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Cross-intersecting non-empty uniform subfamilies of hereditary families

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ABSTRACT

Two families $\mathcal{A}$ and $\mathcal{B}$ of sets are *cross-t-intersecting* if each set in $\mathcal{A}$ intersects each set in $\mathcal{B}$ in at least t elements. A family $\mathcal{H}$ is *hereditary* if for each set A in $\mathcal{H}$, all the subsets of A are in $\mathcal{H}$. Let $\mathcal{H}^{(r)}$ denote the family of r-element sets in $\mathcal{H}$. We show that for any integers t, r, and s with $1 \leq t \leq r \leq s$, there exists an integer $c(r, s, t)$ such that the following holds for any hereditary family $\mathcal{H}$ whose maximal sets are of size at least $c(r, s, t)$. If $\mathcal{A}$ is a non-empty subfamily of $\mathcal{H}^{(r)}$, $\mathcal{B}$ is a non-empty subfamily of $\mathcal{H}^{(s)}$, $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, and $|\mathcal{A}| + |\mathcal{B}|$ is maximum under the given conditions, then for some set I in $\mathcal{H}$ with $t \leq |I| \leq r$, either $\mathcal{A} = \{A \in \mathcal{H}^{(r)} : I \subseteq A\}$ and $\mathcal{B} = \{B \in \mathcal{H}^{(s)} : |B \cap I| \geq t\}$, or $r = s$, $t < |I|$, $\mathcal{A} = \{A \in \mathcal{H}^{(r)} : |A \cap I| \geq t\}$, and $\mathcal{B} = \{B \in \mathcal{H}^{(s)} : I \subseteq B\}$. We give $c(r, s, t)$ explicitly. The result was conjectured by the author for $t = 1$ and generalizes well-known results for the case where $\mathcal{H}$ is a power set.

1. Introduction

1.1. Basic definitions and notation

Unless stated otherwise, we shall use small letters such as x to denote non-negative integers or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose members are sets themselves). Arbitrary sets and families are taken to be finite. An *r-element set* is a set of size r. The set of positive integers is denoted by $\mathbb{N}$. For $n \in \mathbb{N}$, the set $\{i \in \mathbb{N} : i \leq n\}$ is denoted by $[n]$. We take $[0]$ to be the *empty set* $\emptyset$. For a set X, the *power set of* X (that is, $\{A : A \subseteq X\}$) is denoted by 2^X, and the family $\{A \subseteq X : |A| = r\}$ is denoted

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by $\binom{X}{r}$. For a family $\mathcal{F}$, the family $\{A \in \mathcal{F}: |A| = r\}$ is denoted by $\mathcal{F}^{(r)}$ and called the rth *level of* $\mathcal{F}$. For a t-element set T, the family $\{A \in \mathcal{F}: T \subseteq A\}$ is denoted by $\mathcal{F}(T)$ and called a *t-star of* $\mathcal{F}$.

1.2. Intersecting families

We say that a set A *t-intersects* a set B if A and B have at least t common elements. A family $\mathcal{A}$ is said to be *t-intersecting* if for every $A, B \in \mathcal{A}$, A t-intersects B. A 1-intersecting family is also simply called an *intersecting family*. Trivially, t-stars are t-intersecting families. A family $\mathcal{F}$ is said to have the *t-star property* if at least one of the largest t-intersecting subfamilies of $\mathcal{F}$ is a t-star of $\mathcal{F}$.

One of the most popular endeavours in extremal set theory is that of determining the size or the structure of a largest t-intersecting subfamily of a given family $\mathcal{F}$. This originated in [17], which features the classical result referred to as the Erdős–Ko–Rado (EKR) Theorem. The EKR Theorem says that for $1 \le t \le r$ there exists an integer $n_0(r, t)$ such that for $n \ge n_0(r, t)$, the size of a largest t-intersecting subfamily of $\binom{[n]}{r}$ is $\binom{n-t}{r-t}$, meaning that the rth level of $2^{[n]}$ has the t-star property. It also says that the smallest possible $n_0(r, 1)$ is $2r$; among the various proofs of this fact (see [15,17,22,26,30,32,34]) there is a short one by Katona [34], introducing the elegant cycle method, and another one by Daykin [15], using the Kruskal–Katona Theorem [33,35]. Note that $\binom{[n]}{r}$ itself is intersecting if $n < 2r$. The EKR Theorem inspired a sequence of results [1,18,21,43] that culminated in the complete solution of the problem for t-intersecting subfamilies of $\binom{[n]}{r}$. The solution had been conjectured by Frankl [18]. It particularly tells us that the smallest possible $n_0(r, t)$ is $(t + 1)(r - t + 1)$; this was established by Frankl [18] and Wilson [43]. Ahlswede and Khachatrian [1] settled the case $n < (t + 1)(r - t + 1)$. The t-intersection problem for $2^{[n]}$ was solved by Katona [32]. These are among the most prominent results in extremal set theory. The EKR Theorem inspired a wealth of results that establish how large a system of sets can be under certain intersection conditions; see [6,16,19,20,25,27,28].

A set B in a family $\mathcal{F}$ is called a *base of* $\mathcal{F}$ (or a *maximal set of* $\mathcal{F}$) if for each $A \in \mathcal{F}$, B is not a proper subset of A. The size of a smallest base of $\mathcal{F}$ is denoted by $\mu(\mathcal{F})$.

A family $\mathcal{F}$ is said to be *hereditary* if for each $A \in \mathcal{F}$, all the subsets of A are members of $\mathcal{F}$. In the literature, a hereditary family is also called an *ideal*, a *downset*, and an *abstract simplicial complex*. Hereditary families are important combinatorial objects that have attracted much attention. The various interesting examples include the family of *independent sets* of a *graph* or a *matroid*. The power set is the simplest example. In fact, by definition, a family is hereditary if and only if it is a union of power sets. Note that if $X_1, \dots, X_k$ are the bases of a hereditary family $\mathcal{H}$, then $\mathcal{H} = 2^{X_1} \cup \cdots \cup 2^{X_k}$.

The most basic result on intersecting families, also proved in the seminal EKR paper [17], is that the hereditary family $2^{[n]}$ has the 1-star property. One of the central conjectures in extremal set theory, due to Chvátal [14], is that every hereditary family $\mathcal{H}$ has the 1-star property. Several cases have been verified [13,36–41] (see also [12]), many of which are captured by Snevily's result [39] ([4] provides a generalization obtained by means of a self-contained alternative argument). For $t \ge 2$, the t-star property fails already for $\mathcal{H} = 2^{[n]}$ with $n \ge t + 2$; the largest t-intersecting subfamilies of $2^{[n]}$ were determined by Katona [32]. However, for levels of hereditary families, we have the following generalization of the Holroyd–Talbot Conjecture [28, Conjecture 7].

Conjecture 1.1 (*[2]*). *If $1 \le t \le r$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \ge (t + 1)(r - t + 1)$, then $\mathcal{H}^{(r)}$ has the t-star property.*

Note that if $\mathcal{H} = 2^{[n]}$, then $\mathcal{H}^{(r)} = \binom{[n]}{r}$ and $\mu(\mathcal{H}) = n$. It follows by the above-mentioned results for $\binom{[n]}{r}$ that the conjecture is true for $\mathcal{H} = 2^{[n]}$ and that the condition $\mu(\mathcal{H}) \ge (t + 1)(r - t + 1)$ cannot be improved. The author verified the conjecture for $\mu(\mathcal{H})$ sufficiently large depending only on r and t.

Theorem 1.2 (*[2]*). *Conjecture 1.1 is true if $\mu(\mathcal{H}) \ge (r - t)\binom{3r-2t-1}{t+1} + r$.*

By [9, Theorem 1.4 and Lemma 4.4], Conjecture 1.1 is also true if $\mu(\mathcal{H}) \ge (r - t)(r\binom{r}{t} + 1) + r$.

1.3. Cross-intersecting families

A popular variant of the intersection problem described above is the cross-intersection problem. Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-t-intersecting* if each set in $\mathcal{A}$ t-intersects each set in $\mathcal{B}$. Cross-1-intersecting families are also simply called *cross-intersecting families*.

For t-intersecting subfamilies of a given family $\mathcal{F}$, the natural question to ask is how large they can be. For cross-t-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-t-intersecting families. The problem of maximizing the sum or the product of sizes of cross-t-intersecting subfamilies of a given family $\mathcal{F}$ has been attracting much attention (many of the results to date are referenced in [7–9]).

In this paper, we are concerned with the sum problem for the case where $\mathcal{F}$ is a level of a hereditary family, but we also address the problem where the cross-t-intersecting families come from different levels and are non-empty. Thus, it is convenient to introduce the following notation. For two families $\mathcal{F}$ and $\mathcal{G}$, let

$$C(\mathcal{F},\mathcal{G},t)=\{(\mathcal{A},\mathcal{B})\colon \emptyset\neq\mathcal{A}\subseteq\mathcal{F}, \emptyset\neq\mathcal{B}\subseteq\mathcal{G}, \mathcal{A} \text{ and } \mathcal{B} \text{ are cross-}t\text{-intersecting}\},$$

$$m(\mathcal{F},\mathcal{G},t)=\max\{|\mathcal{A}|+|\mathcal{B}|\colon(\mathcal{A},\mathcal{B})\in C(\mathcal{F},\mathcal{G},t)\},$$

$$M(\mathcal{F},\mathcal{G},t)=\{(\mathcal{A},\mathcal{B})\in C(\mathcal{F},\mathcal{G},t)\colon |\mathcal{A}|+|\mathcal{B}|=m(\mathcal{F},\mathcal{G},t)\}.$$

As mentioned above, we consider $\mathcal{F}=\mathcal{H}^{(r)}$ and $\mathcal{G}=\mathcal{H}^{(s)}$ for some hereditary family $\mathcal{H}$. Thus, the setting is analogous to that of Theorem 1.2.

Hilton and Milner [26] showed that if $\mathcal{A}$ and $\mathcal{B}$ are non-empty cross-intersecting subfamilies of $\binom{[n]}{r}$ with $1\leq r\leq n/2$, then $|\mathcal{A}|+|\mathcal{B}|\leq\binom{n}{r}-\binom{n-r}{r}+1$. Equality holds if $\mathcal{A}$ consists of $[r]$ only and $\mathcal{B}$ consists of all the sets in $\binom{[n]}{r}$ that intersect $[r]$. In other words, if $1=t\leq r\leq n/2$ and $\mathcal{F}=\mathcal{G}=\binom{[n]}{r}$, then $(\{[r]\},\{B\in\mathcal{G}\colon B\cap[r]\neq\emptyset\})\in M(\mathcal{F},\mathcal{G},t)$. Frankl and Tokushige [24] showed that the same holds in the more general case where $1=t\leq r\leq s$, $n\geq r+s$, $\mathcal{F}=\binom{[n]}{r}$, and $\mathcal{G}=\binom{[n]}{s}$. Wang and Zhang [42] generalized this for $t\geq 1$. They proved that if $t<\min\{r,s\}$, $n\geq r+s-t+1$, $\binom{n}{r}\leq\binom{n}{s}$, $\mathcal{F}=\binom{[n]}{r}$, and $\mathcal{G}=\binom{[n]}{s}$, then $(\{[r]\},\{B\in\mathcal{G}\colon |B\cap[r]|\geq t\})\in M(\mathcal{F},\mathcal{G},t)$ (an independent proof for $r=s$ has been obtained by Frankl and Kupavskii [23]); they also determined the pairs in $M(\mathcal{F},\mathcal{G},t)$. It immediately follows that if we allow the cross-t-intersecting families $\mathcal{A}$ and $\mathcal{B}$ to be empty, then $|\mathcal{A}|+|\mathcal{B}|$ is maximum if $\mathcal{A}=\emptyset$ and $\mathcal{B}=\binom{[n]}{s}$.

As pointed out above, $\binom{[n]}{r}=\mathcal{H}^{(r)}$ with $\mathcal{H}=2^{[n]}$. Thus, the theorem of Wang and Zhang deals with the rth level and the sth level of the hereditary family $2^{[n]}$. In this paper, we characterize the pairs in $M(\mathcal{H}^{(r)},\mathcal{H}^{(s)},t)$ for any hereditary family $\mathcal{H}$ with $\mu(\mathcal{H})$ sufficiently large depending on r, s, and t (see Theorem 2.1).

Recall that for a family $\mathcal{F}$ and a set X, the family $\{A\in\mathcal{F}\colon X\subseteq A\}$ is denoted by $\mathcal{F}(X)$. A t-intersecting family $\mathcal{A}$ is said to be *trivial* if $\mathcal{A}=\mathcal{A}(T)$ for some t-element set T.

The paper [3] features the following two conjectures for $t=1$.

Conjecture 1.3 (*Weak Form [3]*). *If $1\leq r\leq s$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H})\geq r+s$, then for some $(\mathcal{A},\mathcal{B})\in M(\mathcal{H}^{(r)},\mathcal{H}^{(s)},1)$, $\mathcal{A}$ is a trivial 1-intersecting family.*

Conjecture 1.4 (*Strong Form [3]*). *If $1\leq r\leq s$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H})\geq r+s$, then there exists some $I\in\mathcal{H}$ with $1\leq|I|\leq r$ such that for some $(\mathcal{A},\mathcal{B})\in M(\mathcal{H}^{(r)},\mathcal{H}^{(s)},1)$, $\mathcal{A}=\mathcal{H}^{(r)}(I)$ and $\mathcal{B}=\{B\in\mathcal{H}^{(s)}\colon B\cap I\neq\emptyset\}$.*

These conjectures are true for $r=1$ [3, Theorem 1.4]. Generalizing the above-mentioned result of Frankl and Tokushige [24], the main result in [3] tells us that for *compressed* hereditary families $\mathcal{H}$, Conjecture 1.4 holds with $|I|=r$, in which case $\mathcal{A}$ consists of I only and $\mathcal{B}$ consists of all the sets in $\mathcal{H}^{(s)}$ intersecting I. A question that arises immediately is whether this holds for every hereditary family. This is answered in the negative in [3] too; [3, Proposition 2.1] tells us that for any r, s, and n with $2\leq r\leq s$ and $n\geq r+s$, there are hereditary families $\mathcal{H}$ such that $\mu(\mathcal{H})=n$ and no $(\mathcal{A},\mathcal{B})$ in $M(\mathcal{H}^{(r)},\mathcal{H}^{(s)},1)$ satisfies Conjecture 1.4 with $|I|=r$.

2. Results and conjectures

Throughout the paper, for $t \leq r \leq s$, we take

$$c(r,s,t) = r + (s-t)\max\left\{2\binom{s}{t},\ 2^r(r-t)\binom{r}{t}+1\right\}.$$

Note that Conjecture 1.4 is significantly stronger than Conjecture 1.3. In Section 4, we prove the following generalization for $M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t)$ with $\mu(\mathcal{H}) \geq c(r,s,t)$, hence verifying Conjecture 1.4 for $\mu(\mathcal{H}) \geq c(r,s,1)$.

Theorem 2.1. *If $1 \leq t \leq r \leq s$, $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq c(r,s,t)$, and $(\mathcal{A},\mathcal{B}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t)$, then for some $I \in \mathcal{H}$ with $t \leq |I| \leq r$, either*

$$\mathcal{A} = \mathcal{H}^{(r)}(I) \text{ and } \mathcal{B} = \{B \in \mathcal{H}^{(s)} : |B \cap I| \geq t\},$$

or

$$r = s,\ t < |I|,\ \mathcal{A} = \{A \in \mathcal{H}^{(r)} : |A \cap I| \geq t\}, \text{ and } \mathcal{B} = \mathcal{H}^{(s)}(I).$$

This immediately implies that

$$(\mathcal{H}^{(r)}(I), \{B \in \mathcal{H}^{(s)} : |B \cap I| \geq t\}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t). \tag{1}$$

Theorem 2.1 is the main result of this paper. It is an analogue of Theorem 1.2.

We ask for the smallest possible lower bound for $\mu(\mathcal{H})$ in Theorem 2.1. More precisely, we pose the following problem.

Problem 2.2. For $1 \leq t \leq r \leq s$, let $\eta(r,s,t)$ be the smallest integer n such that for every hereditary family $\mathcal{H}$ with $\mu(\mathcal{H}) \geq n$, $(\mathcal{H}^{(r)}(I), \{B \in \mathcal{H}^{(s)} : |B \cap I| \geq t\}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t)$ for some $I \in \mathcal{H}$ with $t \leq |I| \leq r$. What is the value of $\eta(r,s,t)$?

By Theorem 2.1, $\eta(r,s,t) \leq c(r,s,t)$. Clearly, for $\mathcal{H} = 2^{[n]}$, we have $\mu(\mathcal{H}) = n$, and $\mathcal{H}^{(r)}$ and $\mathcal{H}^{(s)}$ are cross-t-intersecting if and only if $n \leq r+s-t$. Thus, $\eta(r,s,t) \geq r+s-t+1$. We conjecture that equality holds.

Conjecture 2.3. *For $1 \leq t \leq r \leq s$, $\eta(r,s,t) = r+s-t+1$.*

A *graph* G is a pair $(V, \mathcal{E})$ with $\mathcal{E} \subseteq \binom{V}{2}$, and a subset S of V is called an *independent set of* G if $\{i,j\} \notin \mathcal{E}$ for every $i,j \in S$. Let $\mathcal{I}_G$ denote the family of independent sets of G. The EKR problem for $\mathcal{I}_G$ was introduced in [28] and inspired many results [10,11,27–29,44]. Many EKR-type results can be phrased in terms of independent sets of graphs; see [11, page 2878]. Clearly, $\mathcal{I}_G$ is a hereditary family. Kamat [31] conjectured that if $\mu(\mathcal{I}_G) \geq 2r$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting subfamilies of ${\mathcal{I}_G}^{(r)}$, then $|\mathcal{A}| + |\mathcal{B}| \leq |{\mathcal{I}_G}^{(r)}|$. We conjecture that the following strong generalization holds.

Conjecture 2.4. *If $1 \leq t \leq r \leq s$, $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq r+s-t+1$, $\mathcal{A} \subseteq \mathcal{H}^{(r)}$, $\mathcal{B} \subseteq \mathcal{H}^{(s)}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{H}^{(s)}|$.*

In other words, we conjecture that for $\mu(\mathcal{H}) \geq r+s-t+1$, if the cross-$t$-intersecting families $\mathcal{A}$ and $\mathcal{B}$ are allowed to be empty, then their sum of sizes is maximum if $\mathcal{A}$ is empty and $\mathcal{B}$ is $\mathcal{H}^{(s)}$.

In Section 3, we establish some key properties of hereditary families that enable us to prove Theorem 2.1 and the following result.

Lemma 2.5. *If $1 \leq t \leq r \leq s$, $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq r+s-t+1$, I is a set in $\mathcal{H}$ with $t \leq |I| \leq r$, $\mathcal{A} = \mathcal{H}^{(r)}(I)$, and $\mathcal{B} = \{B \in \mathcal{H}^{(s)} : |B \cap I| \geq t\}$, then $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{H}^{(s)}|$, and equality holds only if $t = 1$ and $\mu(\mathcal{H}) = r+s$.*

Lemma 2.5 is proved in Section 3. It immediately gives us the following.

Theorem 2.6. *If Conjecture 2.3 is true, then Conjecture 2.4 is true.*

Lemma 3.1 gives us $|\mathcal{H}^{(s)}| > |\mathcal{H}^{(r)}|$ if $r < s$ and $\mu(\mathcal{H}) > r + s$. Thus, by Theorem 2.1 and Lemma 2.5, we obtain the following.

Theorem 2.7. *If $1 \leq t \leq r \leq s$, $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq \max\{c(r, s, t), t+1\}$, $\mathcal{A} \subseteq \mathcal{H}^{(r)}$, $\mathcal{B} \subseteq \mathcal{H}^{(s)}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{H}^{(s)}|,$$

and if $(s, \mu(\mathcal{H})) \neq (1, 2)$, then equality holds if and only if $(\mathcal{A}, \mathcal{B}) = (\emptyset, \mathcal{H}^{(s)})$ or $r = s$ and $(\mathcal{A}, \mathcal{B}) = (\mathcal{H}^{(r)}, \emptyset)$.

Proof. If $s = t$, then $r = s$ and $\mu(\mathcal{H}) \geq t + 1 = r + s - t + 1$. If $s > t$, then $\mu(\mathcal{H}) \geq c(r, s, t) \geq r + 2s$. Thus, the result is immediate if $\mathcal{A} = \emptyset$ or $\mathcal{B} = \emptyset$. If $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$, then the result follows by Theorem 2.1 and Lemma 2.5. $\square$

By Theorem 2.7, Conjecture 2.4 is true if $\mu(\mathcal{H}) \geq \max\{c(r, s, t), t + 1\}$. Thus, Kamat's conjecture is true if $\mu(\mathcal{I}_G) \geq \max\{c(r, r, 1), 2\}$.

We mention that the analogous problem for cross-intersecting subfamilies of $\mathcal{H}$ is solved in [5].

We now start working towards proving Theorem 2.1 and Lemma 2.5.

3. Key properties of hereditary families

Hereditary families exhibit undesirable phenomena; see, for example, [2, Example 1]. The complete absence of symmetry makes intersection problems like the ones described above difficult to deal with. Many of the well-known techniques in extremal set theory, such as the shifting technique (see [19]), fail to work for hereditary families. The lemmas in this section and in the next section enable us to overcome such difficulties.

The two results below establish the properties of hereditary families that are fundamental to our work. The first one is given by [2, Corollary 3.2].

Lemma 3.1 ([2]). *If $\mathcal{H}$ is a hereditary family and $0 \leq r \leq s \leq \mu(\mathcal{H})$, then*

$$|\mathcal{H}^{(s)}| \geq \frac{\binom{\mu(\mathcal{H})-r}{s-r}}{\binom{s}{s-r}} |\mathcal{H}^{(r)}|.$$

For any family $\mathcal{F}$ and any two sets X and Y, let $\mathcal{F}(X, Y)$ denote the family $\{A \in \mathcal{F} : A \cap Y = X\}$, and let $\mathcal{F}\langle X, Y\rangle$ denote the family $\{A \backslash X : A \in \mathcal{F}(X, Y)\}$.

The new lemma below is used in the proof of Theorem 2.1.

Lemma 3.2. *If $\mathcal{H}$ is a hereditary family, $X \subseteq Y$, and $\mathcal{H}(X, Y) \neq \emptyset$, then*

$$\mu(\mathcal{H}\langle X, Y\rangle) \geq \mu(\mathcal{H}) - |Y|.$$

Proof. Let $\mathcal{F} = \mathcal{H}\langle X, Y\rangle$. Let B be a base of $\mathcal{F}$ of size $\mu(\mathcal{F})$. Let $C = B \cup X$. Then $C \in \mathcal{H}$. Let D be a base of $\mathcal{H}$ such that $C \subseteq D$. Then $X \subseteq D$. Let $E = (D \backslash Y) \cup X$. Since $\mathcal{H}$ is hereditary and $E \subseteq D \in \mathcal{H}$, $E \in \mathcal{H}$. Let $F = E \backslash X$. Since $E \cap Y = X$, $F \in \mathcal{F}$. Since $C \subseteq D$ and $C \cap Y = E \cap Y = X$, $B \subseteq F$. Since B is a base of $\mathcal{F}$, $B = F$. Thus, we have $\mu(\mathcal{F}) = |B| = |F| = |E| - |X| = |D \backslash Y| \geq |D| - |Y| \geq \mu(\mathcal{H}) - |Y|$. $\square$

For $X = Y$, Lemma 3.2 holds even if the family is not hereditary, as established by [3, Lemma 3.2 (i)] for $X = \{a\}$.

Lemma 3.3. *If $\mathcal{F}$ is a family, X is a set, $\mathcal{F}(X) \neq \emptyset$, and $\mathcal{G} = \{F \backslash X : F \in \mathcal{F}(X)\}$, then*

$$\mu(\mathcal{G}) \geq \mu(\mathcal{F}) - |X|.$$

Proof. Let B be a base of $\mathcal{G}$ of size $\mu(\mathcal{G})$. Then $B \cup X$ is a base of $\mathcal{F}$. Thus, $\mu(\mathcal{F}) \leq |B| + |X| = \mu(\mathcal{G}) + |X|$. $\square$

Lemma 3.4. *If $0 \le t \le u \le r$, $s \ge r+t-u$, $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \ge s+u-t$, and T is a t-element subset of a u-element set U such that $\mathcal{H}^{(r)}(U) \neq \emptyset$, then*

$$|\mathcal{H}^{(s)}(T,U)| \ge \frac{\binom{\mu(\mathcal{H})-r}{s+u-t-r}}{\binom{s-t}{s+u-t-r}} |\mathcal{H}^{(r)}(U)|.$$

Proof. Let $\mathcal{S} = \mathcal{H}^{(s)}(T,U)$. Since $\mathcal{H}^{(r)}(U) \neq \emptyset$, $\mathcal{H}(U) \neq \emptyset$. Let $\mathcal{I} = \{H \backslash U : H \in \mathcal{H}(U)\}$. Since $\mathcal{H}$ is hereditary, $\mathcal{I}$ is hereditary. By Lemma 3.3, $\mu(\mathcal{I}) \ge \mu(\mathcal{H}) - u$. Let $p = r-u$ and $q = s-t$. Since $\mu(\mathcal{H}) \ge s+u-t$, $\mu(\mathcal{I}) \ge q$. We have $0 \le p \le q \le \mu(\mathcal{I})$. Therefore, by Lemma 3.1,

$$|\mathcal{I}^{(q)}| \ge \frac{\binom{\mu(\mathcal{I})-p}{q-p}}{\binom{q}{q-p}} |\mathcal{I}^{(p)}|. \tag{2}$$

Clearly, $|\mathcal{I}^{(p)}| = |\mathcal{H}^{(r)}(U)|$. Consider any $A \in \mathcal{I}^{(q)}$. Since $A \cup T \subseteq A \cup U \in \mathcal{H}(U)$ and $\mathcal{H}$ is hereditary, $A \cup T \in \mathcal{H}$. Since $|A \cup T| = s$ and $(A \cup T) \cap U = T$, it follows that $A \cup T \in \mathcal{S}$. Thus, $|\mathcal{I}^{(q)}| \le |\mathcal{S}|$. Therefore, by (2),

$$|\mathcal{S}| \ge \frac{\binom{\mu(\mathcal{I})-p}{q-p}}{\binom{q}{q-p}} |\mathcal{H}^{(r)}(U)| \ge \frac{\binom{(\mu(\mathcal{H})-u)-(r-u)}{(s-t)-(r-u)}}{\binom{s-t}{(s-t)-(r-u)}} |\mathcal{H}^{(r)}(U)| = \frac{\binom{\mu(\mathcal{H})-r}{s+u-t-r}}{\binom{s-t}{s+u-t-r}} |\mathcal{H}^{(r)}(U)|,$$

as required. $\square$

Proof of Lemma 2.5. Let $t' = t-1$. For each $T \in \binom{I}{t'}$, let $\mathcal{S}_T = \mathcal{H}^{(s)}(T,I)$. Consider any $T \in \binom{I}{t'}$. We have $\mathcal{S}_T \cap \mathcal{B} = \emptyset$. Also, by Lemma 3.4,

$$|\mathcal{S}_T| \ge \frac{\binom{\mu(\mathcal{H})-r}{s+|I|-t'-r}}{\binom{s-t'}{s+|I|-t'-r}} |\mathcal{H}^{(r)}(I)| \ge \frac{\binom{s-t+1}{s-t+1+|I|-r}}{\binom{s-t+1}{s-t+1+|I|-r}} |\mathcal{H}^{(r)}(I)| = |\mathcal{A}|,$$

and equality holds throughout only if $\mu(\mathcal{H}) = r+s-t+1$. We have $|\mathcal{H}^{(s)}| \ge |\mathcal{B} \cup \bigcup_{T \in \binom{I}{t'}} \mathcal{S}_T| = |\mathcal{B}| + \sum_{T \in \binom{I}{t'}} |\mathcal{S}_T| \ge |\mathcal{B}| + \binom{|I|}{t'}|\mathcal{A}| \ge |\mathcal{A}| + |\mathcal{B}|$, and equality holds throughout only if $\mu(\mathcal{H}) = r+s-t+1$ and $t' = 0$. The result follows. $\square$

4. Proof of Theorem 2.1

If a set X t-intersects each set in a family $\mathcal{A}$, then we call X a *t-transversal of $\mathcal{A}$*.

Lemma 4.1. *If X is a t-transversal of a family $\mathcal{A}$, then*

$$|\mathcal{A}| \le \binom{|X|}{t} |\mathcal{A}(T)|$$

for some $T \in \binom{X}{t}$.

Proof. Let $\mathcal{X} = \binom{X}{t}$. Let $T \in \mathcal{X}$ such that $|\mathcal{A}(I)| \le |\mathcal{A}(T)|$ for each $I \in \mathcal{X}$. Since $|A \cap X| \ge t$ for each $A \in \mathcal{A}$, we clearly have $\mathcal{A} = \bigcup_{I \in \mathcal{X}} \mathcal{A}(I)$. Thus, $|\mathcal{A}| = \left|\bigcup_{I \in \mathcal{X}} \mathcal{A}(I)\right| \le \sum_{I \in \mathcal{X}} |\mathcal{A}(I)| \le \sum_{I \in \mathcal{X}} |\mathcal{A}(T)| = |\mathcal{X}||\mathcal{A}(T)| = \binom{|X|}{t}|\mathcal{A}(T)|$. $\square$

Lemma 4.2. *If X is a t-transversal of a family $\mathcal{A}$, T is a set of size t, and $T \not\subseteq X$, then*

$$\mathcal{A}(T) = \bigcup_{x \in X \backslash T} \mathcal{A}(T \cup \{x\}).$$

Proof. Obviously, $\bigcup_{x \in X \backslash T} \mathcal{A}(T \cup \{x\}) \subseteq \mathcal{A}(T)$. For each $A \in \mathcal{A}$, we have

$$t \le |A \cap X| = |A \cap (X \cap T)| + |A \cap (X \backslash T)| \le t-1+|A \cap (X \backslash T)|$$

(as $|T| = t$ and $T \not\subseteq X$), and hence $|A \cap (X \backslash T)| \geq 1$. Thus, for each $A \in \mathcal{A}(T)$, we have $a \in A$ for some $a \in X \backslash T$, and hence $A \in \mathcal{A}(T \cup \{a\}) \subseteq \bigcup_{x \in X \backslash T} \mathcal{A}(T \cup \{x\})$. Therefore, we have $\mathcal{A}(T) \subseteq \bigcup_{x \in X \backslash T} \mathcal{A}(T \cup \{x\}) \subseteq \mathcal{A}(T)$. The result follows. $\square$

A family $\mathcal{F}$ is said to be r-*uniform*, or simply *uniform*, if each member of $\mathcal{F}$ is an r-element set. Recall from Section 1.3 that a t-intersecting family is said to be trivial if its members have at least t common elements.

Lemma 4.3. *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty cross-t-intersecting families such that $\mathcal{A}$ is r-uniform, $\mathcal{B}$ is s-uniform, and $\mathcal{B}$ is not a trivial t-intersecting family, then there exist $B, X \in \mathcal{B}$ such that*

$$|\mathcal{A}| \leq s\binom{s}{t}|\mathcal{A}(T \cup \{x\})|$$

for some $T \in \binom{B}{t}$ and some $x \in X \backslash T$.

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, each set in $\mathcal{A}$ is a t-transversal of $\mathcal{B}$, and each set in $\mathcal{B}$ is a t-transversal of $\mathcal{A}$. Let $B \in \mathcal{B}$. By Lemma 4.1, $|\mathcal{A}| \leq \binom{|B|}{t}|\mathcal{A}(T)| = \binom{s}{t}|\mathcal{A}(T)|$ for some $T \in \binom{B}{t}$. Since $\mathcal{B}$ is not a trivial t-intersecting family, $T \not\subseteq X$ for some $X \in \mathcal{B}$. By Lemma 4.2, $\mathcal{A}(T) = \bigcup_{x \in X \backslash T} \mathcal{A}(T \cup \{x\})$, so $|\mathcal{A}(T)| \leq \sum_{x \in X \backslash T} |\mathcal{A}(T \cup \{x\})|$. Let $x^* \in X \backslash T$ such that $|\mathcal{A}(T \cup \{x\})| \leq |\mathcal{A}(T \cup \{x^*\})|$ for each $x \in X \backslash T$. Let $Y = T \cup \{x^*\}$. Thus, $|\mathcal{A}(T)| \leq \sum_{x \in X \backslash T} |\mathcal{A}(Y)| = |X \backslash T||\mathcal{A}(Y)| \leq s|\mathcal{A}(Y)|$, and hence $|\mathcal{A}| \leq \binom{s}{t}s|\mathcal{A}(Y)|$. $\square$

Lemma 4.4. *If $1 \leq t \leq r$, $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq 2r - t$, $\emptyset \neq \mathcal{A} \subseteq \mathcal{H}^{(r)}$, $\mathcal{B}$ is a non-empty s-uniform family that is not a trivial t-intersecting family, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then there exists a t-element set T such that*

$$|\mathcal{A}| < \frac{s(r-t)}{\mu(\mathcal{H}) - r}\binom{s}{t}|\mathcal{H}^{(r)}(T)|$$

and $T \subseteq B$ for some $B \in \mathcal{B}$.

Proof. By Lemma 4.3, there exist $B, X \in \mathcal{B}$ such that $|\mathcal{A}| \leq s\binom{s}{t}|\mathcal{A}(T \cup \{x\})|$ for some $T \in \binom{B}{t}$ and some $x \in X \backslash T$. Since $\mathcal{A} \neq \emptyset$, it follows that $\mathcal{A}(T \cup \{x\}) \neq \emptyset$, so $\mathcal{H}^{(r)}(T \cup \{x\}) \neq \emptyset$. Let $\mathcal{G} = \{H \in \mathcal{H}^{(r)} : H \cap (T \cup \{x\}) = T\}$. We have $|\mathcal{A}(T \cup \{x\})| \leq |\mathcal{H}^{(r)}(T \cup \{x\})| \leq \frac{r-t}{\mu(\mathcal{H})-r}|\mathcal{G}|$ by Lemma 3.4. Since $|\mathcal{H}^{(r)}(T)| = |\mathcal{G}| + |\mathcal{H}^{(r)}(T \cup \{x\})| > |\mathcal{G}|$, we obtain $|\mathcal{A}(T \cup \{x\})| < \frac{r-t}{\mu(\mathcal{H})-r}|\mathcal{H}^{(r)}(T)|$. Since $|\mathcal{A}| \leq s\binom{s}{t}|\mathcal{A}(T \cup \{x\})|$, the result follows. $\square$

We now settle a few calculations so that in the formal proof of Theorem 2.1 we can focus on the combinatorial argument.

Proposition 4.5. *If $1 \leq t \leq r \leq s$, $s > t$, and $n \geq c(r, s, t)$, then the following hold:*

(i) $\dfrac{r(s-t)}{n-s}\dbinom{r}{t} < \dfrac{1}{2}$.

(ii) $\dbinom{s}{t} \leq \dfrac{1}{2}\dfrac{\binom{n-r}{s-r}}{\binom{s-t}{s-r}}$ *if $r < s$.*

Proof. By straightforward induction, $2^a \geq 2a$ for every positive integer a. Since $t \leq r \leq s$ and $s > t$, either $t < r$ or $t = r < s$. If $t < r$, then, since $n \geq 2^r(r-t)(s-t)\binom{r}{t} + r + s - t$, we have $n > 2r(s-t)\binom{r}{t} + s$, which yields (i). If $t = r < s$, then, since $n \geq 2(s-t)\binom{s}{t} + r$, we have $n \geq 2(s-t)\binom{t+1}{t} + t = 2(t+1)(s-t) + t > 2t(s-t) + s = 2r(s-t)\binom{r}{t} + s$ (as $r = t$), which yields (i).

Suppose $s > r$. Since $n \geq 2(s-t)\binom{s}{t} + r$, we have $n - r > s - t > 0$ and $\binom{s}{t} \leq \frac{1}{2}\binom{n-r}{s-t}$. Thus, $\binom{s}{t} \leq \frac{1}{2}\prod_{i=0}^{s-r-1}\binom{n-r-i}{s-t-i} = \frac{1}{2}\frac{\binom{n-r}{s-r}}{\binom{s-t}{s-r}}$, which confirms (ii). $\square$

Proof of Theorem 2.1. Let $n = c(r, s, t)$. Let $(\mathcal{A}, \mathcal{B}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t)$.

Case 1: $\mathcal{A}$ is a trivial t-intersecting family. Let $I = \bigcap_{A\in\mathcal{A}} A$, $\mathcal{C} = \mathcal{H}^{(r)}(I)$, and $\mathcal{D} = \{H \in \mathcal{H}^{(s)} : |H \cap I| \geq t\}$. Then $t \leq |I| \leq r$, $I \in \mathcal{H}$ (as $\mathcal{H}$ is hereditary), and $\mathcal{A} \subseteq \mathcal{C}$.

Suppose $|I| = r$. Then $\mathcal{A} = \{I\}$ and, since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, $\mathcal{B} \subseteq \mathcal{D}$. Since $\{I\}$ and $\mathcal{D}$ are cross-t-intersecting, and since $(\mathcal{A}, \mathcal{B}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t)$, we obtain $\mathcal{B} = \mathcal{D}$, as required.

Now suppose $|I| < r$. Let $\mathcal{A}' = \{A\backslash I : A \in \mathcal{A}\}$, $\mathcal{I} = \{H\backslash I : H \in \mathcal{H}(I)\}$, and $r' = r - |I|$. Then $\mathcal{A}' \subseteq \mathcal{I}^{(r')}$, $\mathcal{I}$ is hereditary, and, by Lemma 3.3, $\mu(\mathcal{I}) \geq \mu(\mathcal{H}) - |I|$. By the definition of I, $\bigcap_{E\in\mathcal{A}'} E = \emptyset$. Thus, $\mathcal{A}'$ is not a trivial 1-intersecting family. For each $i \in \{0\} \cup [t-1]$, let $\mathcal{B}_i = \{B \in \mathcal{B} : |B \cap I| = i\}$. Let $\mathcal{B}_{\geq t} = \{B \in \mathcal{B} : |B \cap I| \geq t\}$. Then $\mathcal{B} = \mathcal{B}_{\geq t} \cup \bigcup_{i=0}^{t-1} \mathcal{B}_i$. Let $J = \{i \in \{0\} \cup [t-1] : \mathcal{B}_i \neq \emptyset\}$.

Suppose $J = \emptyset$. Then $\mathcal{B} = \mathcal{B}_{\geq t}$. Hence $\mathcal{B} \subseteq \mathcal{D}$. Thus, as required, we obtain $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \mathcal{D}$, because $\mathcal{A} \subseteq \mathcal{C}$, $\mathcal{C}$ and $\mathcal{D}$ are cross-t-intersecting, and $(\mathcal{A}, \mathcal{B}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t)$.

We now show that indeed $J = \emptyset$.

Suppose $J \neq \emptyset$. Consider any $j \in J$. For any $S \in \binom{I}{j}$, let $\mathcal{B}_{j,S} = \{B \in \mathcal{B}_j : B \cap I = S\}$. Then $\mathcal{B}_j = \bigcup_{S\in\binom{I}{j}} \mathcal{B}_{j,S}$. Let $\mathcal{S}_j = \{S \in \binom{I}{j} : \mathcal{B}_{j,S} \neq \emptyset\}$. Since $\mathcal{B}_j \neq \emptyset$, $\mathcal{S}_j \neq \emptyset$. Consider any $S \in \mathcal{S}_j$. Let $\mathcal{B}'_{j,S} = \{B\backslash S : B \in \mathcal{B}_{j,S}\}$, $\mathcal{H}_{j,S} = \{H \in \mathcal{H} : H \cap I = S\}$, $\mathcal{J}_{j,S} = \{H\backslash S : H \in \mathcal{H}_{j,S}\}$, $s_j = s - j$, and $t_j = t - j$. Then $\emptyset \neq \mathcal{B}'_{j,S} \subseteq {\mathcal{J}_{j,S}}^{(s_j)}$, $\mathcal{J}_{j,S}$ is hereditary, and, by Lemma 3.2,

$$\mu(\mathcal{J}_{j,S}) \geq \mu(\mathcal{H}) - |I| > n - r \geq 2(s-t)\binom{s}{t} \geq 2s(s-t) \geq 2s > 2s_j - t_j$$

(note that $s > t$ as $t \leq |I| < r \leq s$). Since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, $\mathcal{A}'$ and $\mathcal{B}'_{j,S}$ are cross-t_j-intersecting. Since $t_j \geq 1$ and $\mathcal{A}'$ is not a trivial 1-intersecting family, $\mathcal{A}'$ is not a trivial t_j-intersecting family. By Lemma 4.4, there exists a t_j-element set $X_{j,S}$ such that

$$|\mathcal{B}'_{j,S}| < \frac{r'(s_j - t_j)}{\mu(\mathcal{J}_{j,S}) - s_j}\binom{r'}{t_j}|{\mathcal{J}_{j,S}}^{(s_j)}(X_{j,S})|$$

and $X_{j,S} \subseteq E_{j,S}$ for some $E_{j,S} \in \mathcal{A}'$. We have $|\mathcal{B}'_{j,S}| = |\mathcal{B}_{j,S}|$. Let $T_{j,S} = S \cup X_{j,S}$. Then $|{\mathcal{J}_{j,S}}^{(s_j)}(X_{j,S})| = |{\mathcal{H}_{j,S}}^{(s)}(T_{j,S})|$. Thus,

$$|\mathcal{B}_{j,S}| < \frac{r'(s_j - t_j)}{\mu(\mathcal{J}_{j,S}) - s_j}\binom{r'}{t_j}|{\mathcal{H}_{j,S}}^{(s)}(T_{j,S})| \leq \frac{(r-|I|)(s-t)}{\mu(\mathcal{H}) - |I| + j - s}\binom{r-|I|}{t-j}|{\mathcal{H}_{j,S}}^{(s)}(T_{j,S})|.$$

Since $\mathcal{A}'$ and $\mathcal{B}'_{j,S}$ are cross-t_j-intersecting, we have $r' \geq t_j$, that is, $r - |I| \geq t - j$. Since $0 \leq j \leq t-1$, $t \leq |I| \leq r - 1$, and $\mu(\mathcal{H}) \geq n$, we therefore have

$$|\mathcal{B}_{j,S}| < \frac{(r-t)(s-t)}{n+t-r-s}\binom{r-j}{t-j}|{\mathcal{H}_{j,S}}^{(s)}(T_{j,S})| \leq \frac{1}{2^r}|{\mathcal{H}_{j,S}}^{(s)}(T_{j,S})|$$

as $n \geq (r-t)(s-t)2^r\binom{r}{t} + r + s - t \geq (r-t)(s-t)2^r\binom{r-j}{t-j} + r + s - t$.

Let $j^* \in J$ and $S^* \in \mathcal{S}_{j^*}$ such that for each $j \in J$, $|{\mathcal{H}_{j,S}}^{(s)}(T_{j,S})| \leq |{\mathcal{H}_{j^*,S^*}}^{(s)}(T_{j^*,S^*})|$ for each $S \in \mathcal{S}_j$. We have

$$\begin{aligned}|\mathcal{B}| &= |\mathcal{B}_{\geq t}| + \sum_{j\in J}|\mathcal{B}_j| \leq |\mathcal{D}| + \sum_{j\in J}\sum_{S\in\mathcal{S}_j}|\mathcal{B}_{j,S}| < |\mathcal{D}| + \sum_{j\in J}\sum_{S\in\mathcal{S}_j}\frac{1}{2^r}|{\mathcal{H}_{j,S}}^{(s)}(T_{j,S})| \\ &\leq |\mathcal{D}| + \sum_{j\in J}\sum_{S\in\mathcal{S}_j}\frac{1}{2^r}|{\mathcal{H}_{j^*,S^*}}^{(s)}(T_{j^*,S^*})| \leq |\mathcal{D}| + \frac{1}{2^r}|{\mathcal{H}_{j^*,S^*}}^{(s)}(T_{j^*,S^*})|\sum_{j\in J}\sum_{S\in\mathcal{S}_j}1\end{aligned}$$

and $\sum_{j\in J}\sum_{S\in\mathcal{S}_j} 1 = \sum_{j\in J}|\mathcal{S}_j| < \sum_{j=0}^{|I|}\binom{|I|}{j} = 2^{|I|} \leq 2^{r-1}$. Thus,

$$|\mathcal{B}| < |\mathcal{D}| + \frac{1}{2}|{\mathcal{H}_{j^*,S^*}}^{(s)}(T_{j^*,S^*})|. \tag{3}$$

For convenience, let $j = j^*$ and $S = S^*$. Let $B' \in \mathcal{B}'_{j,S}$. Recall that $\mathcal{A}'$ and $\mathcal{B}'_{j,S}$ are cross-t_j-intersecting, so B' is a t_j-transversal of $\mathcal{A}'$. By Lemma 4.1, $|\mathcal{A}'| \leq \binom{|B'|}{t_j}|\mathcal{A}'(X^*)|$ for some $X^* \in \binom{B'}{t_j}$.

We have

$$0 < |\mathcal{A}| = |\mathcal{A}'| \le \binom{s-j}{t-j}|\mathcal{I}^{(r')}(X^*)| \le \binom{s}{t}|\mathcal{I}^{(r')}(X^*)|. \tag{4}$$

Since $t_j \le r' < s_j$ and $\mu(\mathcal{I}) \ge \mu(\mathcal{H}) - |I| \ge n - |I| \ge r' + 2(s-t)\binom{s}{t} = r' + 2(s_j - t_j)\binom{s}{t}$, Lemma 3.4 (with $T = U = X^*$) gives us

$$\frac{|\mathcal{I}^{(s_j)}(X^*)|}{|\mathcal{I}^{(r')}(X^*)|} \ge \frac{\binom{\mu(\mathcal{I})-r'}{s_j-r'}}{\binom{s_j-t_j}{s_j-r'}} = \prod_{i=0}^{s_j-r'-1}\frac{\mu(\mathcal{I})-r'-i}{s_j-t_j-i} \ge \frac{\mu(\mathcal{I})-r'}{s_j-t_j} \ge 2\binom{s}{t}$$

(note that $\mathcal{I}^{(s_j)}(X^*, X^*) = \mathcal{I}^{(s_j)}(X^*)$), so

$$\binom{s}{t}|\mathcal{I}^{(r')}(X^*)| \le \frac{1}{2}|\mathcal{I}^{(s_j)}(X^*)|. \tag{5}$$

Let $\mathcal{L} = \mathcal{H}^{(|I|+s_j)}(I \cup X^*)$. Then $\mathcal{I}^{(s_j)}(X^*) = \{H\backslash I : H \in \mathcal{L}\}$. Let $\mathcal{L}' = \{L\backslash(I\backslash S) : L \in \mathcal{L}\}$. Since $\mathcal{H}$ is hereditary, $\mathcal{L}' \subseteq \mathcal{H}$. For each $H \in \mathcal{L}'$, we have $|H| = s_j + |I| - (|I| - |S|) = s$, $H \cap I = S$, and $S \cup X^* \subseteq H$. Thus, $\mathcal{L}' \subseteq {\mathcal{H}_{j,S}}^{(s)}(S \cup X^*)$. Let $T_1 = S \cup X^*$. We have $|\mathcal{I}^{(s_j)}(X^*)| = |\mathcal{L}| = |\mathcal{L}'| \le |{\mathcal{H}_{j,S}}^{(s)}(T_1)|$. Thus, by (4) and (5),

$$|\mathcal{A}| \le \frac{1}{2}|{\mathcal{H}_{j,S}}^{(s)}(T_1)|. \tag{6}$$

Let $T_2 = T_{j,S}$. Let $\mathcal{E}$ be a member of $\{{\mathcal{H}_{j,S}}^{(s)}(T_1), {\mathcal{H}_{j,S}}^{(s)}(T_2)\}$ of maximum size. Recall that above we set $j = j^*$ and $S = S^*$. By (3) and (6),

$$|\mathcal{A}| + |\mathcal{B}| < \frac{1}{2}|{\mathcal{H}_{j,S}}^{(s)}(T_1)| + |\mathcal{D}| + \frac{1}{2}|{\mathcal{H}_{j,S}}^{(s)}(T_2)| \le |\mathcal{D}| + |\mathcal{E}|. \tag{7}$$

Let

$$X' = \begin{cases} X^* & \text{if } \mathcal{E} = {\mathcal{H}_{j,S}}^{(s)}(T_1); \\ X_{j,S} & \text{if } \mathcal{E} = {\mathcal{H}_{j,S}}^{(s)}(T_2). \end{cases}$$

Let $F = I \cup X'$. Let $\mathcal{F} = \mathcal{H}^{(r)}(F)$ and $\mathcal{G} = \mathcal{D} \cup \mathcal{E}$. If $X' = X^*$, then, since $|\mathcal{F}| = |\mathcal{H}^{(|I|+r')}(I \cup X^*)| = |\mathcal{I}^{(r')}(X^*)|$, $|\mathcal{F}| > 0$ by (4). If $X' = X_{j,S}$, then, since $X_{j,S} \subseteq E_{j,S} \in \mathcal{A}'$, we have $F \subseteq I \cup E_{j,S} \in \mathcal{A}$, and hence $I \cup E_{j,S} \in \mathcal{F}$. Therefore, $\mathcal{F} \neq \emptyset$. By (7), $\mathcal{G} \neq \emptyset$. For each $G \in \mathcal{D}$, $|G \cap F| \ge |G \cap I| \ge t$. For some $i \in [2]$, $\mathcal{E} = {\mathcal{H}_{j,S}}^{(s)}(T_i)$ and $T_i = S \cup X'$; thus, for each $G \in \mathcal{E}$, $|G \cap F| \ge |T_i \cap F| = |S| + |X'| = j + t_j = t$. For every $G \in \mathcal{G}$ and every $H \in \mathcal{F}$, $|G \cap H| \ge |G \cap F|$, so $|G \cap H| \ge t$. Thus, $\mathcal{F}$ and $\mathcal{G}$ are cross-t-intersecting. For each $H \in \mathcal{E}$, $|H \cap I| = |S| = j < t$. Thus, $\mathcal{D} \cap \mathcal{E} = \emptyset$, and hence $|\mathcal{G}| = |\mathcal{D}| + |\mathcal{E}|$. Bringing all the pieces together, we have that $\emptyset \neq \mathcal{F} \subseteq \mathcal{H}^{(r)}$, $\emptyset \neq \mathcal{G} \subseteq \mathcal{H}^{(s)}$, $\mathcal{F}$ and $\mathcal{G}$ are cross-t-intersecting, and, by (7),

$$|\mathcal{A}| + |\mathcal{B}| < |\mathcal{G}| < |\mathcal{F}| + |\mathcal{G}|,$$

contradicting $(\mathcal{A}, \mathcal{B}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t)$.

Case 2: $\mathcal{A}$ is not a trivial t-intersecting family. If $t = s$, then $t = r = s$ and $n = r = 2s - t$. If $t < s$, then $n > 2s$. Thus, $\mu(\mathcal{H}) \ge 2s - t$. By Lemma 4.4, there exists a t-element set $T_{\mathcal{B}}$ such that

$$|\mathcal{B}| < \frac{r(s-t)}{\mu(\mathcal{H}) - s}\binom{r}{t}|\mathcal{H}^{(s)}(T_{\mathcal{B}})|. \tag{8}$$

Thus, $\mathcal{H}^{(s)}(T_{\mathcal{B}}) \neq \emptyset$. Let T' be a t-element set such that $\mathcal{H}^{(s)}(T')$ is a largest t-star of $\mathcal{H}^{(s)}$. We have

$$0 < |\mathcal{H}^{(s)}(T_{\mathcal{B}})| \le |\mathcal{H}^{(s)}(T')|. \tag{9}$$

Suppose $r < s$. Let $D \in \mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, D is a t-transversal of $\mathcal{A}$. By Lemma 4.1,

$$|\mathcal{A}| \le \binom{|D|}{t}|\mathcal{A}(T_D)| \le \binom{s}{t}|\mathcal{H}^{(r)}(T_D)| \tag{10}$$

for some $T_D \in \binom{D}{t}$. Let $\mathcal{G} = \{H\backslash T_D: H \in \mathcal{H}(T_D)\}$. Then $\mathcal{G}$ is hereditary. Since $|\mathcal{A}| > 0$, (10) yields $\mathcal{H}^{(r)}(T_D) \neq \emptyset$, and hence $\mathcal{G} \neq \emptyset$. By Lemma 3.3, $\mu(\mathcal{G}) \geq \mu(\mathcal{H}) - |T_D| = \mu(\mathcal{H}) - t$. By Lemma 3.1,

$$|\mathcal{G}^{(s-t)}| \geq \frac{\binom{\mu(\mathcal{G})-(r-t)}{(s-t)-(r-t)}}{\binom{s-t}{(s-t)-(r-t)}}|\mathcal{G}^{(r-t)}| = \frac{\binom{\mu(\mathcal{G})+t-r}{s-r}}{\binom{s-t}{s-r}}|\mathcal{G}^{(r-t)}|.$$

Clearly, $|\mathcal{H}^{(r)}(T_D)| = |\mathcal{G}^{(r-t)}|$ and $|\mathcal{H}^{(s)}(T_D)| = |\mathcal{G}^{(s-t)}|$. We have

$$\begin{aligned}\frac{\binom{\mu(\mathcal{H})-r}{s-r}}{\binom{s-t}{s-r}}|\mathcal{H}^{(r)}(T_D)| &\leq \frac{\binom{\mu(\mathcal{G})+t-r}{s-r}}{\binom{s-t}{s-r}}|\mathcal{H}^{(r)}(T_D)| = \frac{\binom{\mu(\mathcal{G})+t-r}{s-r}}{\binom{s-t}{s-r}}|\mathcal{G}^{(r-t)}| \\ &\leq |\mathcal{G}^{(s-t)}| = |\mathcal{H}^{(s)}(T_D)| \leq |\mathcal{H}^{(s)}(T')|. \end{aligned} \tag{11}$$

By any of (9) and (11), $\mathcal{H}^{(s)}(T') \neq \emptyset$. Thus, since $\mathcal{H}$ is hereditary and each member of $\mathcal{H}^{(s)}(T')$ has an r-element subset containing T', $\mathcal{H}^{(r)}(T') \neq \emptyset$. We have

$$\begin{aligned}|\mathcal{A}| + |\mathcal{B}| &< \binom{s}{t}|\mathcal{H}^{(r)}(T_D)| + \frac{r(s-t)}{\mu(\mathcal{H})-s}\binom{r}{t}|\mathcal{H}^{(s)}(T_\mathcal{B})| \quad \text{(by (8) and (10))} \\ &< \frac{1}{2}\frac{\binom{\mu(\mathcal{H})-r}{s-r}}{\binom{s-t}{s-r}}|\mathcal{H}^{(r)}(T_D)| + \frac{1}{2}|\mathcal{H}^{(s)}(T_\mathcal{B})| \quad \text{(by Proposition 4.5)} \\ &\leq \frac{1}{2}|\mathcal{H}^{(s)}(T')| + \frac{1}{2}|\mathcal{H}^{(s)}(T')| \quad \text{(by (9) and (11))} \\ &= |\mathcal{H}^{(s)}(T')| < |\mathcal{H}^{(r)}(T')| + |\mathcal{H}^{(s)}(T')|,\end{aligned}$$

which is a contradiction since $\emptyset \neq \mathcal{H}^{(r)}(T') \subseteq \mathcal{H}^{(r)}$, $\emptyset \neq \mathcal{H}^{(s)}(T') \subseteq \mathcal{H}^{(s)}$, $\mathcal{H}^{(r)}(T')$ and $\mathcal{H}^{(s)}(T')$ are cross-t-intersecting, and $(\mathcal{A}, \mathcal{B}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}, t)$.

Therefore, $r = s$. Suppose that $\mathcal{B}$ is not a trivial t-intersecting family. By Lemma 4.4, there exists a t-element set $T_\mathcal{A}$ such that

$$|\mathcal{A}| < \frac{s(r-t)}{\mu(\mathcal{H})-r}\binom{s}{t}|\mathcal{H}^{(r)}(T_\mathcal{A})|.$$

Thus, $r - t > 0$. We have

$$\begin{aligned}|\mathcal{A}| + |\mathcal{B}| &< \frac{s(r-t)}{\mu(\mathcal{H})-r}\binom{s}{t}|\mathcal{H}^{(r)}(T_\mathcal{A})| + \frac{r(s-t)}{\mu(\mathcal{H})-s}\binom{r}{t}|\mathcal{H}^{(s)}(T_\mathcal{B})| \\ &= \frac{r(r-t)}{\mu(\mathcal{H})-r}\binom{r}{t}\left(|\mathcal{H}^{(r)}(T_\mathcal{A})| + |\mathcal{H}^{(r)}(T_\mathcal{B})|\right) \quad (\text{as } r = s) \\ &< \frac{1}{2}\left(|\mathcal{H}^{(r)}(T_\mathcal{A})| + |\mathcal{H}^{(r)}(T_\mathcal{B})|\right) \quad \text{(by Proposition 4.5(i))} \\ &< |\mathcal{H}^{(r)}(T')| + |\mathcal{H}^{(r)}(T')| \quad (\text{as } r = s),\end{aligned}$$

which is a contradiction because, as in the case $r < s$ above, $\emptyset \neq \mathcal{H}^{(r)}(T') \subseteq \mathcal{H}^{(r)}$, $(\mathcal{H}^{(r)}(T'), \mathcal{H}^{(r)}(T')) \in C(\mathcal{H}^{(r)}, \mathcal{H}^{(r)}, t)$, and $(\mathcal{A}, \mathcal{B}) \in M(\mathcal{H}^{(r)}, \mathcal{H}^{(r)}, t)$.

Therefore, $\mathcal{B}$ is a trivial t-intersecting family. Thus, since $r = s$, we can apply the argument in Case 1 to obtain that there exists some $I \in \mathcal{H}$ such that $t \leq |I| \leq r$, $\mathcal{B} = \mathcal{H}^{(r)}(I)$, and $\mathcal{A} = \{H \in \mathcal{H}^{(r)}: |H \cap I| \geq t\}$. Since $\mathcal{A}$ is not a trivial t-intersecting family, $t < |I|$. It remains to show that $(\mathcal{A}, \mathcal{B}) \neq (\mathcal{H}^{(r)}(I), \{H \in \mathcal{H}^{(r)}: |H \cap I| \geq t\})$ (as the theorem states that the two possibilities resulting from it are mutually exclusive).

Since $t < |I|$, $t < r$. Let $T \in \binom{I}{t}$. Let B be a base of $\mathcal{H}$ such that $I \subseteq B$. Since $\mu(\mathcal{H}) \geq c(r, r, t) \geq r + 2\binom{r}{t} \geq 3r$, $|B| \geq 3r$. Since $|I| \leq r$, $|B\backslash I| \geq 2r$. Let $X \in \binom{B\backslash I}{r-t}$. Since $\mathcal{H}$ is hereditary and $T \cup X \subseteq B \in \mathcal{H}$, $T \cup X \in \mathcal{H}$. Thus, $T \cup X \in \mathcal{A}\backslash\mathcal{H}^{(r)}(I)$, and hence $\mathcal{A} \neq \mathcal{H}^{(r)}(I)$. Therefore, $(\mathcal{A}, \mathcal{B}) \neq (\mathcal{H}^{(r)}(I), \{H \in \mathcal{H}^{(r)}: |H \cap I| \geq t\})$, as required. □

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