial metric. Here the trace and tracelss part of this tensor are split such that $\mathring{B}_{ij} = \frac{1}{3}\mathring{C}_{ij} + \gamma_{ij}\mathring{B}$. The traceless part of the extrinsic curvature, $\mathring{C}_{ij}$ is then conveniently parametrized such that

$$\bar{C}_{ij} = e^{-4\phi}\mathring{C}_{ij} \tag{5.12}$$
$$\bar{C}^{ij} = e^{4\phi}\mathring{C}^{ij}. \tag{5.13}$$

Two properties that are a result of this transformation are that $\bar{C}_{ij}\bar{C}^{ij} = \mathring{C}_{ij}\mathring{C}^{ij}$ and $\mathrm{trc}\left(\bar{C}_{ij}\right) = \bar{\gamma}^{ij}\bar{C}_{ij} = \gamma^{ij}\mathring{C}_{ij} = \mathrm{trc}\left(\mathring{C}_{ij}\right) = 0$.

Having added these new tensors and fields it is necessary to derive a system of evolution equations for them. First the evolution equations for the ϕ and B scalar fields are considered. Taking the trace of the evolution equation of the spatial metric

Eq. (4.67) and substituting Eq.(5.11) one gets

$$\gamma^{ij}\partial_t\gamma_{ij} = 2\alpha\mathring{B} + \beta^k\gamma^{ij}\partial_k\left(\gamma_{ij}\right) + 2\gamma^j{}_k\partial_j\beta^k$$

$$\partial_t Ln\left(\gamma\right) = 2\alpha\mathring{B} + \beta^k\partial_k\left(Ln(\gamma)\right) + 2\partial_j\beta^j,$$

$$\partial_t\phi = \frac{\alpha}{6}\mathring{B} + \beta^k\partial_k\phi + \frac{1}{6}\partial_k\beta^k. \tag{5.14}$$

Similarly taking the trace of the evolution equation of the extrinsic metricity Eq. (4.65) produces an evolution equation dependant on the trace of the lie derivative of $\mathring{B}_{ij}$ with respect to the normal vector Eq. (4.62).

$$\gamma^{ij}\partial_t\mathring{B}_{ij} = \alpha\gamma^{ij}\mathcal{L}_n\mathring{B}_{ij} + \gamma^{ij}\beta^k\partial_\lambda\mathring{B}_{ij} + 2\mathring{B}_{ik}\partial^i\beta^k. \tag{5.15}$$

Noting that

$$\partial_t B = \gamma^{ij}\partial_t B_{ij} + B_{ij}\partial_t\gamma^{ij}, \tag{5.16}$$

that

$$\begin{aligned}\gamma^{ij}\mathcal{L}_n\mathring{B}_{ij} = & -8\pi G\left[-\frac{1}{2}S + \frac{3}{2}\rho\right] + \frac{1}{2}\mathring{Q}_k{}^m{}_m\mathring{L}^k{}_i{}^i + \frac{1}{\alpha}\partial_k\left(\alpha\right)\mathring{L}^k{}_i{}^i + \gamma^{ij}\partial_k\mathring{L}^k{}_{ij} \\ & -\mathring{B}^2 - \partial^i\mathring{L}^k{}_{ki} + \frac{1}{\alpha}\partial^i\partial_i\left(\alpha\right) - \frac{1}{2}\mathring{Q}^{kmi}\mathring{Q}_{[km]i} + \frac{1}{4}\mathring{Q}_{ikm}\mathring{Q}^{ikm} + 2\mathring{B}^{ki}\mathring{B}_{ki},\end{aligned} \tag{5.17}$$

and that

$$\partial_t\gamma^{ij} = -2\alpha\mathring{B}^{ij} + \beta^k\partial_k\gamma^{ij} - \partial^i\beta^j - \partial^j\beta^i, \tag{5.18}$$

and substituting in the Hamiltonian constraint Eq. (4.64) the following evolution equation is obtained

$$\partial_t \mathring{B} = 4\pi G\alpha\,(S - \rho) + \gamma^{ij}\partial_i\partial_j\alpha + \mathring{L}^k{}_i{}^i\partial_k\alpha - \alpha\mathring{B}^{ij}\mathring{B}_{ij} + \beta^k\partial_k\mathring{B}. \tag{5.19}$$

The next step is expanding each of the extrinsic metricity tensors and spatial metrics to their conformal counterparts. This gives the final form of this evolution equation considered here

$$\partial_t \mathring{B} = 4\pi G\alpha\,(S - \rho) + e^{-4\phi}\bar{\gamma}^{ij}\partial_i\partial_j\alpha + \mathring{L}^k{}_i{}^i\partial_k\alpha - \alpha\left(\bar{C}^{ij}\bar{C}_{ij} + \frac{1}{3}\mathring{B}^2\right) + \beta^k\partial_k\mathring{B}. \tag{5.20}$$

The evolution equation of the conformal spatial metric is now considered. Starting from Eq. (4.67) and once again expanding each of the extrinsic metricity tensors and spatial metrics to their conformal counterparts while also subtracting Eq. (5.14) one gets

$$\partial_t\gamma_{ij} = 2\alpha\mathring{B}_{ij} + \beta^k\partial_k\gamma_{ij} + \gamma_{ik}\partial_j\beta^k + \gamma_{jk}\partial_i\beta^k \tag{5.21}$$

$$\partial_t\left(e^{4\phi}\bar{\gamma}_{ij}\right) = 2\alpha\left(\mathring{C}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}\mathring{B}\right) + \beta^k\partial_k\left(e^{4\phi}\bar{\gamma}_{ij}\right) + e^{4\phi}\bar{\gamma}_{ik}\partial_j\beta^k + e^{4\phi}\bar{\gamma}_{jk}\partial_i\beta^k. \tag{5.22}$$

Noting that

$$\partial_t\bar{\gamma}_{ij} + 4\bar{\gamma}_{ij}\partial_t\phi = \partial_t\bar{\gamma}_{ij} + \frac{2}{3}\alpha\bar{\gamma}_{ij}\mathring{B} + 4\bar{\gamma}_{ij}\beta^k\partial_k\phi + \bar{\gamma}_{ij}\frac{2}{3}\partial_k\beta^k, \tag{5.23}$$

one gets

$$\partial_t\bar{\gamma}_{ij} = 2\alpha\bar{C}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij} - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k + \bar{\gamma}_{(i|k}\partial_{|j)}\beta^k. \tag{5.24}$$

Applying a similar procedure to the evolution equation of the extrinsic metricity Eq. (4.65) and subtracting Eq. (5.20) the evolution equation of the trace of the extrinsic metricity is derived

$$\partial_t \mathring{B}_{ij} = \alpha \mathcal{L}_n \mathring{B}_{ij} + \beta^k \partial_k \mathring{B}_{ij} + \mathring{B}_{ik} \partial_j \beta^k + \mathring{B}_{jk} \partial_i \beta^k, \tag{5.25}$$

$$\mathcal{L}_n \mathring{B}_{ij} = -8\pi G \left[S_{ij} - \frac{1}{2}\gamma_{ij} (S - \rho) \right] + H_{ij} - \mathring{B}_{ij}\mathring{B} + 2\mathring{B}^k{}_i \mathring{B}_{kj} + \frac{1}{\alpha}\partial_i \partial_j (\alpha), \tag{5.26}$$

$$\partial_t \mathring{B}_{ij} = \partial_t \left(\mathring{C}_{ij} + \frac{1}{3}\gamma_{ij}\mathring{B} \right) = \partial_t \mathring{C}_{ij} + \frac{1}{3}\gamma_{ij}\partial_t \mathring{B} + \frac{1}{3}\mathring{B}\partial_t \gamma_{ij}, \tag{5.27}$$

$$\partial_t \mathring{C}_{ij} = -\alpha 8\pi G S^{TF}_{ij} + \alpha H^{TF}_{ij} - \frac{1}{3}\alpha \mathring{C}_{ij}\mathring{B} + 2\alpha \mathring{C}^k{}_i \mathring{C}_{kj} + \left(\partial_i \partial_j (\alpha) \right)^{TF} \tag{5.28}$$

$$+ \mathring{C}_{ik}\partial_j \beta^k + \mathring{C}_{jk}\partial_i \beta^k + \beta^k \partial_k \mathring{C}_{ij}.$$

Noting that

$$\partial_t \mathring{C}_{ij} = \partial_t \left(e^{4\phi}\bar{C}_{ij} \right) = \bar{C}_{ij}\partial_t \left(e^{4\phi} \right) + e^{4\phi}\partial_t \left(\bar{C}_{ij} \right), \tag{5.29}$$

the final form of the evolution equation is obtained

$$\partial_t \bar{C}_{ij} = e^{-4\phi} \left[-\alpha 8\pi G S^{\mathrm{TF}}_{ij} + \alpha H^{\mathrm{TF}}_{ij} + \left(\partial_i \partial_j \alpha \right)^{\mathrm{TF}} \right] + \alpha \left(2\bar{C}^k{}_i \bar{C}_{kj} - \bar{C}_{ij}\mathring{B} \right) \tag{5.30}$$

$$+ \bar{C}_{(i|k}\partial_{|j)}\beta^k + \beta^k \partial_k \bar{C}_{ij} - \frac{2}{3}\bar{C}_{ij}\partial_k \beta^k$$

where the superscript TF means trace free and where

$$H_{ij} := \frac{1}{2}\mathring{Q}_k{}^m{}_m \mathring{L}^k{}_{ij} + \frac{1}{\alpha}\partial_k (\alpha) \mathring{L}^k{}_{ij} + \partial_k \mathring{L}^k{}_{ij} - \frac{1}{2}\partial_{(i|}\mathring{L}^k{}_{k|j)} - \frac{1}{2}\mathring{Q}^{km}{}_i \mathring{Q}_{[km]j} + \frac{1}{4}\mathring{Q}_{ikm}\mathring{Q}_j{}^{km}. \tag{5.31}$$

While all of this helps with the separation of the evolution of the radiative and nonradiative degrees of freedom the issue of mixed second derivatives has still not been addressed. In order to tackle this, the conformal variant of the non-metricity tensor and the corresponding disformation tensor are considered. Starting with the non-metricity tensor

$$\begin{aligned}
\overset{\diamond}{Q}_{kij} =& \partial_k \gamma_{ij} \\
=& \partial_k \left(e^{4\phi} \bar{\gamma}_{ij} \right) \\
=& e^{4\phi} \partial_k \bar{\gamma}_{ij} + e^{4\phi} \bar{\gamma}_{ij} \frac{1}{e^{4\phi}} \partial_k e^{4\phi} \\
=& e^{4\phi} \partial_k \bar{\gamma}_{ij} + 4 e^{4\phi} \bar{\gamma}_{ij} \partial_k \phi \\
=& e^{4\phi} \bar{Q}_{kij} + 4 e^{4\phi} \bar{\gamma}_{ij} \partial_k \phi,
\end{aligned} \tag{5.32}$$

and by extension $\overset{\diamond}{Q}{}^{k}{}_{ij} = \bar{Q}^{k}{}_{ij} + 4\bar{\gamma}_{ij}\bar{\gamma}^{km}\partial_m\phi$. Similarly, expanding the disformation tensor one gets

$$\overset{\diamond}{L}_{kij} = e^{4\phi} \bar{L}_{kij} + 2 e^{4\phi} \left(\bar{\gamma}_{ij} \partial_k \phi - \bar{\gamma}_{ki} \partial_j \phi - \bar{\gamma}_{kj} \partial_i \phi \right), \tag{5.33}$$

$$\overset{\diamond}{L}{}^{k}{}_{ij} = \bar{L}^{k}{}_{ij} + 2 \bar{\gamma}^{km} \left(\bar{\gamma}_{ij} \partial_m \phi - \bar{\gamma}_{mi} \partial_j \phi - \bar{\gamma}_{mj} \partial_i \phi \right). \tag{5.34}$$

Through these, the conformal variants of the non-metricity have been defined in such a way that they contract with the new spatial conformal metric instead of the original one, that is, they are compatible with the conformal spatial metric.

It is now possible to construct a vector to parametrize the current evolution equations in such a way that they loose all mixed double derivatives of the spatial metric. The vector θ_i thus has to be some first derivative of the conformal spatial metric.

This is defined as

$$\begin{aligned}\theta_i := \bar{L}_{i\ m}^{m} = \bar{\gamma}^{mn}\bar{L}_{imn} \\ = \frac{1}{2}\bar{\gamma}^{mn}\left(\partial_i\bar{\gamma}_{mn} - 2\partial_m\bar{\gamma}_{in}\right) \\ = \frac{1}{2}\partial_i\left[\mathrm{Ln}\left(\bar{\gamma}\right)\right] - \bar{\gamma}^{mn}\partial_m\bar{\gamma}_{in} \\ = -\bar{Q}^{n}{}_{ni},\end{aligned} \tag{5.35}$$

where the first partial derivative disapears due to the choice that $\bar{\gamma} = 1$. An interesting aside that will be usefull in simplifying equations later is that the only other way to contract the conformal disformation tensor vanishes for the same reason

$$\begin{aligned}\bar{L}^{m}{}_{mi} = \bar{\gamma}^{mn}\bar{L}_{mni} \\ = \frac{1}{2}\bar{\gamma}^{mn}\left(\bar{Q}_{mni} - \bar{Q}_{nmi} - \bar{Q}_{imn}\right) \\ = -\frac{1}{2}\bar{Q}_{im}^{\ \ m} = -\frac{1}{2}\partial_i\mathrm{Ln}\left(\bar{\gamma}\right) \\ = 0.\end{aligned} \tag{5.36}$$

Using these, it is now possible to write the tensor H_{ij} in three parts. A conformal part $\bar{H}_{ij}$, a part purely dependant on the scalar field ϕ, H^{ϕ}_{ij}, and a part with mixed terms $\mathcal{H}_{ij}$

$$\bar{H}_{ij} = \frac{1}{2}\bar{Q}^{k}{}_{im}\left(\bar{Q}^{m}{}_{jk} - \bar{Q}_{jk}{}^{m}\right) + \frac{1}{4}\bar{Q}_{imn}\bar{Q}_{j}{}^{mn} + \partial_k\bar{L}^{k}{}_{ij} + \bar{L}^{k}{}_{ij}\partial_k\left[\mathrm{Ln}\left(\alpha\right)\right], \tag{5.37}$$

$$H^{\phi}_{ij} = 4\left(\partial_i\phi\partial_j\phi - \bar{\gamma}_{ij}\bar{\gamma}^{mn}\partial_m\phi\partial_n\phi\right) - 2\left(\partial_i\partial_j\phi + \bar{\gamma}_{ij}\bar{\gamma}^{mn}\partial_m\partial_n\phi\right), \tag{5.38}$$

$$\mathcal{H}_{ij} = 2\partial_k(\phi)\bar{L}^k{}_{ij} + 4\theta_{(i|}\partial_{|j)}\phi + 2\bar{\gamma}_{ij}\bar{\gamma}^{mn}\theta_n\partial_m\phi \tag{5.39}$$
$$+ 2\left(\partial_{(i|}\phi\partial_{|j)}\left[\mathrm{Ln}\left(\alpha\right)\right] - \bar{\gamma}_{ij}\bar{\gamma}^{mn}\partial_m\phi\partial_n\left[\mathrm{Ln}\left(\alpha\right)\right]\right).$$

The only problematic term in these three equations is $\partial_k\bar{L}^k{}_{ij}$. The issue is that if expanded in its current form it will result in mixed double derivatives of the conformal spatial metric. This can be solved through the following expansion

$$\partial_k\bar{L}^k{}_{ij} = \frac{1}{2}\bar{\gamma}^{km}\partial_k\partial_m\bar{\gamma}_{ij} + \bar{L}_{mij}\partial_k\bar{\gamma}^{mk} + \frac{1}{2}\partial_{(i|}\bar{\gamma}^{km}\partial_k\bar{\gamma}_{m|j)} + \frac{1}{2}\partial_{(i|}\theta_{|j)}, \tag{5.40}$$

where the issue terms were substituted with the new vector defined in Eq. (5.35) and where the last double derivative of the conformal spatial metric left is a Laplacian type double derivative. This finally solves the mixed derivative issue.

The last thing to consider is the evolution equation for the new vector θ_i. Expanding it by definition and employing a switch of time and spatial partial derivatives one gets

$$\partial_t\theta_i = -\partial_t\bar{\gamma}^{mn}\partial_m\bar{\gamma}_{in} + \bar{\gamma}^{mn}\partial_m\partial_t\bar{\gamma}_{in}, \tag{5.41}$$

where the first term can be substituted for through raising the indices of Eq. (5.24)

$$\partial_t\bar{\gamma}^{mn} = -\bar{\gamma}^{mp}\bar{\gamma}^{nq}\partial_t\bar{\gamma}_{pq} \tag{5.42}$$
$$= -2\alpha\bar{C}^{mn} + \beta^k\partial_k\bar{\gamma}^{mn} + \frac{2}{3}\bar{\gamma}^{mn}\partial_k\beta^k - \bar{\gamma}^{(m|p}\partial_p\beta^{|n)}.$$

Considering the second term in Eq. (5.41) and once again starting from Eq. (5.24) one gets

$$\bar{\gamma}^{mn}\partial_m\partial_t\bar{\gamma}_{in} = 2\alpha\bar{\gamma}^{mn}\partial_m\bar{C}_{in} + 2\bar{\gamma}^{mn}\bar{C}_{in}\partial_m\alpha + \bar{\gamma}^{mn}\partial_m\beta^k\partial_k\bar{\gamma}_{in} + \beta^k\bar{\gamma}^{mn}\partial_m\partial_k\bar{\gamma}_{in} \\ - \frac{2}{3}\bar{\gamma}^{mn}\partial_m\bar{\gamma}_{in}\partial_k\beta^k - \frac{2}{3}\partial_i\partial_k\beta^k + \bar{\gamma}^{mn}\partial_m\bar{\gamma}_{(i|k}\partial_{|n)}\beta^k + \bar{\gamma}^{mn}\bar{\gamma}_{(i|k}\partial_m\partial_{|n)}\beta^k. \tag{5.43}$$

There are two terms that will pose numerical issue in this expression. The first, $\beta^k\bar{\gamma}^{mn}\partial_m\partial_k\bar{\gamma}_{in}$, can be shown to be equal to $-\beta^k\left(\partial_k\bar{\gamma}^{mn}\partial_m\bar{\gamma}_{im} + \partial_k\theta_i\right)$. Being written in terms of first order derivatives of the conformal spatial metric and the new vector θ_i this term is no longer a concern. The second issue term is the divergence of the trace of the conformal extrinsic metricity $\bar{\gamma}^{mn}\partial_m\bar{C}_{in}$. Using the momentum constraint this term can be written as

$$\bar{\gamma}^{mn}\partial_m\bar{C}_{in} = 8\pi G S_i + \frac{1}{2}\bar{C}^{mn}\bar{Q}_{imn} + \frac{2}{3}B\partial_i\phi - \theta^m\bar{C}_{im} - 6\bar{\gamma}^{mp}\bar{C}_{im}\partial_p\phi + \frac{2}{3}\partial_i\mathring{B}. \tag{5.44}$$

Using all of these it is possible to write the final form of Eq. (5.41) as

$$\partial_t\theta_i = 2\alpha\left[\bar{C}^{mn}\left(\bar{Q}_{min} + \frac{1}{2}\bar{Q}_imn\right) - \bar{C}_{im}\left(\theta^m + 6\bar{\gamma}^{mn}\partial_n\phi\right) + \frac{2}{3}\left(\mathring{B}\partial_i\phi + \partial_i\mathring{B}\right)\right] \\ + \beta^k\left(2\bar{Q}_k{}^{mn}\bar{Q}_{min} + \bar{\gamma}^{mn}\partial_m\partial_k\bar{\gamma}_{in}\right) + \frac{4}{3}\theta_i\partial_k\beta^k - \beta^k\partial_k\theta_i - \theta_k\partial_i\beta^k \\ + 2\bar{\gamma}^{mn}\partial_m\beta^k\left(\bar{Q}_{kim} + \bar{Q}_{mik}\right) + 2\bar{\gamma}^{mn}\bar{C}_{in}\partial_m\alpha - \frac{2}{3}\partial_i\partial_k\beta^k + \bar{\gamma}^{mn}\bar{\gamma}_{(i|k}\partial_m\partial_{|n)}\beta^k. \tag{5.45}$$

With this done, all of the necessary evolution equations for all of the evolving tensors and scalar fields have been derived. The only remaining mixed and double derivatives are either Laplace derivatives or mixed derivatives of non evolving vectors and scalars. As in standard gravity this form of the evolution equations, the STEGR BSSN equations, increase the number of variables to solve from 12 to 15. That being said, the advantage of having of having stable numerical evolution out ways the minor increase in variable count [9, 8].

6 Discussion and Conclusion

6.1 Discussion

In this dissertation, the problem of producing numerical structures in alternative theories of gravity was considered in the context of teleparallel gravity. Due to the innovative work that has been done in order to detect and observe gravitational waves in the past couple of years, gravitational wave simulations were of specific interest. These detections, [3, 4], are leading us to a new era of data collection especially data relating to massive gravity interactions. The analysis of such data will help us better understand the fabric of the universe we inhabit. In order for such simulations to be carried out the lack of appropriate systems of equations needed to be addressed. As such, the development of such formulations was the main product of this work. In Chapter 1 the importance of such simulations and formulations as well as an explanation of the base of such formulations, know as the 3+1 decomposition, was addressed. A brief overview of the following chapters was also presented.

In Chapter 2 an analysis of the link between geometry and gravity was presented focusing on the two fundamental variables considered in this work, the metric and the tetrad. The properties and behaviour of a general covariant derivative were also explored paving the way for Chapter 3. An in depth explanation of the two alternative theories of gravity that were considered in Chapters 4 and 5, TEGR and STEGR, was then given highlighting the effects the choice of their gravitating tensors, the torsion tensors and non-metricity tensor respectively, had on the theories. Their equivalence to GR was also considered.

The basis for a 3+1 formalism based on a general linear affine connection while assuming non-metricity was set up in Chapter 3. Following the methodology used in attaining a metric 3+1 formalism for GR in Refs.[8, 9, 22], a general form of the 3+1 was set up both for a metric and a tetrad formalism. A number of fundamental definitions were given and a number of new generalized equations were derived. Some of the more notable ones being the definition of the purely spatial tetrad Eq.(3.39), an equation as a basis on which the gauss equation for gravity expressing tensors can be built Eq.(3.64) and the necessary general evolution equations Eqs.(3.78,3.74,3.75-3.77,3.79). While this paved the way for the three gravitational theories considered in Chapter 4, this general formulation has a much broader impact beyond what is done in this work. Due to the general nature of the derivation and the lack of assumptions, the equations derived here are also applicable for theories that are extensions and modifications of GR, TEGR and STEGR. While the most obvious candidates would be $f(R)$, $f(T)$ and $f(Q)$, other more exotic theories would also be applicable so long as they are built around one or more of the gravitating tensors considered here.

In the first section of Chapter 4 it was shown that when the Levi-Civita connection is specified, the full metric 3+1 formalism for GR can be retrieved from the equations for a general linear affine connection derived in the Chapter 3. This result is confirmed to be in full agreement with the standard ones given by Refs.[8, 9, 22]. This derivation was carried out and presented in order to test the general form of the equations against known results and thus providing a validation of the method used.

Through this result using these general equations while specifying other gravitating tensors, and thus alternative theories, was justified.

In the second section of Chapter 4 a tetrad TEGR 3+1 formalisms was attempted. Due to the lack of a purely tetrad based equation for the spin connection like there is in GR, a spin zero TEGR 3+1 was derived. The absence of such an equation should not come as a surprise as in this theory the spin connection is an independent fundamental variable in its own right. Initially a torsion Gauss-like equation relating the fully spatial torsion tensor and the 4-dimensional torsion tensor was derived Eq.(4.34). The torsion vector and torsion scalar Gauss-like equations Eq.(4.35,4.36) were also derived and presented here. Choosing the teleparallel connection for TEGR and assuming spin zero, the evolution equations for a general linear affine connection for the tetrad reduce to the their teleparallel counterparts. The two constraint equations were derived and the teleparallel field equations were then substituted into the evolution equations. All these resulting equations were then simplified and modified to consist only of terms of either the first order Lie derivative or other purely spatial tensors in order to obtain a fully closed system of equations.

Unfortunately, the last form of the second evolution equation derived in this work, Eq. (4.41), is not the final form needed in order to start producing simulations. As explained in detail in this Section, one term that is expected to be substituted out through the inclusion of the field equations, survives. This results in the right hand side of the second evolution equation containing a single term that is impossible to write in terms of purely spatial tensors. The reason for this is analysed, discussed in detail and a number of possible solutions to this issue are presented. Unfortunately, most of the solutions are ruled out. The consideration of the spin connection as an independent non-zero fundamental variable is the most promising. If it indeed turns out to be the solution it might shed some light on the nature of the spin connection itself. Another is the use of some yet unknown constraint equation.

In the final section of this chapter the 3+1 formulation for STEGR was derived.

Assuming non-metricity, vanishing torsion and curvature terms and the coincident gauge, all covariant derivatives become partial derivatives resulting in a 3+1 that is simpler, derivative wise, than the previous theories. The Gauss equations for this theory, Eqs. (4.52,4.53), had already been derived in Chapter 3 as they are independent of what theory is being considered. In preparation for the derivation of the evolution equations a new expression for the non-metricity scalar was derived that was dependant on the energy momentum scalar. This was substituted into the field equations, Eqs. (3.74, 3.76). The generalized evolution equations were then reduced to consist of only non-metricity related terms and the field equations were substituted into the second evolution equation. By contracting the field equations in particular ways the Hamiltonian and Momemntum constraints were also obtained Eqs. (4.63,4.64). Simplifying this system of equations and converting all terms on the right hand side of the second evolution equation and the constraint equations to purely spatial tensors and derivatives, the final form of the ADM-like system of equations for STEGR was fully acquired.

Due to obtaining a full and consistent STEGR 3+1 system of equations in Chapter 4 the next step was to validate these equations and re-write them in such a way that any simulations carried out would be stable. In Chapter 5 similarities and differences of the GR system of equations and the STEGR equivalents are analysed in detail. The STEGR system of equations was then tested by generating the first and second evolution equations as well as the scalar extrinsic curvature when choosing four known solutions to the GR equations. The results can be seen in Table 5.1 and are all in agreement with their GR counterparts up to a negative sign that was found to be a consequence of a choise of convention.

In the second section of Chapter 5 the issue of hyperbolisity and well-posednes was tackled. If the ADM system of equations in Gr and STEGR are left in their original formulation, once simulations are carried out, both will suffer from long term numerical instability due to them being non-hyperbolic and thus not well-posed system of equations. On of the most standard ways of fixing these issues in GR is to derive what is known as the BSSN formulation. Given the similarity at the level of differential equations of the STEGR equations to the GR counterparts, this method is also carried out in STEGR. In this section the equations are successfully conformally parametrized and cast into a first order system of equations Eqs. (5.7 - 5.45) converting the differential equations into a hyperbolic form.

6.2 Future Works

The results obtained up to this point lay the groundwork for constructing the 3+1 formalism for a general linear affine connection with non-metricity both with respect to the tetrad and with respect to the metric in any theory that is based on curvature, torsion, non-metricity or any combination of these plausible geometric manifestations of gravity. In particular it has shown that such a general foundation is fully capable of producing the correct GR ADM equations if the Levi-Civita connection is chosen and a fully consistent system of new STEGR equations if the coincident gauge is chosen. With regards to torsional gravity and the choice of teleparallel connection, more work has to be done in order to reconcile the single remaining problem term.

The dynamic STEGR equations were put in a form more useful for numerical simulations, their equivalent of the BSSN formalism. Since these differential equations are now in a well-posed and hyperbolic from, the next steps are for them to be initially used in a simple numerical framework developed in house in order to produce something similar to Ref.[59] and then within a community code such as Cactus using the Einstein Toolkit. Gravitational wave profiles are intended to be achieved in each of these cases with the Cactus implementation opening the project up to a myriad of possible research paths. The following is a summary of the state of the research leading to the intended way forward.

- Generating numerical simulations

  Since hyperbolic and well-posed equations have been achieved for STEGR in Chapter 5 the next step is to introduce such equations into numerical codes that can solve them. Up till this point some work has been carried out during this project in order to acquire a modest understanding of a particular community code called Cactus [60, 61]. Cactus is an open source program for generating simulations in areas varying from numerical relativity to fluid dynamics to quantum gravity. An example of its use in generating simulations for gravitational waves can be found in Ref.[62]. Here gravitational radiation simulations were carried out stemming from the collapse of neutron stars and rotating black holes.

  Currently the knowledge about how to successfully build such a code both on a personal computer and on a cluster has been acquired and a number of simulations have been successfully run in GR. There are two main sides to the Cactus code. The Cactus flesh is the base code of cactus which one initially builds and which all other parts interact with. The second are the thorns. Although not an exact description, thorns can be thought of as packages. In general thorns are packets of code written in C and Fortran 90 that have a particular purpose, some take care of the numerics of the simulation, some define the grid on top of which our gravitating sources will exist during the simulation as well as its refinement and its resolution, others define what type of output is generated. Other thorns are not directly related to the simulation itself like the thorns that convert a system of tensor differential equations and their initial conditions into a thorn in its own right written in C and Fortran 90. This allows the rest of the thorns to interpret it and use it correctly during simulations. The Einstein toolkit [61] which is currently being used in this work, provides a collection of thorns that are ideal for the kind of work

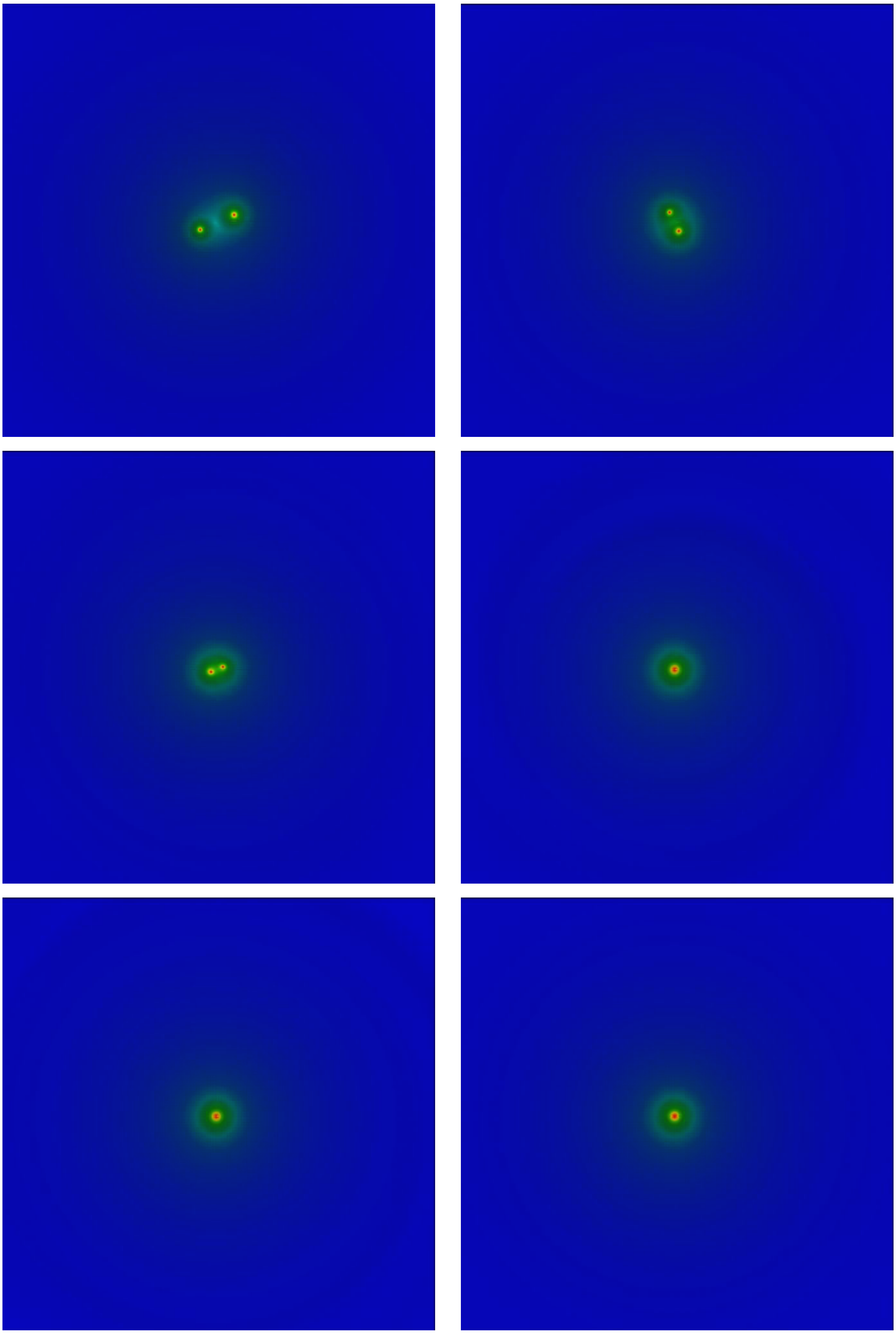

Figure 6.1: These images are snapshots generated from a simulation of the original binary black hole merger first observed in 2015. The instructions and parameter file used to in order to run the simulation can be found on the Einstein toolkit website and a detailed analysis in Ref.[1]

that is intended to be produced as a continuation of this dissertation.

Once all equations are set and all the necessary thorns are built, the simulation itself is initiated by running a parameter file which tells cactus which thorns to run and provides the required thorns with all the necessary numerical values. A number of simulations have been run during the coarse of this project with the most visually presentable one being a simulation of the original binary black hole merger first observed in 2015 [63, 1]. The results of this simulation can be seen in Fig.(6.1) where the gravitational waves are most visible in sub figure 4.

Apart from this simulation new simple thorns were written from scratch in order to learn how the process works and what cactus accepts and does not accept. The thorn Krank [64] was also used in order to generate a thorn from a Mathematica file containing differential equations. This process seemed to be successful as replacing the original given thorn with the new thorn generated the same exact output in simulation.

The final work that is being produced is the development of a simple numeric code that is capable of reproducing a proof of concept result similar to what was achieved in Ref.[59]. In this work the GR BSSN formalism is put head to head against the original ADM formalsim in a numerical stability sense. In both cases a two level iterative Crank-Nicholson method is used to numerically integrate the evolution equations assuming that most fundamental variables behave like outgoing waves at the boundaries. Using this method a simple wave could be simulated and compared. As it stands two separate pieces of Mathematica code have been developed with this aim. The first is capable of converting any type of partial derivative term generated through the xAct package into its correct finite element form, in this case the Crank-Nicholson finite element method. The second is capable of using the residual method to solve differential equations over a purely spatial slice in time. Once these two codes are fully developed, integrated with the equations derived in Chapter

5 and appropriate boundary conditions studied and implemented, an analysis similar to Ref.[59] should be possible.

- Finalization of 3+1 formalisms

In the case of TEGR, the tetrad system of equations is very close to being fully consistent. While most torsion terms have been successfully converted either into spatial torsion terms or into terms equivalent to the first evolution equation of the respective formalisms, a single space-time covariant derivative of torsion remains. The most plausible solution to this issue is the consideration of a non-vanishing spin connection which would require the derivation of a spin Gauss-like equation or the inclusion of some constraint equation that is currently not known or being considered in this work. Once the right hand side of the second evolution equation is fully spatial and written in terms of known variables than the system of equation will become fully closed and a similar procedure to what was done in Chapter 5 for STEGR would be carried out in TEGR.

One other possible extension to this work is a tetrad 3+1 formalism in Gr. While the development of this formalism was not presented in this work due to its issues, it still has potential and merits mentioning. Currently the issue lies with the spin terms that are remaining in the second order tetrad lie derivative equation and the lack of a working spin Gauss-like equation. As with the other formalisms, in order to obtain the final form of this equation the left hand side must be written purely in terms of know purely spatial terms such as those in the first evolution equation and the purely spatial tetrad itself. There are a number of ways that such remaining 4-dimensional spin terms could be modified in order to achieve this. Primarily a spin Gauss-like equation needs to be developed so as to be used to modify all of the 4-spin terms into 3-spin terms. Since the 3-spin can be purely written in terms of 3-tetrads this would place the evolution equation in the required form. Unfortunately, it is unlikely that it is possible to do this with all of the spin terms and as such it needs to be seen whether the remaining terms can be written in terms of the extrinsic curvature, in terms of the the second term of the first tetrad evolution equation or turned into torsion terms, effectively eliminating them due to GR being a torsion free theory.

6.3 Impact of this research

Having achieved a BSSN formalism in STEGR the most promising plan for future work is finding numerical solutions for ever more complicated and relevant physical systems such as the interaction of massive objects. This would allow a direct comparison of such simulations with current data and if in agreement, this would validate the method further. A second point that needs to be considered is that at this point in time no tetrad 3+1 formulation as described in Chapters 3 and 4 exits in GR and TEGR. Once all of the steps discussed above are achieved it is expected that the simulations for both should agree with those generated by the metric 3+1 formulation in GR and STEGR up to numerical error.

But why are these formulations important if they should produce the same results as those already done in GR? As we have seen in Chapter 5, the form the system of equations takes matters a lot in terms of numerical stability. As such one of the primary reasons for all of these versions is that the formulation one ends up with, while producing the same solutions, is always somewhat different. It is very important to consider such different formulations as some might prove to be more numerically efficient and stable than others. This of course can be done by simply taking the existing equations and manipulating them in a random manner in an attempt at finding a better formulation, however, linking the attempts to different geometric interpretations has the potential benefit of a better physical understanding of the equations through geometry. Another major reason is paving the way for modified theories of gravity. As pointed out in the introduction Gr has its limits, and while there are methods to compensate for such limits such as the inclusion of dark energy and dark matter, it is also important to consider modifications in the geometric interpretation of gravity rather than just in the matter part. That is, testing Lagrangians that do not produce equivalent theories like the ones discussed here. The work done here as well as all of the endeavours mentioned above all pave the way towards a numerical analysis of such theories. Questions such as if such theories would predict wave forms that are different from those produced in GR and to what extent they differ could be answered. If found, these differences could help in constraining such a theory or possibly even show that some class of the theory is more apt at predicting such wave forms than GR itself.

Having the opportunity to explore the derived STEGR BSSN numerically first will serve as a good starting point for future numerical projects in the other theories. The process of obtaining a 3+1 formalism in itself as well as modifying the existing codes to work with new differential equations is no simple feat. Starting this process as close to the original formulation as possible and with as much of the code that already exists as possible is the best way to train oneself into achieving the correct results and reduce the risk of mistakes being made. It is also a benefit that the solution is something that is already known so that if a different outcome is achieved the issues can be rectified.

In the case of the tetrad formulations, starting with the one in GR itself would be the best way forward. Firstly it is once again a matter of familiarization. The second reason is that there might be benefits in using this formalism instead of the existing metric one. The tetrad is a more fundamental variable than the metric and it is possible that the equations generated could be solved more quickly numerically. While this is not guaranteed it is something that is wise to look into given the major simulations that are being attempted and the amount of time they take to run. Any advantage in computing time would be of great help not only to get answers quicker but also for running costs. Another advantage is related to the fully spatial spin connection and its relation to the physical spin of the system. Through this formalism the 3-spin would be trivially obtained. The same can be said for the tetrad fromulation in TEGR which is one step further from the original formulation that is fully coded.

Apart from the obvious academic merit that will come from future publications of this work beyond the current publication [65], this project has been an incredible experience with regards to building a more holistic view of research and the scientific community as a whole. Great interest has been shown from our international collaborators, some of which even offering their time and resources to aid with the research. Among others, this work has involved two STSM's to Frankfurt in Germany with one of the leaders of this field, Prof. Luciano Rezzola. Here a greater understanding of the basis of the GR metric 3+1 formalism was obtained and has greatly helped with the development of the generalized formalism in Chapter 3. Another very important experience was attending the Lost In Gravity Conference in Saint Flour, France, where the importance and possible methodology of obtaining well-posed and hyperbolic evolution equations hit home leading to the current state of the BSSN equations in STEGR. During this conference, multiple contacts were made that will be invaluable for carrying out the modification of the relevant parts of the cactus code. Finally, and most importantly, being a committee member of the COST Action CA16104 Gravitational Waves, Black Holes and Fundamental Physics which has allowed for most of the above visits to be possible and the building of a network for future collaborations that will outlast both the cost action and this part of the research.

As stated in the beginning of this work, this is a very exiting time to be studying the fundamental formalisms in various theories that will lead to relevant numerical simulations as these are now moving to the forefront of Astrophysical event detection systems and observation projects. As such it is very important that in our quest for understanding the Universe they are studied holistically including through alternative theories of gravity.

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Appendices

A A General Second Order Lie Derivative of the spatial Metric

One starts by considering the first lie derivative of the spatial metric along the global normal vector for a general affine connection with non-metricity as derived in Eq.(3.74)

$$\begin{aligned}\mathcal{L}_n\gamma_{\mu\nu} &= \gamma_{\sigma(\nu|}\gamma^{\beta}{}_{|\mu).}n^{\lambda}T^{\sigma}{}_{\lambda\beta} + 2\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n_{\lambda}L^{\lambda}{}_{\alpha\beta} - k_{(\mu\nu)} \\ &= A_{(\mu\nu)} + B_{(\mu\nu)} - k_{(\mu\nu)}\,.\end{aligned}$$

Starting from the torsion term $A_{\mu\nu}$, taking its Lie derivative results in

$$\begin{aligned}\mathcal{L}_nA_{\mu\nu} =& n^{\lambda}\partial_{\lambda}A_{\mu\nu} + A_{\lambda\mu}\partial_{\nu}n^{\lambda} + A_{\nu\lambda}\partial_{\mu}n^{\lambda} &(1)\\ =&\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}A_{\alpha\beta} + A_{\lambda\mu}\gamma^{\lambda}{}_{\sigma}\gamma^{\alpha}{}_{\nu}\nabla_{\alpha}n^{\sigma} + A_{\nu\lambda}\gamma^{\lambda}{}_{\sigma}\gamma^{\beta}{}_{\mu}\nabla_{\beta}n^{\sigma} \\ &+ A_{\epsilon\mu}\gamma^{\epsilon}{}_{\sigma}\gamma^{\alpha}{}_{\nu}n^{\lambda}T^{\sigma}{}_{\lambda\alpha} + A_{\nu\epsilon}\gamma^{\epsilon}{}_{\sigma}\gamma^{\beta}{}_{\mu}n^{\lambda}T^{\sigma}{}_{\lambda\beta} \\ =&\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}A_{\alpha\beta} - A_{\lambda\mu}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\alpha}{}_{\nu}n_{\epsilon}Q_{\alpha}{}^{\epsilon\sigma} + k_{\nu}{}^{\lambda}\right) \\ &- A_{\nu\lambda}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\beta}{}_{\mu}n_{\epsilon}Q_{\beta}{}^{\epsilon\sigma} + k_{\mu}{}^{\lambda}\right) + A_{\sigma\mu}A^{\sigma}{}_{\nu} + A_{\nu\sigma}A^{\sigma}{}_{\mu}\,, &(2)\end{aligned}$$

Here, the partial derivatives are expanded and given that the lie derivative of a lower index tensor has been shown to be itself spatial a spatially mapping tensor is extracted for each index. In the second line all first lie terms are substituted for by their respective $A_{\mu\nu}$ form and Eq.(3.57) is used to expand the second and third terms. with regards to the first term, the $A_{\alpha\beta}$ is expanded and the Leibniz rule is applied to it as seen below,

$$\begin{aligned}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}A_{\alpha\beta} =&\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}\left(\gamma^{\epsilon}{}_{\alpha}\gamma^{\chi}{}_{\beta}n_{\sigma}T_{\epsilon}{}^{\sigma}{}_{\chi}\right) \\ =&-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\nabla_{\lambda}T_{\alpha}{}^{\lambda}{}_{\beta}+D_{\lambda}T_{\nu}{}^{\lambda}{}_{\mu}+\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}a_{\lambda}T_{\alpha}{}^{\lambda}{}_{\beta} \\ &+\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}T_{\epsilon}{}^{\sigma}{}_{\chi}n^{\lambda}n_{\sigma}\nabla_{\lambda}\left(\delta^{\epsilon}_{\alpha}n^{\chi}n_{\beta}+\delta^{\chi}_{\beta}n^{\epsilon}n_{\alpha}\right) \\ =&-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\nabla_{\lambda}T_{\alpha}{}^{\lambda}{}_{\beta}+D_{\lambda}T_{\nu}{}^{\lambda}{}_{\mu}+\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}a_{\lambda}T_{\alpha}{}^{\lambda}{}_{\beta} \\ &\gamma^{\beta}{}_{\mu}n_{\sigma}n^{\epsilon}T_{\epsilon}{}^{\sigma}{}_{\beta}\gamma^{\alpha}{}_{\nu}a_{\alpha}.\end{aligned} \tag{3}$$

Here, in the last line, the anti-symmetric nature of the torsion tensor tensor was used to eliminate one of the terms in the bracket. As a result the following is the final form of the lie of the torsion term

$$\begin{aligned}\mathcal{L}_{n}A_{\mu\nu}=&-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\nabla_{\lambda}T_{\alpha}{}^{\lambda}{}_{\beta}+D_{\lambda}T_{\nu}{}^{\lambda}{}_{\mu} \\ &+\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}a_{\sigma}T_{\alpha}{}^{\sigma}{}_{\beta}+\gamma^{\beta}{}_{\mu}n_{\sigma}n^{\epsilon}T_{\epsilon}{}^{\sigma}{}_{\beta}\gamma^{\alpha}{}_{\nu}a_{\alpha} \\ &-A_{\lambda\mu}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\alpha}{}_{\nu}n_{\epsilon}Q_{\alpha}{}^{\epsilon\sigma}+k_{\nu}{}^{\lambda}\right) \\ &-A_{\nu\lambda}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\beta}{}_{\mu}n_{\epsilon}Q_{\beta}{}^{\epsilon\sigma}+k_{\mu}{}^{\lambda}\right) \\ &+A_{\sigma\mu}A^{\sigma}{}_{\nu}+A_{\nu\sigma}A^{\sigma}{}_{\mu}\,.\end{aligned} \tag{4}$$

Considering the disformation term $\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n_{\lambda}L^{\lambda}{}_{\alpha\beta}$ as $B_{\mu\nu}$ and applying a lie derivative to it with respect to the normal vector and employing the same procedure as above, one acquires

$$\begin{aligned}\mathcal{L}_{n}B_{\mu\nu} =&n^{\lambda}\partial_{\lambda}B_{\mu\nu}+B_{\lambda\mu}\partial_{\nu}n^{\lambda}+B_{\nu\lambda}\partial_{\mu}n^{\lambda} \\ =&n^{\lambda}\nabla_{\lambda}B_{\nu\mu}+B_{\lambda(\mu}\nabla_{\nu)}n^{\lambda}+n^{\lambda}B_{\sigma(\mu|}T^{\sigma}{}_{\lambda|\nu)} \\ =&\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}B_{\alpha\beta}-\gamma^{\epsilon}{}_{(\nu|}\gamma^{\chi}{}_{|\mu)}B_{\alpha\chi}\left[n_{\sigma}Q_{\epsilon}{}^{\sigma\alpha}+k_{\epsilon}{}^{\alpha}\right]+B_{\sigma(\mu|}A^{\sigma}{}_{|\nu)}.\end{aligned} \tag{5}$$

Taking the first term and expanding it one gets,

$$\begin{aligned}
\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}B_{\alpha\beta} =&\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}\left(\gamma^{\epsilon}{}_{\alpha}\gamma^{\chi}{}_{\beta}n_{\sigma}L^{\sigma}{}_{\epsilon\chi}\right) \\
=& -\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\nabla_{\lambda}L^{\lambda}{}_{\alpha\beta} + D_{\lambda}L^{\lambda}{}_{\mu\nu} + \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}a_{\lambda}L^{\lambda}{}_{\alpha\beta} \\
&+ \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}L^{\sigma}{}_{\epsilon\chi}n^{\lambda}n_{\sigma}\nabla_{\lambda}\left(\delta^{\epsilon}_{\alpha}n^{\chi}n_{\beta} + \delta^{\chi}_{\beta}n^{\epsilon}n_{\alpha}\right) \\
=& -\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\nabla_{\lambda}L^{\lambda}{}_{\alpha\beta} + D_{\lambda}L^{\lambda}{}_{\mu\nu} \\
&\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(a_{\sigma}L^{\sigma}{}_{\alpha\beta} + a_{(\beta|}L^{\sigma}{}_{\epsilon|\alpha)n_{\sigma}n^{\epsilon}}\right)
\end{aligned} \tag{6}$$

Combining this with Eq.(5) the final form of the non-metricity contribution to the second evolution equation is achieved

$$\begin{aligned}
\mathcal{L}_{n}B_{\mu\nu} =& -\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\nabla_{\lambda}L^{\lambda}{}_{\alpha\beta} + D_{\lambda}L^{\lambda}{}_{\mu\nu} \\
&+ \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(a_{\sigma}L^{\sigma}{}_{\alpha\beta} + a_{(\beta|}L^{\sigma}{}_{\epsilon|\alpha)}n_{\sigma}n^{\epsilon}\right) \\
&- \gamma^{\epsilon}{}_{(\nu|}\gamma^{\chi}{}_{|\mu)}B_{\alpha\chi}\left(n_{\sigma}Q_{\epsilon}{}^{\sigma\alpha} + k_{\epsilon}{}^{\alpha}\right) \\
&+ B_{\sigma(\mu|}A^{\sigma}{}_{|\nu)}\,.
\end{aligned} \tag{7}$$

Finally the extrinsic curvature term is considered. As with the other two cases the definition of the lie derivative along a vector applied to a tensor is used to get

$$\begin{aligned}
\mathcal{L}_{n}k_{\mu\nu} =& n^{\lambda}\partial_{\lambda}k_{\mu\nu} + k_{\lambda\mu}\partial_{\nu}n^{\lambda} + k_{\nu\lambda}\partial_{\mu}n^{\lambda} \\
=& n^{\lambda}\nabla_{\lambda}k_{\nu\mu} + k_{\lambda(\mu}\nabla_{\nu)}n^{\lambda} + n^{\lambda}k_{\sigma\mu}T^{\sigma}{}_{\lambda\nu} + n^{\lambda}k_{\nu\sigma}T^{\sigma}{}_{\lambda\mu} \\
=& \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}k_{\alpha\beta} - k_{\lambda\mu}\left(n_{\epsilon}\gamma^{\lambda}{}_{\sigma}\gamma^{\alpha}{}_{\nu}Q_{\alpha}{}^{\epsilon\sigma} + k_{\nu}{}^{\lambda}\right) \\
&- k_{\nu\lambda}\left(n_{\epsilon}\gamma^{\lambda}{}_{\sigma}\gamma^{\beta}{}_{\mu}Q_{\beta}{}^{\epsilon\sigma} + k_{\mu}{}^{\lambda}\right) + k_{\sigma\mu}A^{\sigma}{}_{\nu} + k_{\nu\sigma}A^{\sigma}{}_{\mu}.
\end{aligned} \tag{8}$$

Taking the first term and expanding one obtaines

$$\begin{aligned}
\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}k_{\alpha\beta} =& \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}\nabla_{\alpha}n_{\beta} \\
&+ \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left[\nabla_{\alpha}\left(n_{\chi}\right)n^{\lambda}n^{\chi}\nabla_{\lambda}\left(n_{\beta}\right) + \nabla_{\epsilon}\left(n_{\beta}\right)n^{\lambda}n^{\epsilon}\nabla_{\lambda}\left(n_{\alpha}\right)\right] \\
=& \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}\nabla_{\alpha}n_{\beta}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(\frac{1}{2}n^{\chi}n^{\epsilon}Q_{\alpha\chi\epsilon}a_{\beta} + a_{\beta}a_{\alpha}\right).
\end{aligned} \tag{9}$$

Combining these two parts one obtaines the final form of the curvature contribution to the second evolution equation given below

$$\begin{aligned}\mathcal{L}_n k_{\mu\nu} =&\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}n^{\lambda}\nabla_{\lambda}\nabla_{\alpha}n_{\beta} \\ &+\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(\frac{1}{2}n^{\chi}n^{\epsilon}Q_{\alpha\chi\epsilon}a_{\beta}+a_{\beta}a_{\alpha}\right) \\ &-k_{\lambda\mu}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\alpha}{}_{\nu}n_{\epsilon}Q_{\alpha}{}^{\epsilon\sigma}+k_{\nu}{}^{\lambda}\right) \\ &-k_{\nu\lambda}\left(\gamma^{\lambda}{}_{\sigma}\gamma^{\beta}{}_{\mu}n_{\epsilon}Q_{\beta}{}^{\epsilon\sigma}+k_{\mu}{}^{\lambda}\right) \\ &+k_{\sigma\mu}A^{\sigma}{}_{\nu}+k_{\nu\sigma}A^{\sigma}{}_{\mu}\,.\end{aligned} \tag{10}$$

B TEGR Second Evolution Equation and Constraints

B.1 TEGR Second Evolution Equation

Starting where Eq.(4.40) left off and combining Eq.(4.37) and Eq.(4.38) to get the final form of the TEGR field equations, one gets,

$$\begin{aligned}&\hat{S}_{\alpha\sigma}{}^{\rho}\hat{T}^{\lambda}{}_{\rho\lambda}-\hat{S}^{\rho\lambda}{}_{\sigma}\hat{T}_{\rho\lambda\alpha}-\frac{1}{2}\hat{S}_{\alpha}{}^{\rho\lambda}\hat{T}_{\sigma\lambda\rho} \\ &\qquad+\frac{1}{2}g_{\alpha\sigma}\left(\Theta+2\hat{T}^{\beta}{}_{\rho\beta}\hat{T}^{\lambda}{}_{\lambda}{}^{\rho}+2\hat{\nabla}_{\rho}\hat{T}^{\lambda}{}_{\lambda}{}^{\rho}\right)-\hat{\nabla}^{\lambda}\hat{S}_{\alpha\lambda\sigma}=\Theta_{\alpha\sigma}\,.\end{aligned} \tag{11}$$

This Final form was then expanded and simplified using Mathematica through the package XAct [66]. This package is capable of tensor manipulation and simplification, it also gives the user the ability to definite rules for tensor substitution while maintaining the correct index placement and notation. The code used here can be found in Appendix D.1. Taking up the derivation from after this simplification one obtains

$$\begin{aligned}e^{A(3)}_{\ \sigma}\gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\hat{\nabla}_{\rho}\hat{T}^{\lambda}{}_{\alpha}{}^{\rho} =&e^{A(3)}_{\ \sigma}\gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\left(-\frac{1}{2}\hat{T}_{\alpha}{}^{\rho\chi}\hat{T}^{\lambda}{}_{\rho\chi}+\hat{T}_{\rho}{}^{\lambda}{}_{\chi}\hat{T}^{\rho}{}_{\alpha}{}^{\chi}+\hat{T}^{\rho}{}_{\alpha}{}^{\chi}\hat{T}_{\lambda}{}^{\lambda}{}_{\rho}-\hat{T}_{\alpha}{}^{\lambda\rho}\hat{T}^{\chi}{}_{\rho\chi}\right. \\ &\left.-\hat{T}^{\lambda}{}_{\alpha}{}^{\rho}\hat{T}^{\chi}{}_{\rho\chi}+\hat{\nabla}_{\alpha}\hat{T}^{\rho\lambda}{}_{\rho}+\hat{\nabla}^{\lambda}\hat{T}^{\rho}{}_{\alpha\rho}-\hat{\nabla}_{\rho}\hat{T}_{\alpha}{}^{\lambda\rho}-\delta^{\lambda}_{\alpha}\theta+2\theta_{\alpha}{}^{\lambda}\right)\end{aligned} \tag{12}$$

At this point each of the terms in the equation above need to be converted into terms that are purely spatial. Starting from the top left and moving to the bottom right the main method used here is the expansion of the contracted indices. The fact that contracting the antisymmetric indices of the torsion tensor together or contracting both with the normal vector results in a zero term is used throughout and the Gauss Equation for the torsion tensor, Eq.(4.34), is also used for conversion. Another important relation to note while simplifying these terms is

$$\begin{aligned}\partial_\nu n_\epsilon =& -\partial_\nu \left(\alpha \partial_\epsilon [t]\right) \\ =& -\partial_\epsilon [t]\, \partial_\nu [\alpha] \\ =& \frac{1}{\alpha} n_\epsilon \partial_\nu [\alpha],\end{aligned} \tag{13}$$

apart from this, it should also be noted that through Eq.(3.61) and Eq.(4.30) terms with the following shape also vanish, $\gamma^\nu{}_\alpha \gamma^\mu{}_\beta n_\sigma T^\sigma{}_{\nu\mu} = 0$. Finally, as a consequence of a vanishing spin connection, the following relation will also be used to simplify these terms

$$\begin{aligned}n_\rho \hat{T}^\rho{}_{\mu\nu} =& n_\rho \left(\hat{\Gamma}^\rho_{\mu\nu} - \hat{\Gamma}^\rho_{\nu\mu}\right) \\ =& \partial_\nu n_\mu - \partial_\mu n_\nu\end{aligned} \tag{14}$$

Omitting the spatial tetrad for the time being one obtains

$$\begin{aligned}-\frac{1}{2}\gamma^\sigma{}_\lambda \gamma^\alpha{}_\nu \hat{T}_\alpha{}^{\rho\chi} \hat{T}^\lambda{}_{\rho\chi} =& -\frac{1}{2}\gamma^\sigma{}_\lambda \gamma^\alpha{}_\nu \left(\gamma^\rho{}_\mu - n^\rho n_\mu\right)\left(\gamma^\chi{}_\epsilon - n^\chi n_\epsilon\right) \hat{T}_\alpha{}^{\mu\epsilon} \hat{T}^\lambda{}_{\rho\chi} \\ =& -\frac{1}{2}\hat{T}^{(3)\rho\chi}_\nu \hat{T}^{\sigma(3)}_{\rho\chi} + \hat{A}_\nu{}^\rho \hat{A}^\sigma{}_\rho,\end{aligned} \tag{15}$$

$$\begin{aligned}\gamma^\sigma{}_\lambda \gamma^\alpha{}_\nu \hat{T}_\rho{}^\lambda{}_\chi \hat{T}^\rho{}_\alpha{}^\chi =& \gamma^\sigma{}_\lambda \gamma^\alpha{}_\nu \left(\gamma^\rho{}_\mu - n^\rho n_\mu\right)\left(\gamma^\chi{}_\epsilon - n^\chi n_\epsilon\right) \hat{T}_\rho{}^\lambda{}_\chi \hat{T}^\mu{}_\alpha{}^\epsilon \\ =& \hat{T}_\rho{}^{\sigma(3)}{}_\chi \hat{T}^{\rho(3)\chi}_\mu - \hat{A}^\rho{}_\nu \hat{A}_\rho{}^\sigma + \gamma^{\sigma\lambda}\gamma^\alpha{}_\nu n^\epsilon \left(\partial_\alpha n_\epsilon - \partial_\epsilon n_\alpha\right) n^\chi \left(\partial_\lambda n_\chi - \partial_\chi n_\lambda\right) \\ =& \hat{T}_\rho{}^{\sigma(3)}{}_\chi \hat{T}^{\rho(3)\chi}_\mu - \hat{A}^\rho{}_\nu \hat{A}_\rho{}^\sigma + \frac{1}{\alpha^2}\partial^{(3)}_\nu (\alpha)\, \partial^\sigma_{(3)} (\alpha)\end{aligned} \tag{16}$$

$$\begin{aligned}\gamma^\sigma{}_\lambda \gamma^\alpha{}_\nu \hat{T}_\chi{}^\lambda{}_\rho \hat{T}^\rho{}_\alpha{}^\chi =& \gamma^\sigma{}_\lambda \gamma^\alpha{}_\nu \left(\gamma^\rho{}_\mu - n^\rho n_\mu\right)\left(\gamma^\chi{}_\epsilon - n^\chi n_\epsilon\right) \hat{T}_\chi{}^\lambda{}_\rho \hat{T}^\mu{}_\alpha{}^\epsilon \\ =& \hat{T}_\chi{}^{\sigma(3)}{}_\rho \hat{T}^{\rho(3)\chi}_\nu + \frac{1}{\alpha^2}\partial^{(3)}_\nu (\alpha)\, \partial^\sigma_{(3)} (\alpha)\end{aligned} \tag{17}$$

$$
\begin{aligned}
-\gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\hat{T}_{\alpha}{}^{\lambda\rho}\hat{T}^{\chi}{}_{\rho\chi} =& \gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\left(\gamma^{\rho}{}_{\mu} - n^{\rho}n_{\mu}\right)\left(\gamma^{\chi}{}_{\epsilon} - n^{\chi}n_{\epsilon}\right)\hat{T}_{\alpha}{}^{\lambda\mu}\hat{T}^{\epsilon}{}_{\rho\chi} \qquad (18)\\
=& \hat{T}_{\nu(3)}{}^{\sigma\rho}\hat{T}^{\chi(3)}_{\rho\chi} - \frac{1}{\alpha}\hat{T}^{(3)\sigma\rho}_{\nu}\partial^{(3)}_{\rho}(\alpha) - \hat{A}_{\nu}{}^{\sigma}\hat{A}\\
-\gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\hat{T}^{\lambda}{}_{\alpha}{}^{\rho}\hat{T}^{\chi}{}_{\rho\chi} =& \gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\left(\gamma^{\rho}{}_{\mu} - n^{\rho}n_{\mu}\right)\left(\gamma^{\chi}{}_{\epsilon} - n^{\chi}n_{\epsilon}\right)\hat{T}^{\lambda}{}_{\alpha}{}^{\mu}\hat{T}^{\epsilon}{}_{\rho\chi} \qquad (19)\\
=& \hat{T}^{\sigma(3)\rho}_{\nu}\hat{T}^{\chi(3)}_{\rho\chi} - \frac{1}{\alpha}\hat{T}^{\sigma(3)\rho}_{\nu}\partial^{(3)}_{\rho}(\alpha) - \hat{A}^{\sigma}{}_{\nu}\hat{A}\\
\gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\hat{\nabla}_{\alpha}\hat{T}^{\rho\lambda}{}_{\rho} =& \gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\hat{\nabla}_{\alpha}\left(\gamma^{\rho}{}_{\mu} - n^{\rho}n_{\mu}\right)\hat{T}^{\mu\lambda}{}_{\rho} \qquad (20)\\
=& \hat{D}^{\sigma}\hat{T}^{\rho\sigma}_{(3)\rho} - \gamma_{\sigma\lambda}\gamma^{\alpha}{}_{\nu}\hat{\nabla}_{\alpha}\left(n^{\rho}n_{\mu}T^{\mu}{}_{\lambda\rho}\right)\\
=& \hat{D}^{\sigma}\hat{T}^{\rho\sigma}_{(3)\rho} + \hat{D}_{\nu}\left(\frac{1}{\alpha}\partial^{\sigma}_{(3)}[\alpha]\right)\\
\gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\hat{\nabla}^{\lambda}\hat{T}^{\rho}{}_{\alpha\rho} =& \gamma^{\sigma}{}_{\lambda}\gamma^{\alpha}{}_{\nu}\hat{\nabla}_{\alpha}\left(\gamma^{\rho}{}_{\mu} - n^{\rho}n_{\mu}\right)\hat{T}^{\mu}{}_{\alpha\rho} \qquad (21)\\
=& \hat{D}^{\sigma}\hat{T}^{\rho(3)}_{\nu\rho} + \hat{D}^{\sigma}\left(\frac{1}{\alpha}\partial^{(3)}_{\nu}[\alpha]\right)
\end{aligned}
$$

Combining all these terms and substituting into the second evolution equation gives Eq.(4.42).

B.2 TEGR Constraint Equations

In this subsection the two constraint equations are considered. Starting from the Hamiltonian constraint one needs to contract the Symmetric part of the TEGR Field equations with two normal vectors as seen below

$$
n^{\lambda}n^{\alpha}FE_{(\alpha\lambda)} = 2n^{\lambda}n^{\alpha}\theta_{\alpha\lambda} = 16\pi n^{\lambda}n^{\alpha}\mathcal{T}_{\alpha\lambda} = 16\pi\rho. \qquad (22)
$$

where the tensor $FE_{\alpha\lambda}$ represents the geometry part of the field equations, $\mathcal{T}_{\alpha\lambda}$ is the energy momentum tensor and ρ is the energy density.

Expanding and simplifying through Mathematica (Appendix D.1) as with the evolution equation in Appendix A, one obtains

$$n^\lambda n^\alpha \Big(\frac{1}{2} \hat{T}_\alpha{}^{\rho\sigma} \hat{T}_{\lambda\rho\sigma} - \hat{T}_{\rho\lambda\sigma} \hat{T}^\rho{}_\alpha{}^\sigma - \hat{T}^\rho{}_\alpha{}^\sigma \hat{T}_{\sigma\lambda\rho} + 2\hat{T}_{\alpha\lambda}{}^\rho \hat{T}^\sigma{}_{\rho\sigma} - 2\hat{\nabla}_\lambda \hat{T}^\rho{}_{\alpha\rho} - 2\hat{\nabla}_\rho \hat{T}_{\alpha\lambda}{}^\rho \Big) \tag{23}$$

$$+ 2\hat{T}^\alpha{}_\alpha{}^\lambda \hat{T}^\rho{}_{\lambda\rho} - \hat{T} + 2\hat{\nabla}_\lambda \hat{T}^\alpha{}_\alpha{}^\lambda = 16\pi\rho.$$

Considering the terms individually and expanding the dummy indices one gets

$$\begin{aligned} \frac{1}{2} n^\alpha n^\lambda \hat{T}_\alpha{}^{\rho\sigma} \hat{T}_{\lambda\rho\sigma} =& \frac{1}{2} n^\alpha n^\lambda \left(\gamma^\rho{}_\mu - n^\rho n_\mu\right)\left(\gamma^\sigma{}_\epsilon - n^\sigma n_\epsilon\right) \hat{T}_\alpha{}^{\mu\epsilon} \hat{T}_{\lambda\rho\sigma} \\ =& -\frac{1}{2}\Big[\gamma^{\rho\mu} n^\epsilon \left(\partial_\mu n_\epsilon - \partial_\epsilon n_\mu\right) n^\sigma \left(\partial_\rho n_\sigma - \partial_\sigma n_\rho\right) \\ & + \gamma^{\sigma\epsilon} n^\mu \left(\partial_\mu n_\epsilon - \partial_\epsilon n_\mu\right) n^\rho \left(\partial_\rho n_\sigma - \partial_\sigma n_\rho\right) \Big] \\ =& -\frac{1}{\alpha^2} \partial^\rho_{(3)}(\alpha)\, \partial^{(3)}_\rho(\alpha), \end{aligned} \tag{24}$$

$$\begin{aligned} -n^\alpha n^\lambda \hat{T}_{\rho\lambda\sigma} \hat{T}^\rho{}_\alpha{}^\sigma =& - n^\alpha n^\lambda \left(\gamma^\rho{}_\mu - n^\rho n_\mu\right)\left(\gamma^\sigma{}_\epsilon - n^\sigma n_\epsilon\right) \hat{T}_{\rho\lambda\sigma} \hat{T}^\mu{}_\alpha{}^\epsilon \\ =& -\hat{A}_{\rho\sigma} \hat{A}^{\rho\sigma} + \frac{1}{\alpha^2} \partial^\rho_{(3)}(\alpha)\, \partial^{(3)}_\rho(\alpha), \end{aligned} \tag{25}$$

$$\begin{aligned} -n^\alpha n^\lambda \hat{T}^\rho{}_\alpha{}^\sigma \hat{T}_{\sigma\lambda\rho} =& - n^\alpha n^\lambda \left(\gamma^\rho{}_\mu - n^\rho n_\mu\right)\left(\gamma^\sigma{}_\epsilon - n^\sigma n_\epsilon\right) \hat{T}^\mu{}_\alpha{}^\epsilon \hat{T}_{\sigma\lambda\rho} \\ =& -\hat{A}^{\rho\sigma} \hat{A}_{\sigma\rho}, \end{aligned} \tag{26}$$

$$\begin{aligned} 2n^\alpha n^\lambda \hat{T}_{\alpha\lambda}{}^\rho \hat{T}^\sigma{}_{\rho\sigma} =& 2\gamma^{\rho\mu} n_\alpha n^\lambda \hat{T}^\alpha{}_{\lambda\mu} \hat{T}^\sigma{}_{\rho\sigma} \\ =& 2\gamma^{\rho\mu} n^\lambda \left(\partial_\lambda n_\mu - \partial_\mu n_\lambda\right) \hat{T}^\sigma{}_{\rho\sigma} \\ =& \frac{2}{\alpha} \partial^\rho_{(3)}(\alpha)\, \hat{T}^{\sigma(3)}_{\rho\sigma} + \frac{2}{\alpha^2} \partial^\rho_{(3)}(\alpha)\, \partial^{(3)}_\rho(\alpha), \end{aligned} \tag{27}$$

$$\begin{aligned} -2n^\alpha n^\lambda \hat{\nabla}_\lambda \hat{T}^\rho{}_{\alpha\rho} =& -2\left(\gamma^{\alpha\lambda} - g^{\alpha\lambda}\right)\left(\gamma^\rho{}_\mu - n^\alpha n_\mu\right) \hat{\nabla}_\lambda \hat{T}^\mu{}_{\alpha\rho} \\ =& -2\hat{D}_\lambda \hat{T}^{\rho\lambda}_{(3)\rho} - 2\hat{D}_\lambda \left[\frac{1}{\alpha} \partial^\lambda_{(3)}(\alpha)\right] + \hat{\nabla}_\lambda \hat{T}^{\rho\lambda}{}_\rho, \end{aligned} \tag{28}$$

$$-2n^{\alpha}n^{\lambda}\hat{\nabla}_{\rho}\hat{T}_{\alpha\rho}^{\ \ \rho} = -2\hat{D}_{\rho}\left[\frac{1}{\alpha}\partial^{\rho}_{(3)}(\alpha)\right], \tag{29}$$

$$2\hat{T}^{\alpha}{}_{\alpha}{}^{\lambda}\hat{T}^{\rho}{}_{\lambda\rho} = 2\left(\gamma^{\alpha}{}_{\epsilon} - n^{\alpha}n_{\epsilon}\right)\left(\gamma^{\lambda}{}_{\chi} - n^{\lambda}n_{\chi}\right)\hat{T}^{\epsilon}{}_{\alpha}{}^{\chi}\hat{T}^{\rho}{}_{\lambda\rho}. \tag{30}$$

Taking each all of the resulting terms from the Eq.(30)

$$2\hat{T}^{\alpha(3)\lambda}_{\alpha}\left(\gamma^{\rho}{}_{\mu} - n^{\rho}n_{\mu}\right)\hat{T}^{\mu}{}_{\lambda\rho} = 2\hat{T}^{\alpha(3)\lambda}_{\alpha}\hat{T}^{\rho(3)}_{\lambda\rho} + \frac{2}{\alpha}\hat{T}^{\alpha(3)\lambda}_{\alpha}\partial^{(3)}_{\lambda}(\alpha) \tag{31}$$

$$2\hat{A}\left(\gamma^{\rho}{}_{\mu} - n^{\rho}n_{\mu}\right)n^{\lambda}\hat{T}^{\mu}{}_{\lambda\rho} = 2\hat{A}^{2} \tag{32}$$

$$-2\gamma^{\lambda\chi}n^{\alpha}n_{\epsilon}\hat{T}^{\epsilon}{}_{\lambda\rho} = -\frac{2}{\alpha}\partial^{\lambda}_{(3)}(\alpha)\hat{T}^{\rho(3)}_{\lambda\rho} - \frac{2}{\alpha^{2}}\partial^{\lambda}_{(3)}(\alpha)\partial^{(3)}_{\lambda}(\alpha) \tag{33}$$

Combining all of the above, noting that the last term of Eq.(28) cuts out with the last term of Eq.(23) and substituting Eq.(4.36), leaves a purely spatial Hamiltonian constraint equation, Eq.(4.45).

With regards to the momentum constraint, one once again needs to start from the symmetric part of the field equations with one of the free indices contracted with a normal vector and one with the spatial mapping tensor

$$n^{\alpha}\gamma^{\sigma}{}_{\chi}FE_{(\alpha\sigma)} = 2n^{\alpha}\gamma^{\sigma}{}_{\chi}\theta_{\alpha\sigma} = 16\pi n^{\alpha}\gamma^{\sigma}{}_{\chi}\mathcal{T}_{\alpha\sigma} = 16\pi S_{\chi}. \tag{34}$$

Here S_{χ} is the momentum density. Expanding and simplifying this equation using Mathematica (Appendix D.1) gives the following

$$\begin{aligned} n^{\alpha}\gamma^{\sigma}{}_{\chi}\Big[& -\hat{T}_{\lambda\sigma\rho}\hat{T}^{\lambda}{}_{\alpha}{}^{\rho} - 2g_{\alpha\sigma}\left(\hat{T}^{\beta}{}_{\rho\beta}\hat{T}^{\lambda}{}_{\lambda}{}^{\rho} - \frac{1}{2}\hat{T} + \hat{\nabla}_{\rho}\hat{T}^{\lambda}{}_{\lambda}{}^{\rho}\right) - \hat{T}^{\lambda}{}_{\alpha}{}^{\rho}\hat{T}_{\rho\sigma\lambda} + \hat{T}_{\alpha\sigma}{}^{\lambda}\hat{T}^{\rho}{}_{\lambda\rho} \\ & + \hat{T}^{\rho}{}_{\lambda\rho}\hat{T}_{\sigma\alpha}{}^{\lambda} + \frac{1}{2}\hat{T}_{\alpha}{}^{\lambda\rho}\hat{T}_{\sigma\lambda\rho} + \hat{\nabla}_{\alpha}\hat{T}^{\lambda}{}_{\sigma\lambda} + \hat{\nabla}_{\lambda}\hat{T}_{\alpha\sigma}{}^{\lambda} + \hat{\nabla}_{\lambda}\hat{T}_{\sigma\alpha}{}^{\lambda} - \hat{\nabla}_{\sigma}\hat{T}^{\lambda}{}_{\alpha\lambda}\Big] \\ & = 16\pi S_{\chi}. \end{aligned} \tag{35}$$

Considering each term individually and noting that $n^{\alpha}\gamma^{\sigma}{}_{\chi}g_{\alpha\sigma}=0$ one gets

$$-n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{T}_{\lambda\sigma\rho}\hat{T}^{\lambda}{}_{\alpha}{}^{\rho} = -n^{\alpha}\gamma^{\sigma}{}_{\chi}\gamma^{\rho}{}_{\mu}\left(\gamma^{\lambda}{}_{\nu}-n^{\lambda}n_{\nu}\right)\hat{T}_{\lambda\sigma\rho}\hat{T}^{\nu}{}_{\alpha}{}^{\mu} \tag{36}$$

$$= -\hat{T}^{(3)}_{\lambda\chi\rho}\hat{A}^{\lambda\rho},$$

$$-n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{T}^{\lambda}{}_{\alpha}{}^{\rho}\hat{T}_{\rho\sigma\lambda} = -n^{\alpha}\gamma^{\sigma}{}_{\chi}\gamma^{\rho}{}_{\mu}\left(\gamma^{\lambda}{}_{\nu}-n^{\lambda}n_{\nu}\right)\hat{T}^{\nu}{}_{\alpha}{}^{\mu}\hat{T}_{\rho\sigma\lambda} \tag{37}$$

$$= -\hat{A}^{\lambda\rho}\hat{T}^{(3)}_{\rho\chi\lambda} - n^{\alpha}\left(\partial_{\chi}n_{\mu}-\partial_{\mu}n_{\alpha}\right)\hat{A}^{\nu}{}_{\chi}$$

$$= -\hat{A}^{\lambda\rho}\hat{T}^{(3)}_{\rho\chi\lambda} - \frac{1}{\alpha}\partial^{(3)}_{\rho}(\alpha)\hat{A}^{\rho}{}_{\chi},$$

$$n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{T}_{\alpha\sigma}{}^{\lambda}\hat{T}^{\rho}{}_{\lambda\rho} = n^{\alpha}\gamma^{\sigma}{}_{\chi}\left(\gamma^{\lambda}{}_{\nu}-n^{\lambda}n_{\nu}\right)\left(\gamma^{\rho}{}_{\mu}-n^{\rho}n_{\mu}\right)\hat{T}_{\alpha\sigma}{}^{\nu}\hat{T}^{\mu}{}_{\lambda\rho} \tag{38}$$

$$= -n^{\alpha}\gamma^{\sigma}{}_{\chi}n^{\lambda}n_{\nu}\gamma^{\rho}{}_{\mu}\hat{T}_{\alpha\sigma}{}^{\nu}\hat{T}^{\mu}{}_{\lambda\rho}$$

$$= \frac{1}{\alpha}\partial^{(3)}_{\chi}(\alpha)\hat{A},$$

$$\frac{1}{2}n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{T}_{\alpha}{}^{\lambda\rho}\hat{T}_{\sigma\lambda\rho} = \frac{1}{2}n^{\alpha}\gamma^{\sigma}{}_{\chi}\left(\gamma^{\lambda}{}_{\nu}-n^{\lambda}n_{\nu}\right)\left(\gamma^{\rho}{}_{\mu}-n^{\rho}n_{\mu}\right)\hat{T}_{\alpha}{}^{\lambda\rho}\hat{T}_{\sigma\lambda\rho} \tag{39}$$

$$= -\frac{1}{\alpha}\partial^{(3)}_{\nu}(\alpha)\hat{A}_{\chi}{}^{\nu},$$

$$-n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{\nabla}_{\alpha}\hat{T}^{\lambda}{}_{\sigma\lambda} = -n^{\alpha}\gamma^{\sigma}{}_{\chi}\left(\hat{\nabla}_{\sigma}\hat{T}^{\lambda}{}_{\alpha\lambda}-\hat{\nabla}_{\lambda}\hat{T}^{\lambda}{}_{\alpha\sigma}-\hat{T}^{\beta}{}_{\alpha\sigma}\hat{T}^{\lambda}{}_{\beta\lambda}\right). \tag{40}$$

In the case of the last term the Bianchi identity in TEGR as given by Pereira [28] was used.

Considering the individual terms in this equation separately one obtaines

$$-n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{\nabla}_{\sigma}\hat{T}^{\lambda}{}_{\alpha\lambda} = -n^{\alpha}\gamma^{\sigma}{}_{\chi}\left(\gamma^{\lambda}{}_{\nu}-n^{\lambda}n_{\nu}\right)\hat{\nabla}_{\sigma}\hat{T}^{\lambda}{}_{\alpha\lambda} \tag{41}$$

$$= -\hat{D}_{\chi}\hat{A},$$

$$n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{\nabla}_{\lambda}\hat{T}^{\lambda}{}_{\alpha\sigma} = n^{\alpha}\gamma^{\sigma}{}_{\chi}\left(\gamma^{\lambda}{}_{\nu}-n^{\lambda}n_{\nu}\right)\hat{\nabla}_{\lambda}\hat{T}^{\lambda}{}_{\alpha\sigma} \tag{42}$$

$$= \hat{D}_{\lambda}\hat{A}^{\lambda}{}_{\chi} - n^{\lambda}\hat{\nabla}_{\lambda}\left[\frac{1}{\alpha}\partial_{\chi}(\alpha)\right],$$

$$n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{T}^{\beta}{}_{\alpha\sigma}\hat{T}^{\lambda}{}_{\beta\lambda} = n^{\alpha}\gamma^{\sigma}{}_{\chi}\left(\gamma^{\beta}{}_{\nu} - n^{\beta}n_{\nu}\right)\hat{T}^{\nu}{}_{\alpha\sigma}\hat{T}^{\lambda}{}_{\beta\lambda} \tag{43}$$

$$= \hat{A}^{\beta}{}_{\chi}\hat{T}^{\lambda(3)}_{\beta\lambda} + \frac{1}{\alpha}\partial^{(3)}_{\beta}(\alpha)\hat{A}^{\beta}{}_{\chi} - \frac{1}{\alpha}\partial^{(3)}_{\chi}(\alpha)\hat{A},$$

Continuing with the terms from Eq.(35)

$$n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{\nabla}_{\lambda}\hat{T}_{\alpha\sigma}{}^{\lambda} = n^{\lambda}\hat{\nabla}\left[\frac{1}{\alpha}\partial^{(3)}_{\chi}(\alpha)\right], \tag{44}$$

$$n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{\nabla}_{\lambda}\hat{T}_{\sigma\alpha}{}^{\lambda} = \hat{D}_{\lambda}\hat{A}_{\chi}{}^{\lambda}, \tag{45}$$

$$-n^{\alpha}\gamma^{\sigma}{}_{\chi}\hat{\nabla}_{\lambda}\hat{\nabla}_{\sigma}\hat{T}^{\lambda}{}_{\alpha\lambda} = -\hat{D}_{\chi}\hat{A}. \tag{46}$$

Combining all of the above and noting that the last term of Eq.(43) cuts out with Eq.(44), leaves a purely spatial Momentum constraint equation, Eq.(4.44).

C STEGR Second Evolution Equation and Constraints

C.1 STEGR Second Evolution Equation

After substituting the STEGR field equations into the second evolution equation one is left with

$$\mathcal{L}_n\mathring{B}_{\mu\nu} = -\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\Big(\Theta_{\beta\alpha} - \frac{1}{2}g_{\beta\alpha}\Theta + \mathring{L}_{\sigma}{}^{\sigma}{}_{\lambda}\mathring{L}^{\lambda}{}_{\beta\alpha} + \frac{1}{2}\partial_{(\beta|}\mathring{L}^{\epsilon}{}_{\epsilon|\alpha)} + \mathring{Q}^{\lambda\sigma}{}_{\alpha}\mathring{L}_{\lambda\sigma\beta} \tag{47}$$

$$- \frac{1}{2}\mathring{Q}_{\beta\lambda\sigma}\mathring{L}_{\alpha}{}^{\lambda\sigma}\Big) + \partial^{(3)}_{\lambda}\mathring{L}^{\lambda}{}_{\beta\alpha} + \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\mathring{a}_{\sigma}\mathring{L}^{\sigma}{}_{\alpha\beta} - \gamma^{\epsilon}{}_{(\nu|}\gamma^{\chi}{}_{|\mu)}\mathring{B}_{\alpha\chi}n_{\sigma}\mathring{Q}_{\epsilon}{}^{\sigma\alpha}.$$

Before expanding the terms above and converting them to purely spatial terms a number of relations should be considered.

$$-n^{\lambda}\partial_{\epsilon}n_{\lambda} = n^{\lambda}\partial_{\epsilon}\left(\alpha\partial_{\lambda}t\right) \tag{48}$$
$$= \frac{\alpha}{\alpha}\partial_{\lambda}\left(t\right)n^{\lambda}\partial_{\epsilon}\left(\alpha\right)$$
$$= \frac{1}{\alpha}\partial_{\epsilon}\left(\alpha\right),$$

$$a_{\epsilon} = n^{\lambda}\partial_{\lambda}n_{\epsilon} \tag{49}$$
$$= -n\lambda\partial_{\lambda}\left[\alpha\partial_{\epsilon}\left(t\right)\right]$$
$$= \frac{1}{\alpha}n_{\epsilon}n^{\lambda}\partial_{\lambda}\left(\alpha\right).$$

Considering the third term in the bracket of Eq.(47) one obtains

$$-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\mathring{L}_{\sigma}{}^{\sigma}{}_{\lambda}\mathring{L}^{\lambda}{}_{\beta\alpha} = \frac{1}{2}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\mathring{Q}_{\epsilon}{}^{\sigma}{}_{\sigma}\mathring{L}^{\lambda}{}_{\beta\alpha} \tag{50}$$
$$= \frac{1}{2}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}g^{\sigma\lambda}\mathring{\nabla}_{\epsilon}\left(g_{\lambda\sigma}\right)\mathring{L}^{\epsilon}{}_{\beta\alpha}$$
$$= \frac{1}{2}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(\mathring{\nabla}_{\epsilon}\gamma^{\lambda}{}_{\lambda} - \gamma_{\lambda\sigma}\mathring{\nabla}g^{\sigma\lambda} - 2n^{\lambda}\mathring{\nabla}n_{\lambda}\right)$$
$$= \gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left[\frac{1}{2}\gamma_{\lambda\sigma}\mathring{Q}_{\epsilon}{}^{\sigma\lambda} + \frac{1}{\alpha}\partial^{(3)}_{\epsilon}\left(\alpha\right) - a_{\epsilon}\right]\mathring{L}^{\epsilon}{}_{\alpha\beta}.$$

labeling each of the three terms inside the square brackets in order as 1a, 1b and 1c respectively one notes that 1c and $\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\mathring{a}_{\sigma}\mathring{L}^{\sigma}{}_{\alpha\beta}$ cancel out. Considering 1a

$$\frac{1}{2}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\gamma_{\lambda\sigma}\mathring{Q}_{\epsilon}{}^{\sigma\lambda}\mathring{L}^{\epsilon}{}_{\alpha\beta} = \frac{1}{2}\mathring{Q}_{\epsilon}{}^{\sigma(3)}{}_{\sigma}\mathring{L}^{\epsilon(3)}_{\mu\nu} - \frac{1}{2}\gamma_{\lambda\sigma}n^{\epsilon}\mathring{Q}_{\epsilon}{}^{\lambda\sigma}\mathring{B}_{\mu\nu}, \tag{51}$$

such that one is left with

$$\frac{1}{2}\mathring{Q}_{\epsilon}{}^{\sigma(3)}{}_{\sigma}\mathring{L}^{\epsilon(3)}_{\mu\nu} + \frac{1}{\alpha}\partial^{(3)}_{\epsilon}\left(\alpha\right)\mathring{L}^{\epsilon(3)}_{\mu\nu} - \frac{1}{2}\gamma_{\lambda\sigma}n^{\epsilon}\mathring{Q}_{\epsilon}{}^{\lambda\sigma}\mathring{B}_{\mu\nu}. \tag{52}$$

Noting now the first term outside of the brackets in Eq.(47)

$$\begin{aligned}\mathring{D}_\lambda \mathring{L}^\lambda{}_{\nu\mu} =& \gamma^\alpha{}_\nu \gamma^\beta{}_\mu \gamma^\sigma{}_\lambda \mathring{\nabla}_\sigma \mathring{L}^\lambda{}_{\alpha\beta} \\ =& \gamma^\epsilon{}_\nu \gamma^\chi{}_\mu \gamma^\sigma{}_\gamma \mathring{\nabla}_\sigma \mathring{L}^{\gamma(3)}_{\epsilon\chi} - \gamma^\epsilon{}_\nu \gamma^\chi{}_\mu \gamma^\sigma{}_\gamma \mathring{L}^\lambda{}_{\alpha\beta} \mathring{\nabla}_\sigma \left(\gamma^\alpha{}_\epsilon \gamma^\beta{}_\chi \gamma^\gamma{}_\lambda\right) \\ =& \mathring{D}_\sigma \mathring{L}^\sigma{}_{\nu\mu} + B_{\nu\mu} \gamma^\sigma{}_\gamma n_\epsilon \mathring{Q}_\sigma{}^{\epsilon\gamma},\end{aligned} \tag{53}$$

combining it with Eq.(51) and using the definitions of the disformation tensor, Eq.(2.35), and the scalar extrinsic metricity one concludes with

$$\begin{aligned}&\frac{1}{2}\mathring{Q}_\epsilon{}^{\sigma(3)}{}_\sigma \mathring{L}^{\epsilon(3)}_{\mu\nu} + \frac{1}{\alpha}\partial^{(3)}_\epsilon(\alpha)\mathring{L}^{\epsilon(3)}_{\mu\nu} + \mathring{D}_\sigma \mathring{L}^\sigma{}_{\nu\mu} + \mathring{B}_{\mu\nu}\gamma^{\lambda\sigma} n^\epsilon \left(\mathring{Q}_{\sigma\epsilon\lambda} - \frac{1}{2}\mathring{Q}_{\epsilon\sigma\lambda}\right) \\ =&\frac{1}{2}\mathring{Q}_\epsilon{}^{\sigma(3)}{}_\sigma \mathring{L}^{\epsilon(3)}_{\mu\nu} + \frac{1}{\alpha}\partial^{(3)}_\epsilon(\alpha)\mathring{L}^{\epsilon(3)}_{\mu\nu} + \mathring{D}_\sigma \mathring{L}^\sigma{}_{\nu\mu} + \mathring{B}_{\mu\nu}\mathring{B}.\end{aligned} \tag{54}$$

Taking the fourth term inside the brackets of Eq.(47), noting that the symmetries can be shifted to the two outer spatial mapping terms and considering one of the symmetries on its own one obtains

$$\begin{aligned}-\frac{1}{2}\gamma^\alpha{}_\nu \gamma^\beta{}_\mu \partial_\beta \mathring{L}^\epsilon{}_{\epsilon\alpha} =& -\frac{1}{2}\gamma^\alpha{}_\nu \gamma^\beta{}_\mu \partial_\alpha \left[\left(\gamma^\epsilon{}_\chi - n^\epsilon n_\chi\right)\mathring{L}^\chi{}_{\epsilon\alpha}\right] \\ =& -\frac{1}{2}\gamma^\alpha{}_\nu \gamma^\sigma{}_\mu \partial_\alpha \left[\gamma^\beta{}_\sigma \left(\gamma^\epsilon{}_\chi - n^\epsilon n_\chi\right)\mathring{L}^\chi{}_{\epsilon\alpha}\right] + \frac{1}{2}\gamma^\alpha{}_\nu \gamma^\sigma{}_\mu \left(\gamma^\epsilon{}_\chi - n^\epsilon n_\chi\right)\mathring{L}^\chi{}_{\epsilon\alpha}\partial_\alpha \gamma^\beta{}_\sigma \\ =& -\frac{1}{2}\mathring{D}_\nu \mathring{L}^{\epsilon(3)}_{\epsilon\mu} + \frac{1}{4}\gamma^\alpha{}_\nu \gamma^\sigma{}_\mu \partial_\alpha \left(\gamma^\beta{}_\sigma n^\epsilon n^\chi \mathring{L}_\chi \epsilon\alpha\right) \\ =& -\frac{1}{2}\mathring{D}_\nu \mathring{L}^{\epsilon(3)}_{\epsilon\mu} + \frac{1}{2}\partial^{(3)}_\nu \left[\frac{1}{\alpha}\partial^{(3)}_\mu(\alpha)\right].\end{aligned} \tag{55}$$

Combining what has been obtained up to this point one gets

$$\frac{1}{2}\mathring{Q}_\epsilon{}^{\sigma(3)}{}_\sigma \mathring{L}^{\epsilon(3)}_{\mu\nu} + \frac{1}{\alpha}\partial^{(3)}_\epsilon(\alpha)\mathring{L}^{\epsilon(3)}_{\mu\nu} + \mathring{D}_\sigma \mathring{L}^\sigma{}_{\nu\mu} + \mathring{B}_{\mu\nu}\mathring{B} - \frac{1}{2}\mathring{D}_\nu \mathring{L}^{\epsilon(3)}_{\epsilon\mu} + \frac{1}{2}\partial^{(3)}_\nu \left[\frac{1}{\alpha}\partial^{(3)}_\mu(\alpha)\right],$$

which is fully spatial.

Considering the last two terms in the brackets, expanding the disformation tensors and re-combining them gives

$$-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(\mathring{Q}^{\lambda\sigma}{}_{\alpha}\mathring{L}_{\lambda\sigma\beta}-\frac{1}{2}\mathring{Q}_{\beta\lambda\sigma}\mathring{L}_{\alpha}{}^{\lambda\sigma}\right) \tag{56}$$

$$=-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(\mathring{Q}^{\lambda\sigma}{}_{\alpha}\mathring{Q}_{[\lambda\sigma]\beta}-\frac{1}{4}\mathring{Q}_{\beta\lambda\sigma}\mathring{Q}_{\alpha}{}^{\lambda\sigma}\right)$$

$$=-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(\gamma^{\sigma}{}_{\chi}\gamma^{\lambda}{}_{\epsilon}-\gamma^{\sigma}{}_{\chi}n^{\lambda}n_{\epsilon}-\gamma^{\lambda}{}_{\epsilon}n^{\sigma}n_{\chi}+n^{\lambda}n_{\epsilon}n^{\sigma}n_{\chi}\right)\left[\mathring{Q}^{\epsilon\chi}{}_{\alpha}\mathring{Q}_{[\lambda\sigma]\beta}-\frac{1}{4}\mathring{Q}_{\beta\lambda\sigma}\mathring{Q}_{\alpha}{}^{\epsilon\chi}\right]$$

$$=-\frac{1}{2}\mathring{Q}^{\lambda\sigma(3)}{}_{\nu}\mathring{Q}^{(3)}_{[\lambda\sigma]\mu}-\frac{1}{2}\mathring{Q}^{(3)}_{\mu\lambda\sigma}\mathring{Q}^{(3)\lambda\sigma}_{\nu}$$

$$-\frac{1}{2}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\gamma^{\sigma\chi}n^{\lambda}n^{\epsilon}\left(\mathring{Q}_{[\chi\epsilon]\alpha}\mathring{Q}_{[\lambda\sigma]\beta}+\mathring{Q}_{\beta\lambda\sigma}\mathring{Q}_{\alpha\epsilon\chi}\right)-\frac{1}{\alpha^{2}}\partial^{(3)}_{\nu}(\alpha)\,\partial^{(3)}_{\mu}(\alpha)\,, \tag{57}$$

where the following was used

$$\gamma^{\alpha}{}_{\nu}n^{\lambda}n^{\sigma}\partial_{\alpha}g_{\lambda\sigma}=\gamma^{\alpha}{}_{\nu}\left(n^{\lambda}\partial_{\alpha}n_{\lambda}-n_{\lambda}\partial_{\alpha}n^{\lambda}\right) \tag{58}$$

$$=2\gamma^{\alpha}{}_{\nu}n^{\lambda}\partial_{\alpha}n_{\lambda}$$

$$=-2\gamma^{\alpha}{}_{\nu}\frac{\alpha}{\alpha}\partial_{\lambda}(t)\,n^{\lambda}\partial_{\alpha}n_{\lambda}$$

$$=-\frac{2}{\alpha}\partial^{(3)}_{\nu}(\alpha)$$

Considering the first term in the the brackets in the above equation,

$$-\frac{1}{2}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\gamma^{\sigma\chi}n^{\lambda}n^{\epsilon}\mathring{Q}_{[\chi\epsilon]\alpha}\mathring{Q}_{[\lambda\sigma]\beta}=-\frac{1}{2}\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\gamma^{\sigma\chi}n^{\lambda}n^{\epsilon}\mathring{L}_{[\chi\epsilon]\alpha}\mathring{L}_{[\lambda\sigma]\beta} \tag{59}$$

$$=-\frac{1}{2}\gamma^{\alpha}{}_{(\nu|}\gamma^{\beta}{}_{|\mu)}\mathring{B}^{\sigma}{}_{\alpha}n^{\lambda}\mathring{L}_{\sigma\lambda\beta}+\frac{1}{2}\mathring{B}^{\sigma}{}_{\nu}\mathring{B}_{\sigma\mu}+\frac{1}{4}\gamma^{\alpha}{}_{(\nu|}\gamma^{\beta}{}_{|\mu)}n^{\lambda}n^{\epsilon}\mathring{L}_{\chi\epsilon\alpha}\mathring{L}_{\sigma\lambda\beta},$$

and then considering the last term here, one gets

$$\frac{1}{4}\gamma^{\alpha}{}_{(\nu|}\gamma^{\beta}{}_{|\mu)}n^{\lambda}n^{\epsilon}\mathring{L}_{\chi\epsilon\alpha}\mathring{L}_{\sigma\lambda\beta} = \frac{1}{16}\gamma^{\alpha}{}_{(\nu|}\gamma^{\beta}{}_{|\mu)}n^{\lambda}n^{\epsilon}\left(\mathring{Q}_{\chi\epsilon\alpha}-\mathring{Q}_{\epsilon\chi\alpha}-\mathring{Q}_{\alpha\chi\epsilon}\right)\left(\mathring{Q}_{\sigma\lambda\beta}-\mathring{Q}_{\sigma\lambda\beta}-\mathring{Q}_{\lambda\sigma\beta}\right) \tag{60}$$

$$=\frac{1}{16}\gamma^{\alpha}{}_{(\nu|}\gamma^{\beta}{}_{|\mu)}n^{\lambda}n^{\epsilon}\left(-2\mathring{L}_{\epsilon\chi\alpha}-2\mathring{Q}_{\alpha\chi\epsilon}\right)\left(-2\mathring{L}_{\lambda\sigma\beta}-2\mathring{Q}_{\beta\sigma\lambda}\right)$$

$$=\frac{1}{4}\mathring{B}^{\sigma}{}_{(\nu|}\mathring{B}_{\sigma|\mu)}+\frac{1}{2}\gamma^{\alpha}{}_{(\nu|}\gamma^{\beta}{}_{|\mu)}\mathring{B}^{\sigma}{}_{\alpha}n^{\lambda}\mathring{Q}_{\beta\sigma\lambda}+\frac{1}{4}\gamma^{\alpha}{}_{(\nu|}\gamma^{\beta}{}_{|\mu)}\gamma^{\sigma\chi}n^{\lambda}n^{\epsilon}\mathring{Q}_{\alpha\chi\epsilon}\mathring{Q}_{\beta\sigma\lambda}.$$

Updating Eq.(56) and combining it with the last term in Eq.(47) outside of the brackets one obtains

$$-\frac{1}{2}\mathring{Q}^{\lambda\sigma(3)}{}_{\nu}\mathring{Q}^{(3)}_{[\lambda\sigma]\mu}-\frac{1}{2}\mathring{Q}^{(3)}_{\mu\lambda\sigma}\mathring{Q}^{(3)\lambda\sigma}_{\mu}+\mathring{B}^{\sigma}{}_{(\nu|}\mathring{B}_{\sigma|\mu)}-\frac{1}{\alpha^{2}}\partial^{(3)}_{\nu}(\alpha)\,\partial^{(3)}_{\mu}(\alpha)\,. \tag{61}$$

Finally Updating Eq.(47) with this and Eq.(51) one obtains the final, fully spatial form of the second evolution equation in STEGR

$$\begin{aligned}\mathcal{L}_{n}\mathring{B}_{\mu\nu} = &-\gamma^{\alpha}{}_{\nu}\gamma^{\beta}{}_{\mu}\left(\Theta_{\beta\alpha}-\frac{1}{2}g_{\beta\alpha}\Theta\right)\\ &+\frac{1}{2}\mathring{Q}_{\epsilon}{}^{\sigma(3)}{}_{\sigma}\mathring{L}^{\epsilon(3)}_{\mu\nu}+\frac{1}{\alpha}\partial^{(3)}_{\epsilon}(\alpha)\,\mathring{L}^{\epsilon(3)}_{\mu\nu}+\partial^{(3)}_{\sigma}\mathring{L}^{\sigma(3)}_{\mu\nu}\\ &-\mathring{B}_{\mu\nu}\mathring{B}-\frac{1}{2}\partial^{(3)}_{(\nu|}\mathring{L}^{\sigma(3)}{}_{\sigma|\mu)}+\frac{1}{2}\partial^{(3)}_{(\nu|}\left(\frac{1}{\alpha}\partial^{(3)}_{|\mu)}\alpha\right)\\ &-\frac{1}{2}\mathring{Q}^{\lambda\sigma(3)}{}_{\nu}\mathring{Q}^{(3)}_{[\lambda\sigma]\mu}+\frac{1}{4}\mathring{Q}^{(3)}_{\mu\lambda\sigma}\mathring{Q}^{(3)\lambda\sigma}{}_{\nu}+\mathring{B}^{\sigma}{}_{(\nu|}\mathring{B}_{\sigma|\mu)}\\ &+\frac{1}{\alpha^{2}}\partial^{(3)}_{\nu}\alpha\,\partial^{(3)}_{\mu}\alpha\,.\end{aligned} \tag{62}$$

C.2 STEGR Constraint Equations

Starting from the momentum constraint, Eq.(4.57) is contracted with a normal vector and a spatial mapping tensor

$$
\begin{aligned}
n^\nu \gamma^\mu{}_\beta \Theta_{\mu\nu} = n^\nu \gamma^\mu{}_\beta \Bigg(& -\overset{\diamond}{L}_\sigma{}^\sigma{}_\alpha \overset{\diamond}{L}^\alpha{}_{\mu\nu} - \partial_\alpha \overset{\diamond}{L}^\alpha{}_{\mu\nu} - \frac{1}{2}\partial_{(\mu|}\overset{\diamond}{L}^\epsilon{}_{\epsilon|\nu)} \\
& - \overset{\diamond}{Q}^{\sigma\epsilon}{}_\nu \overset{\diamond}{L}_{\sigma\epsilon\mu} + \frac{1}{2}\overset{\diamond}{Q}_{\mu\nu\epsilon}\overset{\diamond}{L}_\nu{}^{\sigma\epsilon} + \frac{1}{2} g_{\mu\nu}\Theta \Bigg) .
\end{aligned} \tag{63}
$$

At this point this was put into a Mathematica code that expanded and simplified the equation (Appendix D.2). The code then split the terms into terms with no derivatives and others containing derivatives. Starting with the non-derivative parts one gets.

$$
\begin{aligned}
&-\frac{1}{4}n^{\alpha}\left[\mathring{Q}_{\alpha}^{\ \epsilon\lambda}\mathring{Q}^{(3)}_{\beta\epsilon\lambda}+2\mathring{Q}^{\epsilon\ \ \lambda}_{\ \alpha}\left(\mathring{Q}^{(3)}_{\epsilon\beta\lambda}-\mathring{Q}^{(3)}_{\lambda\beta\epsilon}\right)\right] \qquad (64)\\
=&-\frac{1}{4}n^{\alpha}\left[2\mathring{L}_{\alpha}^{\ \epsilon\lambda}\mathring{Q}^{(3)}_{\beta\epsilon\lambda}+\mathring{Q}^{(3)}_{\beta\epsilon\lambda}\left(\mathring{Q}^{\epsilon\ \lambda}_{\ \alpha}+\mathring{Q}^{\lambda\ \epsilon}_{\ \alpha}\right)+2\mathring{Q}^{\epsilon\ \ \lambda}_{\ \alpha}\left(\mathring{Q}^{(3)}_{\epsilon\beta\lambda}-\mathring{Q}^{(3)}_{\lambda\beta\epsilon}\right)\right]\\
=&-\frac{1}{2}\mathring{B}^{\epsilon\lambda}\mathring{Q}^{(3)}_{\beta\epsilon\lambda}-\frac{1}{2}n^{\alpha}\mathring{Q}^{\epsilon\ \lambda}_{\ \alpha}\left(\mathring{Q}^{(3)}_{\beta\epsilon\lambda}+\mathring{Q}^{(3)}_{\epsilon\beta\lambda}-\mathring{Q}^{(3)}_{\lambda\beta\epsilon}\right)\\
=&-\frac{1}{2}\mathring{B}^{\epsilon\lambda}\mathring{Q}^{(3)}_{\beta\epsilon\lambda}+n^{\alpha}\mathring{Q}^{\epsilon\ \lambda}_{\ \alpha}\mathring{L}^{(3)}_{\lambda\beta\epsilon},
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2\alpha}n^{\alpha}\partial^{\epsilon}_{(3)}(\alpha)\left[\gamma_{\beta\theta}\gamma_{\epsilon\lambda}\left(\mathring{Q}^{\lambda\ \theta}_{\ \alpha}+\mathring{Q}^{\theta\ \lambda}_{\ \alpha}-\mathring{Q}_{\alpha}^{\ \lambda\theta}\right)+\gamma_{\beta\epsilon}\gamma_{\lambda\theta}\left(\mathring{Q}_{\alpha}^{\ \lambda\theta}-2\mathring{Q}^{\lambda\ \theta}_{\ \alpha}\right)\right] \qquad (65)\\
=&\frac{1}{\alpha}\partial^{\epsilon}_{(3)}(\alpha)\,\mathring{B}_{\epsilon\beta}-\frac{1}{\alpha}\partial^{(3)}_{\beta}(\alpha)\,\mathring{B},
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{4}n^{\alpha}\left[\gamma_{\beta\epsilon}\gamma_{\lambda\theta}\left(-\mathring{Q}^{\theta\rho(3)}_{\ \ \ \rho}+2\mathring{Q}^{\rho\ \ \theta}_{\ \rho(3)}\right)\mathring{Q}_{\alpha}^{\ \epsilon\lambda}+\mathring{Q}^{\theta\rho}_{\ \ \rho}\gamma_{\beta\lambda}\gamma_{\epsilon\theta}\left(\mathring{Q}^{\epsilon\ \lambda}_{\ \alpha}-\mathring{Q}^{\lambda\ \theta}_{\ \alpha}\right)\right. \qquad (66)\\
&\qquad\left.+2\mathring{Q}^{\rho\ \ \theta}_{\ \rho(3)}\gamma_{\beta\lambda}\gamma_{\epsilon\theta}\left(-\mathring{Q}^{\epsilon\ \lambda}_{\ \alpha}+\mathring{Q}^{\lambda\ \epsilon}_{\ \alpha}\right)\right]\\
=&\frac{1}{4}n^{\alpha}\left[\gamma_{\beta\epsilon}\gamma_{\lambda\theta}2\mathring{L}^{\theta\rho(3)}_{\ \ \ \rho}\mathring{Q}_{\alpha}^{\ \epsilon\lambda}+\gamma_{\beta\lambda}\gamma_{\epsilon\theta}\left(\mathring{Q}^{\epsilon\ \lambda}_{\ \alpha}-\mathring{Q}^{\lambda\ \epsilon}_{\ \alpha}\right)\left(\mathring{Q}^{\theta\rho(3)}_{\ \ \ \rho}-2\mathring{Q}^{\rho\ \ \theta}_{\ \rho(3)}\right)\right]\\
=&\frac{1}{4}n^{\alpha}\gamma_{\beta\epsilon}\gamma_{\lambda\theta}2\mathring{L}^{\theta\rho(3)}_{\ \ \ \rho}\left(-2\mathring{L}_{\alpha}^{\ \epsilon\lambda}-2\mathring{Q}^{\epsilon\ \lambda}_{\ \alpha}\right)\\
=&-\mathring{L}^{\theta\rho(3)}_{\ \ \ \rho}\mathring{B}_{\beta\theta}-\mathring{L}_{\alpha\ \theta}^{\ \theta(3)}\gamma^{\epsilon}_{\ \beta}n_{\sigma}\mathring{Q}_{\epsilon}^{\ \sigma\alpha}.
\end{aligned}
$$

Considering the derivative terms one gets

$$
\begin{aligned}
&\frac{1}{2}n^{\alpha}\gamma_{\beta}^{\ \epsilon}\gamma^{\mu\nu}\left[\partial_{\epsilon}\mathring{Q}_{\alpha\mu\nu}+\partial_{\nu}\left(-\mathring{Q}_{\alpha\epsilon\mu}-\mathring{Q}_{\epsilon\alpha\mu}+\mathring{Q}_{\mu\alpha\epsilon}\right)\right] \qquad (67)\\
=&\frac{1}{2}n^{\alpha}\gamma_{\beta}^{\ \epsilon}\gamma^{\mu\nu}\left[\partial_{\epsilon}\left(\mathring{Q}_{\alpha\mu\nu}-2\mathring{Q}_{\nu\alpha\mu}\right)-2\partial_{\nu}\left(\mathring{L}_{\alpha\epsilon\mu}\right)\right]\\
=&n^{\alpha}\gamma_{\beta}^{\ \epsilon}\gamma^{\mu\nu}\left[\partial_{\epsilon}\left(\mathring{L}_{\alpha\mu\nu}\right)-\partial_{\nu}\left(\mathring{L}_{\alpha\epsilon\mu}\right)\right].
\end{aligned}
$$

Considering the two terms separately one obtains

$$\begin{aligned}
n^{\alpha}\gamma^{\epsilon}{}_{\beta}\gamma^{\mu\nu}\partial_{\epsilon}\left(\mathring{L}_{\alpha\mu\nu}\right) =& n^{\alpha}\gamma^{\epsilon}{}_{\beta}\gamma^{\sigma\rho}\gamma^{\rho}{}_{\mu}\gamma^{\sigma}{}_{\nu}\partial_{\epsilon}\left(\mathring{L}_{\alpha\mu\nu}\right) \\
=& n^{\alpha}\gamma^{\epsilon}{}_{\beta}\gamma^{\sigma\rho}\partial_{\epsilon}\left(\gamma^{\rho}{}_{\mu}\gamma^{\sigma}{}_{\nu}\mathring{L}_{\alpha\mu\nu}\right) \\
=& \gamma^{\epsilon}{}_{\beta}\gamma^{\sigma\rho}\partial_{\epsilon}\mathring{B}_{\rho\sigma} - \gamma^{\epsilon}{}_{\beta}\gamma^{\mu\nu}\mathring{L}_{\alpha\mu\nu}\partial_{\epsilon}\left(n^{\alpha}\right) \\
=& \gamma^{\sigma\rho}\partial^{(3)}_{\beta}\mathring{B}_{\rho\sigma} - \frac{1}{\alpha}\mathring{B}\partial^{(3)}_{\beta}\left(\alpha\right) - \gamma^{\epsilon}{}_{\beta}\gamma^{\mu\nu}n^{\alpha}\mathring{L}_{\alpha\mu\nu}n_{\lambda}n_{\sigma}\mathring{Q}^{\lambda\sigma}_{\epsilon} + \mathring{L}_{\alpha}{}^{\theta(3)}{}_{\theta}\gamma^{\epsilon}{}_{\beta}n_{\sigma}\mathring{Q}_{\epsilon}{}^{\alpha\sigma} \\
=& \gamma^{\sigma\rho}\partial^{(3)}_{\beta}\mathring{B}_{\rho\sigma} + \frac{1}{\alpha}\mathring{B}\partial^{(3)}_{\beta}\left(\alpha\right) + \mathring{L}_{\alpha}{}^{\theta(3)}{}_{\theta}\gamma^{\epsilon}{}_{\beta}n_{\sigma}\mathring{Q}_{\epsilon}{}^{\alpha\sigma},
\end{aligned} \tag{68}$$

where the following relation was used

$$\begin{aligned}
\gamma^{\sigma\rho}\partial_{\epsilon}\gamma_{\mu}{}^{\rho} =& -\gamma^{\sigma\rho}\partial_{\epsilon}n_{\mu}n^{\rho} \\
=& -\gamma^{\sigma\rho}n_{\mu}\partial_{\epsilon}n^{\rho} \\
=& \gamma^{\sigma\rho}n_{\mu}\partial_{\epsilon}\left(\alpha\partial_{\rho}t\right) \\
=& \frac{\alpha}{\alpha}\gamma^{\sigma\rho}\partial_{\rho}\left(t\right)n_{\mu}\partial_{\epsilon}\left(\alpha\right) \\
=& -\frac{1}{\alpha}\gamma^{\sigma\rho}n_{\rho}n_{\mu}\partial_{\epsilon}\left(\alpha\right) \\
=& 0.
\end{aligned} \tag{69}$$

Considering the second term of Eq.(67) and following the exact same procedure as with the first term one gets

$$\begin{aligned}
-n^{\alpha}\gamma^{\epsilon}{}_{\beta}\gamma^{\mu\nu}\partial_{\nu}\left(\mathring{L}_{\alpha\epsilon\mu}\right) =& -n^{\alpha}\gamma^{\lambda}{}_{\beta}\gamma^{\rho\nu}\partial_{\nu}\left(\gamma^{\epsilon}{}_{\lambda}\gamma^{\mu}{}_{\rho}\mathring{L}_{\alpha\epsilon\mu}\right) \\
=& -\gamma^{\rho\nu}\partial^{(3)}_{\nu}\mathring{B}_{\beta\rho} - \frac{1}{\alpha}\mathring{B}^{\nu}_{\beta}\partial^{(3)}_{\nu}\left(\alpha\right) + \mathring{L}_{\alpha\beta}{}^{(3)\nu}n_{\sigma}\mathring{Q}_{\nu}{}^{\sigma\alpha}.
\end{aligned} \tag{70}$$

At this point it is noted that the non-spatial parts of Eq.(64) and Eq.(65) cut out with the non-spatial parts of Eq.(68) and Eq.(70). Combining the rest and defining the momentum density as $-S_{\beta} = n^{\nu}\gamma^{\mu}{}_{\beta}\left(\Theta_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Theta\right)$ one obtains Eq.(4.64).

$$8\pi G S_{\beta} = \frac{1}{2}\mathring{B}^{\epsilon\lambda}\mathring{Q}^{(3)}_{\beta\epsilon\lambda} + \mathring{L}^{\theta\rho(3)}_{\rho}\mathring{B}_{\beta\theta} - \gamma^{\sigma\rho}\partial^{(3)}_{\beta}\left(\mathring{B}_{\rho\sigma}\right) + \gamma^{\sigma\rho}\partial^{(3)}_{\rho}\left(\mathring{B}_{\beta\sigma}\right). \tag{71}$$

Moving forward the Hamiltonian constraint is considered. Once again starting from the field equations and contracting them fully with normal vectors one gets

$$n^\nu n^\mu \Theta_{\mu\nu} = n^\nu n^\mu \Big(-\mathring{L}_\sigma{}^\sigma{}_\alpha \mathring{L}^\alpha{}_{\mu\nu} - \partial_\alpha \mathring{L}^\alpha{}_{\mu\nu} - \partial_\mu \mathring{L}^\epsilon{}_{\epsilon\nu} - \mathring{Q}^{\sigma\epsilon}{}_\nu \mathring{L}_{\sigma\epsilon\mu} + \frac{1}{2}\mathring{Q}_{\mu\nu\epsilon}\mathring{L}_\nu{}^{\sigma\epsilon} + \frac{1}{2} g_{\mu\nu}\Theta \Big) . \tag{72}$$

Switching to Mathematica (Appendix D.2), expanding and simplifying, the derivative terms are considered first,

$$\frac{1}{2} n^\alpha n^\beta \gamma^{\mu\nu} \left(\partial_\beta \mathring{Q}_{\alpha\mu\nu} - 2\partial_\nu \mathring{Q}_{\beta\alpha\mu} + \partial_\mu \mathring{Q}_{\nu\alpha\beta} \right). \tag{73}$$

Considering the first two terms and noting that the partial derivative index and the first non-metricity index can be exchanged as it amounts to a partial derivative swap, one gets

$$\begin{aligned}
\frac{1}{2} n^\alpha n^\beta \gamma^{\mu\nu} \partial_\beta \left(\mathring{Q}_{\alpha\mu\nu} - 2\mathring{Q}_{\nu\alpha\mu} \right) =& n^\alpha n^\beta \gamma^{\mu\nu} \partial_\beta \left(L_{\alpha\nu\mu} \right) \\
=& n^\alpha n^\beta \gamma^{\sigma\lambda} \gamma^\mu{}_\lambda \gamma^\nu{}_\sigma \partial_\beta \left(\mathring{L}_{\alpha\nu\mu} \right) \\
=& n^\beta \gamma^{\sigma\lambda} \partial_\beta \left(n^\alpha \gamma^\mu{}_\lambda \gamma^\nu{}_\sigma \mathring{L}_{\alpha\nu\mu} \right) - n^\beta \gamma^{\mu\nu} \mathring{L}_{\alpha\nu\mu} \partial_\beta \left(n^\alpha \right) \\
=& n^\beta \gamma^{\sigma\lambda} \partial_\beta \left(\mathring{B}_{\sigma\lambda} \right) - n^\beta \gamma^{\mu\nu} \mathring{L}_{\alpha\nu\mu} \partial_\beta \left(g^{\sigma\alpha} n_\sigma \right) \\
=& n^\beta \gamma^{\sigma\lambda} \partial_\beta \left(\mathring{B}_{\sigma\lambda} \right) - n^\beta n_\sigma \gamma^{\mu\nu} \mathring{L}_{\alpha\nu\mu} \mathring{Q}_\beta{}^{\sigma\alpha} - \frac{1}{\alpha} \gamma^{\nu\mu} \mathring{L}^\sigma{}_{\nu\mu} n^\beta n_\sigma \partial_\beta (\alpha) \\
=& n^\beta \gamma^{\sigma\lambda} \partial_\beta \left(\mathring{B}_{\sigma\lambda} \right) - n^\beta n_\sigma \gamma^{\mu\nu} \mathring{L}_{\alpha\nu\mu} \mathring{Q}_\beta{}^{\sigma\alpha} - \frac{1}{\alpha} \mathring{B} n^\beta \partial_\beta (\alpha) .
\end{aligned} \tag{74}$$

Considering the third term and noting that

$$\begin{aligned}
\gamma^\mu{}_\lambda n^\alpha n^\beta \mathring{Q}_{\mu\alpha\beta} =& \gamma^\mu{}_\lambda \left(n^\alpha \partial_\mu n_\alpha - n_\beta \partial_\nu n^\beta \right) \\
=& 2\gamma^\mu{}_\lambda n^\alpha \partial_\mu n_\alpha \\
=& -\frac{2}{\alpha} \partial^{(3)}_\lambda (\alpha) ,
\end{aligned} \tag{75}$$

one gets

$$\begin{aligned}
\frac{1}{2}n^{\alpha}n^{\beta}\gamma^{\nu\mu}\partial_{\nu}\mathring{Q}_{\mu\alpha\beta} =&\frac{1}{2}n^{\alpha}n^{\beta}\gamma^{\sigma\lambda}\gamma^{\mu}{}_{\lambda}\gamma^{\nu}{}_{\sigma}\partial_{\nu}\mathring{Q}_{\mu\alpha\beta} \\
=&\frac{1}{2}\gamma^{\sigma\lambda}\gamma^{\nu}{}_{\sigma}\left[\partial_{\nu}\left(\gamma^{\mu}{}_{\lambda}n^{\alpha}n^{\beta}\mathring{Q}_{\mu\alpha\beta}\right)-2\gamma^{\mu}{}_{\lambda}\mathring{Q}_{\mu\alpha\beta}n^{\alpha}\partial_{\nu}n^{\beta}\right] \\
=&-\partial^{\nu}_{(3)}\left[\frac{1}{\alpha}\partial^{(3)}_{\nu}(\alpha)\right]-\gamma^{\nu\mu}n^{\alpha}\mathring{Q}_{\mu\alpha\beta}\left(-n_{\sigma}\mathring{Q}_{\nu}{}^{\beta\sigma}+g^{\beta\sigma}\partial_{\nu}n_{\sigma}\right) \\
=&-\partial^{\nu}_{(3)}\left[\frac{1}{\alpha}\partial^{(3)}_{\nu}(\alpha)\right]+\gamma^{\nu\mu}n^{\alpha}\mathring{Q}_{\mu\alpha\beta}n_{\sigma}\mathring{Q}_{\nu}{}^{\beta\sigma}+\frac{2}{\alpha^{2}}\partial^{(3)}_{\nu}(\alpha)\,\partial^{\nu}_{(3)}(\alpha) \\
&-\partial^{\nu}_{(3)}\left[\frac{1}{\alpha}\partial^{(3)}_{\nu}(\alpha)\right]+\gamma^{\nu\mu}n^{\alpha}\mathring{Q}_{\mu\alpha\beta}(\gamma^{\beta\lambda}-n^{\beta}n^{\lambda})n^{\sigma}\mathring{Q}_{\nu\lambda\sigma}+\frac{2}{\alpha^{2}}\partial^{(3)}_{\nu}(\alpha)\,\partial^{\nu}_{(3)}(\alpha) \\
&-\partial^{\nu}_{(3)}\left[\frac{1}{\alpha}\partial^{(3)}_{\nu}(\alpha)\right]+\gamma^{\nu\mu}n^{\alpha}\mathring{Q}_{\mu\alpha\beta}\gamma^{\beta\lambda}n^{\sigma}\mathring{Q}_{\nu\lambda\sigma}-\frac{2}{\alpha^{2}}\partial^{(3)}_{\nu}(\alpha)\,\partial^{\nu}_{(3)}(\alpha)\,.
\end{aligned} \tag{76}$$

Combining with Eq.(74) and substituting into the current form of the constraint equation one gets

$$\begin{aligned}
n^{\mu}n^{\nu}\left(\Theta_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\Theta\right)=&-\frac{1}{\alpha}\mathring{L}^{\alpha\sigma}_{(3)\ \sigma}\partial_{\alpha}(\alpha)-\frac{1}{\alpha^{2}}\partial^{(3)}_{\nu}(\alpha)\,\partial^{\nu}_{(3)}(\alpha)-\partial^{\nu}_{(3)}\left[\frac{1}{\alpha}\partial_{\nu}(\alpha)\right] \\
&+\frac{1}{2}\gamma^{\sigma\mu}\gamma^{\lambda\nu}n^{\alpha}n^{\beta}\left(\mathring{Q}_{\lambda\alpha\mu}\mathring{Q}_{\sigma\beta\nu}-\frac{1}{2}\mathring{Q}_{\alpha\sigma\lambda}\mathring{Q}_{\beta\mu\nu}+\mathring{Q}_{\sigma\alpha\lambda}\mathring{Q}_{\mu\beta\nu}\right)+\gamma^{\lambda\sigma}n^{\beta}\partial_{\beta}\left(\mathring{B}_{\lambda\sigma}\right).
\end{aligned} \tag{77}$$

Considering the non-derivative part

$$\frac{1}{2}\gamma^{\sigma\mu}\gamma^{\lambda\nu}n^{\alpha}n^{\beta}\left(\mathring{Q}_{\lambda\alpha\mu}\mathring{Q}_{\sigma\beta\nu}-\frac{1}{2}\mathring{Q}_{\alpha\sigma\lambda}\mathring{Q}_{\beta\mu\nu}+\mathring{Q}_{\sigma\alpha\lambda}\mathring{Q}_{\mu\beta\nu}\right) \tag{78}$$

$$=\frac{1}{2}\gamma^{\sigma\mu}\gamma^{\lambda\nu}n^{\alpha}n^{\beta}\left[\mathring{Q}_{\mu\beta\nu}\left(\mathring{Q}_{\lambda\alpha\sigma}+\mathring{Q}_{\sigma\alpha\lambda}\right)-\frac{1}{2}\mathring{Q}_{\alpha\sigma\lambda}\mathring{Q}_{\beta\mu\nu}\right]$$

$$=\frac{1}{2}\gamma^{\sigma\mu}\gamma^{\lambda\nu}n^{\alpha}n^{\beta}\left[\mathring{Q}_{\mu\beta\nu}\left(-2\mathring{L}_{\alpha\lambda\sigma}+\mathring{Q}_{\alpha\lambda\sigma}\right)-\frac{1}{2}\mathring{Q}_{\alpha\sigma\lambda}\mathring{Q}_{\beta\mu\nu}\right]$$

$$=\frac{1}{2}\gamma^{\sigma\mu}\gamma^{\lambda\nu}n^{\alpha}n^{\beta}\left[-2\mathring{Q}_{\mu\beta\nu}\mathring{L}_{\alpha\lambda\sigma}+\mathring{Q}_{\alpha\lambda\sigma}\left(\mathring{Q}_{\mu\beta\nu}-\frac{1}{2}\mathring{Q}_{\beta\mu\nu}\right)\right]$$

$$=\frac{1}{2}\gamma^{\sigma\mu}\gamma^{\lambda\nu}n^{\alpha}n^{\beta}\left[-2\mathring{Q}_{\mu\beta\nu}\mathring{L}_{\alpha\lambda\sigma}-\mathring{Q}_{\alpha\lambda\sigma}\mathring{L}_{\beta\mu\nu}\right]$$

$$=\frac{1}{2}\gamma^{\sigma\mu}\gamma^{\lambda\nu}n^{\alpha}n^{\beta}\mathring{L}_{\alpha\lambda\sigma}\left(-2\mathring{Q}_{\mu\beta\nu}-\mathring{Q}_{\beta\mu\nu}\right)$$

$$=\frac{1}{2}\mathring{B}^{\mu\nu}n^{\beta}\left(2\mathring{L}_{\beta\mu\nu}-2\mathring{Q}_{\beta\mu\nu}\right)$$

$$=\mathring{B}^{\mu\nu}\mathring{B}_{\mu\nu}-\mathring{B}^{\mu\nu}n^{\beta}\partial_{\beta}\gamma_{\mu\nu}.$$

At this point the Hamiltonian constraint looks like this

$$n^{\mu}n^{\nu}\left(\Theta_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\Theta\right)=-\frac{1}{\alpha}\mathring{L}^{\alpha\sigma}_{(3)\ \ \sigma}\partial_{\alpha}(\alpha)-\frac{1}{\alpha^{2}}\partial^{(3)}_{\nu}(\alpha)\,\partial^{\nu}_{(3)}(\alpha)-\partial^{\nu}_{(3)}\left[\frac{1}{\alpha}\partial_{\nu}(\alpha)\right] \tag{79}$$

$$+\mathring{B}^{\mu\nu}\mathring{B}_{\mu\nu}-\mathring{B}^{\mu\nu}n^{\beta}\partial_{\beta}\left(\gamma_{\mu\nu}\right)+\gamma^{\lambda\sigma}n^{\beta}\partial_{\beta}\left(\mathring{B}_{\lambda\sigma}\right).$$

Manipulating the last two non spatial terms gives

$$
\begin{aligned}
&\gamma^{\lambda\sigma} n^{\beta} \partial_{\beta} \left(\mathring{B}_{\lambda\sigma} \right) - \mathring{B}^{\mu\nu} n^{\beta} \partial_{\beta} \left(\gamma_{\mu\nu} \right) \\
=&\gamma^{\lambda\sigma} n^{\beta} \partial_{\beta} \left(\mathring{B}_{\lambda\sigma} \right) - \mathring{B}^{\mu\nu} n^{\beta} \mathring{Q}_{\beta\mu\nu} \\
=&\gamma^{\lambda\sigma} n^{\beta} \partial_{\beta} \left(\mathring{B}_{\lambda\sigma} \right) - \mathring{B}^{\mu\nu} n^{\beta} \left(2\mathring{L}_{\beta\mu\nu} + 2\mathring{Q}_{\mu\beta\nu} \right) \\
=&\gamma^{\lambda\sigma} n^{\beta} \partial_{\beta} \left(\mathring{B}_{\lambda\sigma} \right) - 2\mathring{B}^{\mu\nu} \mathring{B}_{\mu\nu} + 2\mathring{B}^{\mu\nu} \gamma_{\beta\nu} \partial_{\mu} \left(n^{\beta} \right) \\
=& - 2\mathring{B}^{\mu\nu} \mathring{B}_{\mu\nu} + \gamma^{\mu\nu} \left[n^{\beta} \partial_{\beta} \left(\mathring{B}_{\mu\nu} \right) + \mathring{B}_{\beta\nu} \partial_{\mu} \left(n^{\beta} \right) + \mathring{B}_{\beta\mu} \partial_{\nu} \left(n^{\beta} \right) \right],
\end{aligned}
\tag{80}
$$

here it is noted that the last term is equivalent to the lie derivative of the extrinsic metricity which can substituted from Eq.(4.62). After this and some simplification the final form of the Hamiltonian constraint is obtained.

$$
\begin{aligned}
16\pi G\rho =& \frac{1}{2} \mathring{Q}_{\epsilon\ \ \sigma}^{\ \sigma(3)} \mathring{L}^{\epsilon\ \ \nu}_{\ \ \nu(3)} + \gamma^{\mu\nu} \partial_{\epsilon}^{(3)} \mathring{L}^{\epsilon(3)}_{\ \ \mu\nu} - \mathring{B}^{2} - \partial^{\nu}_{(3)} \mathring{L}^{\epsilon(3)}_{\ \ \epsilon\nu} \\
& - \frac{1}{2} \mathring{Q}^{\lambda\sigma\nu}_{(3)} \mathring{Q}^{(3)}_{[\lambda\sigma]\nu} + \frac{1}{4} \mathring{Q}^{\lambda\sigma\nu}_{(3)} \mathring{Q}^{(3)}_{\lambda\sigma\nu} + \mathring{B}^{\sigma\nu} \mathring{B}_{\sigma\nu} ,
\end{aligned}
\tag{81}
$$

here the energy density term on the left hand side is obtained through

$$
\begin{aligned}
&8\pi G \left(n^{\mu} n^{\nu} + \gamma^{\mu\nu} \right) \left(\mathcal{T}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} \mathcal{T} \right) \\
=&8\pi G \left(2n^{\mu} n^{\nu} + g^{\mu\nu} \right) \left(\mathcal{T}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} \mathcal{T} \right) \\
=&8\pi G \left(2\rho + \mathcal{T} + \mathcal{T} - \frac{4}{2} \mathcal{T} \right) \\
=&16\pi G\rho .
\end{aligned}
\tag{82}
$$

D Mathematica code

D.1 Torsional Gravity

```
Remove["Global`*"]

(*Importing xAct packages*)

<< xAct`xTensor`
<< xAct`xCoba`
<< xAct`xPert` ;
<< xAct`xTras` ;
<< xAct`TexAct`;

(*Defining geometric entities*)
DefManifold[Global, 4, {\[Alpha], \[Beta], \[Zeta], \[Eta], \[Iota],
\[Lambda], \[Mu], \[Nu], \[Xi], \[Rho], \[Sigma], \[Upsilon], \[Chi],
\[CapitalTheta], \[CapitalLambda], \[CapitalXi], \[CapitalPi],
\[CapitalSigma], \[CapitalUpsilon], \[CapitalPhi], \[CapitalPsi],
\[CapitalOmega], \[CapitalSampi], \[CapitalStigma], \[CapitalKoppa]}]

DefManifold[Local, 4, {a, b, c, d, e, g, i, j, k, l, m, q, u, A, B, G,
   H, J, L, M, P, Q, R, U, V, W, X, Y, Z}]

(* Define the Metric Tensor *)
DefMetric[-1, metric[-\[Mu], -\[Nu]], CD, PrintAs -> "g"]
DefMetric[-1, MinkMet[-a, -b], CDm, PrintAs -> "\[Eta]"]

DefChart[Ba, Global, {0, 1, 2, 3}, {t[], r[], \[Theta][], \[Phi][]},
 ChartColor -> Blue]
DefChart[Bb, Local, {0, 1, 2, 3}, {\[Tau][], x[], y[], z[]},
 ChartColor -> Red]

Ba /: CIndexForm[0, Ba] := "t";
Ba /: CIndexForm[1, Ba] := "r";
Ba /: CIndexForm[2, Ba] := "\[Theta]";
Ba /: CIndexForm[3, Ba] := "\[Phi]";

$PrePrint = ScreenDollarIndices;

(*Defining all necessary Tensors*)

DefTensor[h[a, -\[Alpha]], {Local, Global}, PrintAs -> "h"]
DefTensor[T[\[Lambda], -\[Mu], -\[Nu]], Global,
 Antisymmetric[{-\[Mu], -\[Nu]}]]
DefTensor[Deth[], Global]
DefTensor[\[Kappa][-\[Nu], -\[Mu], -\[Lambda]], Global,
 Antisymmetric[{-\[Nu], -\[Mu]}]]
DefTensor[S[\[Nu], -\[Mu], -\[Lambda]], Global,
 Antisymmetric[{-\[Mu], -\[Lambda]}]]
DefTensor[\[Omega][-\[Alpha], -\[Beta], -\[Lambda]], Global,
 Antisymmetric[{-\[Alpha], -\[Beta]}]]
DefTensor[{TS[], \[Theta]s[]}, Global]
DefTensor[\[Theta]T[\[Alpha], \[Sigma]], Local,
 Symmetric[{\[Alpha], \[Sigma]}]]

(*Defining the Torsion, Contortion and Super Potential tensor*)

TorTens =
 MakeRule[{T[\[Nu], -\[Rho], -\[Mu]],
   h[-a, \[Nu]] PD[-\[Rho]][h[a, -\[Mu]]] -
    h[-a, \[Nu]] PD[-\[Mu]][
      h[a, -\[Rho]]] + \[Omega][\[Nu], -\[Mu], -\[Rho]] - \[Omega][
```

```
\[Nu], -\[Rho], -\[Mu]]}, MetricOn -> All, ContractMetrics -> False]

T[\[Nu], -\[Rho], -\[Mu]] /. TorTens

KonTens =
 MakeRule[{\[Kappa][\[Rho], \[Sigma], \[Mu]],
   1/2 (T[\[Mu], \[Rho], \[Sigma]] + T[\[Sigma], \[Rho], \[Mu]] -
      T[\[Rho], \[Sigma], \[Mu]])}, MetricOn -> All,
  ContractMetrics -> False]

\[Kappa][\[Nu], -\[Rho], -\[Mu]] /. KonTens

SupPot =
 MakeRule[{S[-\[Mu], \[Sigma], \[Rho]], \[Kappa][\[Sigma], \[Rho], -
\[Mu]] - delta[\[Rho], -\[Mu]] T[\[Nu], \[Sigma], -\[Nu]] +
    delta[\[Sigma], -\[Mu]] T[\[Nu], \[Rho], -\[Nu]]},
  MetricOn -> All, ContractMetrics -> False]

S[\[Rho], -\[Mu], -\[Nu]] /. SupPot

(*Defining the Field Equations*)

FE = -CD[\[Lambda]][S[-\[Alpha], -\[Lambda], -\[Sigma]]] -
  1/2 S[-\[Alpha], \[Rho], \[Lambda]] T[-\[Sigma], -\[Lambda],
-\[Rho]] +
  S[-\[Alpha], -\[Sigma], \[Rho]] T[\[Lambda], -\[Rho], -\[Lambda]] -
  S[\[Rho], \[Lambda], -\[Sigma]] T[-\[Rho], -\[Lambda], -\[Alpha]] +
(1/2) metric[-\[Alpha], -\[Sigma]] TS[] (*= \[Theta]T[\[Sigma],\[Alpha]] *)

FE metric[\[Alpha], \[Sigma]] // ContractMetric // Simplification
ScalarFE = % /. KonTens // ContractMetric // Simplification

TSdef = MakeRule[{S[-\[Alpha], -\[Beta], -\[Nu]] T[\[Alpha], \[Beta],
\[Nu]], 2 TS[]}, MetricOn -> All,
   ContractMetrics -> False]; (*Since Ts = 1/2TS*)

ScalarFE = (ScalarFE /. TSdef /. SupPot /. KonTens) //
    ContractMetric // Simplification // Expand

TSSol = Solve[ScalarFE == \[Theta]s[], TS[]] // Simplification

TSRule =
 MakeRule[{TS[],
   2 T[\[Alpha], -\[Rho], -\[Alpha]]
     T[\[Lambda], -\[Lambda], \[Rho]] + \[Theta]s[] + 2 (
CD[-\[Rho]][
T[\[Lambda], -\[Lambda], \[Rho]]])}, MetricOn -> All,
  ContractMetrics -> False]

(FE /. TSRule) //
  Simplification // ContractMetric (*= \[Theta]T[-\[Sigma],-\[Alpha]] *)
FE3 = - S[\[Lambda], -\[Sigma], \[Rho]]
   T[-\[Lambda], -\[Alpha], -\[Rho]] +
  metric[-\[Alpha], -\[Sigma]]   T[\[Beta], -\[Rho], -\[Beta]]
   T[\[Lambda], -\[Lambda], \[Rho]] +
  S[-\[Alpha], -\[Sigma], \[Lambda]]
   T[\[Rho], -\[Lambda], -\[Rho]] +
  1/2  S[-\[Alpha], \[Lambda], \[Rho]]
   T[-\[Sigma], -\[Lambda], -\[Rho]] + CD[-\[Lambda]][
S[-\[Alpha], -\[Sigma], \[Lambda]]] + metric[-\[Alpha], -\[Sigma]]  (
CD[-\[Rho]][
T[\[Lambda], -\[Lambda], \[Rho]]])(*= \[Theta]T[-\[Sigma],-\[Alpha]] - 1/2 g[\[Alpha],\[
    Sigma]]\[Theta]s*)

(*Generating Evolution Equation*)
```

```
FE4 = (FE3 /. SupPot /. KonTens) // ContractMetric // Simplification //
    Expand
Print["Elimination of antisymmetric Parts"]
FE5 = ((FE4 + (FE4 /. \[Alpha] -> \[Mu] /. \[Sigma] -> \[Alpha] /.
\[Mu] -> \[Sigma]))) // Simplification // Expand

Print["AntiSymmetric part just in case"]
((FE4 - (FE4 /. \[Alpha] -> \[Mu] /. \[Sigma] -> \[Alpha] /. \[Mu] -> \[Sigma]))) //
     Simplification // Expand

FET1 = (FE5 -
        2 (\[Theta]T[-\[Sigma], -\[Alpha]] -
           1/2  metric[-\[Alpha], -\[Sigma]]  \[Theta]s[]) //
       Expand) /. \[Lambda] -> \[Chi] /. \[Sigma] -> -\[Lambda] /.
\[Beta] -> \[Rho]

EET = CD[-\[Rho]][T[\[Lambda], -\[Alpha], \[Rho]]]

TetSubst = Solve[FET1 == 0, CD[-\[Rho]][
T[\[Lambda], -\[Alpha], \[Rho]]]][[1, 1]]

(EET /. TetSubst) // Simplification // Expand

(*Generating the Hamiltonian constraint*)

(*=0*)
(((FE + (FE /. \[Alpha] -> \[Mu] /. \[Sigma] -> \[Alpha] /. \[Mu] ->
\[Sigma])) - 2 \[Theta]T[-\[Sigma], -\[Alpha]] ) /. SupPot /.
      KonTens) // ContractMetric // Simplification // Expand
HC = (((n[\[Sigma]] n[\[Alpha]] %) // Expand // ContractMetric) /.
      nn) // Simplification // Expand
Collect[HC, n[\[Lambda]] n[\[Alpha]]]
% // Length

(*Generating the Momentum Constraint*)
(*=0*)
(((FE + (FE /. \[Alpha] -> \[Mu] /. \[Sigma] -> \[Alpha] /. \
\[Mu] -> \[Sigma])) - 2 \[Theta]T[-\[Sigma], -\[Alpha]] ) /.
       SupPot /. KonTens) // ContractMetric //
  Simplification // Expand
MC = (((\[Gamma][\[Sigma], -\[Chi]] n[\[Alpha]] %) // Expand //
           ContractMetric) /. nn /. n\[Gamma]) // \[Gamma]\[Gamma] //
     Simplification // Expand // \[Gamma]\[Gamma]
Length[%]
```

D.2 Symmetric Teleparallel Gravity

```
Remove["Global`*"]

(*Importing Packages*)

<< xAct`xTensor`
<< xAct`xCoba`
<< xAct`xPert` ;
<< xAct`xTras` ;
<< xAct`TexAct` ;

(*Defining geometric entities*)

DefManifold[Global, 4, {\[Alpha], \[Beta], \[Zeta], \[Eta], \[Iota], \
```

```
\[Lambda], \[Mu], \[Nu], \[Xi], \[Rho], \[Sigma], \[Upsilon], \[Chi], \
\[CapitalTheta], \[CapitalLambda], \[CapitalXi], \[CapitalPi], \
\[CapitalSigma], \[CapitalUpsilon], \[CapitalPhi], \[CapitalPsi], \
\[CapitalOmega], \[CapitalSampi], \[CapitalStigma], \[CapitalKoppa]}]

DefManifold[Local, 4, {a, b, c, d, e, i, j, k, l, m, q, u, B, G, H, J,
   M, R, U, V, W, X, Y, Z}]

(* Define the Metric Tensor *)

DefMetric[-1, metric[-\[Mu], -\[Nu]], CD, PrintAs -> "g"]
DefMetric[-1, MinkMet[-a, -b], CDm, PrintAs -> "\[Eta]"]

DefChart[Ba, Global, {0, 1, 2, 3}, {t[], r[], \[Theta][], \[Phi][]},
 ChartColor -> Blue]
DefChart[Bb, Local, {0, 1, 2, 3}, {\[Tau][], x[], y[], z[]},
 ChartColor -> Red]

Ba /: CIndexForm[0, Ba] := "t";
Ba /: CIndexForm[1, Ba] := "r";
Ba /: CIndexForm[2, Ba] := "\[Theta]";
Ba /: CIndexForm[3, Ba] := "\[Phi]";

$PrePrint = ScreenDollarIndices;


DefTensor[Q[-\[Lambda], -\[Nu], -\[Mu]], Global,
 Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[L[-\[Lambda], -\[Nu], -\[Mu]], Global,
 Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[Q3[-\[Lambda], -\[Nu], -\[Mu]], Global,
 Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[L3[-\[Lambda], -\[Nu], -\[Mu]], Global,
 Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[P[-\[Alpha], -\[Mu], -\[Nu]], Global,
 Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[A[-\[Mu], -\[Nu]], Global, Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[AS[], Global]

(*Defining the Disformation tensor and the P tensor*)

Disform =
 MakeRule[{L[-\[Lambda], -\[Nu], -\[Mu]],
    1/2 (Q[-\[Lambda], -\[Nu], -\[Mu]] -
       Q[-\[Nu], -\[Lambda], -\[Mu]] - Q[-\[Mu], -\[Lambda], -\[Nu]])},
    MetricOn -> All, ContractMetrics -> False]
L[-\[Nu], -\[Rho], -\[Mu]] /. Disform

Ptensor =
 MakeRule[{P[-\[Alpha], -\[Mu], -\[Nu]],
    1/2 L[-\[Alpha], -\[Mu], -\[Nu]] +
     1/4 metric[-\[Nu], -\[Mu]] (L[\[Sigma], -\[Sigma], -\[Alpha]] -
        L[-\[Alpha], -\[Sigma], \[Sigma]]) -
     1/4 metric[-\[Alpha], -\[Mu]] L[\[Sigma], -\[Sigma], -\[Nu]] -
     1/4 metric[-\[Alpha], -\[Nu]] L[\[Sigma], -\[Sigma], -\[Mu]]},
   MetricOn -> All, ContractMetrics -> False]
P[-\[Alpha], -\[Mu], -\[Nu]] /. Ptensor

DefTensor[\[Gamma][\[Nu], \[Lambda]], Global,
 Symmetric[{\[Nu], \[Lambda]}]]
DefTensor[n[-\[Nu]], {Global}]

(*Defining Substitution Rules*)

Gn = MakeRule[{n[-\[Mu]] n[\[Mu]], -1}, ContractMetrics -> False,
   MetricOn -> All]
```

```

SimpQ = MakeRule[{Q[-\[Lambda], \[Sigma], -\[Sigma]],
   metric[\[Sigma], \[Beta]] Q[-\[Lambda], -\[Beta], -\[Sigma]]},
  ContractMetrics -> False, MetricOn -> All]


\[Gamma]1 = MakeRule[{\[Gamma][\[Lambda], -\[Nu]] n[-\[Lambda]], 0}]
\[Gamma]2 = MakeRule[{\[Gamma][\[Lambda], -\[Nu]] n[\[Nu]], 0}]
\[Gamma]3 =
 MakeRule[{\[Gamma][\[Alpha], \[Beta]] \[Gamma][-\[Alpha], -\[Nu]],
\[Gamma][\[Beta], -\[Nu]]}]
meto3 = MakeRule[{metric[-\[Mu], -\[Nu]], \[Gamma][-\[Mu], -\[Nu]] -
    n[-\[Mu]] n[-\[Nu]]}]
metricity1 =
 MakeRule[{PD[-\[Lambda]][metric[-\[Mu], -\[Nu]]],
   Q[-\[Lambda], -\[Mu], -\[Nu]]}, ContractMetrics -> False,
  MetricOn -> None]
metricity2 =
 MakeRule[{PD[-\[Lambda]][
    metric[\[Mu], \[Nu]]], -Q[-\[Lambda], \[Mu], \[Nu]]},
  ContractMetrics -> False, MetricOn -> None]

DefScalarFunction[\[Alpha]1]
DefTensor[PD\[Alpha][-\[Lambda]], Global]

firstlie1 =
 MakeRule[{\[Gamma][\[Nu], -\[Alpha]] \[Gamma][\[Mu], -\[Beta]] n[
\[Lambda]] L[-\[Lambda], -\[Nu], -\[Mu]], A[-\[Alpha], -\[Beta]]},
  ContractMetrics -> False, MetricOn -> All]
firstlie2 =
 MakeRule[{\[Gamma][\[Nu], \[Mu]] n[\[Lambda]] L[-\[Lambda], -\[Nu],
-\[Mu]], AS[]}, ContractMetrics -> False, MetricOn -> All]
case1 = MakeRule[{n[\[Nu]] n[\[Mu]] Q[-\[Lambda], -\[Mu], -\[Nu]],
-2/\[Alpha]1 PD\[Alpha][-\[Lambda]]}, ContractMetrics -> False,
  MetricOn -> All]
case2 = MakeRule[{\[Gamma][\[Lambda], \[Alpha]] \[Gamma][\[Mu],
\[Beta]] \[Gamma][\[Nu], \[Sigma]] Q[-\[Lambda], -\[Mu], -\[Nu]],
   Q3[\[Alpha], \[Beta], \[Sigma]]}, ContractMetrics -> True,
  MetricOn -> All]
case3 = MakeRule[{\[Gamma][\[Lambda], \[Alpha]] \[Gamma][\[Nu],
\[Sigma]] Q[-\[Lambda], -\[Alpha], -\[Nu]],
   Q3[\[Alpha], -\[Alpha], \[Sigma]]}, ContractMetrics -> False,
  MetricOn -> All]
case4 = MakeRule[{\[Gamma][\[Lambda], \[Alpha]] \[Gamma][\[Nu],
\[Sigma]] Q[-\[Lambda], -\[Nu], -\[Alpha]],
   Q3[\[Alpha], \[Sigma], -\[Alpha]]}, ContractMetrics -> False,
  MetricOn -> All]
case5 = MakeRule[{\[Gamma][\[Lambda], \[Alpha]] \[Gamma][\[Nu],
\[Sigma]] Q[-\[Nu], -\[Lambda], -\[Alpha]],
   Q3[\[Sigma], \[Alpha], -\[Alpha]]}, ContractMetrics -> False,
  MetricOn -> All]

case6 = MakeRule[{n[\[Sigma]] Q3[-\[Sigma], -\[Beta], -\[Alpha]], 0},
  ContractMetrics -> False, MetricOn -> All]
case7 = MakeRule[{n[\[Sigma]] Q3[-\[Beta], -\[Sigma], -\[Alpha]], 0},
  ContractMetrics -> False, MetricOn -> All]
case8 = MakeRule[{n[\[Sigma]] Q3[-\[Alpha], -\[Beta], -\[Sigma]], 0},
  ContractMetrics -> False, MetricOn -> All]
case9 = MakeRule[{n[\[Sigma]] Q3[-\[Alpha], \[Alpha], -\[Sigma]], 0},
  ContractMetrics -> False, MetricOn -> All]
case10 =
 MakeRule[{n[\[Sigma]] Q3[-\[Alpha], -\[Sigma], \[Alpha]], 0},
  ContractMetrics -> False, MetricOn -> All]
case11 =
 MakeRule[{n[\[Sigma]] Q3[-\[Sigma], \[Alpha], -\[Alpha]], 0},
  ContractMetrics -> False, MetricOn -> All]
```

```


case12 =
 MakeRule[{\[Gamma][-\[Lambda], -\[Alpha]] \[Gamma][-\[Mu], -\[Beta]]
\[Gamma][-\[Nu], -\[Sigma]] Q[\[Lambda], \[Mu], \[Nu]],
    Q3[-\[Alpha], -\[Beta], -\[Sigma]]}, ContractMetrics -> False,
   MetricOn -> All]

Qtg = MakeRule[{Q[-\[Lambda], -\[Mu], -\[Nu]],
    PD[-\[Lambda]][metric[-\[Mu], -\[Nu]]]}, ContractMetrics -> False,
   MetricOn -> None]

(*Setting Up The Evolution Equation*)

FE=-(1/4)(2CD[-\[Alpha]][Q[\[Alpha],-\[Mu],-\[Nu]]+Q[\[Alpha],-\[Mu],-\[Nu]] Ig[\[Lambda]
    ],\[Beta]] g[-\[Sigma],-\[Alpha]]Q[\[Sigma],-\[Lambda],-\[Beta]]+Q[-\[Mu],-\[Alpha]
    ],-\[Beta]]Q[-\[Nu],\[Alpha],\[Beta]]-2Q[-\[Alpha],-\[Beta],-\[Mu]]Q[\[Alpha],\[Beta
    ],-\[Nu]]-1/2 g[-\[Mu],-\[Nu]]Q[-\[Alpha],-\[Beta],-\[Lambda]]Q[\[Alpha],\[Beta],\[
    Lambda]])+1/2 (CD[-\[Alpha]][(Q[-\[Nu],\[Alpha],-\[Mu]]+Q[-\[Mu],\[Alpha],-\[Nu]])
    ]+1/2 (Q[-\[Nu],\[Alpha],-\[Mu]]+Q[-\[Mu],\[Alpha],-\[Nu]]) Ig[\[Lambda],\[Beta]]Q
    [-\[Alpha],-\[Lambda],-\[Beta]]-Q[-\[Alpha],-\[Beta],-\[Mu]]Q[\[Beta],\[Alpha],-\[Nu
    ]]-1/2 g[-\[Mu],-\[Nu]]Q[-\[Alpha],-\[Beta],-\[Lambda]]Q[\[Alpha],\[Beta],\[Lambda
    ]])+1/4 (2CD[-\[Alpha]][g[-\[Nu],-\[Mu]] Ig[\[Lambda],\[Beta]]Q[\[Alpha],-\[Lambda
    ],-\[Beta]]]+Ig[\[Lambda],\[Beta]]Q[-\[Nu],-\[Lambda],-\[Beta]] Ig[\[Sigma],\[Rho]]Q
    [-\[Mu],-\[Sigma],-\[Rho]]-2 Q[-\[Alpha],-\[Mu],-\[Nu]] Ig[\[Lambda],\[Beta]]Q[\[
    Alpha],-\[Lambda],-\[Beta]]+1/2 g[-\[Nu],-\[Mu]] Ig[\[Rho],\[Xi]]Q[-\[Alpha],-\[Rho
    ],-\[Xi]] Ig[\[Lambda],\[Beta]]Q[\[Alpha],-\[Lambda],-\[Beta]])-1/2 (1/2 (CD[-\[Nu]][
    Ig[\[Rho],\[Xi]]Q[-\[Mu],-\[Rho],-\[Xi]]]+CD[-\[Mu]][Ig[\[Rho],\[Xi]]Q[-\[Nu],-\[Rho
    ],-\[Xi]]])+CD[-\[Alpha]][g[-\[Nu],-\[Mu]] Ig[\[Lambda],\[Beta]] Ig[\[Sigma],\[Alpha]]
    Q[-\[Lambda],-\[Beta],-\[Sigma]]]+1/2 Ig[\[Lambda],\[Beta]]Q[-\[Nu],-\[Lambda],-\[
    Beta]] Ig[\[Sigma],\[Rho]]Q[-\[Mu],-\[Sigma],-\[Rho]]-Ig[\[Lambda],\[Beta]]Q[-\[
    Lambda],-\[Beta],-\[Alpha]]Q[\[Alpha],-\[Nu],-\[Mu]])

FE // SameDummies // Simplify

(*Setting up the Hamiltonian Constraint*)

(*Non Derivative Parts NDP*)
(n[\[Nu]]n[\[Mu]](-L[-\[Sigma],\[Sigma],-\[Alpha]]L[\[Alpha],-\[Mu],-\[Nu]]-Q[\[Alpha
    ],\[Sigma],-\[Nu]]L[-\[Alpha],-\[Sigma],-\[Mu]]+1/2 Q[-\[Mu],-\[Alpha],-\[Sigma]]L
    [-\[Nu],\[Alpha],\[Sigma]])//Expand)
%/.Disform//Expand
%//Simplification//Expand

(*NDP - Extracted metrics are then split*)
(n[\[Nu]]n[\[Mu]](\[Gamma][\[Sigma],\[Chi]]-n[\[Sigma]]n[\[Chi]])(\[Gamma][\[Alpha],\[
    Lambda]]-n[\[Alpha]]n[\[Lambda]])(-L[-\[Sigma],-\[Chi],-\[Alpha]]L[-\[Lambda],-\[Mu
    ],-\[Nu]]-Q[-\[Lambda],-\[Chi],-\[Nu]]L[-\[Alpha],-\[Sigma],-\[Mu]]+1/2 Q[-\[Mu],-\[
    Alpha],-\[Sigma]]L[-\[Nu],-\[Lambda],-\[Chi]])//Expand);
%/.Disform//Expand;
%//Simplification;
NDP=%//Expand

(*Partial Derivative Parts*)
(n[\[Nu]]n[\[Mu]](\[Gamma][\[Alpha],-\[Lambda]]-n[\[Alpha]]n[-\[Lambda]])(PD[-\[Alpha]][
    L[\[Lambda],-\[Mu],-\[Nu]]]-PD[-\[Mu]][L[\[Lambda],-\[Alpha],-\[Nu]]]))/.Disform;
%//Expand;
%/.metricity1/.metricity2;
(((%//SameDummies)/.meto3)//Expand)/.\[Gamma]1/.\[Gamma]2/.Gn/.\[Gamma]3

(*Parts from the derivative parts that do not have derivatives, Non Derivative
    Derivative Part, NDDP*)

NDDP = -(1/2)   n[\[Alpha]]   n[\[Beta]]   n[\[Zeta]]   n[\[Eta]]
    n[\[Mu]]   n[\[Nu]]   Q[-\[Alpha], -\[Zeta], -\[Eta]]
    Q[-\[Beta], -\[Mu], -\[Nu]] +
```

```
    1/2 n[\[Alpha]] n[\[Beta]] n[\[Zeta]] n[\[Eta]] n[\[Mu]]
       n[\[Nu]] Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Mu], -\[Beta], -\[Nu]] -
    1/2 n[\[Alpha]] n[\[Beta]] n[\[Zeta]] n[\[Eta]] n[\[Mu]]
       n[\[Nu]] Q[-\[Alpha], -\[Beta], -\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]] +
    1/2 n[\[Alpha]] n[\[Beta]] n[\[Zeta]] n[\[Eta]] n[\[Mu]]
       n[\[Nu]] Q[-\[Beta], -\[Alpha], -\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]] -
    1/2 n[\[Alpha]] n[\[Beta]] n[\[Zeta]] n[\[Eta]] n[\[Mu]]
       n[\[Nu]] Q[-\[Mu], -\[Zeta], -\[Eta]]
     Q[-\[Nu], -\[Beta], -\[Alpha]] +
    1/2 n[\[Alpha]] n[\[Beta]] n[\[Zeta]] n[\[Eta]] n[\[Mu]]
       n[\[Nu]] Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Nu], -\[Beta], -\[Mu]] +
    1/2 n[\[Beta]] n[\[Eta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Beta], -\[Mu], -\[Nu]] \[Gamma][\[Alpha], \[Zeta]] -
    1/2 n[\[Beta]] n[\[Eta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Mu], -\[Beta], -\[Nu]] \[Gamma][\[Alpha], \[Zeta]] +
    1/2 n[\[Beta]] n[\[Eta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Alpha], -\[Beta], -\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]] \[Gamma][\[Alpha], \[Zeta]] -
    1/2 n[\[Beta]] n[\[Eta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Beta], -\[Alpha], -\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]] \[Gamma][\[Alpha], \[Zeta]] +
    1/2 n[\[Beta]] n[\[Eta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]]
     Q[-\[Nu], -\[Beta], -\[Alpha]] \[Gamma][\[Alpha], \[Zeta]] -
    1/2 n[\[Beta]] n[\[Eta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Nu], -\[Beta], -\[Mu]] \[Gamma][\[Alpha], \[Zeta]] +
    1/2 n[\[Alpha]] n[\[Zeta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Beta], -\[Mu], -\[Nu]] \[Gamma][\[Beta], \[Eta]] -
    1/2 n[\[Alpha]] n[\[Zeta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Mu], -\[Beta], -\[Nu]] \[Gamma][\[Beta], \[Eta]] +
    1/2 n[\[Alpha]] n[\[Zeta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Alpha], -\[Beta], -\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]] \[Gamma][\[Beta], \[Eta]] -
    1/2 n[\[Alpha]] n[\[Zeta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Beta], -\[Alpha], -\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]] \[Gamma][\[Beta], \[Eta]] +
    1/2 n[\[Alpha]] n[\[Zeta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]]
     Q[-\[Nu], -\[Beta], -\[Alpha]] \[Gamma][\[Beta], \[Eta]] -
    1/2 n[\[Alpha]] n[\[Zeta]] n[\[Mu]] n[\[Nu]]
     Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Nu], -\[Beta], -\[Mu]] \[Gamma][\[Beta], \[Eta]] -
    1/2 n[\[Mu]] n[\[Nu]] Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Beta], -\[Mu], -\[Nu]] \[Gamma][\[Alpha], \[Zeta]]
\[Gamma][\[Beta], \[Eta]] +
    1/2 n[\[Mu]] n[\[Nu]] Q[-\[Alpha], -\[Zeta], -\[Eta]]
     Q[-\[Mu], -\[Beta], -\[Nu]] \[Gamma][\[Alpha], \[Zeta]]
\[Gamma][\[Beta], \[Eta]] -
    1/2 n[\[Mu]] n[\[Nu]] Q[-\[Alpha], -\[Beta], -\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]] \[Gamma][\[Alpha], \[Zeta]]
\[Gamma][\[Beta], \[Eta]] +
    1/2 n[\[Mu]] n[\[Nu]] Q[-\[Beta], -\[Alpha], -\[Nu]]
     Q[-\[Mu], -\[Zeta], -\[Eta]] \[Gamma][\[Alpha], \[Zeta]]
\[Gamma][\[Beta], \[Eta]] -
    1/2 n[\[Mu]] n[\[Nu]] Q[-\[Mu], -\[Zeta], -\[Eta]]
     Q[-\[Nu], -\[Beta], -\[Alpha]] \[Gamma][\[Alpha], \[Zeta]]
\[Gamma][\[Beta], \[Eta]] +
```

```
   1/2  n[\[Mu]]   n[\[Nu]]   Q[-\[Alpha], -\[Zeta], -\[Eta]]
    Q[-\[Nu], -\[Beta], -\[Mu]]   \[Gamma][\[Alpha], \[Zeta]]
\[Gamma][\[Beta], \[Eta]] // Simplification

(*Parts from derivative part that have derivatives, Derivative Derivative Parts, DDP*)

DDP = -(1/2)  n[\[Alpha]]   n[\[Beta]]   n[\[Mu]]   n[\[Nu]]
    xAct`xTensor`PD[-\[Alpha]][
Q[-\[Beta], -\[Mu], -\[Nu]]] +
  1/2  n[\[Mu]]   n[\[Nu]]   \[Gamma][\[Alpha], \[Beta]]
    xAct`xTensor`PD[-\[Alpha]][
Q[-\[Beta], -\[Mu], -\[Nu]]] +
  1/2  n[\[Alpha]]   n[\[Beta]]   n[\[Mu]]   n[\[Nu]]
    xAct`xTensor`PD[-\[Alpha]][
Q[-\[Mu], -\[Beta], -\[Nu]]] -
  1/2  n[\[Mu]]   n[\[Nu]]   \[Gamma][\[Alpha], \[Beta]]
    xAct`xTensor`PD[-\[Alpha]][
Q[-\[Mu], -\[Beta], -\[Nu]]] +
  1/2  n[\[Alpha]]   n[\[Beta]]   n[\[Mu]]   n[\[Nu]]
    xAct`xTensor`PD[-\[Alpha]][
Q[-\[Nu], -\[Beta], -\[Mu]]] -
  1/2  n[\[Mu]]   n[\[Nu]]   \[Gamma][\[Alpha], \[Beta]]
    xAct`xTensor`PD[-\[Alpha]][
Q[-\[Nu], -\[Beta], -\[Mu]]] -
  1/2  n[\[Alpha]]   n[\[Beta]]   n[\[Mu]]   n[\[Nu]]
    xAct`xTensor`PD[-\[Mu]][
Q[-\[Alpha], -\[Beta], -\[Nu]]] +
  1/2  n[\[Mu]]   n[\[Nu]]   \[Gamma][\[Alpha], \[Beta]]
    xAct`xTensor`PD[-\[Mu]][
Q[-\[Alpha], -\[Beta], -\[Nu]]] +
  1/2  n[\[Alpha]]   n[\[Beta]]   n[\[Mu]]   n[\[Nu]]
    xAct`xTensor`PD[-\[Mu]][
Q[-\[Beta], -\[Alpha], -\[Nu]]] -
  1/2  n[\[Mu]]   n[\[Nu]]   \[Gamma][\[Alpha], \[Beta]]
    xAct`xTensor`PD[-\[Mu]][
Q[-\[Beta], -\[Alpha], -\[Nu]]] -
  1/2  n[\[Alpha]]   n[\[Beta]]   n[\[Mu]]   n[\[Nu]]
    xAct`xTensor`PD[-\[Mu]][
Q[-\[Nu], -\[Beta], -\[Alpha]]] +
  1/2  n[\[Mu]]   n[\[Nu]]   \[Gamma][\[Alpha], \[Beta]]
    xAct`xTensor`PD[-\[Mu]][
Q[-\[Nu], -\[Beta], -\[Alpha]]]

(*All the non derivative parts*)
AllNDP=NDDP+NDP//Simplification//Expand

(*All the derivative parts*)
ALLDDP = DDP // Simplification // Expand

(*Simplification of all not derivative parts*)
(((AllNDP /. case1 /.
                case2 /. case3 /. case4 /. case5) /. meto3 //
          Expand) /. \[Gamma]1 /. \[Gamma]2 /. Gn /. \[Gamma]3) /.
    case6 /. case7 /. case8 /. case1
FAllNDP = % // Simplification // Expand

(*Manual term Manipulation*)

Collect[Collect[Collect[FAllNDP,PD\[Alpha][\[Alpha]]/(\[Alpha]1)],n[\[Alpha]]n[\[Beta]]
    (PD\[Alpha][-\[Alpha]]\[Gamma][-\[Zeta],-\[Eta]])/(\[Alpha]1)],n[\[Alpha]]n[\[Beta]]
    Q[-\[Alpha],-\[Beta],\[Zeta]]]

(*Simplifying the first part*)
(
 PD\[Alpha][\[Alpha]]  (-(1/2)  Q3[-\[Alpha], \[Beta], -\[Beta]] +
    Q3[\[Beta], -\[Alpha], -\[Beta]]))/xAct`xTensor`Scalar[\[Alpha]1]
```

```
// InputForm

(PD[\[Alpha]][\[Alpha]]*(-L3[-\[Alpha], \[Beta], -\[Beta]]))/Scalar[\[Alpha]1]

(*Simplifiying the second part*)
(
 n[\[Alpha]]   n[\[Beta]]
  PD\[Alpha][-\[Alpha]]  (-(1/2)  Q[-\[Beta], \[Zeta], \[Eta]] +
     Q[\[Zeta], -\[Beta], \[Eta]])  \[Gamma][-\[Zeta], \
-\[Eta]])/xAct`xTensor`Scalar[\[Alpha]1] // InputForm

(n[\[Alpha]]*n[\[Beta]]*PD\[Alpha][-\[Alpha]]*(-L[-\[Beta], \[Zeta], \[Eta]])*\[Gamma
     ][-\[Zeta], -\[Eta]])/Scalar[\[Alpha]1]

-((L[-\[Beta], \[Zeta], \[Eta]]   n[\[Alpha]]   n[\[Beta]]
    PD\[Alpha][-\[Alpha]]   \[Gamma][-\[Zeta], -\[Eta]])/
  xAct`xTensor`Scalar[\[Alpha]1]) // InputForm

-((A[]*n[\[Alpha]]*PD\[Alpha][-\[Alpha]])/Scalar[\[Alpha]1])

(*Simplifying the third part*)

n[\[Alpha]]   n[\[Beta]]
  Q[-\[Alpha], -\[Beta], \[Zeta]]  (-(1/2)
     Q3[\[Eta], \[Lambda], -\[Lambda]]   \[Gamma][-\[Zeta], -\[Eta]] +
     Q3[\[Eta], -\[Eta], \[Lambda]]   \[Gamma][-\[Zeta], -\[Lambda]])
// InputForm

n[\[Alpha]]*n[\[Beta]]*Q[-\[Alpha], -\[Beta], \[Zeta]]*\[Gamma][-\[Zeta], -\[Eta]]*(-(L3
     [\[Eta], \[Lambda], -\[Lambda]]))//Simplify

(*Considering the rest of the terms*)
+(1/2)  n[\[Alpha]]   n[\[Beta]]
  Q[\[Zeta], -\[Alpha], \[Eta]]
  Q[\[Lambda], -\[Beta], \[Mu]]   \[Gamma][-\[Zeta], -\[Mu]]
\[Gamma][-\[Eta], -\[Lambda]] -
 1/4  n[\[Alpha]]   n[\[Beta]]   Q[-\[Alpha], \[Zeta], \[Eta]]
  Q[-\[Beta], \[Lambda], \[Mu]]   \[Gamma][-\[Zeta], -\[Lambda]]
\[Gamma][-\[Eta], -\[Mu]] -
 1/2  n[\[Alpha]]   n[\[Beta]]   Q[\[Zeta], -\[Alpha], \[Eta]]
  Q[\[Lambda], -\[Beta], \[Mu]]   \[Gamma][-\[Zeta], -\[Lambda]]
\[Gamma][-\[Eta], -\[Mu]]

(*Final Non Derivative part*)

-((L3[-\[Alpha], \[Beta], -\[Beta]]   PD\[Alpha][\[Alpha]])/
  xAct`xTensor`Scalar[\[Alpha]1]) + (
 PD\[Alpha][\[Alpha]]
  PD\[Alpha][\[Beta]]   \[Gamma][-\[Beta],
-\[Alpha]])/xAct`xTensor`Scalar[\[Alpha]1]^2 - (
 A[]  n[\[Alpha]]
  PD\[Alpha][-\[Alpha]])/xAct`xTensor`Scalar[\[Alpha]1] -
 L3[\[Eta], \[Lambda], -\[Lambda]]   n[\[Alpha]]   n[\[Beta]]
  Q[-\[Alpha], -\[Beta], \[Zeta]]   \[Gamma][-\[Zeta], -\[Eta]] +
 1/2  n[\[Alpha]]   n[\[Beta]]   Q[\[Zeta], -\[Alpha], \[Eta]]
  Q[\[Lambda], -\[Beta], \[Mu]]   \[Gamma][-\[Zeta], -\[Mu]]
\[Gamma][-\[Eta], -\[Lambda]] -
 1/4  n[\[Alpha]]   n[\[Beta]]   Q[-\[Alpha], \[Zeta], \[Eta]]
  Q[-\[Beta], \[Lambda], \[Mu]]   \[Gamma][-\[Zeta], -\[Lambda]]
\[Gamma][-\[Eta], -\[Mu]] -
 1/2  n[\[Alpha]]   n[\[Beta]]   Q[\[Zeta], -\[Alpha], \[Eta]]
  Q[\[Lambda], -\[Beta], \[Mu]]   \[Gamma][-\[Zeta], -\[Lambda]]
\[Gamma][-\[Eta], -\[Mu]]

(ALLDDP // Simplification) /. Qtg
% // Simplification
```

```
(*Momentum Constraint*)
(*Non Derivative Parts NDP Extracted metrics and 3+1 split them*)
(n[\
\[Nu]] \[Gamma][\[Mu], -\[Beta]] (\[Gamma][\[Sigma], \[Chi]] -
      n[\[Sigma]] n[\[Chi]]) (\[Gamma][\[Alpha], \[Lambda]] -
      n[\[Alpha]] n[\[Lambda]]) (-L[-\[Sigma], -\[Chi], -\[Alpha]]
L[-\[Lambda], -\[Mu], -\[Nu]] -
      Q[-\[Lambda], -\[Chi], -\[Nu]] L[-\[Alpha], -\[Sigma], -\[Mu]]
+ 1/2 Q[-\[Mu], -\[Alpha], -\[Sigma]] L[-\[Nu], -\[Lambda], -\[Chi]])
// Expand);
% /. Disform // Expand;
% // Simplification;
NDP = % // Expand

(*Derivative part DP *)
(n[\[Nu]] \[Gamma][\[Mu], -\[Beta]] \
(\[Gamma][\[Alpha], -\[Lambda]] -
      n[\[Alpha]] n[-\[Lambda]]) (PD[-\[Alpha]][
       L[\[Lambda], -\[Mu], -\[Nu]]] -
      1/2 (PD[-\[Mu]][L[\[Lambda], -\[Alpha], -\[Nu]]] +
         PD[-\[Nu]][L[\[Lambda], -\[Alpha], -\[Mu]]]))) /. Disform;
% // Expand;
% /. metricity1 /. metricity2;
(((% // SameDummies) /. meto3) //
      Expand) /. \[Gamma]1 /. \[Gamma]2 /. Gn /. \[Gamma]3


(*Non derivative part of derivative part  NDDP*)

NDDP = (-(1/2)  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]
      n[\[CapitalTheta]]   n[\[Nu]]
      Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
      Q[-\[Zeta], -\[Mu], -\[Nu]]   \[Gamma][\[Mu], -\[Beta]] +
     1/2  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]   n[\[CapitalTheta]]
      n[\[Nu]]   Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
      Q[-\[Mu], -\[Zeta], -\[Nu]]   \[Gamma][\[Mu], -\[Beta]] -
     1/4  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]   n[\[CapitalTheta]]
      n[\[Nu]]   Q[-\[Alpha], -\[Zeta], -\[Nu]]
      Q[-\[Mu], -\[Eta], -\[CapitalTheta]]   \[Gamma][\[Mu],
-\[Beta]] +
     1/4  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]   n[\[CapitalTheta]]
      n[\[Nu]]   Q[-\[Zeta], -\[Alpha], -\[Nu]]
      Q[-\[Mu], -\[Eta], -\[CapitalTheta]]   \[Gamma][\[Mu],
-\[Beta]] -
     1/4  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]   n[\[CapitalTheta]]
      n[\[Nu]]   Q[-\[Mu], -\[Eta], -\[CapitalTheta]]
      Q[-\[Nu], -\[Zeta], -\[Alpha]]   \[Gamma][\[Mu], -\[Beta]] +
     1/2  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]   n[\[CapitalTheta]]
      n[\[Nu]]   Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
      Q[-\[Nu], -\[Zeta], -\[Mu]]   \[Gamma][\[Mu], -\[Beta]] -
     1/4  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]   n[\[CapitalTheta]]
      n[\[Nu]]   Q[-\[Alpha], -\[Zeta], -\[Mu]]
      Q[-\[Nu], -\[Eta], -\[CapitalTheta]]   \[Gamma][\[Mu],
-\[Beta]] +
     1/4  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]   n[\[CapitalTheta]]
      n[\[Nu]]   Q[-\[Zeta], -\[Alpha], -\[Mu]]
      Q[-\[Nu], -\[Eta], -\[CapitalTheta]]   \[Gamma][\[Mu],
-\[Beta]] -
     1/4  n[\[Alpha]]   n[\[Zeta]]   n[\[Eta]]   n[\[CapitalTheta]]
      n[\[Nu]]   Q[-\[Mu], -\[Zeta], -\[Alpha]]
      Q[-\[Nu], -\[Eta], -\[CapitalTheta]]   \[Gamma][\[Mu],
-\[Beta]] +
     1/2  n[\[Zeta]]   n[\[CapitalTheta]]   n[\[Nu]]
      Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
      Q[-\[Zeta], -\[Mu], -\[Nu]]   \[Gamma][\[Alpha], \[Eta]]
```

```
\[Gamma][\[Mu], -\[Beta]] -
      1/2 n[\[Zeta]] n[\[CapitalTheta]] n[\[Nu]]
       Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
       Q[-\[Mu], -\[Zeta], -\[Nu]] \[Gamma][\[Alpha], \[Eta]]
\[Gamma][\[Mu], -\[Beta]] +
      1/4 n[\[Zeta]] n[\[CapitalTheta]] n[\[Nu]]
       Q[-\[Alpha], -\[Zeta], -\[Nu]]
       Q[-\[Mu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Alpha],
\[Eta]] \[Gamma][\[Mu], -\[Beta]] -
      1/4 n[\[Zeta]] n[\[CapitalTheta]] n[\[Nu]]
       Q[-\[Zeta], -\[Alpha], -\[Nu]]
       Q[-\[Mu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Alpha],
\[Eta]] \[Gamma][\[Mu], -\[Beta]] +
      1/4 n[\[Zeta]] n[\[CapitalTheta]] n[\[Nu]]
       Q[-\[Mu], -\[Eta], -\[CapitalTheta]]
       Q[-\[Nu], -\[Zeta], -\[Alpha]] \[Gamma][\[Alpha], \[Eta]]
\[Gamma][\[Mu], -\[Beta]] -
      1/2 n[\[Zeta]] n[\[CapitalTheta]] n[\[Nu]]
       Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
       Q[-\[Nu], -\[Zeta], -\[Mu]] \[Gamma][\[Alpha], \[Eta]]
\[Gamma][\[Mu], -\[Beta]] +
      1/4 n[\[Zeta]] n[\[CapitalTheta]] n[\[Nu]]
       Q[-\[Alpha], -\[Zeta], -\[Mu]]
       Q[-\[Nu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Alpha],
\[Eta]] \[Gamma][\[Mu], -\[Beta]] -
      1/4 n[\[Zeta]] n[\[CapitalTheta]] n[\[Nu]]
       Q[-\[Zeta], -\[Alpha], -\[Mu]]
       Q[-\[Nu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Alpha],
\[Eta]] \[Gamma][\[Mu], -\[Beta]] +
      1/4 n[\[Zeta]] n[\[CapitalTheta]] n[\[Nu]]
       Q[-\[Mu], -\[Zeta], -\[Alpha]]
       Q[-\[Nu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Alpha],
\[Eta]] \[Gamma][\[Mu], -\[Beta]] +
      1/2 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
       Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
       Q[-\[Zeta], -\[Mu], -\[Nu]] \[Gamma][\[Zeta],
\[CapitalTheta]] \[Gamma][\[Mu], -\[Beta]] -
      1/2 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
       Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
       Q[-\[Mu], -\[Zeta], -\[Nu]] \[Gamma][\[Zeta],
\[CapitalTheta]] \[Gamma][\[Mu], -\[Beta]] +
      1/4 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
       Q[-\[Alpha], -\[Zeta], -\[Nu]]
       Q[-\[Mu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Zeta],
\[CapitalTheta]] \[Gamma][\[Mu], -\[Beta]] -
      1/4 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
       Q[-\[Zeta], -\[Alpha], -\[Nu]]
       Q[-\[Mu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Zeta],
\[CapitalTheta]] \[Gamma][\[Mu], -\[Beta]] +
      1/4 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
       Q[-\[Mu], -\[Eta], -\[CapitalTheta]]
       Q[-\[Nu], -\[Zeta], -\[Alpha]] \[Gamma][\[Zeta],
\[CapitalTheta]] \[Gamma][\[Mu], -\[Beta]] -
      1/2 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
       Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
       Q[-\[Nu], -\[Zeta], -\[Mu]] \[Gamma][\[Zeta],
\[CapitalTheta]] \[Gamma][\[Mu], -\[Beta]] +
      1/4 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
       Q[-\[Alpha], -\[Zeta], -\[Mu]]
       Q[-\[Nu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Zeta],
\[CapitalTheta]] \[Gamma][\[Mu], -\[Beta]] -
      1/4 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
       Q[-\[Zeta], -\[Alpha], -\[Mu]]
       Q[-\[Nu], -\[Eta], -\[CapitalTheta]] \[Gamma][\[Zeta],
\[CapitalTheta]] \[Gamma][\[Mu], -\[Beta]] +
      1/4 n[\[Alpha]] n[\[Eta]] n[\[Nu]]
```

```
        Q[-\[Mu], -\[Zeta], -\[Alpha]]
        Q[-\[Nu], -\[Eta], -\[CapitalTheta]]    \[Gamma][\[Zeta],
\[CapitalTheta]]    \[Gamma][\[Mu], -\[Beta]] -
       1/2  n[\[Nu]]    Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
        Q[-\[Zeta], -\[Mu], -\[Nu]]    \[Gamma][\[Alpha], \[Eta]]
\[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -\[Beta]] +
       1/2  n[\[Nu]]    Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
        Q[-\[Mu], -\[Zeta], -\[Nu]]    \[Gamma][\[Alpha], \[Eta]]
\[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -\[Beta]] -
       1/4  n[\[Nu]]    Q[-\[Alpha], -\[Zeta], -\[Nu]]
        Q[-\[Mu], -\[Eta], -\[CapitalTheta]]    \[Gamma][\[Alpha],
\[Eta]]    \[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -
\[Beta]] +
       1/4  n[\[Nu]]    Q[-\[Zeta], -\[Alpha], -\[Nu]]
        Q[-\[Mu], -\[Eta], -\[CapitalTheta]]    \[Gamma][\[Alpha],
\[Eta]]    \[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -
\[Beta]] -
       1/4  n[\[Nu]]    Q[-\[Mu], -\[Eta], -\[CapitalTheta]]
        Q[-\[Nu], -\[Zeta], -\[Alpha]]    \[Gamma][\[Alpha], \[Eta]]
\[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -\[Beta]] +
       1/2  n[\[Nu]]    Q[-\[Alpha], -\[Eta], -\[CapitalTheta]]
        Q[-\[Nu], -\[Zeta], -\[Mu]]    \[Gamma][\[Alpha], \[Eta]]
\[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -\[Beta]] -
       1/4  n[\[Nu]]    Q[-\[Alpha], -\[Zeta], -\[Mu]]
        Q[-\[Nu], -\[Eta], -\[CapitalTheta]]    \[Gamma][\[Alpha],
\[Eta]]    \[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -
\[Beta]] +
       1/4  n[\[Nu]]    Q[-\[Zeta], -\[Alpha], -\[Mu]]
        Q[-\[Nu], -\[Eta], -\[CapitalTheta]]    \[Gamma][\[Alpha],
\[Eta]]    \[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -
\[Beta]] -
       1/4  n[\[Nu]]    Q[-\[Mu], -\[Zeta], -\[Alpha]]
        Q[-\[Nu], -\[Eta], -\[CapitalTheta]]    \[Gamma][\[Alpha],
\[Eta]]    \[Gamma][\[Zeta], \[CapitalTheta]]    \[Gamma][\[Mu], -
\[Beta]]) // Simplification // Expand

(*Derivative Derivative part DDP*)

DDP = (-(1/2)  n[\[Alpha]]    n[\[Zeta]]
        n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Alpha]][
Q[-\[Zeta], -\[Mu], -\[Nu]]] +
       1/2
        n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Alpha]][
Q[-\[Zeta], -\[Mu], -\[Nu]]] +
       1/2  n[\[Alpha]]    n[\[Zeta]]
        n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Alpha]][
Q[-\[Mu], -\[Zeta], -\[Nu]]] -
       1/2
        n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Alpha]][
Q[-\[Mu], -\[Zeta], -\[Nu]]] +
       1/2  n[\[Alpha]]    n[\[Zeta]]
        n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Alpha]][
Q[-\[Nu], -\[Zeta], -\[Mu]]] -
       1/2
        n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Alpha]][
Q[-\[Nu], -\[Zeta], -\[Mu]]] -
       1/4  n[\[Alpha]]    n[\[Zeta]]
        n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Mu]][
Q[-\[Alpha], -\[Zeta], -\[Nu]]] +
       1/4
        n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Mu]][
Q[-\[Alpha], -\[Zeta], -\[Nu]]] +
```

```
      1/4  n[\[Alpha]]   n[\[Zeta]]
       n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Mu]][
Q[-\[Zeta], -\[Alpha], -\[Nu]]] -
      1/4
       n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Mu]][
Q[-\[Zeta], -\[Alpha], -\[Nu]]] -
      1/4  n[\[Alpha]]   n[\[Zeta]]
       n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Mu]][
Q[-\[Nu], -\[Zeta], -\[Alpha]]] +
      1/4
       n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Mu]][
Q[-\[Nu], -\[Zeta], -\[Alpha]]] -
      1/4  n[\[Alpha]]   n[\[Zeta]]
       n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Nu]][
Q[-\[Alpha], -\[Zeta], -\[Mu]]] +
      1/4
       n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Nu]][
Q[-\[Alpha], -\[Zeta], -\[Mu]]] +
      1/4  n[\[Alpha]]   n[\[Zeta]]
       n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Nu]][
Q[-\[Zeta], -\[Alpha], -\[Mu]]] -
      1/4
       n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Nu]][
Q[-\[Zeta], -\[Alpha], -\[Mu]]] -
      1/4  n[\[Alpha]]   n[\[Zeta]]
       n[\[Nu]]    \[Gamma][\[Mu], -\[Beta]]   xAct`xTensor`PD[-\[Nu]][
Q[-\[Mu], -\[Zeta], -\[Alpha]]] +
      1/4
       n[\[Nu]]    \[Gamma][\[Alpha], \[Zeta]]    \[Gamma][\[Mu], -
\[Beta]]   xAct`xTensor`PD[-\[Nu]][
Q[-\[Mu], -\[Zeta], -\[Alpha]]]) // Simplification // Expand

(*All non derivative parts*)
AllNDP = (NDP + NDDP) // Simplification //
   Expand

   (*Simplification of all not derivative parts*)
SALLNDP = ((((AllNDP \
/. case1 /. case2 /. case3 /. case4 /. case5) /. meto3 //
              Expand) /. \[Gamma]1 /. \[Gamma]2 /.
          Gn /. \[Gamma]3) /. case6 /. case7 /. case8 /. case1) //
  SameDummies

(*Manual manipulation of terms*)

-(1/4)  n[\[Alpha]]   Q[-\[Alpha], \[Zeta], \[Eta]]
   Q3[-\[Beta], -\[Zeta], -\[Eta]] -
  1/2  n[\[Alpha]]   Q[\[Zeta], -\[Alpha], \[Eta]]
   Q3[-\[Zeta], -\[Beta], -\[Eta]] +
  1/2  n[\[Alpha]]   Q[\[Zeta], -\[Alpha], \[Eta]]
   Q3[-\[Eta], -\[Beta], -\[Zeta]] // Simplification

-((n[\[Alpha]]   PD\[Alpha][-\[CapitalTheta]]
    Q[-\[Alpha], \[Zeta], \[Eta]]    \[Gamma][-\[Beta],
\[CapitalTheta]]    \[Gamma][-\[Zeta], -\[Eta]])/(2 \[Alpha]1)) + (
  n[\[Alpha]]   PD\[Alpha][-\[CapitalTheta]]
   Q[\[Zeta], -\[Alpha], \[Eta]]    \[Gamma][-\[Beta],
\[CapitalTheta]]    \[Gamma][-\[Zeta], -\[Eta]])/\[Alpha]1 + (
  n[\[Alpha]]   PD\[Alpha][-\[CapitalTheta]]
   Q[-\[Alpha], \[Zeta], \[Eta]]    \[Gamma][-\[Beta], -\[Zeta]]
\[Gamma][-\[Eta], \[CapitalTheta]])/(2 \[Alpha]1) - (
  n[\[Alpha]]   PD\[Alpha][-\[CapitalTheta]]
```

```
   Q[\[Zeta], -\[Alpha], \[Eta]]    \[Gamma][-\[Beta], -\[Eta]]
\[Gamma][\[CapitalTheta], -\[Zeta]])/(2 \[Alpha]1) - (
  n[\[Alpha]]    PD\[Alpha][-\[CapitalTheta]]
   Q[\[Zeta], -\[Alpha], \[Eta]]    \[Gamma][-\[Beta], -\[Zeta]]
\[Gamma][\[CapitalTheta], -\[Eta]])/(2 \[Alpha]1) // Simplification

-(1/2)  n[\[Alpha]]    Q[\[Zeta], -\[Alpha], \[Eta]]
     Q3[\[CapitalTheta], -\[CapitalTheta], \[Lambda]]    \[Gamma][-
\[Beta], -\[Eta]]    \[Gamma][-\[Lambda], -\[Zeta]] +
    1/4  n[\[Alpha]]    Q[\[Zeta], -\[Alpha], \[Eta]]
     Q3[\[Lambda], \[CapitalTheta], -\[CapitalTheta]]    \[Gamma][-
\[Beta], -\[Eta]]    \[Gamma][-\[Lambda], -\[Zeta]] +
    1/2  n[\[Alpha]]    Q[-\[Alpha], \[Zeta], \[Eta]]
     Q3[\[CapitalTheta], -\[CapitalTheta], \[Lambda]]    \[Gamma][-
\[Beta], -\[Zeta]]    \[Gamma][-\[Lambda], -\[Eta]] +
    1/2  n[\[Alpha]]    Q[\[Zeta], -\[Alpha], \[Eta]]
     Q3[\[CapitalTheta], -\[CapitalTheta], \[Lambda]]    \[Gamma][-
\[Beta], -\[Zeta]]    \[Gamma][-\[Lambda], -\[Eta]] -
    1/4  n[\[Alpha]]    Q[-\[Alpha], \[Zeta], \[Eta]]
     Q3[\[Lambda], \[CapitalTheta], -\[CapitalTheta]]    \[Gamma][-
\[Beta], -\[Zeta]]    \[Gamma][-\[Lambda], -\[Eta]] -
    1/4  n[\[Alpha]]    Q[\[Zeta], -\[Alpha], \[Eta]]
     Q3[\[Lambda], \[CapitalTheta], -\[CapitalTheta]]    \[Gamma][-
\[Beta], -\[Zeta]]    \[Gamma][-\[Lambda], -\[Eta]] //
  SameDummies // Simplification

DDP // SameDummies // Simplification

(1/4  n[\[Alpha]]   (n[\[Zeta]]
       n[\[Mu]]    \[Gamma][-\[Beta], \[Nu]]   (xAct`xTensor`PD[-\[Mu]][
Q[-\[Nu], -\[Alpha], -\[Zeta]]] - xAct`xTensor`PD[-\[Nu]][
Q[-\[Alpha], -\[Zeta], -\[Mu]]])) /. Qtg) // Simplification

(1/4  n[\[Alpha]]    \[Gamma][-\[Beta], \[Zeta]]    \[Gamma][\[Mu],
\[Nu]]   (xAct`xTensor`PD[-\[Alpha]][
Q[-\[Zeta], -\[Mu], -\[Nu]]] + xAct`xTensor`PD[-\[Zeta]][
Q[-\[Alpha], -\[Mu], -\[Nu]]] - 2 (xAct`xTensor`PD[-\[Nu]][
Q[-\[Alpha], -\[Zeta], -\[Mu]]] + xAct`xTensor`PD[-\[Nu]][
Q[-\[Zeta], -\[Alpha], -\[Mu]]] - xAct`xTensor`PD[-\[Nu]][
Q[-\[Mu], -\[Alpha], -\[Zeta]]])) /. Qtg) // Simplification
```

D.3 Testing of Four Known Metrics

Setup

```
Remove["Global`*"]

(*Making sure the code never exceeds allotted ram*)

Dynamic[Refresh[

  If[N[MemoryInUse[]*10^-9] >= N[10],


   Print[
    "Max Memory Reached: " <> ToString[N[MemoryInUse[]*10^-9, 10]] <>
     "GB"] ; Quit[],

```

```
    N[MemoryInUse[]*10^-9, 10]


    ], UpdateInterval -> 0]]

(*Modifying notebook limits*)
SetOptions[Simplify,
 TimeConstraint -> Infinity]
SetOptions[EvaluationNotebook[], OutputSizeLimit -> 1040000]

(*Importing xAct packages*)

<< xAct`xTensor`

<< xAct`xCoba`

<< xAct`xPert`;

<< xAct`xTras`;

(*Defining geometric entities*)
$CVVerbose = False;
DefManifold[Global, 3, {\[Zeta], \[Eta], \[Iota], \[Lambda], \[Mu], \
\[Nu], \[Xi], \[Sigma], \[Upsilon], \[Chi], \[CapitalTheta], \
\[Epsilon], \[CapitalLambda], \[CapitalXi], \[CapitalPi], \
\[CapitalSigma], \[CapitalUpsilon], \[CapitalPhi], \[CapitalPsi], \
\[CapitalOmega], \[CapitalSampi], \[CapitalStigma], \[CapitalKoppa]}]
DefManifold[Local, 3, {a, b, c, d, e, g, h, i, s, j, k, l}]

DefChart[Ba, Global, {1, 2, 3}, {r[], \[Theta][], \[Phi][]},
 ChartColor -> Blue]
DefChart[Bb, Local, {1, 2, 3}, {x[], y[], z[]}, ChartColor -> Red]

DefBasis[global, TangentGlobal, {1, 2, 3}]
DefBasis[local, TangentLocal, {1, 2, 3}]

(*The metric is defined same as any other tensor to keep control over \
automated simplifications that do not assume modified theories*)

DefTensor[{metric[\[Nu], \[Mu]], metricI[-\[Mu], -\[Nu]],
    DeltaG3[\[Nu], -\[Mu]], Mink[a, b], MinkI[-a, -b], Mink3[a, b],
    Mink3I[-a, -b], DeltaL3[a, -b], FirstLie[-\[Nu], -\[Mu]],
    SecondLie[-\[Nu], -\[Mu]]}, Global];

DefTensor[metric3[\[Mu], \[Nu]], Global, PrintAs -> "\[Gamma]"]
DefTensor[metric3I[-\[Mu], -\[Nu]], Global, PrintAs -> "\[Gamma]"]

DefTensor[\[Beta]eta[\[Nu]], Global, PrintAs -> "\[Beta]"]
DefTensor[\[Beta]etaI[-\[Nu]], Global, PrintAs -> "\[Beta]"]


DefTensor[Q3[-\[Lambda], -\[Nu], -\[Mu]], Global,
 Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[L3[-\[Lambda], -\[Nu], -\[Mu]], Global,
 Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[B[-\[Mu], -\[Nu]], Global, Symmetric[{-\[Nu], -\[Mu]}]]
DefTensor[\[CapitalGamma]3[\[Sigma], -\[Mu], -\[Nu]], Global]
DefTensor[EK[-\[Mu], -\[Nu]], Global]
DefTensor[R[-\[Mu], -\[Nu]], Global]
DefTensor[EKs[], Global]
DefTensor[SL[-\[Mu], -\[Nu]], Global]


DefTensor[S[-\[Nu], -\[Mu]], Global]
DefTensor[Ss[], Global]
```

```
DefTensor[Sv[-\[CapitalStigma]], Global]
DefTensor[Bs[], Global]
DefTensor[HC[], Global]
DefTensor[MC[], Global]
DefTensor[G[], Global]
DefTensor[\[Rho][], Global]


DefScalarFunction[\[ScriptCapitalM]]
DefScalarFunction[\[Alpha]]

ToCoba = {\[Zeta] -> {\[Zeta], Ba}, \[Eta] -> {\[Eta],
     Ba}, \[Iota] -> {\[Iota], Ba}, \[Lambda] -> {\[Lambda],
     Ba}, \[Mu] -> {\[Mu], Ba}, \[Nu] -> {\[Nu],
     Ba}, \[Epsilon] -> {\[Epsilon], Ba}, \[Xi] -> {\[Xi],
     Ba}, \[Sigma] -> {\[Sigma], Ba}, \[Upsilon] -> {\[Upsilon],
     Ba}, \[Chi] -> {\[Chi], Ba}, \[CapitalTheta] -> {\[CapitalTheta],
      Ba}, \[CapitalLambda] -> {\[CapitalLambda],
     Ba}, \[CapitalXi] -> {\[CapitalXi],
     Ba}, \[CapitalPi] -> {\[CapitalPi],
     Ba}, \[CapitalSigma] -> {\[CapitalSigma],
     Ba}, \[CapitalUpsilon] -> {\[CapitalUpsilon],
     Ba}, \[CapitalPhi] -> {\[CapitalPhi],
     Ba}, \[CapitalPsi] -> {\[CapitalPsi],
     Ba}, \[CapitalOmega] -> {\[CapitalOmega],
     Ba}, \[CapitalSampi] -> {\[CapitalSampi],
     Ba}, \[CapitalStigma] -> {\[CapitalStigma],
     Ba}, \[CapitalKoppa] -> {\[CapitalKoppa], Ba}};

(*Custom method for storing tensor components*)

StoreTensor[Tens_, Matrix_] := (

  CompArrayofTensor = ComponentArray[Tens] // Flatten;
  ElementsOfTensor = Matrix // Flatten;

  IterLenght = CompArrayofTensor // Length;

  If[IterLenght == 0,

   RuelsOfTensor = {Tens -> Matrix},

   RuelsOfTensor =
     Map[
      Evaluate[ComponentArray[CompArrayofTensor][[#]]] ->
        ElementsOfTensor[[#]] &, Range[IterLenght]];

   ];

  If[ValueQ[RulesOfTensors] == False, RulesOfTensors = {}];

  Print[Matrix // MatrixForm];

  RulesOfTensors = Join[RuelsOfTensor, RulesOfTensors];

  )

(*Custom method for outputing tensor components*)

TensorElements[
  Tens_] := (((Tens // TraceBasisDummy // ComponentArray) /.
     RulesOfTensors) // MatrixForm)
```

General Definition of Tensor Components

```
 Clear[RulesOfTensors]

$CVVerbose = False;

(*Defining the Shift vector components*)
MatrixForm[Betaarray = ( {
     {\[Beta]1},
     {\[Beta]2},
     {\[Beta]3}
    } )];

StoreTensor[\[Beta]eta[{\[Nu], Ba}], Betaarray]

(*Defining the metric components*)
Style["metric3", 20, Bold]

MatrixForm[metric3array = ( {
     {\[Gamma]11, \[Gamma]12, \[Gamma]13},
     {\[Gamma]21, \[Gamma]22, \[Gamma]23},
     {\[Gamma]31, \[Gamma]32, \[Gamma]33}
    } )];

StoreTensor[metric3[{\[Mu], Ba}, {\[Nu], Ba}], metric3array]

(*Defining the inverse metric components*)
Style["metricI3", 20, Bold]

MatrixForm[metric3Iarray = ( {
     {\[Gamma]I11, \[Gamma]I12, \[Gamma]I13},
     {\[Gamma]I21, \[Gamma]I22, \[Gamma]I23},
     {\[Gamma]I31, \[Gamma]I32, \[Gamma]I33}
    } )];

StoreTensor[metric3I[{-\[Mu], -Ba}, {-\[Nu], -Ba}], metric3Iarray]

(*Defining the co shift vector components*)
Style["\[Beta]etaI", 20, \
Bold]

\[Beta]etaIarray = ((
     metric3I[{-\[Mu], -Ba}, {-\[Nu], -Ba}] \[Beta]eta[{\[Nu], Ba}] //
        TraceBasisDummy // ComponentArray) /. RulesOfTensors) //
  FullSimplify

StoreTensor[\[Beta]etaI[{-\[Nu], -Ba}], \[Beta]etaIarray]

(*Defining the spatial mapping tensor components*)
Style["DeltaG3", \
20, Bold]

DeltaG3array = ((
      metric3[{\[Mu], Ba}, {\[Nu],
           Ba}] metric3I[{-\[Mu], -Ba}, {-\[Lambda], -Ba}] //
        TraceBasisDummy // ComponentArray) /. RulesOfTensors) //
   FullSimplify;

StoreTensor[DeltaG3[{\[Nu], Ba}, {-\[Mu], -Ba}], DeltaG3array]

Style["Streamlining components", 20, Bold]

\[Beta]I1 = \[Beta]1 \[Gamma]I11 + \[Beta]2 \[Gamma]I12 + \[Beta]3 \
\[Gamma]I13;

\[Beta]I2 = \[Beta]1 \[Gamma]I21 + \[Beta]2 \[Gamma]I22 + \[Beta]3 \
```

```
\[Gamma]I23;

\[Beta]I3 = \[Beta]1 \[Gamma]I31 + \[Beta]2 \[Gamma]I32 + \[Beta]3 \
\[Gamma]I33;

{
    {\[Gamma]11, \[Gamma]12, \[Gamma]13},
    {\[Gamma]21, \[Gamma]22, \[Gamma]23},
    {\[Gamma]31, \[Gamma]32, \[Gamma]33}
   } = {
    {(\[Gamma]I23 \[Gamma]I32 - \[Gamma]I22 \[Gamma]I33)/(\[Gamma]I13 \
\[Gamma]I22 \[Gamma]I31 - \[Gamma]I12 \[Gamma]I23 \[Gamma]I31 - \
\[Gamma]I13 \[Gamma]I21 \[Gamma]I32 + \[Gamma]I11 \[Gamma]I23 \
\[Gamma]I32 + \[Gamma]I12 \[Gamma]I21 \[Gamma]I33 - \[Gamma]I11 \
\[Gamma]I22 \[Gamma]I33), (\[Gamma]I13 \[Gamma]I32 - \[Gamma]I12 \
\[Gamma]I33)/(-\[Gamma]I13 \[Gamma]I22 \[Gamma]I31 + \[Gamma]I12 \
\[Gamma]I23 \[Gamma]I31 + \[Gamma]I13 \[Gamma]I21 \[Gamma]I32 - \
\[Gamma]I11 \[Gamma]I23 \[Gamma]I32 - \[Gamma]I12 \[Gamma]I21 \
\[Gamma]I33 + \[Gamma]I11 \[Gamma]I22 \[Gamma]I33), (\[Gamma]I13 \
\[Gamma]I22 - \[Gamma]I12 \[Gamma]I23)/(\[Gamma]I13 \[Gamma]I22 \
\[Gamma]I31 - \[Gamma]I12 \[Gamma]I23 \[Gamma]I31 - \[Gamma]I13 \
\[Gamma]I21 \[Gamma]I32 + \[Gamma]I11 \[Gamma]I23 \[Gamma]I32 + \
\[Gamma]I12 \[Gamma]I21 \[Gamma]I33 - \[Gamma]I11 \[Gamma]I22 \
\[Gamma]I33)},
    {(\[Gamma]I23 \[Gamma]I31 - \[Gamma]I21 \[Gamma]I33)/(-\[Gamma]I13 \
\[Gamma]I22 \[Gamma]I31 + \[Gamma]I12 \[Gamma]I23 \[Gamma]I31 + \
\[Gamma]I13 \[Gamma]I21 \[Gamma]I32 - \[Gamma]I11 \[Gamma]I23 \
\[Gamma]I32 - \[Gamma]I12 \[Gamma]I21 \[Gamma]I33 + \[Gamma]I11 \
\[Gamma]I22 \[Gamma]I33), (\[Gamma]I13 \[Gamma]I31 - \[Gamma]I11 \
\[Gamma]I33)/(\[Gamma]I13 \[Gamma]I22 \[Gamma]I31 - \[Gamma]I12 \
\[Gamma]I23 \[Gamma]I31 - \[Gamma]I13 \[Gamma]I21 \[Gamma]I32 + \
\[Gamma]I11 \[Gamma]I23 \[Gamma]I32 + \[Gamma]I12 \[Gamma]I21 \
\[Gamma]I33 - \[Gamma]I11 \[Gamma]I22 \[Gamma]I33), (\[Gamma]I13 \
\[Gamma]I21 - \[Gamma]I11 \[Gamma]I23)/(-\[Gamma]I13 \[Gamma]I22 \
\[Gamma]I31 + \[Gamma]I12 \[Gamma]I23 \[Gamma]I31 + \[Gamma]I13 \
\[Gamma]I21 \[Gamma]I32 - \[Gamma]I11 \[Gamma]I23 \[Gamma]I32 - \
\[Gamma]I12 \[Gamma]I21 \[Gamma]I33 + \[Gamma]I11 \[Gamma]I22 \
\[Gamma]I33)},
    {(\[Gamma]I22 \[Gamma]I31 - \[Gamma]I21 \[Gamma]I32)/(\[Gamma]I13 \
\[Gamma]I22 \[Gamma]I31 - \[Gamma]I12 \[Gamma]I23 \[Gamma]I31 - \
\[Gamma]I13 \[Gamma]I21 \[Gamma]I32 + \[Gamma]I11 \[Gamma]I23 \
\[Gamma]I32 + \[Gamma]I12 \[Gamma]I21 \[Gamma]I33 - \[Gamma]I11 \
\[Gamma]I22 \[Gamma]I33), (\[Gamma]I12 \[Gamma]I31 - \[Gamma]I11 \
\[Gamma]I32)/(-\[Gamma]I13 \[Gamma]I22 \[Gamma]I31 + \[Gamma]I12 \
\[Gamma]I23 \[Gamma]I31 + \[Gamma]I13 \[Gamma]I21 \[Gamma]I32 - \
\[Gamma]I11 \[Gamma]I23 \[Gamma]I32 - \[Gamma]I12 \[Gamma]I21 \
\[Gamma]I33 + \[Gamma]I11 \[Gamma]I22 \[Gamma]I33), (\[Gamma]I12 \
\[Gamma]I21 - \[Gamma]I11 \[Gamma]I22)/(\[Gamma]I13 \[Gamma]I22 \
\[Gamma]I31 - \[Gamma]I12 \[Gamma]I23 \[Gamma]I31 - \[Gamma]I13 \
\[Gamma]I21 \[Gamma]I32 + \[Gamma]I11 \[Gamma]I23 \[Gamma]I32 + \
\[Gamma]I12 \[Gamma]I21 \[Gamma]I33 - \[Gamma]I11 \[Gamma]I22 \
\[Gamma]I33)}
   };
```

Evolution and Constraint Equations

```

G[] = 0

Style["First Lie - Extrinsic Metricity - B[-\[Mu],-\[Nu]]", 20, Bold]
```

```

EBarray = (((1/(
            2 \[Alpha][
              r[]]) (D[
               metric3I[-\[Mu], -\[Nu]], \[Tau][]] - \[Beta]eta[\
\[Lambda]] PD[-\[Lambda]][metric3I[-\[Mu], -\[Nu]]] -
              metric3I[-\[Mu], -\[Lambda]] PD[-\[Nu]][\[Beta]eta[\
\[Lambda]]] -
              metric3I[-\[Lambda], -\[Nu]] PD[-\[Mu]][\[Beta]eta[\
\[Lambda]]]) /. ToCoba) // TraceBasisDummy // ComponentArray) /.
      RulesOfTensors) // FullSimplify;

EBarray =
  Map[FullSimplify[#,
      Assumptions -> {\[ScriptCapitalM][] \[Element] Reals,
        r[] \[Element] Reals, r[] > 0}] &, EBarray, {2}];

StoreTensor[B[{-\[Mu], -Ba}, {-\[Nu], -Ba}], EBarray]

Style["Extrinsic Metricity scalar", 20, Bold]

EBsarray =
   FullSimplify[((((B[-\[Mu], -\[Nu]] metric3[\[Mu], \[Nu]]) /.
             ToCoba) // TraceBasisDummy // ComponentArray) /.
        RulesOfTensors) // FullSimplify,
     Assumptions -> {\[ScriptCapitalM][] \[Element] Reals,
       r[] \[Element] Reals, r[] > 0}];

StoreTensor[Bs[], EBsarray]

Style["Second Lie", 20, Bold]

SecondLiearray = ((((HoldForm[(\[Alpha][
                      r[]] (-(B[-\[Mu], -\[Nu]]*Bs[]) +
                       2*B[-\[Mu], -\[Epsilon]]*B[-\[Nu], -\[Zeta]]*
                        metric3[\[Epsilon], \[Zeta]] -
                       8*Pi*G[]*S[-\[Nu], -\[Mu]] +
                       4*Pi*G[]*metric3I[-\[Mu], -\[Nu]]*Ss[] -
                       4*Pi*G[]*
                        metric3I[-\[Mu], -\[Nu]]*\[Rho][] + \
(PD[-\[Epsilon]][metric3I[-\[Mu], -\[Nu]]]*
                         metric3[\[Epsilon], \[Lambda]] PD[-\[Lambda]][\
\[Alpha][r[]]])/(2*\[Alpha][r[]]) + (metric3[\[Epsilon], \[Zeta]]*
                         metric3[\[Eta], \[CapitalTheta]]*
                        PD[-\[Epsilon]][metric3I[-\[Mu], -\[Nu]]]*
                        PD[-\[Zeta]][
                         metric3I[-\[Eta], -\[CapitalTheta]]])/
                       4 - (metric3[\[Epsilon], \[Zeta]]*
                         metric3[\[Eta], \[CapitalTheta]]*
                        PD[-\[Epsilon]][metric3I[-\[Mu], -\[Eta]]]*
                        PD[-\[Zeta]][metric3I[-\[Nu], -\[CapitalTheta]]])/
                       2 + (metric3[\[Epsilon], \[Zeta]]*
                        PD[-\[Zeta]][
                        PD[-\[Epsilon]][metric3I[-\[Mu], -\[Nu]]]])/
                       2 - (metric3[\[Epsilon], \[Zeta]]*
                        PD[-\[Zeta]][
                        PD[-\[Mu]][metric3I[-\[Nu], -\[Epsilon]]]])/
                       2 - (metric3[\[Epsilon], \[Zeta]]*
                        PD[-\[Zeta]][
                        PD[-\[Nu]][metric3I[-\[Mu], -\[Epsilon]]]])/
                       2 - (metric3[\[Epsilon], \[Zeta]]*
                         metric3[\[Eta], \[CapitalTheta]]*
                        PD[-\[Epsilon]][metric3I[-\[Mu], -\[Nu]]]*
                        PD[-\[CapitalTheta]][
                         metric3I[-\[Zeta], -\[Eta]]])/
                       2 + (metric3[\[Epsilon], \[Zeta]]*
```

```
                          metric3[\[Eta], \[CapitalTheta]]*
                          PD[-\[Epsilon]][metric3I[-\[Mu], -\[Eta]]]*
                          PD[-\[CapitalTheta]][metric3I[-\[Nu], -\[Zeta]]])/
                         2 - (metric3[\[Epsilon], \[Lambda]] \
PD[-\[Lambda]][\[Alpha][r[]]]*
                          PD[-\[Mu]][
                          metric3I[-\[Nu], -\[Epsilon]]])/(2*\[Alpha][
                          r[]]) - (metric3[\[Epsilon], \[Zeta]]*
                          metric3[\[Eta], \[CapitalTheta]]*
                          PD[-\[Zeta]][metric3I[-\[Eta], -\[CapitalTheta]]]*
                          PD[-\[Mu]][metric3I[-\[Nu], -\[Epsilon]]])/
                         4 + (metric3[\[Epsilon], \[Zeta]]*
                          metric3[\[Eta], \[CapitalTheta]]*
                          PD[-\[CapitalTheta]][metric3I[-\[Zeta], -\[Eta]]]*
                          PD[-\[Mu]][metric3I[-\[Nu], -\[Epsilon]]])/
                         2 + (metric3[\[Epsilon], \[Zeta]]*
                          PD[-\[Mu]][
                          PD[-\[Nu]][metric3I[-\[Epsilon], -\[Zeta]]]])/
                         4 - (metric3[\[Epsilon], \[Zeta]]*
                          metric3[\[Eta], \[CapitalTheta]]*
                          PD[-\[Mu]][metric3I[-\[Zeta], -\[CapitalTheta]]]*
                          PD[-\[Nu]][metric3I[-\[Epsilon], -\[Eta]]])/
                         4 - (metric3[\[Epsilon], \[Lambda]] \
PD[-\[Lambda]][\[Alpha][r[]]]*
                          PD[-\[Nu]][
                          metric3I[-\[Mu], -\[Epsilon]]])/(2*\[Alpha][
                          r[]]) - (metric3[\[Epsilon], \[Zeta]]*
                          metric3[\[Eta], \[CapitalTheta]]*
                          PD[-\[Zeta]][metric3I[-\[Eta], -\[CapitalTheta]]]*
                          PD[-\[Nu]][metric3I[-\[Mu], -\[Epsilon]]])/
                         4 + (metric3[\[Epsilon], \[Zeta]]*
                          metric3[\[Eta], \[CapitalTheta]]*
                          PD[-\[CapitalTheta]][metric3I[-\[Zeta], -\[Eta]]]*
                          PD[-\[Nu]][metric3I[-\[Mu], -\[Epsilon]]])/
                         2 + (metric3[\[Epsilon], \[Zeta]]*
                          PD[-\[Nu]][
                          PD[-\[Mu]][metric3I[-\[Epsilon], -\[Zeta]]]])/4 +
                       PD[-\[Nu]][PD[-\[Mu]][\[Alpha][r[]]]]/\[Alpha][
                          r[]]) + \[Beta]eta[\[Lambda]] PD[-\[Lambda]][
                      B[-\[Mu], -\[Nu]]] +
                    B[-\[Mu], -\[Lambda]] \
PD[-\[Nu]][\[Beta]eta[\[Lambda]]] +
                    B[-\[Nu], -\[Lambda]] \
PD[-\[Mu]][\[Beta]eta[\[Lambda]]])] /. ToCoba) // ComponentArray //
           ReleaseHold) /. RulesOfTensors) // TraceBasisDummy) /.
     RulesOfTensors;


SecondLiearray =
  Map[FullSimplify[#,
      Assumptions -> {\[ScriptCapitalM][] \[Element] Reals,
        r[] \[Element] Reals, r[] > 0}] &, SecondLiearray, {2}];

StoreTensor[SecondLie[{-\[Nu], -Ba}, {-\[Mu], -Ba}], SecondLiearray]

Style["Hamiltonian Constraint", 20, Bold]

HamiltonianConstraintarray = (((((-Bs[]^2 +
                  B[-\[Epsilon], -\[Zeta]]*B[-\[Eta], -\[CapitalTheta]]*
                   metric3[\[Epsilon], \[Eta]]*
                   metric3[\[Zeta], \[CapitalTheta]] -
                  16*Pi*
                   G[]*\[Rho][] - (3*metric3[\[Epsilon], \[Zeta]]*
                      metric3[\[Eta], \[CapitalTheta]]*
                      metric3[\[Iota], \[Lambda]]*
                     PD[-\[Epsilon]][metric3I[-\[Eta], -\[Iota]]]*
```

```
                  PD[-\[Zeta]][
                    metric3I[-\[CapitalTheta], -\[Lambda]]])/
                 4 + (metric3[\[Epsilon], \[Zeta]]*
                   metric3[\[Eta], \[CapitalTheta]]*
                   metric3[\[Iota], \[Lambda]]*
                   PD[-\[Epsilon]][metric3I[-\[Eta], -\[CapitalTheta]]]*
                   PD[-\[Zeta]][metric3I[-\[Iota], -\[Lambda]]])/4 +
                metric3[\[Epsilon], \[Zeta]]*
                 metric3[\[Eta], \[CapitalTheta]]*
                 PD[-\[Zeta]][
                  PD[-\[Epsilon]][
                   metric3I[-\[Eta], -\[CapitalTheta]]]] + (metric3[\
\[Epsilon], \[Zeta]]*metric3[\[Eta], \[CapitalTheta]]*
                   metric3[\[Iota], \[Lambda]]*
                   PD[-\[Epsilon]][metric3I[-\[Eta], -\[Iota]]]*
                   PD[-\[CapitalTheta]][
                    metric3I[-\[Zeta], -\[Lambda]]])/
                 2 - (metric3[\[Epsilon], \[Zeta]]*
                   metric3[\[Eta], \[CapitalTheta]]*
                   metric3[\[Iota], \[Lambda]]*
                   PD[-\[Epsilon]][metric3I[-\[Zeta], -\[Eta]]]*
                   PD[-\[CapitalTheta]][
                    metric3I[-\[Iota], -\[Lambda]]])/2 -
                metric3[\[Epsilon], \[Zeta]]*
                 metric3[\[Eta], \[CapitalTheta]]*
                 PD[-\[CapitalTheta]][
                  PD[-\[Epsilon]][
                   metric3I[-\[Zeta], -\[Eta]]]] - (metric3[\[Epsilon], \
\[Zeta]]*metric3[\[Eta], \[CapitalTheta]]*metric3[\[Iota], \[Lambda]]*
                   PD[-\[Epsilon]][metric3I[-\[Eta], -\[CapitalTheta]]]*
                   PD[-\[Lambda]][metric3I[-\[Zeta], -\[Iota]]])/2 +
                metric3[\[Epsilon], \[Zeta]]*
                 metric3[\[Eta], \[CapitalTheta]]*
                 metric3[\[Iota], \[Lambda]]*
                 PD[-\[Epsilon]][metric3I[-\[Zeta], -\[Eta]]]*
                 PD[-\[Lambda]][
                  metric3I[-\[CapitalTheta], -\[Iota]]]) /. ToCoba) //
          ComponentArray // ReleaseHold) /. RulesOfTensors) //
      TraceBasisDummy) /. RulesOfTensors;

StoreTensor[HC[], HamiltonianConstraintarray // FullSimplify]

Style["Momentum Constraint", 20, Bold]

MomentumConstraintarray = (((((-8*Pi*G[]*
                 Sv[-\[CapitalStigma]] + (B[-\[CapitalStigma], -\
\[CapitalTheta]]*metric3[\[Epsilon], \[CapitalTheta]]*
                   metric3[\[Zeta], \[Eta]]*
                   PD[-\[Epsilon]][metric3I[-\[Zeta], -\[Eta]]])/2 +
                metric3[\[Epsilon], \[Zeta]]*
                 PD[-\[Zeta]][B[-\[CapitalStigma], -\[Epsilon]]] -
                B[-\[CapitalStigma], -\[CapitalTheta]]*
                 metric3[\[Epsilon], \[CapitalTheta]]*
                 metric3[\[Zeta], \[Eta]]*
                 PD[-\[Zeta]][metric3I[-\[Epsilon], -\[Eta]]] -
                metric3[\[Epsilon], \[Zeta]]*
                 PD[-\[CapitalStigma]][
                  B[-\[Epsilon], -\[Zeta]]] + (B[-\[Eta], -\
\[CapitalTheta]]*metric3[\[Epsilon], \[Eta]]*
                   metric3[\[Zeta], \[CapitalTheta]]*
                   PD[-\[CapitalStigma]][
                    metric3I[-\[Epsilon], -\[Zeta]]])/2) /. ToCoba) //
          ComponentArray // ReleaseHold) /. RulesOfTensors) //
      TraceBasisDummy) /. RulesOfTensors;

StoreTensor[MC[{-\[CapitalStigma], -Ba}],
```

```
MomentumConstraintarray // FullSimplify]
```

Schwarzchild Metric

```

Clear[\[Gamma]I11, \[Gamma]I12, \[Gamma]I13, \[Gamma]I21, \
\[Gamma]I22, \[Gamma]I23, \[Gamma]I31, \[Gamma]I32, \[Gamma]I33, \
\[Beta]1, \[Beta]2, \[Beta]3, \[Alpha]]

DefScalarFunction[\[Alpha]]

Style["Schwarzchild", 30, Bold, Red]

Style["Laps Function \[Alpha][r[]]", 20, Bold]

\[Alpha][x_] := (1 - 2 \[ScriptCapitalM][]/x)^(1/2)

\[Alpha][r[]]

\[Beta]1 = 0;
\[Beta]2 = 0;
\[Beta]3 = 0;

{
   {\[Gamma]I11, \[Gamma]I12, \[Gamma]I13},
   {\[Gamma]I21, \[Gamma]I22, \[Gamma]I23},
   {\[Gamma]I31, \[Gamma]I32, \[Gamma]I33}
  } = ( {
    {(1 - (2 \[ScriptCapitalM][])/r[])^-1, 0, 0},
    {0, r[]^2, 0},
    {0, 0, r[]^2 Sin[\[Theta][]]^2}
   } );

Style["Shift vector", 20, Bold]

\[Beta]etaIarray // Simplify

Style["Metric and Inverse Metric", 20, Bold]

metric3array // Simplify // MatrixForm
metric3Iarray // Simplify // MatrixForm
```

Isotropic

```
Clear[\[Gamma]I11, \[Gamma]I12, \[Gamma]I13, \[Gamma]I21, \
\[Gamma]I22, \[Gamma]I23, \[Gamma]I31, \[Gamma]I32, \[Gamma]I33, \
\[Beta]1, \[Beta]2, \[Beta]3, \[Alpha]]
DefScalarFunction[\[Alpha]]
DefScalarFunction[\[Gamma]I11, \[Gamma]I12, \[Gamma]I13, \[Gamma]I21, \
\[Gamma]I22, \[Gamma]I23, \[Gamma]I31, \[Gamma]I32, \[Gamma]I33, \
\[Beta]1, \[Beta]2, \[Beta]3]

Style["Isotropic", 30, Bold, Red]

```

```
Style["Laps Function \[Alpha][r[]]", 20, Bold]

\[Alpha][x_] := (1 - \[ScriptCapitalM][]/(2 x))/(
 1 + \[ScriptCapitalM][]/(2 x))

\[Alpha][r[]]

\[Beta]1 = 0;
\[Beta]2 = 0;
\[Beta]3 = 0;

{
   {\[Gamma]I11, \[Gamma]I12, \[Gamma]I13},
   {\[Gamma]I21, \[Gamma]I22, \[Gamma]I23},
   {\[Gamma]I31, \[Gamma]I32, \[Gamma]I33}
  } = (1 + \[ScriptCapitalM][]/(2 r[]))^4 ( {
     {1, 0, 0},
     {0, r[]^2, 0},
     {0, 0, r[]^2 Sin[\[Theta][]]^2}
    } );

Style["Shift vector", 20, Bold]

\[Beta]etaIarray // Simplify

Style["Metric and Inverse Metric", 20, Bold]

metric3array // Simplify // MatrixForm
metric3Iarray // Simplify // MatrixForm
```

Painleve-Gullstrand

```
Clear[\[Gamma]I11, \[Gamma]I12, \[Gamma]I13, \[Gamma]I21, \
\[Gamma]I22, \[Gamma]I23, \[Gamma]I31, \[Gamma]I32, \[Gamma]I33, \
\[Beta]1, \[Beta]2, \[Beta]3, \[Alpha]]
DefScalarFunction[\[Alpha]]
Style["Painleve-Gullstrand", 30, Bold, Red]

Style["Laps Function \[Alpha][r[]]", 20, Bold]

\[Alpha][x_] := 1

\[Alpha][r[]]

\[Beta]1 = ((2 \[ScriptCapitalM][])/r[])^(1/2);
\[Beta]2 = 0;
\[Beta]3 = 0;

{
   {\[Gamma]I11, \[Gamma]I12, \[Gamma]I13},
   {\[Gamma]I21, \[Gamma]I22, \[Gamma]I23},
   {\[Gamma]I31, \[Gamma]I32, \[Gamma]I33}
  } = ( {
    {1, 0, 0},
    {0, r[]^2, 0},
    {0, 0, r[]^2 Sin[\[Theta][]]^2}
   } );

Style["Shift vector and lower index", 20, Bold]

```

```
Betaarray // Simplify // MatrixForm
\[Beta]etaIarray // Simplify // MatrixForm


Style["Metric and Inverse Metric", 20, Bold]

metric3array // Simplify // MatrixForm
metric3Iarray // Simplify // MatrixForm
```

Kerr-Schild

```
Clear[\[Gamma]I11, \[Gamma]I12, \[Gamma]I13, \[Gamma]I21, \
\[Gamma]I22, \[Gamma]I23, \[Gamma]I31, \[Gamma]I32, \[Gamma]I33, \
\[Beta]1, \[Beta]2, \[Beta]3, \[Alpha]]
DefScalarFunction[\[Alpha]]
Style["Kerr−Schild", 30, Bold, Red]

Style["Laps Function \[Alpha][r[]]", 20, Bold]

\[Alpha][x_] := (1 + 2 \[ScriptCapitalM][]/x)^(−(1/2))

\[Alpha][r[]]

\[Beta]1 = \[Alpha][r[]]^2 (2 \[ScriptCapitalM][])/r[];
\[Beta]2 = 0;
\[Beta]3 = 0;

{
    {\[Gamma]I11, \[Gamma]I12, \[Gamma]I13},
    {\[Gamma]I21, \[Gamma]I22, \[Gamma]I23},
    {\[Gamma]I31, \[Gamma]I32, \[Gamma]I33}
   } = ( {
     {1 + (2 \[ScriptCapitalM][])/r[], 0, 0},
     {0, r[]^2, 0},
     {0, 0, r[]^2 Sin[\[Theta][]]^2}
    } );

Style["Shift vector and lower index", 20, Bold]

Betaarray // Simplify // MatrixForm
\[Beta]etaIarray // Simplify // MatrixForm


Style["Metric and Inverse Metric", 20, Bold]

metric3array // Simplify // MatrixForm
metric3Iarray // Simplify // MatrixForm
```