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An equivalent formulation of the Fan-Raspaud Conjecture and related problems

Giuseppe Mazzuoccolo

Dipartimento di Informatica, Università di Verona, Strada Le Grazie 15, Verona, Italy

Jean Paul Zerafa

Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Via Campi 213/B, Modena, Italy

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Abstract

In 1994, it was conjectured by Fan and Raspaud that every simple bridgeless cubic graph has three perfect matchings whose intersection is empty. In this paper we answer a question recently proposed by Mkrtchyan and Vardanyan, by giving an equivalent formulation of the Fan-Raspaud Conjecture. We also study a possibly weaker conjecture originally proposed by the first author, which states that in every simple bridgeless cubic graph there exist two perfect matchings such that the complement of their union is a bipartite graph. Here, we show that this conjecture can be equivalently stated using a variant of Petersen-colourings, we prove it for graphs having oddness at most four and we give a natural extension to bridgeless cubic multigraphs and to certain cubic graphs having bridges.

Keywords: Cubic graph, perfect matching, oddness, Fan-Raspaud Conjecture, Berge-Fulkerson Conjecture, Petersen-colouring.

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1 Introduction and terminology

Many interesting problems in graph theory are about the behaviour of perfect matchings in cubic graphs. One of the early classical results was made by Petersen [28] and states that every bridgeless cubic graph has at least one perfect matching. Some years ago, one of the most prominent conjectures in this area was completely solved by Esperet et al. in [5]: the conjecture, proposed by Lovász and Plummer in the 1970s, stated that the number

E-mail addresses: giuseppe.mazzuoccolo@univr.it (Giuseppe Mazzuoccolo), jeanpaul.zerafa@unimore.it (Jean Paul Zerafa)

of perfect matchings in a bridgeless cubic graph grows exponentially with its order (see [20]). However, many others are still open, such as Conjecture 2.1 proposed independently by Berge and Fulkerson in the 1970s as well, and Conjecture 2.2 by Fan and Raspaud (see [10] and [7], respectively). These two conjectures are related to the behaviour of the union and intersection of sets of perfect matchings, and properties of this kind are already largely studied: see, amongst others, [1, 2, 15, 16, 17, 19, 22, 23, 25, 30, 31]. In this paper we prove that a seemingly stronger version of the Fan-Raspaud Conjecture is equivalent to the classical formulation (Theorem 3.3), and so to another interesting formulation proposed in [21] (see also [18]). In the second part of the paper (Section 4 and Section 5), we study a weaker conjecture proposed by the first author in [24]: we show how we can state it in terms of a variant of Petersen-colourings (Proposition 4.1) and we prove it for cubic graphs of oddness four (Theorem 5.4). Although all mentioned conjectures are about simple cubic graphs without bridges, we extend our study of the union of two perfect matchings to bridgeless cubic multigraphs and to particular cubic graphs having bridges (Section 6.1 and Section 6.2).

Graphs considered in the sequel, unless otherwise stated, are simple connected bridgeless cubic graphs and so do not contain loops and parallel edges. Graphs that may contain parallel edges will be referred to as *multigraphs*. For a graph G, let $V(G)$ and $E(G)$ be the set of vertices and the set of edges of G, respectively. A *matching* of G is a subset of $E(G)$ such that any two of its edges do not share a common vertex. For an integer $k \geq 0$, a *k-factor* of G is a spanning subgraph of G (not necessarily connected) such that the degree of every vertex is k. The edge-set of a 1-factor is said to be a *perfect matching*. The least number of odd cycles amongst all 2-factors of G, denoted by $\omega(G)$, is called the *oddness* of G, and is clearly even for a cubic graph since G has an even number of vertices. For $M \subseteq E(G)$, we denote the graph $G \setminus M$ by $\overline{M}$. In particular, when M is a perfect matching of G, then $\overline{M}$ is a 2-factor of G. In this case, following the terminology used for instance in [8], if $\overline{M}$ has $\omega(G)$ odd cycles, then M is said to be a *minimal perfect matching*.

A *cut* in G is any set $X \subseteq E(G)$ such that $\overline{X}$ has more components than G, and no proper subset of X has this property, i.e. for any $X' \subset X$, $\overline{X'}$ does not have more components than G. The set of edges with precisely one end in $W \subseteq V(G)$ is denoted by $\partial_G W$, or just ∂W when it is obvious to which graph we are referring. Moreover, a cut X is said to be *odd* if there exists a subset W of $V(G)$ having odd cardinality such that $X = \partial W$.

We next define some standard operations on graphs that will be useful in the sequel. Let G_1 and G_2 be two bridgeless graphs (not necessarily cubic), and let e_1 and e_2 be two edges such that $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. A 2-*cut connection* on u_1v_1 and u_2v_2 is a graph operation that consists of constructing the new graph

$$[G_1 - e_1] \cup [G_2 - e_2] \cup \{u_1u_2, v_1v_2\},$$

and denoted by $G_1(u_1v_1) * G_2(u_2v_2)$. It is clear that another possible graph obtained by a 2-cut connection on e_1 and e_2 is $G_1(u_1v_1) * G_2(v_2u_2)$. Clearly, the two graphs obtained are bridgeless, and, unless otherwise stated, if it is not important which of these two graphs is obtained, we use the notation $G_1(e_1) * G_2(e_2)$ and we say that it is a graph obtained by a 2-cut connection on e_1 and e_2.

Now, let G_1 and G_2 be two bridgeless cubic graphs, $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that the vertices adjacent to v_1 are x_1, y_1 and z_1, and those adjacent to v_2 are x_2, y_2 and z_2. A 3-*cut connection* (sometimes also known as the star product, see for instance [11]) on v_1

and v_2 is a graph operation that consists of constructing the new graph

$$[G_1 - v_1] \cup [G_2 - v_2] \cup \{x_1x_2, y_1y_2, z_1z_2\},$$

and denoted by $G_1(x_1y_1z_1)*G_2(x_2y_2z_2)$. The 3-edge-cut $\{x_1x_2, y_1y_2, z_1z_2\}$ is referred to as the *principal* 3-*edge cut* (see for instance [9]). As in the case of 2-cut connections, other graphs can be obtained by a 3-cut connection on v_1 and v_2, and, unless otherwise stated, if it is not important how the adjacencies in the principal 3-edge cut look like, we use the notation $G_1(v_1) * G_2(v_2)$ and we say that it is a graph obtained by a 3-cut connection on v_1 and v_2. It is clear that any resulting graph is also bridgeless and cubic.

2 A list of relevant conjectures

One of the aims of this paper is to study the behaviour of perfect matchings in cubic graphs, more specifically the union of two perfect matchings (see Section 4 and Section 5). We relate this to well-known conjectures stated here below, in particular: the Berge-Fulkerson Conjecture and the Fan-Raspaud Conjecture.

Conjecture 2.1 (Berge-Fulkerson [10]). *Every bridgeless cubic graph G admits six perfect matchings $M_1, \ldots, M_6$ such that any edge of G belongs to exactly two of them.*

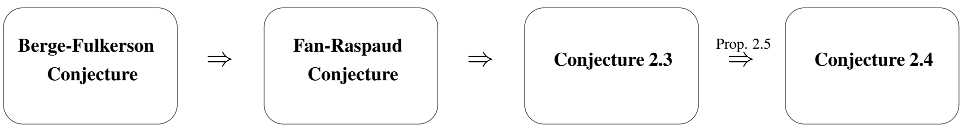

Figure 1: Conjectures mentioned and how they are related.

We also state here other (possibly weaker) conjectures implied by the above conjecture.

Conjecture 2.2 (Fan-Raspaud [7]). *Every bridgeless cubic graph admits three perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.*

In the sequel we will refer to three perfect matchings satisfying Conjecture 2.2 as an *FR-triple*. We can see that Conjecture 2.2 is immediately implied by the Berge-Fulkerson Conjecture, since we can take any three perfect matchings out of the six which satisfy Conjecture 2.1. A still weaker statement implied by the Fan-Raspaud Conjecture is the following:

Conjecture 2.3 ([21]). *For each bridgeless cubic graph G, there exist two perfect matchings M_1 and M_2 such that $M_1 \cap M_2$ contains no odd-cut of G.*

We claim that any two perfect matchings out of the three in an FR-triple have no odd-cut in their intersection, in other words that Conjecture 2.2 implies Conjecture 2.3. For, suppose not. Then, without loss of generality, suppose that $M_2 \cap M_3$ contains an odd-cut X. Hence, since every perfect matching has to intersect an odd-cut at least once, $|M_1 \cap (M_2 \cap M_3)| \geq |M_1 \cap X| \geq 1$, a contradiction, since we assumed that $M_1 \cap M_2 \cap M_3$ is empty. In relation to the above, the first author proposed the following conjecture:

Conjecture 2.4 (S_4-Conjecture [24]). *For any bridgeless cubic graph G, there exist two perfect matchings such that the deletion of their union leaves a bipartite subgraph of G.*

For reasons which shall be obvious in Section 4 we let such a pair of perfect matchings be called an S_4-*pair* of G and shall refer to Conjecture 2.4 as the S_4-Conjecture. We will first proceed by showing that this conjecture is implied by Conjecture 2.3, and so, by what we have said so far, is a consequence of the Berge-Fulkerson Conjecture. In particular, we can see the S_4-Conjecture as Conjecture 2.3 restricted to odd-cuts $\partial V(C)$, where C is an odd cycle of G.

Proposition 2.5. *Conjecture 2.3 implies the S_4-Conjecture.*

Proof. Let M_1 and M_2 be two perfect matchings such that their intersection does not contain any odd-cut. Consider $\overline{M_1 \cup M_2}$, and suppose that it contains an odd cycle C. Then, all the edges of $\partial V(C)$ belong to $M_1 \cap M_2$. If $\overline{\partial V(C)}$ has exactly two components, then $\partial V(C)$ is an odd-cut belonging to $M_1 \cap M_2$, a contradiction. Therefore, $\overline{\partial V(C)}$ must have more than two components, say k, denoted by $C_1, C_2, \ldots, C_k$, where the first component C_1 is the cycle C. Let $[C_1, C_j]$ denote the set of edges between C_1 and C_j, for $j \in \{2, \ldots, k\}$. Since $\sum_{j=2}^{k} |[C_1, C_j]| = |\partial V(C)| \equiv 1 \pmod 2$, there exists $j' \in \{2, \ldots, k\}$, such that $|[C_1, C_{j'}]| \equiv 1 \pmod 2$. However, $[C_1, C_{j'}]$ is an odd-cut which belongs to $M_1 \cap M_2$, a contradiction. □

3 Statements equivalent to the Fan-Raspaud Conjecture

Let $M_1, \ldots, M_t$ be a list of perfect matchings of G, and let $a \in E(G)$. We denote the number of times a occurs in this list by $\nu_G[a : M_1, \ldots, M_t]$. When it is obvious which list of perfect matchings or which graph we are referring to, we will denote this as $\nu(a)$ and refer to it as the *frequency of* a. We will sometimes need to refer to the frequency of an ordered list of edges, say (a, b, c), and we will do this by saying that the frequency of (a, b, c) is (i, j, k), for some integers i, j and k. Mkrtchyan et al. [27] showed that the Fan-Raspaud Conjecture, i.e. Conjecture 2.2, is equivalent to the following:

Conjecture 3.1 ([27])**.** *For each bridgeless cubic graph G, any edge $a \in E(G)$ and any $i \in \{0, 1, 2\}$, there exist three perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$ and $\nu_G[a : M_1, M_2, M_3] = i$.*

In other words they show that if a graph has an FR-triple then, for every i in $\{0, 1, 2\}$, there exists an FR-triple in which the frequency of a pre-chosen edge is exactly i. In the same paper, Mkrtchyan et al. state the following seemingly stronger version of the Fan-Raspaud Conjecture:

Conjecture 3.2 ([27])**.** *Let G be a bridgeless cubic graph, w a vertex of G and i, j and k three integers in $\{0, 1, 2\}$ such that $i + j + k = 3$. Then, G has an FR-triple in which the edges incident to w in a given order have frequencies (i, j, k).*

This means that we can prescribe the frequencies to the three edges incident to a given vertex. At the end of [27], the authors remark that it would be interesting to show that Conjecture 3.2 is equivalent to the Fan-Raspaud Conjecture. We prove here that this is actually the case.

Theorem 3.3. *Conjecture 3.2 is equivalent to the Fan-Raspaud Conjecture.*

Proof. Since the Fan-Raspaud Conjecture is equivalent to Conjecture 3.1, it suffices to show the equivalence of Conjectures 3.1 and 3.2. The latter clearly implies the former, so assume Conjecture 3.1 is true and let a, b and c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). It is sufficient to show that there exist two FR-triples in which the frequencies of (a, b, c) are $(2, 1, 0)$ in one FR-triple (Case 1 below) and $(1, 1, 1)$ in the other FR-triple (Case 2 below).

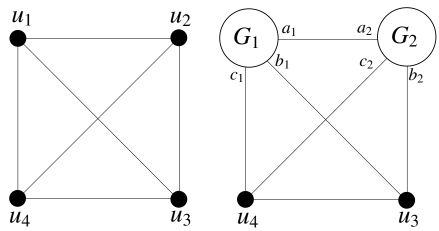

Figure 2: The graphs K_4 and K_4^* in Case 1 of the proof of Theorem 3.3.

Case 1. Let u_1, u_2, u_3 and u_4 be the vertices of the complete graph K_4 as in Figure 2. Consider two copies of G, and let the vertex w in the i^{th} copy of G be denoted by w_i, for each $i \in \{1, 2\}$. We apply a 3-cut connection between u_i and w_i, for each $i \in \{1, 2\}$. With reference to this resulting graph, denoted by K_4^*, we refer to the copy of the graph $G - w$ at u_1 as G_1, and to the corresponding edges a, b and c as a_1, b_1 and c_1, respectively. The graph G_2 and the edges a_2, b_2 and c_2 are defined in a similar way, and the 3-cut connection is done in such a way that b_1 and b_2 are adjacent, and also c_1 and c_2, as Figure 2 shows. Note also that a_1 and a_2 coincide in K_4^*. By our assumption, there exists an FR-triple M_1, M_2 and M_3 of K_4^* in which the edge u_3u_4 has frequency 2. Without loss of generality, let $u_3u_4 \in M_1 \cap M_2$. Then, a_1 (and so a_2) must belong to $M_1 \cap M_2$. Clearly, a_1 (and so a_2) cannot belong to M_3, and so the principal 3-edge-cuts with respect to G_1 and G_2 do not belong to M_3. If $b_1 \in M_3$, then we are done, as then M_1, M_2 and M_3 restricted to G_1, together with a and b having the same frequencies as a_1 and b_1, induce an FR-triple of G such that the frequencies of (a, b, c) are $(2, 1, 0)$. So suppose $c_1 \in M_3$. Then, $b_2 \in M_3$, and so by a similar argument applied to G_2 and the corresponding edges, M_1, M_2 and M_3 induce an FR-triple in G such that the frequencies of (a, b, c) are $(2, 1, 0)$.

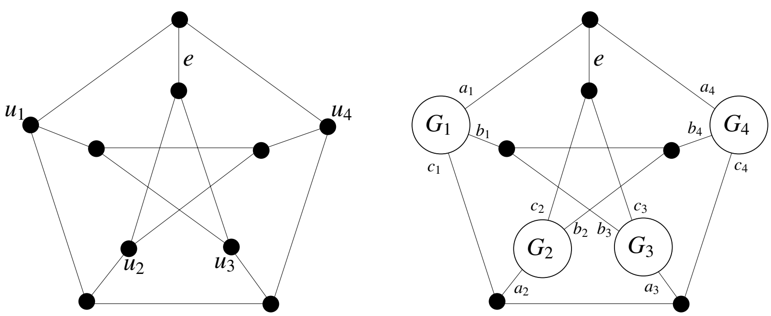

Figure 3: The graphs P and P^* in Case 2 of the proof of Theorem 3.3.

Case 2. Let P be the Petersen graph and $\{u_1, u_2, u_3, u_4\}$ be a maximum independent set of vertices in P as in Figure 3. Consider four copies of G. Let the vertex w in the i^{th} copy of G be denoted by w_i, for each $i \in \{1, \ldots, 4\}$. Let P^* be the graph obtained by applying a 3-cut connection between each u_i and w_i, as shown in Figure 3. Similar to Case 1 we refer to the copy of $G - w$ at u_i as G_i and to the corresponding edges a, b and c as a_i, b_i and c_i, respectively. Since we are assuming that Conjecture 3.1 is true, we can consider an FR-triple M_1, M_2 and M_3 of P^* in which the edge e incident to both a_1 and a_4 has frequency 2. Without loss of generality, let the two perfect matchings containing e be M_1 and M_2. The edges a_1, c_2, c_3 and a_4 are not contained in M_1 and neither M_2, since they are all incident to e, and so no principal 3-edge-cut leaving G_i belongs to M_1 or M_2. Then, M_1 and M_2 induce perfect matchings of P (clearly distinct), and since there are exactly two perfect matchings of P containing e, we can assume that M_1 contains $\{e, b_1, a_2, a_3, b_4\}$, and M_2 contains $\{e, c_1, b_2, b_3, c_4\}$.

If the third perfect matching M_3 induces a perfect matching of the Petersen graph then the induced perfect matching cannot be one of the perfect matchings induced by M_1 and M_2 in P. Hence, since every two distinct perfect matchings of P intersect in exactly one edge of P, there exists $i \in \{1, 2, 3, 4\}$ such that the frequencies of (a_i, b_i, c_i) are $(1, 1, 1)$ and so, M_1, M_2 and M_3 restricted to G_i, together with a, b and c having the same frequencies as a_i, b_i and c_i, induce an FR-triple in G with the needed property.

Therefore, suppose M_3 contains the principal 3-edge-cut of one of the G_is, say G_1 by symmetry of P^*. Thus, a_1, b_1 and c_1 belong to M_3. The perfect matching M_3 can intersect the principal 3-edge-cut at G_2 either in b_2 or c_2 (not both). If $c_2 \in M_3$ we are done by the same reasoning above applied to G_2 and the corresponding edges. So suppose $b_2 \in M_2 \cap M_3$. Then, $c_4 \in M_3$, and M_3 can only intersect the principal 3-edge-cut at G_3 in c_3, implying that the frequencies of (a_3, b_3, c_3) are $(1, 1, 1)$ in P^* and that M_1, M_2 and M_3 restricted to G_3, together with a, b and c having the same frequencies as a_3, b_3 and c_3, induce an FR-triple in G with the needed property. □

In [27] it is also shown that a minimal counterexample to Conjecture 3.2 is cyclically 4-edge-connected. It remains unknown whether a smallest counterexample to the original formulation of the Fan-Raspaud Conjecture has the same property. Indeed, we only prove that the two assertions are equivalent, but we cannot say whether a possible counterexample to Conjecture 3.2 is itself a counterexample to the original formulation.

4 Statements equivalent to the S_4-Conjecture

All conjectures presented in Section 2 are implied by a conjecture made by Jaeger in the late 1980s. In order to state it we need the following definitions. Let G and H be two graphs. An *H-colouring* of G is a proper edge-colouring f of G with edges of H, such that for each vertex $u \in V(G)$, there exists a vertex $v \in V(H)$ with $f(\partial_G\{u\}) \subseteq \partial_H\{v\}$. If G admits an H-colouring, then we will write $H \prec G$. In this paper we consider S_4-colourings of bridgeless cubic graphs, where S_4 is the multigraph shown in Figure 4.

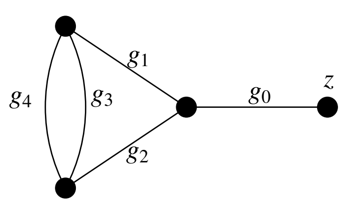

Figure 4: The multigraph S_4.

The importance of H-colourings is mainly due to Jaeger's Conjecture [14] which states that each bridgeless cubic graph G admits a P-colouring (where P is again the Petersen graph). For recent results on P-colourings, known as Petersen-colourings, see for instance [12, 13, 26, 29]. The following proposition shows why we choose to refer to a pair of perfect matchings whose deletion leaves a bipartite subgraph as an S_4-pair.

Proposition 4.1. *Let G be a bridgeless cubic graph, then $S_4 \prec G$ if and only if G has an S_4-pair.*

Proof. Along the entire proof we denote the edges of S_4 by using the same labelling as in Figure 4. Let M_1 and M_2 be an S_4-pair of G. The graph induced by $M_1 \cup M_2$, denoted by $G[M_1 \cup M_2]$, is made up of even cycles and isolated edges, whilst the bipartite graph $\overline{M_1 \cup M_2}$ is made up of even cycles and paths. We obtain an S_4-colouring of G as follows:

- the isolated edges in $M_1 \cup M_2$ are given colour g_0,
- the edges of the even cycles in $M_1 \cup M_2$ are properly edge-coloured with g_3 and g_4, and
- the edges of the paths and even cycles in $\overline{M_1 \cup M_2}$ are properly edge-coloured with g_1 and g_2.

One can clearly see that this gives an S_4-colouring of G. Conversely, assume that $S_4 \prec G$. We are required to show that there exists an S_4-pair of G. Let M_1 be the set of edges of G coloured g_3 and g_0, and let M_2 be the set of edges of G coloured g_4 and g_0. If e and f are edges of G coloured g_3 (or g_4) and g_0, respectively, then e and f cannot be adjacent, otherwise we contradict the S_4-colouring of G. Thus, M_1 and M_2 are matchings. Next, we show that they are in fact perfect matchings. This follows since for every vertex v of G, $f(\partial_G\{v\})$ is equal to $\{g_1, g_3, g_4\}$, or $\{g_2, g_3, g_4\}$, or $\{g_0, g_1, g_2\}$. Thus, $\overline{M_1 \cup M_2}$ is the graph induced by the edges coloured g_1 and g_2, which clearly cannot induce an odd cycle. □

Hence, by the previous proof, Conjecture 2.4 can be stated in terms of S_4-colourings, which clearly shows why we choose to refer to it as the S_4-Conjecture. In analogy to what we did for FR-triples, here we prove that for S_4-pairs we can prescribe the frequency of an edge and the frequencies of the edges leaving a vertex (the proof of the latter also implies that we can prescribe the frequencies of the edges of each 3-cut). Consider the following conjecture, analogous to Conjecture 3.1:

Conjecture 4.2. *For any bridgeless cubic graph G, any edge $a \in E(G)$ and any $i \in \{0, 1, 2\}$, there exists an S_4-pair, say M_1 and M_2, such that $\nu_G[a : M_1, M_2] = i$.*

In Theorem 4.3 we show that the latter conjecture is actually equivalent to the S_4-Conjecture. The proof given in [27] to show the equivalence of the Fan-Raspaud Conjecture and Conjecture 3.1 is very similar to the proof we give here for the analogous case for the S_4-Conjecture, however, we need a slightly more complicated tool in our context.

Theorem 4.3. *Conjecture 4.2 is equivalent to the S_4-Conjecture.*

Proof. Clearly, Conjecture 4.2 implies the S_4-Conjecture so it suffices to show the converse. Assume the S_4-Conjecture to be true and let f_1, f_2, f_3 be three consecutive edges of K_4 inducing a path. Consider two copies of G. Let the edge a in the i^{th} copy of G

be denoted by a_i, for each $i \in \{1, 2\}$. Let K_4' be the graph obtained by applying a 2-cut connection between f_i and a_i for each $i \in \{1, 2\}$. We refer to the copy of the graph $G - a$ on f_i as G_i.

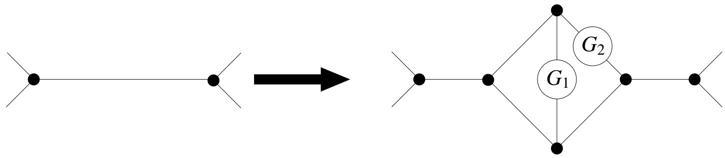

Figure 5: An edge in P transformed into the corresponding structure in H.

Let $\{e_1, \ldots, e_{15}\}$ be the edges of the Petersen graph and let $T_1, \ldots, T_{15}$ be fifteen copies of K_4'. For every $i \in \{1, \ldots, 15\}$, apply a 2-cut connection on e_i and the edge f_3 of T_i. Consequently, every edge e_i of the Petersen graph is transformed into the structure E_i as in Figure 5, and we refer to G_1 and G_2 on E_i as G_1^i and G_2^i, respectively. Let H be the resulting graph. By our assumption, there exists an S_4-pair of H, say M_1 and M_2, which induces a pair of two distinct perfect matchings in P, say N_1 and N_2, respectively. There exists an edge of P, say e_j, for some $j \in \{1, \ldots, 15\}$, such that $\nu_P[e_j : N_1, N_2] = 1$, since every two distinct perfect matchings of P have exactly one edge of P in common. Hence, the restriction of M_1 and M_2 to the edge set of G_1^j, together with the edge a having the same frequency as e_j, gives rise to an S_4-pair of G in which the frequency of a is 1.

Moreover, there exists an edge of P, say e_k, for some $k \in \{1, \ldots, 15\}$, such that $\nu_P[e_k : N_1, N_2] = 2$. Restricting M_1 and M_2 to the edge set of G_1^k, together with the edge a having the same frequency as e_k, gives rise to an S_4-pair of G, in which the frequency of a is 2. Also, the restriction of M_1 and M_2 to the edge set of G_2^k gives rise to an S_4-pair of G (G_2^k together with a), in which the frequency of a is 0, because if not, then there exists an odd cycle in G, say of length α, passing through a and having all its edges with frequency 0. However, this would mean that there is an odd cycle of length $\alpha + 4$ on E_k in $\overline{M_1 \cup M_2}$ (in H), a contradiction. □

As in Section 3, we state an analogous conjecture to Conjecture 3.2, but for S_4-pairs:

Conjecture 4.4. *Let G be a bridgeless cubic graph, w a vertex of G and i, j and k three integers in $\{0, 1, 2\}$ such that $i + j + k = 2$. Then, G has an S_4-pair in which the edges incident to w in a given order have frequencies (i, j, k).*

The following theorem shows that this conjecture is actually equivalent to Conjecture 4.2, and so to the S_4-Conjecture by Theorem 4.3.

Theorem 4.5. *Conjecture 4.4 is equivalent to the S_4-Conjecture.*

Proof. Since the S_4-Conjecture is equivalent to Conjecture 4.2, it suffices to show the equivalence of Conjectures 4.2 and 4.4. Clearly, Conjecture 4.4 implies Conjecture 4.2 and so we only need to show the converse. Let a, b and c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). We only need to prove the case when (i, j, k) is equal to $(1, 1, 0)$, as all other cases follow from Conjecture 4.2.

Consider the graph $G(w) * P(v)$, where P is the Petersen graph and v is any vertex of P. We refer to the edges corresponding to a, b and c in $G(w) * P(v)$, as a_w, b_w and c_w.

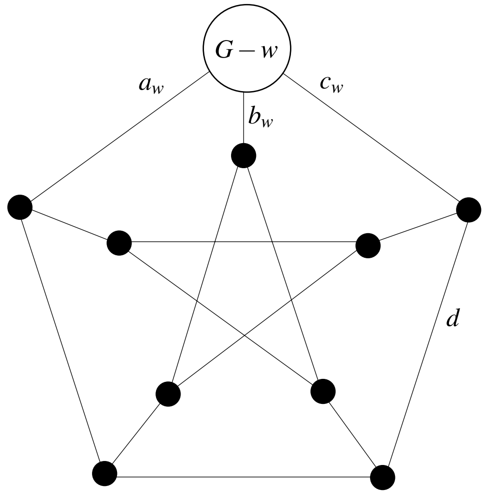

Figure 6: The graph $G(w) * P(v)$ from Theorem 4.5.

Let d be an edge originally belonging to P and adjacent to c_w in $G(w) * P(v)$. Since we are assuming Conjecture 4.2 to be true, there exists an S_4-pair in $G(w) * P(v)$ in which d has frequency 2. If the frequencies of (a_w, b_w, c_w) are $(1, 1, 0)$, then we are done, because the S_4-pair for $G(w) * P(v)$ restricted to the edges in $G - w$, together with a and b having the same frequencies as a_w and b_w, give an S_4-pair for G with the desired property. We claim that this must be the case. For, suppose not. Then, without loss of generality, the frequencies of (a_w, b_w, c_w) are $(2, 0, 0)$. This implies that all the edges of $G(w) * P(v)$ originally in P have either frequency 0 or 2, since the two perfect matchings in the S_4-pair induce the same perfect matching in P. However, this implies that P has a perfect matching whose complement is bipartite, a contradiction since P is not 3-edge-colourable. □

As in [27], a minimal counterexample to Conjecture 4.4 (but not necessarily to the S_4-Conjecture) is cyclically 4-edge-connected. We omit the proof of this result as it is very similar to the proof of Theorem 2 in [27].

5 Further results on the S_4-Conjecture

Little progress has been made on the Fan-Raspaud Conjecture so far. Bridgeless cubic graphs which trivially satisfy this conjecture are those which can be edge-covered by four perfect matchings. In this case, every three perfect matchings from a cover of this type form an FR-triple since every edge has frequency one or two with respect to this cover. Therefore, a possible counterexample to the Fan-Raspaud Conjecture should be searched for in the class of bridgeless cubic graphs whose edge-set cannot be covered by four perfect matchings, see for instance [6]. In 2009, Máčajová and Škoviera [22] shed some light on the Fan-Raspaud Conjecture by proving it for bridgeless cubic graphs having oddness two. One of the aims of this paper is to show that even if the S_4-Conjecture is still open, some results are easier to extend than the corresponding ones for the Fan-Raspaud Conjecture. Clearly, the result by Máčajová and Škoviera in [22] implies the following result:

Theorem 5.1. *Let G be a bridgeless cubic graph of oddness two. Then, G has an S_4-pair.*

We first give a proof of Theorem 5.1 in the same spirit of that used in [22], however much shorter since we are proving a weaker result.

Proof 1 of Theorem 5.1. Let M_1 be a minimal perfect matching of G, and let C_1 and C_2 be the two odd cycles in $\overline{M_1}$. Colour the even cycles in $\overline{M_1}$ using two colours, say 1 and 2. For each $i \in \{1, 2\}$, let E_i be the set of edges belonging to the even cycles in $\overline{M_1}$ and having colour i. In G, there must exist a path Q whose edges alternate in M_1 and E_1 and whose end-vertices belong to C_1 and C_2, respectively, since C_1 and C_2 are odd cycles. Note that since the edges of C_1 and C_2 are not edges in $M_1 \cup E_1$, every other vertex on Q which is not an end-vertex does not belong to C_1 and C_2.

For each $i \in \{1, 2\}$, let v_i be the end-vertex of Q belonging to C_i, and let M_{C_i} be the unique perfect matching of $C_i - v_i$. Let $M_2 := (M_1 \cap Q) \cup (E_1 \setminus Q) \cup M_{C_1} \cup M_{C_2}$. Clearly, M_2 is a perfect matching of G which intersects C_1 and C_2, and so $\overline{M_1 \cup M_2}$ is bipartite. □

We now give a second alternative proof of the same theorem using fractional perfect matchings, which we will show to be easier to use for graphs having larger oddness. Let w be a vector in $\mathbb{R}^{|E(G)|}$. The entry of w corresponding to $e \in E(G)$ is denoted by $w(e)$, and for $A \subseteq E(G)$, we let the weight of A, denoted by $w(A)$, to be equal to $\sum_{e\in A} w(e)$. The vector w is said to be a *fractional perfect matching* of G if:

1. $w(e) \in [0, 1]$ for each $e \in E(G)$,
2. $w(\partial\{v\}) = 1$ for each $v \in V(G)$, and
3. $w(\partial W) \geq 1$ for each $W \subseteq V(G)$ of odd cardinality.

The following lemma is presented in [16] and it is a consequence of Edmonds' characterisation of perfect matching polytopes in [3].

Lemma 5.2. *If w is a fractional perfect matching in a graph G, and $c \in \mathbb{R}^{|E(G)|}$, then G has a perfect matching N such that*

$$c \cdot \chi^N \geq c \cdot w,$$

where $\cdot$ denotes the inner product. Moreover, there exists a perfect matching satisfying the above inequality and which contains exactly one edge of each odd-cut X with $w(X) = 1$.

Remark 5.3. If we let $w(e) = 1/3$ for all $e \in E(G)$, for some graph G, then we know that w is a fractional perfect matching of G. Also, since the weight of every 3-cut is one, then by Lemma 5.2 there exists a perfect matching of G containing exactly one edge of each 3-cut of G.

Proof 2 of Theorem 5.1. Let M_1 be a minimal perfect matching of G, and let C_1 and C_2 be the two odd cycles in $\overline{M_1}$. For each $i \in \{1, 2\}$, let e_1^i and e_2^i be two adjacent edges belonging to C_i. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^{2}\{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

By Remark 5.3, we also know that if we let $w(e) = 1/3$ for all $e \in E(G)$, then w is a fractional perfect matching of G. Hence, by Lemma 5.2, there exists a perfect matching M_2 such that $c \cdot \chi^{M_2} \geq c \cdot w$, which implies that

$$|\cup_{i=1}^{2}\{e_1^i, e_2^i\} \cap M_2| \geq 1/3 \times 2 \times 2 = 4/3 > 1.$$

Therefore, for each $i \in \{1, 2\}$, there exists exactly one $j \in \{1, 2\}$ such that $e_j^i \in M_2$. Hence, M_2 intersects C_1 and C_2 and so $\overline{M_1 \cup M_2}$ is bipartite. □

Using the same idea as in Proof 2 of Theorem 5.1, we also prove that the S_4-Conjecture is true for graphs having oddness four.

Theorem 5.4. *Let G be a bridgeless cubic graph of oddness four. Then, G has an S_4-pair.*

Proof. Let M_1 be a minimal perfect matching of G, and let C_1, C_2, C_3 and C_4 be the four odd cycles in $\overline{M_1}$. By Remark 5.3, there exists a perfect matching N of G such that if G has any 3-cuts, then N intersects every 3-cut of G in one edge. Moreover, for every $i \in \{1, \ldots, 4\}$, there exists at least a pair of adjacent edges e_1^i and e_2^i belonging to $C_i \cap \overline{N}$. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^{4}\{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the vector $w \in \mathbb{R}^{|E(G)|}$ as follows:

$$w(e) = \begin{cases} 1/5 & \text{if } e \in N, \\ 2/5 & \text{otherwise.} \end{cases}$$

The vector w is clearly a fractional perfect matching of G because, in particular, N intersects every 3-cut in one edge and so $w(X) \geq 1$ for each odd-cut X of G. Hence, by Lemma 5.2, there exists a perfect matching M_2 such that $c \cdot \chi^{M_2} \geq c \cdot w$, which implies that

$$|\cup_{i=1}^{4}\{e_1^i, e_2^i\} \cap M_2| \geq 2/5 \times 2 \times 4 = 16/5 > 3.$$

Therefore, for each $i \in \{1, \ldots, 4\}$, there exists exactly one $j \in \{1, 2\}$ such that $e_j^i \in M_2$. Hence, M_2 intersects C_1, C_2, C_3 and C_4 and so $\overline{M_1 \cup M_2}$ is bipartite. □

As the above proofs show us, extending results with respect to the S_4-Conjecture is easier than in the case of the Fan-Raspaud Conjecture and this is why we believe that a proof of the S_4-conjecture could be a first feasible step towards a solution of the Fan-Raspaud Conjecture. For graphs having oddness at least six we are not able to prove the existence of an S_4-pair and we wonder how many perfect matchings we need such that the complement of their union is bipartite. In the next proposition we use the technique used in Theorem 5.4 and show that given a bridgeless cubic graph G, if $\omega(G) \leq 5^{k-1} - 1$ for some positive integer k, then there exist k perfect matchings such that the complement of their union is bipartite. Note that for $k = 2$ we obtain $\omega(G) \leq 4$.

Proposition 5.5. *Let G be a bridgeless cubic graph and let $\mathcal{C}$ be a collection of disjoint odd cycles in G such that $|\mathcal{C}| \leq 5^{k-1} - 1$ for some positive integer k. Then, there exist $k - 1$ perfect matchings of G, say $M_1, \ldots, M_{k-1}$, such that for every $C \in \mathcal{C}$, there exists $j \in \{1, \ldots, k-1\}$ for which $C \cap M_j \neq \emptyset$. Moreover, if $\omega(G) \leq 5^{k-1} - 1$, then there exist k perfect matchings such that the complement of their union is bipartite.*

Proof. We proceed by induction on k. For $k = 1$, the assertion trivially holds since $\mathcal{C}$ is the empty set. Assume the result is true for some $k \geq 1$ and consider $k+1$. Let $C_1, C_2, \dots, C_t$, with $t \leq 5^k - 1$, be the odd cycles of G in $\mathcal{C}$. Let N be a perfect matching of G which intersects every 3-cut of G once. For every $i \in \{1, \dots, t\}$, there exists at least a pair of adjacent edges e_1^i and e_2^i belonging to $C_i \cap \overline{N}$. We define the vector $c \in \mathbb{R}^{|E(G)|}$ such that

$$c(e) = \begin{cases} 1 & \text{if } e \in \cup_{i=1}^{t} \{e_1^i, e_2^i\}, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the vector $w \in \mathbb{R}^{|E(G)|}$ as follows:

$$w(e) = \begin{cases} {}^1\!/\!_5 & \text{if } e \in N, \\ {}^2\!/\!_5 & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 5.4, w is a fractional perfect matching of G and by Lemma 5.2 there exists a perfect matching M_k such that $c \cdot \chi^{M_k} \geq c \cdot w$. This implies that

$$| \cup_{i=1}^{t} \{e_1^i, e_2^i\} \cap M_k| \geq 2 \times {}^2\!/\!_5 \times t.$$

Let $\mathcal{C}'$ be the subset of $\mathcal{C}$ which contains the odd cycles of $\mathcal{C}$ with no edge of M_k. Then, $|\mathcal{C}'| \leq |\mathcal{C}| - \frac{4}{5}t = t - \frac{4}{5}t = \frac{t}{5} \leq 5^{k-1} - \frac{1}{5}$, and so $|\mathcal{C}'| \leq 5^{k-1} - 1$. By induction, there exist $k - 1$ perfect matchings of G, say $M_1, \dots, M_{k-1}$, having the required property with respect to $\mathcal{C}'$. Therefore, $M_1, \dots, M_k$ intersect all odd cycles in $\mathcal{C}$. The second part of the statement easily follows by considering $\mathcal{C}$ to be the set of odd cycles in the complement of a minimal perfect matching M of G, since the union of M with the $k-1$ perfect matchings which intersect all the odd cycles in $\mathcal{C}$ has a bipartite complement. □

Remark 5.6. We note that with every step made in the proof of Proposition 5.5, one could update the weight w of the edges using the methods presented in [16, 23] which gives a slightly better upper bound for $\omega(G)$. For reasons of simplicity and brevity, we prefer the present weaker version of Proposition 5.5.

6 Extension of the S_4-Conjecture to larger classes of cubic graphs

6.1 Multigraphs

In this section we discuss natural extensions of some previous conjectures to bridgeless cubic multigraphs. We note that bridgeless cubic multigraphs cannot contain any loops. We will make use of the following standard operation on parallel edges, referred to as *smoothing*. Let G' be a bridgeless cubic multigraph. Let u and v be two vertices in G' such that there are exactly two parallel edges between them.

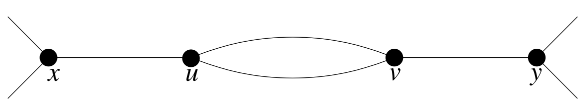

Figure 7: Vertices x, u, v and y in G'.

Let x and y be the vertices adjacent to u and v, respectively (see Figure 7). We say that we *smooth* uv if we delete the vertices u and v from G' and add an edge between x

and y (even if x and y are already adjacent in G'). One can easily see that the resulting multigraph, say G, after smoothing uv is again bridgeless and cubic.

In what follows, we will say that a perfect matching M of G and a perfect matching M' of G' are *corresponding* perfect matchings if the following holds:

$$M = \begin{cases} M' \cup xy - \{xu, vy\} & \text{if } xu \in M', \\ M' - uv & \text{otherwise.} \end{cases}$$

The following theorem can be easily proved by using smoothing operations.

Theorem 6.1. *The S_4-Conjecture is true if and only if every bridgeless cubic multigraph has an S_4-pair.*

Now we show that Conjecture 4.4 can also be extended to multigraphs.

Theorem 6.2. *Let i, j and k be three integers in $\{0, 1, 2\}$ such that $i + j + k = 2$ and let w be a vertex in a bridgeless cubic multigraph G'. Then, the S_4-Conjecture is true if and only if G' has an S_4-pair in which the edges incident to w in a given order have frequencies (i, j, k).*

Proof. It suffices to assume that the S_4-Conjecture is true and only show the forward direction, by Theorem 6.1. Let G' be a minimal counterexample and suppose it has some parallel edges. If $G' = C_{2,3}$ then the result clearly follows. So assume $G' \neq C_{2,3}$. Let a, b and c be the edges incident to w such that the frequencies (i, j, k) are to be assigned to (a, b, c). We proceed by considering two cases: when w has two parallel edges incident to it (Figure 8) and otherwise (Figure 9).

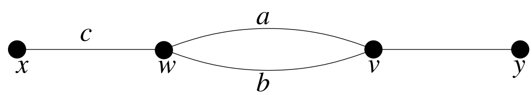

Figure 8: Case 1 in the proof of Theorem 6.2.

Case 1. Let G be the resulting multigraph after smoothing wv. By minimality of G', G has an S_4-pair (say M_1 and M_2) in which $\nu(xy) = k$. It is easy to see that a pair of corresponding perfect matchings in G' give $\nu_{G'}(c) = \nu_{G'}(vy) = k$ and can be chosen in such a way such that $\nu_{G'}(a) = i$ and $\nu_{G'}(b) = j$, a contradiction to our initial assumption. Therefore, we must have Case 2.

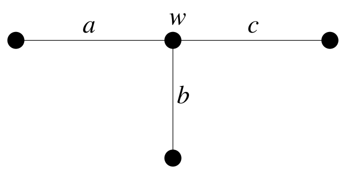

Figure 9: Case 2 in the proof of Theorem 6.2.

Case 2. Let G be the resulting multigraph after smoothing some parallel edge in G' and let a_w, b_w and c_w be the corresponding edges incident to w in G after smoothing is done.

In G, there exists an S_4-pair such that the frequencies of (a_w, b_w, c_w) are equal to (i, j, k). Clearly, the corresponding perfect matchings in G' form an S_4-pair in which the frequencies of (a, b, c) are (i, j, k), a contradiction, proving Theorem 6.2. □

Using the same ideas as in Theorem 6.1 and Theorem 6.2 one can also state analogous results for the Fan-Raspaud Conjecture in terms of multigraphs.

6.2 Graphs having bridges

Since every perfect matching must intersect every bridge of a cubic graph, then the Fan-Raspaud Conjecture cannot be extended to cubic graphs containing bridges. The situation is quite different for the S_4-Conjecture as Theorem 6.3 shows. By Errera's Theorem [4] we know that if all the bridges of a connected cubic graph lie on a single path, then the graph has a perfect matching. We use this idea to show that there can be graphs with bridges that can have an S_4-pair.

Theorem 6.3. *Let G be a connected cubic graph having k bridges, all of which lie on a single path, for some positive integer k. If the S_4-Conjecture is true, then G admits an S_4-pair.*

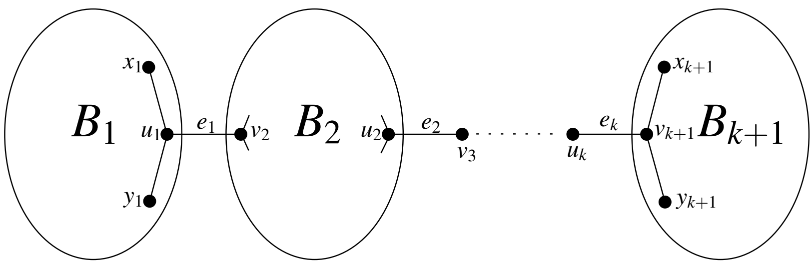

Figure 10: G with k bridges lying all on the same path.

Proof. Let $B_1, B_2, \ldots, B_{k+1}$ be the 2-connected components of G and let $e_1, \ldots, e_k$ be the bridges of G such that $e_i = u_i v_{i+1}$ for each $i \in \{1, \ldots, k\}$, where $u_i \in V(B_i)$ and $v_{i+1} \in V(B_{i+1})$. Let x_1 and y_1 be the two vertices adjacent to u_1 in B_1, and let x_{k+1} and y_{k+1} be the two vertices adjacent to v_{k+1} in B_{k+1}. Let $B'_1 = (B_1 - u_1) \cup x_1 y_1$ and $B'_{k+1} = (B_{k+1} - v_{k+1}) \cup x_{k+1} y_{k+1}$. Also, let $B'_i = B_i \cup v_i u_i$ for every $i \in \{2, \ldots, k\}$. Clearly, $B'_1, \ldots, B'_{k+1}$ are bridgeless cubic multigraphs. Since we are assuming that the S_4-Conjecture holds, then, by Theorem 6.1, for every $i \in \{1, \ldots, k+1\}$, B'_i has an S_4-pair, say M_1^i and M_2^i. Using Theorem 6.2, we choose the S_4-pair in:

- B'_1, such that the two edges originally incident to x_1 (not $x_1 u_1$) both have frequency 1,
- B'_i, for each $i \in \{2, \ldots, k\}$, such that $\nu_{B'_i}(v_i u_i) = 2$, and
- B'_{k+1}, such that the two edges originally incident to x_{k+1} (not $x_{k+1} v_{k+1}$) both have frequency 1.

Let $M_1 := (\cup_{i=1}^{k+1} M_1^i) \cup (\cup_{j=1}^{k} e_j) - \cup_{l=2}^{k} v_l u_l$, and let $M_2 := (\cup_{i=1}^{k+1} M_2^i) \cup (\cup_{j=1}^{k} e_j) - \cup_{l=2}^{k} v_l u_l$. Then, M_1 and M_2 are an S_4-pair of G. □

Finally, we remark that there exist cubic graphs which admit a perfect matching however do not have an S_4-pair. For example, since the edges u_iv_i in Figure 11 are bridges, then they must be in any perfect matching. Consequently, every pair of perfect matchings do not intersect the edges of the odd cycle T. This shows that it is not possible to extend the S_4-Conjecture to the entire class of cubic graphs.

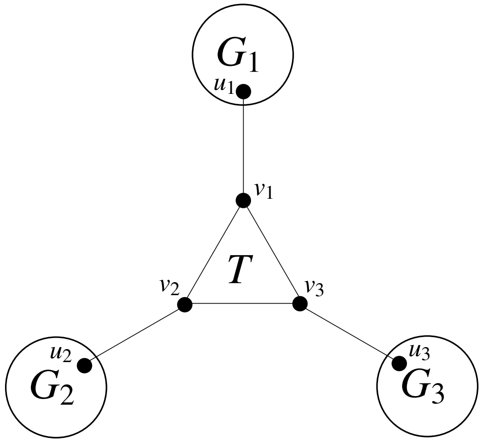

Figure 11: A cubic graph with bridges having no S_4-pair.

7 Remarks and problems

Many problems about the topics presented above remain unsolved: apart from asking if we can solve the Fan-Raspaud Conjecture and the S_4-Conjecture completely, or at least partially for higher oddness, we do not know which are those graphs containing bridges which admit an S_4-pair and we do not know either if the S_4-Conjecture is equivalent to Conjecture 2.3. Here we would like to add some other specific open problems.

For a positive integer k, we define ω_k to be the largest integer such that any graph with oddness at most ω_k, admits k perfect matchings with a bipartite complement. Clearly, for $k = 1$, we have $\omega_1 = 0$, since the existence of a perfect matching of G with a bipartite complement is equivalent to the 3-edge-colourability of G. Moreover, the S_4-Conjecture is equivalent to $\omega_k = \infty$, for $k \geq 2$, but a complete result to this is still elusive. Proposition 5.5 (see also Remark 5.6) gives a lower bound for ω_k and it would be interesting if this lower bound can be significantly improved. We believe that the following problem, weaker than the S_4-Conjecture, is another possible step forward.

Problem 7.1. Prove the existence of a constant k such that every bridgeless cubic graph admits k perfect matchings whose union has a bipartite complement.

It is also known that not every perfect matching can be extended to an FR-triple and neither to a Berge-Fulkerson cover, where the latter is a collection of six perfect matchings which cover the edge set exactly twice. We do not see a way how to produce a similar argument for S_4-pairs and so we also suggest the following problem.

Problem 7.2. Establish whether any perfect matching of a bridgeless cubic graph can be extended to an S_4-pair.

It can be shown that Problem 7.2 is equivalent to saying that given any collection of disjoint odd cycles in a bridgeless cubic graph, then there exists a perfect matching which intersects all the odd cycles in this collection.

ORCID iDs

Giuseppe Mazzuoccolo https://orcid.org/0000-0001-7775-065X
Jean Paul Zerafa https://orcid.org/0000-0002-3159-2980

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