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J. Math. Anal. Appl. 318 (2006) 199–210
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Sequential convergence of regular measures on prehilbert space logics ☆

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Received 25 November 2004
Available online 16 June 2005
Submitted by J.D.M. Wright

Abstract

This paper investigates Nikodym-type and Cafiero-type convergence theorems for regular charges in the general set-up of projection logics of prehilbert spaces. For this aim we also characterize bounded regular charges.

Keywords: Inner product (= prehilbert) space; Splitting subspace; Charge; Regular charge; Exhaustive charge; Nikodym convergence theorem; Cafiero theorem

1. Introduction

The present paper contributes to the study of sequential convergence in 'quantum measure theory' (see [1,3,8,12] and many others). We investigate some measure-theoretic

☆ The authors acknowledge the support of the grant VEGA 2/3163/23 SAV, the Slovak–Italian Project No. 15, the Center of Excellence SAS Physics of Information, and the Science and Technology Assistance Agency under the contract No. APVT-51-032002.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.05.041

properties of the projection logic of an inner product space (see [4,9,13]). Let S be an inner product space. Unless otherwise stated we shall not assume that S is (topologically) complete. On the other hand, we assume that the linear structure of S is defined over either the field of real or complex numbers, or the division ring of quaternions. For any subspace M of S we shall write $\overline{M}$ for the completion of M and we shall denote by $[x]$ the one-dimensional subspace of S generated by the non-zero vector x. Denote by $E(S)$ the set of all the subspaces M of S for which the projection theorem holds, i.e., $E(S) = \{M \subset S\colon\ M \oplus M^{\perp} = S\}$. Observe that $E(S)$ includes $\{0\}$, S and all the complete subspaces of S. When endowed with the order induced by set-theoretic inclusion $\subset$ and with the orthocomplementation $\perp$ the system $E(S)$ is an orthomodular poset—the projection logic of S [9,12,14]. In the particular case when S is a Hilbert space, $E(S)$ is a complete orthomodular lattice—the 'standard' quantum logic $L(H)$ associated with a quantum system [16]. Let us recall that there are various algebraic conditions that when imposed on $E(S)$ imply the completeness of S [9].

A mapping $m : E(S) \to \mathbb{R}$ is said to be a *charge* if

$$m\Big(\bigvee_{i\in I} A_i\Big) = \sum_{i\in I} m(A_i), \tag{1.1}$$

whenever $\{A_i\colon\ i \in I\}$ is a finite family of pairwise orthogonal (i.e., $A_i \perp A_j$ for $i \neq j$) elements of $E(S)$. The charge m is *completely additive* (respectively *σ-additive*) whenever Eq. (1.1) holds for any family (respectively countable family) $\{A_i\colon\ i \in I\}$ of pairwise orthogonal elements of $E(S)$ satisfying that the supremum $\bigvee_{i\in I} A_i$ exists in $E(S)$. A charge m is *bounded* if there exists a positive number K such that $|m(A)| \leqslant K$ for all $A \in E(S)$. We say that a charge m is *regular* if for any positive real number ϵ and $A \in E(S)$ there exists a finite dimensional subspace M of A such that $|m(A) - m(M)| < \epsilon$. A charge m which satisfies $m(M) = 0$ for all finite dimensional subspaces M of S is said to be a *free charge*. A *state* s is a normalized (i.e., $s(S) = 1$) positive charge. When S is a Hilbert space, the set of completely additive charges and the set of regular bounded charges on $E(S)$ coincide [9, Theorem 3.7.7]. Gleason theorem [10] (see also [9]) states that if S is an infinite dimensional Hilbert space and m is a completely additive charge on $E(S)$ (or a regular bounded charge), then there exists a unique Hermitian trace class operator T on S such that $m(M) = \mathrm{tr}(TP_M)$ for every $M \in E(S)$.[1] (Here P_M denotes the projection of S on M.) It is worth observing that the way Gleason theorem is formulated here relies on the significant result of S.V. Dorofeev and A.N. Sherstnev which states that every completely additive charge on $E(S)$ is bounded whenever $\dim S = \infty$ [6]. We also remark that in [4] it was shown that the set of regular charges on $E(S)$ is always larger than the set of completely additive ones.

Let us denote by $\Delta(E(S))$ the set of all charges on $E(S)$ when viewed as a (topological) subspace of $\mathbb{R}^{E(S)}$. (Here $\mathbb{R}^{E(S)}$ is endowed with the topology of pointwise convergence.) We shall denote by $\Omega_r(E(S))$ the linear space (over $\mathbb{R}$) of all regular bounded charges on $E(S)$. The state space of $E(S)$, presently denoted by $\mathcal{S}(E(S))$, is a convex subset of

[1] Gleason theorem was first proved for σ-additive states for the case when S is a separable Hilbert space of dimension at least three [10].

the cube $[0,1]^{E(S)}$. One can easily verify that $\mathcal{S}(E(S))$ is closed in $[0,1]^{E(S)}$ and hence $\mathcal{S}(E(S))$ is a convex compact (topological) space. For any unit vector x in S, the equation $s(A) = \|P_{\overline{A}}x\|^2$ defines a regular state on $E(S)$—the *vector state* associated with x.

In the present article we first extend Gleason theorem to regular bounded charges on the projection logic of any inner product space. Then, we show that in general the set of regular states need not be sequentially closed in the state space of $E(S)$; i.e., there is no Nikodym convergence kind of theorem for the set of regular states on $E(S)$. This is in contrast to the case when S is complete and gives a negative answer to an open problem (see [9, Problem 4.3.15]). In Theorem 3.6 a sufficient condition is given, under which the limit of a pointwise convergent sequence of regular bounded charges on $E(S)$ is regular. We finally put some observations concerning exhaustivity of charges. We exhibit a regular charge that is not exhaustive—this sharpens further the difference between the notion of σ—additivity and regularity. We then go on investigating whether Cafiero theorem is true in the set-up of $E(S)$ when S is not forced to be complete. (It is known that Cafiero theorem is true when S is a Hilbert space.)

2. Gleason theorem revisited

When S is incomplete, $E(S)$ does not admit any completely additive charge [9, Theorem 4.2.3]. However, $E(S)$ always allows for a separating set of regular bounded charges. For any Hermitian trace class operator T defined on $\overline{S}$, consider the map

$$m_T : E(S) \to [0,1], \quad M \mapsto \operatorname{tr}(T P_{\overline{M}}). \tag{2.1}$$

It is clear that Eq. (2.1) defines a bounded charge on $E(S)$. To prove regularity, use is made of the Amemiya–Araki principle of approximating orthogonal vectors in $\overline{S}$ by orthogonal vectors in S. Explicitly, this amounts to the fact that for every finite orthonormal system $\{y_1, y_2, \ldots, y_n\}$ in $\overline{S}$ and $\delta > 0$, there exists a finite orthonormal system $\{x_1, x_2, \ldots, x_n\}$ in S such that $\|y_i - x_i\| < \delta$ (for each $i = 1, 2, \ldots, n$). Let M be in $E(S)$ and $\{y_i\colon i \in I\}$ be an orthonormal basis of $\overline{M}$. There exists a finite subset I_0 of I such that

$$\left|\operatorname{tr}(T P_{\overline{M}}) - \sum_{i \in I_0} \langle T y_i, y_i \rangle\right| < \epsilon/3.$$

Let $K \in \mathbb{N}$ be the cardinality of I_0. There exists a finite orthonormal system $\{x_i\colon i \in I_0\}$ in M such that $\|x_i - y_i\| < \epsilon/(3K\|T\|)$ (for each $i \in I_0$). Then

$$\begin{aligned}
&\left|\operatorname{tr}(T P_{\overline{M}}) - \sum_{i \in I_0} \langle T x_i, x_i \rangle\right| \\
&\quad \leqslant \left|\operatorname{tr}(T P_{\overline{M}}) - \sum_{i \in I_0} \langle T y_i, y_i \rangle\right| + \left|\sum_{i \in I_0} \langle T y_i, y_i \rangle - \sum_{i \in I_0} \langle T y_i, x_i \rangle\right| \\
&\qquad + \left|\sum_{i \in I_0} \langle T y_i, x_i \rangle - \sum_{i \in I_0} \langle T x_i, x_i \rangle\right| \\
&\quad < \epsilon/3 + \epsilon/3 + \epsilon/3,
\end{aligned}$$

i.e., m_T is regular. In the following theorem we show that every regular bounded charge on $E(S)$ arises in this way. It is a proper extension to [9, Theorem 4.3.5]. As one would immediately reckon, the proof builds on Gleason theorem. First, let us prove the following elementary lemma that we will need in the proof of the theorem.

Lemma 2.1. *Let m be a bounded charge on $E(S)$. For any $M \in E(S)$ and $\epsilon > 0$ there exists a finite dimensional subspace M_0 of M such that $|m(N)| < \epsilon$ for every finite dimensional subspace $N \subset M_0^\perp \cap M$.*

Proof. If the statement of Lemma 2.1 is false, a sequence $\{N_i\colon i \in \mathbb{N}\}$ of pairwise orthogonal finite dimensional subspaces of M can be found with the property $|m(N_i)| \geqslant \epsilon$ (for each i). This would lead to a contradiction in view of the fact that m is bounded and that one of the sets $\{i \in \mathbb{N}\colon m(N_i) \geqslant \epsilon\}$ and $\{i \in \mathbb{N}\colon m(N_i) \leqslant -\epsilon\}$ is infinite. □

Theorem 2.2. *Let m be a regular bounded charge on $E(S)$ ($\dim S \geqslant 3$). There exists a unique Hermitian trace class operator T on $\overline{S}$ such that $m(M) = \operatorname{tr}(TP_{\overline{M}})$ for all $M \in E(S)$.*

Proof. Restrict m to $E(N)$, where N is any subspace of S with a finite dimension greater than 2. By Gleason theorem, there exists a bounded Hermitian conjugate-bilinear form t_N on $N \times N$ such that $m([x]) = t_N(x,x)$ for every unit vector x of N. Define a conjugate-bilinear form t on $S \times S$ as follows: $t(x,y) = t_N(x,y)$, where N is any subspace of S with a finite dimension greater than 2 that contains x and y. In view of the polarization identity of Hermitian conjugate-bilinear forms, it is clear that the definition of t depends only on x and y, i.e., t is well defined. Since m is bounded, t is also bounded and therefore t extends continuously to a unique Hermitian conjugate-bilinear form, again denoted by t, on $\overline{S} \times \overline{S}$. Let T denote the unique Hermitian operator on $\overline{S}$ such that $t(x,y) = \langle Tx, y\rangle$ for all $x, y \in \overline{S}$.

We claim that T is a trace class operator. For this we need to exploit once again the 'Amemiya–Araki principle' described before. Let $\{y_i\colon i \in I\}$ be any orthonormal basis of $\overline{S}$. Let I_0 be a finite subset of I. For any pre-assigned positive number ϵ we can find a finite orthonormal system $\{x_i\colon i \in I_0\}$ in S such that

$$\left|\sum_{i\in I_0}\langle Ty_i, y_i\rangle - \sum_{i\in I_0}\langle Tx_i, x_i\rangle\right| < \epsilon.$$

This implies that

$$\begin{aligned}\left|\sum_{i\in I_0}\langle Ty_i, y_i\rangle\right| &\leqslant \left|\sum_{i\in I_0}\langle Tx_i, x_i\rangle\right| + \left|\sum_{i\in I_0}\langle Ty_i, y_i\rangle - \sum_{i\in I_0}\langle Tx_i, x_i\rangle\right| \\ &< \left|m\big(\operatorname{span}\{x_i\colon i \in I_0\}\big)\right| + \epsilon.\end{aligned}$$

Since m is bounded, it follows that $\sum_{i\in I}\langle Ty_i, y_i\rangle$ is summable, i.e., T is a trace class operator.

Remains to be shown that $m(M) = \operatorname{tr}(TP_{\overline{M}})$ for all $M \in E(S)$. This is evident in the case when $\dim M < \infty$. For every $M \in E(S)$ and $\epsilon > 0$ there exists a finite dimensional subspace M_0 of M such that $|m(N)| < \epsilon$ for every finite dimensional subspace N in

$M_0^\perp \cap M$ (Lemma 2.1). Since m is regular, it follows that $|m(M_0^\perp \cap M)| \leqslant \epsilon$. In addition $|\mathrm{tr}(T P_{\overline{M_0^\perp \cap M}})| \leqslant \epsilon$ since the charge m_T defined via Eq. (2.1) is regular. This implies that

$$\begin{aligned}&\bigl|m(M) - \mathrm{tr}(T P_{\overline{M}})\bigr| \\ &\quad\leqslant \bigl|m(M_0) - \mathrm{tr}(T P_{M_0})\bigr| + \bigl|m\bigl(M_0^\perp \cap M\bigr) - \mathrm{tr}\bigl(T P_{\overline{M_0^\perp \cap M}}\bigr)\bigr| \leqslant 2\epsilon. \qquad \square\end{aligned}$$

From Theorem 2.2 it follows that every regular bounded charge on $E(S)$ is the restriction of a (unique) completely additive charge on $E(\overline{S})$. As a by-product of Theorem 2.2 we have the following.

Corollary 2.3. *Every bounded charge on $E(S)$ (*$\dim S \geqslant 3$*) can be (uniquely) expressed as a sum of a regular bounded charge and a free bounded charge. In addition, every regular bounded charge can be expressed as a difference of two positive regular charges.*

The following proposition will be useful in Section 4. We recall that by a convex subset of $\mathbb{R}$ we mean a set X satisfying that if $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, then $x = \lambda x_1 + (1 - \lambda)x_2 \in X$.

Proposition 2.4. *Let S be an infinite dimensional inner product space and let m be a regular bounded charge on $E(S)$. Then* $\mathrm{Range}(m)$ *is a convex subset containing (λ, μ), where $\lambda = \inf\{m(A)\colon A \in E(S)\}$ and $\mu = \sup\{m(A)\colon A \in E(S)\}$.*

Proof. By Theorem 2.2, the charge m is determined by some Hermitian trace class operator T on $\overline{S}$, i.e., $m(M) = \mathrm{tr}(T P_{\overline{M}})$, for every $M \in E(S)$. Without loss of generality we can assume $\mu > 0$. We show that $[0, \mu) \subset \mathrm{Range}(m)$. For any $\epsilon > 0$ there exist finite dimensional subspaces A and B of S such that $\dim A = \dim B$, $A \perp B$, $m(A) > \mu - \epsilon$ and $m(B) < \epsilon$. Let $\{a_i\colon i \leqslant n\}$ and $\{b_i\colon i \leqslant n\}$ be orthonormal bases of A and B, respectively, and let $c_i = \cos\phi\, a_i + \sin\phi\, b_i$, where $\phi \in [0, \pi/2]$. Set $C_\phi = \mathrm{span}\{c_i\colon i \leqslant n\}$ and compute

$$\begin{aligned}m(C_\phi) &= \mathrm{tr}(T P_{C_\phi}) = \sum_{i \leqslant n} \langle T c_i, c_i\rangle \\ &= \cos^2\phi\, m(A) + \sin^2\phi\, m(B) + \sin 2\phi\, \mathrm{Re}\biggl(\sum_{i \leqslant n} \langle T a_i, b_i\rangle\biggr),\end{aligned}$$

which implies that $[m(B), m(A)] \subset \mathrm{Range}(m)$, and thus, the range of m contains $[0, \mu)$.
In exactly the same way, it can be shown that $(\lambda, 0] \subset \mathrm{Range}(m)$. $\square$

3. Nikodym convergence theorem

It is not difficult to verify that $\Omega_r(E(S))$ is never closed in $\Delta(E(S))$. Indeed, let $\{x_i\colon i \in \mathbb{N}\}$ be an orthonormal system in $\overline{S}$. For each $i \in \mathbb{N}$, let s_i be the vector state on $E(S)$ associated with the vector x_i, i.e., $s_i(M) = \|P_{\overline{M}} x_i\|^2$ for every $M \in E(S)$. The state space $\mathcal{S}(E(S))$ is compact and therefore $\{s_i\colon i \in \mathbb{N}\}$ has a convergent subnet, say $\{s_k\colon k \in D\}$

(D is a directed countable set). Let s denote the limit-state to which the subnet $\{s_k\colon k \in D\}$ converges. For any unit vector u of S, we have

$$s([u]) = \lim_k \{s_k([u])\colon k \in D\} = \lim_k \{|\langle u, x_k\rangle|^2\colon k \in D\} = 0.$$

This implies that s is a free state and is certainly not regular.

Is $\Omega_r(E(S))$ sequentially closed in $\Delta(E(S))$? That is, if given a sequence of regular bounded charges $\{m_i\colon i \in \mathbb{N}\}$ on $E(S)$, converging pointwise to some other charge m—does it follow that m is regular and bounded? It is worthwhile observing that this question is the natural analogue to the one which is answered in Nikodym convergence theorem for measures on σ-fields [7]. We recall that Nikodym convergence theorem asserts that the pointwise limit of a sequence of σ-additive measures $\{\mu_i\colon i \in \mathbb{N}\}$ on a σ-field Σ is a σ-additive measure and the set $\{\mu_i\colon i \in \mathbb{N}\}$ is uniformly countably additive (i.e., for any disjoint sequence $\{X_k\colon k \in \mathbb{N}\}$ in Σ, and for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that $|\sum_{k \geqslant K} \mu_i(X_k)| < \epsilon$ for each μ_i). We further remark that convergence theorems of completely additive charges on $E(S)$, in the particular case that S is a Hilbert space, were originally studied by R. Jajte [13]. We say that a sequence $\{m_i\colon i \in \mathbb{N}\}$ of regular bounded charges on $E(S)$ is *uniformly regular* if for any $M \in E(S)$ and for any $\epsilon > 0$ there exists a finite dimensional subspace M_0 of M such that $|m_i(M) - m_i(M_0)| < \epsilon$ for each m_i. The following theorem is Nikodym convergence theorem in the $E(S)$-set-up, where S is a Hilbert space. This follows directly from [9, Theorem 3.10.1] since when S is complete, $\Omega_r(E(S))$ coincides precisely with the set of completely additive charges.

Theorem 3.1. *Let S be a Hilbert space and $\{m_i\colon i \in \mathbb{N}\}$ be a sequence of regular bounded charges on $E(S)$ converging pointwise to some charge $m \in \Delta(E(S))$. Then m is bounded and regular. (This means that $\Omega_r(E(S))$ is sequentially closed in $\Delta(E(S))$.) Moreover, the sequence $\{m_i\colon i \in \mathbb{N}\}$ is uniformly regular.*

We shall now investigate the problem for the case when S is incomplete. As we shall see, this case does not allow for such a clear answer. In the following theorem it is shown that the set of regular bounded charges on $E(S)$ need not be sequentially closed in $\Delta(E(S))$ for an incomplete S—not even if we restrict ourselves to states. For incomplete inner product spaces, the projection logic can be extremely poor and consequently the pointwise topology on $\Delta(E(S))$ can be very coarse. The proof relies on the construction of P. Pták and H. Weber [14]. This provides a negative answer to the problem posed by A. Dvurečenskij (see [9, Problem 4.3.15]).

Theorem 3.2. *Let H be an infinite dimensional, separable Hilbert space. There exists a dense hyperplane S of H such that the set of regular states on $E(S)$ is not sequentially closed in $\mathcal{S}(E(S))$.*

Proof. In [14, Theorem 2.2.8] a dense hyperplane S of H was constructed such that $E(S)$ consists merely of the finite dimensional subspaces and their respective orthogonal complements.

Let $\{e_i\colon i \in \mathbb{N}\}$ be an orthonormal basis of S and let $s_i : E(S) \to [0, 1]$, $M \mapsto \|P_{\overline{M}} e_i\|^2$ ($i \in \mathbb{N}$) be the associated vector states on $E(S)$. We claim that $\{s_i\colon i \in \mathbb{N}\}$ converges (pointwise) to the unique two-valued state s in $\mathcal{S}(E(S))$ assigning 0 to each complete subspace and 1 to every cocomplete subspace of S. Let u be a unit vector of S. Then $\lim_i s_i([u]) = \lim_i |\langle u, e_i\rangle|^2 = 0$. This implies that $\lim_i s_i(M) = 0$ for every finite dimensional subspace M of S, i.e., s is the unique free state in $\mathcal{S}(E(S))$. □

Nikodym convergence theorem in the $E(S)$-set-up, when S is a Hilbert space, is a consequence of the σ-orthocompleteness of $E(S)$ and the classical Nikodym theorem. (When S is Hilbert, $E(S)$ is a complete lattice.) However, it is well known that $E(S)$ is σ-orthocomplete if and only if S is a Hilbert space (see [9] and many others). In spite of this, in the following theorem we show that when S has a countable linear dimension then the limit of any sequence of regular bounded charges on $E(S)$ is regular.

Theorem 3.3. *Let S be an inner product space with a countable linear dimension. If $\{m_i\colon i \in \mathbb{N}\}$ is a sequence of regular bounded charges on $E(S)$ converging pointwise to some charge $m \in \Delta(E(S))$, then m is regular. Moreover, the sequence $\{m_i\colon i \in \mathbb{N}\}$ is uniformly regular.*

Proof. If $\dim S < \infty$, then S is a Hilbert space and result follows from Theorem 3.1. So assume that $\dim S = \infty$. It is not difficult to verify that every two infinite dimensional inner product spaces having a countable linear dimension are unitarily equivalent (see, for example, [14, Proposition 2.1.1]).

Fix an arbitrary element M in $E(S)$ with infinite dimension. Let $\{u_k\colon k \in \mathbb{N}\}$ be an orthonormal linear basis of M. Let $T_1, T_2, \ldots$ be the Hermitian trace class operators defined on $\overline{S}$ associated (in view of Theorem 2.2) with the regular bounded charges $m_1, m_2, \ldots$. For each $i \in \mathbb{N}$ define

$$\mu_i : 2^{\mathbb{N}} \to \mathbb{R}, \quad J \mapsto \operatorname{tr}(T_i P_J),$$

where P_J denotes the projection of $\overline{S}$ on the subspace $\overline{\operatorname{span}}\{u_k\colon k \in J\}$. It is clear that each μ_i is a σ-additive measure on $2^{\mathbb{N}}$, and $\{\mu_i\colon i \in \mathbb{N}\}$ converges pointwise on $2^{\mathbb{N}}$ since for each $J \subset \mathbb{N}$, the subspace $\operatorname{span}\{u_k\colon k \in J\}$ is an element of $E(S)$. Put

$$\mu(J) = \lim_i \mu_i(J).$$

By the classical Nikodym convergence theorem, it follows that μ is σ-additive and therefore

$$m(M) = \mu(\mathbb{N}) = \sum_{k\in\mathbb{N}} \mu(\{k\}) = \sum_{k\in\mathbb{N}} m([u_k]).$$

This implies that m is regular. Since the sequence $\{\mu_i\colon i \in \mathbb{N}\}$ is uniformly countably additive, it follows that $\{m_i\colon i \in \mathbb{N}\}$ is uniformly regular on $E(S)$. The proof of Theorem 3.3 is complete. □

Observe that in this case we are not in a position to guarantee that the limit-charge m is bounded. (We recall that $E(S)$ admits regular charges that are unbounded.) If we restrict ourselves to states, then we have the following Nikodym convergence type of theorem.

Theorem 3.4. *Let S be an inner product space with a countable linear dimension. If $\{s_i\colon i \in \mathbb{N}\}$ is a sequence of regular states on $E(S)$ converging pointwise to some state $s \in \mathcal{S}(E(S))$, then s is regular. Moreover, the sequence $\{s_i\colon i \in \mathbb{N}\}$ is uniformly regular.*

Finally, we give a sufficient condition which is independent of S, under which the limit of a convergent sequence of regular bounded charges on $E(S)$ is regular and bounded. First of all, observe that the limit of a convergent, uniformly regular sequence of regular charges on $E(S)$, is necessarily regular. The following proposition will be used in the proof of Theorem 3.6. It was first proved by R. Jajte [13]. Here we give a simpler proof without using Shur theorem.

Proposition 3.5 *(Jajte). Let H be a Hilbert space. The sequence $\{m_i\colon i \in \mathbb{N}\}$ of regular bounded charges on $E(H)$ converges pointwise if and only if the following two conditions hold*:

(i) $\{m_i([x])\colon i \in \mathbb{N}\}$ *converges for each unit vector x of H*;
(ii) *for any orthonormal sequence $\{x_k\colon k \in \mathbb{N}\}$ in H, the series $\sum_k m_i([x_k])$ converges uniformly with respect to i.*

In such case, the limit charge m is regular and bounded. The sequence $\{m_i\colon i \in \mathbb{N}\}$ is uniformly regular.

Proof. The 'only if' part of the statement follows directly from Nikodym convergence theorem (Theorem 3.1). Suppose that (i) and (ii) are true. Fix Y in $E(H)$ and let $\{y_n\}$ be an orthonormal basis of Y. In view of (ii), for any given positive ϵ, there exists a positive integer N, such that for all $i \in \mathbb{N}$, we have

$$\left|m_i(\overline{M}) - m_i\left(\operatorname{span}\{y_n\colon n < N\}\right)\right| < \epsilon/3.$$

Let $I \in \mathbb{N}$ such that, for any $p > q \geqslant I$, we have

$$\left|m_p\left(\operatorname{span}\{y_n\colon n < N\}\right) - m_q\left(\operatorname{span}\{y_n\colon n < N\}\right)\right| < \epsilon/3.$$

(This can be done in view of condition (i).) Then, for any $p > q \geqslant I$, we have

$$\begin{aligned}\left|m_p(Y) - m_q(Y)\right| &\leqslant \left|m_p(Y) - m_p\left(\operatorname{span}\{y_n\colon n < N\}\right)\right| \\ &\quad + \left|m_p\left(\operatorname{span}\{y_n\colon n < N\}\right) - m_q\left(\operatorname{span}\{y_n\colon n < N\}\right)\right| \\ &\quad + \left|m_q(Y) - m_q\left(\operatorname{span}\{y_n\colon n < N\}\right)\right| \\ &< \epsilon.\end{aligned}$$

This implies that $\lim_i m_i(Y)$ exists for every $Y \in E(H)$. □

Theorem 3.6. *Let $\{m_i\colon i \in \mathbb{N}\}$ be a sequence of regular bounded charges on $E(S)$ converging pointwise to some charge $m \in \Delta(E(S))$. Suppose that there exists $m_0 \in \Omega_r(E(S))$ such that $|m_i([x])| \leqslant |m_0([x])|$ for each $i \in \mathbb{N}$ and for every unit vector x of S. Then $m \in \Omega_r(E(S))$. Moreover, the sequence $\{m_i\colon i \in \mathbb{N}\}$ is uniformly regular.*

Proof. Denote by T_i $(i = 0, 1, \ldots)$ the corresponding Hermitian trace class operator on $\overline{S}$ associated with m_i, i.e., $m_i(M) = \mathrm{tr}(T_i P_{\overline{M}})$ for all $M \in E(S)$. First we prove that the sequence $\{\langle T_i z, z\rangle \colon i \in \mathbb{N}\}$ converges for each unit vector z of $\overline{S}$. From the following inequalities

$$\|T_i\| = \sup_{x\in S,\ \|x\|=1} \bigl\{\bigl|\langle T_i x, x\rangle\bigr|\bigr\} \leqslant \sup_{x\in S,\ \|x\|=1} \bigl\{\langle T_0 x, x\rangle\bigr\} = \|T_0\|,$$

it follows that $\|T_i\| \leqslant \|T_0\|$ for all $i = 1, 2, \ldots.$ Fix a unit vector z in $\overline{S}$ and let $\epsilon > 0$ be arbitrary. There exists a unit vector x of S such that $\|z - x\| < (\|T_0\|\epsilon)/6$. In addition, we are guaranteed that for some $I \in \mathbb{N}$, $|\langle (T_p - T_q)x, x\rangle| < \epsilon/3$ for all $p > q \geqslant I$. Thus, for $p > q \geqslant I$, we have

$$\begin{aligned}
&\bigl|\bigl\langle (T_p - T_q)z, z\bigr\rangle\bigr| \\
&\quad \leqslant \bigl|\bigl\langle (T_p - T_q)z - x, z\bigr\rangle\bigr| + \bigl|\bigl\langle (T_p - T_q)x, z - x\bigr\rangle\bigr| + \bigl|\bigl\langle (T_p - T_q)x, x\bigr\rangle\bigr| \\
&\quad < 4\|T_0\| \cdot \|z - x\| + \epsilon/3 < \epsilon,
\end{aligned}$$

and therefore $\{\langle T_i z, z\rangle \colon i \in \mathbb{N}\}$ is convergent.

Let $\{z_k \colon k \in \mathbb{N}\}$ be an arbitrary orthonormal system in $\overline{S}$. Since $\sum_{k\in\mathbb{N}} |\langle T_0 z_k, z_k\rangle| < \infty$, there exists a positive integer K such that

$$\sum_{k\geqslant K} \bigl|\langle T_i z_k, z_k\rangle\bigr| \leqslant \sum_{k\geqslant K} \bigl|\langle T_0 z_k, z_k\rangle\bigr| < \delta,$$

for any $\delta > 0$ and $i \in \mathbb{N}$. If we denote by $\tilde{m}_i$ the charge on $E(\overline{S})$ defined by $\tilde{m}_i(N) = \mathrm{tr}(T_i P_N)$, for $N \in E(\overline{S})$, then it is clear that the sequence $\{\tilde{m}_i \colon i \in \mathbb{N}\}$ satisfies conditions (i) and (ii) of Proposition 3.5. This implies that $\{\tilde{m}_i \colon i \in \mathbb{N}\}$ converges pointwise on $E(\overline{S})$ to some $\tilde{m}$ in $\Omega_r(E(\overline{S}))$. It is clear that m is the restriction of $\tilde{m}$, i.e., $m(M) = \tilde{m}(\overline{M})$ for all $M \in E(S)$. Since $\tilde{m}$ is determined by some Hermitian trace class operator, it follows that m is in $\Omega_r(E(S))$. The uniform regularity of $\{m_i \colon i \in \mathbb{N}\}$ follows from that of $\{\tilde{m}_i \colon i \in \mathbb{N}\}$. Indeed, for any $M \in E(S)$ and $\epsilon > 0$ there exists a finite dimensional subspace M_0 of $\overline{M}$ such that $|\tilde{m}_i(\overline{M}) - \tilde{m}_i(M_0)| < \epsilon$ for each i. Using the Amemiya–Araki technique of approximating a finite orthonormal system in $\overline{S}$ by a finite orthonormal system in S (as was described in the beginning of this paper), a finite dimensional subspace N of M can be found with the property that $|m_i(M) - m_i(N)| \leqslant \epsilon$ for each i. □

At the end of this section we formulate the following open problem.

Problem 3.7. *Suppose that a sequence* $\{m_i \colon i \in \mathbb{N}\}$ *of regular bounded charges on* $E(S)$ *converges to some charge* m. *Is* m *bounded*? *We all know that this is the case when the inner product space is Hilbert*; *but how is it in the general case*?

4. Exhaustive charges on $E(S)$

A charge $m \in \Delta(E(S))$ is said to be *exhaustive* if $\lim_j m(M_j) = 0$ for every sequence $\{M_j \colon j \in \mathbb{N}\}$ of pairwise orthogonal subspaces of $E(S)$. Clearly, every bounded charge on

$E(S)$ is exhaustive. We will exhibit a regular (unbounded) charge that is not exhaustive. The ideas follow those in [4].

First we define a Hamel discontinuous function on $\mathbb{R}$ as follows. (See also [11].) Let $\mathfrak{B} = \{x_s\colon s \in \Sigma\}$ be a Hamel basis in $\mathbb{R}$ over the field of rational numbers. It is harmless to assume that $x_s > 0$ for each $s \in \Sigma$. Fix an element $x_{s_0} \in \mathfrak{B}$. Then every real number $x \in \mathbb{R}$ can be uniquely expressed in the form

$$x = \beta_{s_0} x_{s_0} + \sum_{s \in \sigma} \beta_s x_s, \tag{4.1}$$

where σ is a finite subset of $\Sigma \setminus \{s_0\}$ and β's are rational numbers. We define a Hamel discontinuous function $\phi : \mathbb{R} \to \mathbb{Q}$ by $\phi(x) = \beta_{s_0}$ whenever $x \in \mathbb{R}$ is of the form (4.1).

Let s be any regular state on $E(S)$. We claim to show that $\phi \circ s$ is a regular charge. Let $\epsilon > 0$ and $A \in E(S)$ be given. If $\phi(s(A)) = 0$, we take $M = \{0\}$, which yields $|\phi(s(A)) - \phi(s(M))| < \epsilon$. So let $0 \neq s(A) = \beta_{s_0} x_{s_0} + \sum_{s \in \sigma} \beta_s x_s$, where $\beta_{s_0} \neq 0$. There is an integer $n \geqslant 1$ such that $1/n < \epsilon$ and $x_{s_0}/n < s(A)$. Then $0 < (\beta_{s_0} - 1/n) x_{s_0} + \sum_{s \in \sigma} \beta_s x_s < s(A)$. By [4, Corollary 3.7], there is a finite dimensional subspace M of A such that $s(M) = (\beta_{s_0} - 1/n) x_{s_0} + \sum_{s \in \sigma} \beta_s x_s$. Hence, $|\phi(s(A)) - \phi(s(M))| = 1/n < \epsilon$ which proves that $\phi \circ s$ is a regular charge on $E(S)$.

Since Range($\phi \circ s$) is contained in the set of rational numbers, it follows (by Proposition 2.4) that $\phi \circ s$ is unbounded on $E(S)$. Since $\phi \circ s$ is regular, there exists a finite dimensional subspace M_1 such that $|\phi \circ s(M_1)| > 1$. Let $S_1 = M_1^{\perp}$. Then $\dim S_1$ is infinite and the restriction of $\phi \circ s$ on $E(S_1)$ is regular and (again by Proposition 2.4) unbounded. Hence, we can find a finite dimensional subspace M_2 in $E(S_1)$ such that $|\phi \circ s(M_2)| > 2$. We can keep on repeating this and get an infinite sequence $\{M_i\colon i \in \mathbb{N}\}$ of pairwise orthogonal finite dimensional subspaces of S such that $|\phi \circ s(M_i)| > i$, i.e., $\phi \circ s$ is not exhaustive.

A sequence $\{m_i\colon i \in \mathbb{N}\}$ of charges on $E(S)$ is *uniformly exhaustive* if $\lim_j m_i(M_j) = 0$ uniformly in $\{m_i\colon i \in \mathbb{N}\}$ for every sequence $\{M_j\colon j \in \mathbb{N}\}$ of pairwise orthogonal splitting subspaces of S. The theorem of Cafiero gives a sufficient condition for a sequence of finitely additive exhaustive set functions defined on some σ-algebra Σ to be uniformly exhaustive (see, for example, [5, Chapter 4, Section 2.7]). It is known that Cafiero theorem holds when Σ is replaced with $E(S)$ for the case when S is a Hilbert space. In fact, the following theorem was proved in [2]. (Refer to [2] for necessary notions.)

Theorem 4.1 *(Cafiero). Let L be an orthomodular lattice with the subsequential interpolation property*[2] *and $\{\mu_n\colon n \in \mathbb{N}\}$ a sequence of exhaustive* (*real*) *measures on L. Then*

[2] An orthomodular lattice L has the subsequential interpolation property if for every countable orthogonal set K in L and every infinite subset $K_0 \subset K$ there exist $b \in L$ and an infinite subset $H \subset K_0$ such that $a \leqslant b$ for all $a \in H$ and $a \leqslant b^{\perp}$ for all $a \in K \setminus H$. Clearly, if L is a σ-complete lattice then L enjoys the subsequential interpolation property. Thus, Cafiero theorem is true in the set-up of the projection lattice of a Hilbert space. However, it follows very easily from [15] that $E(S)$ enjoys the subsequential interpolation property if and only if S is a Hilbert space.

$\{\mu_n\colon n \in \mathbb{N}\}$ *is uniformly exhaustive if and only if for every orthogonal sequence* $\{a_i\colon i \in \mathbb{N}\}$ *in* L *and for every* $\epsilon > 0$ *there exist* $N, I \in \mathbb{N}$ *such that*

$$\left|\mu_n(a_I)\right| < \epsilon \quad \text{for all } n \geqslant N. \tag{4.2}$$

We show that the same need not be true if we consider incomplete prehilbert spaces. Indeed, we show that Cafiero theorem does not hold for $E(S)$ when we take S to be the dense hyperplane considered in Theorem 3.2. We recall that in this particular case $E(S)$ consists merely of the finite/co-finite subspaces of S and therefore any non-trivial orthogonal sequence in $E(S)$ must consist of finite dimensional subspaces. Let $\{s_i\colon i \in \mathbb{N}\}$ be the sequence of vector states on $E(S)$ corresponding to an orthonormal basis $\{e_i\colon i \in \mathbb{N}\}$ of S. Clearly, each s_i is exhaustive. Let $\epsilon > 0$ be given. If A is a finite dimensional subspace of S and $\{x_k\colon 1 \leqslant k \leqslant n\}$ is an orthonormal basis of A, then there exists $N \in \mathbb{N}$ such that $|\langle x_k, e_i\rangle|^2 < \epsilon/n$ for all $k = 1, 2, \ldots, n$ and for all $i \geqslant N$. Thus it follows that $\{s_i\colon i \in \mathbb{N}\}$ satisfy condition (4.2) of Cafiero theorem. To show the $\{s_i\colon i \in \mathbb{N}\}$ is not uniformly exhaustive consider the orthogonal sequence $\{[e_j]\colon j \in \mathbb{N}\}$ of the rays spanned by the orthonormal basis of S. It is clear that there is no $J \in \mathbb{N}$ satisfying that $s_i([e_j]) < 1/2$ for each $j \geqslant J$ and for all $i \in \mathbb{N}$.

Acknowledgment

The authors are indebted to the referee for his valuable remarks which improved the readability of the paper.

References

[1] T.A. Cooke, The Nikodym–Hahn–Vitali–Saks theorem for states on a quantum logic, in: A.R. Marlow (Ed.), Mathematical Foundations of Quantum Theory, Academic Press, London, 1978, pp. 275–285.
[2] A.B. D'Andrea, P. de Lucia, The Brooks–Jewett theorem on an orthomodular lattice, J. Math. Anal. Appl. 154 (1991) 507–522.
[3] A.B. D'Andrea, P. de Lucia, P. Morales, The Lebesgue decomposition and the Nikodym convergence theorem on an orthomodular poset, Atti Sem. Mat. Fis. Univ. Modena 39 (1991) 137–158.
[4] E. Chetcuti, A. Dvurečenskij, Measures on the splitting subspaces of an inner product space, Internat. J. Theoret. Phys. 43 (2004) 369–384.
[5] P. de Lucia, E. Pap, Convergence theorems for set functions, in: E. Pap (Ed.), Handbook of Measure Theory, vol. I, North-Holland, Amsterdam, 2002, pp. 125–178.
[6] S.V. Dorofeev, A.N. Sherstnev, Frame-type functions and their applications, Izv. Vuzov Matem. 4 (1990) 23–29 (in Russian).
[7] N. Dunford, J. Schwartz, Linear Operators, vols. 1, 2, Wiley, New York, 1957.
[8] A. Dvurečenskij, Signed states on a logic, Math. Slovaca 28 (1978) 33–40.
[9] A. Dvurečenskij, Gleason's Theorem and Its Applications, Kluwer Academic/Ister Science Press, Dordrecht/Bratislava, 1992.
[10] A.M. Gleason, Measures on the closed subspaces of a Hilbert space, J. Math. Mech. 6 (1957) 885–893.
[11] G. Hamel, Eine Basis allen Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x + y) = f(x) + f(y)$, Math. Ann. 60 (1905) 459–462.
[12] J. Hamhalter, Quantum Measure Theory, Kluwer Academic, Dordrecht, 2003.
[13] R. Jajte, On convergence of Gleason measures, Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys. 20 (1972) 211–214.

[14] P. Pták, H. Weber, Lattice properties of subspace families of inner product spaces, Proc. Amer. Math. Soc. 129 (2000) 2111–2117.
[15] P. Pták, H. Weber, Order properties of splitting subspaces in an inner product space, Math. Slovaca 54 (2004) 119–126.
[16] V.S. Varadarajan, Geometry of Quantum Theory, Springer-Verlag, New York, 1985.