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The infinite finite

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Science | Entertainment

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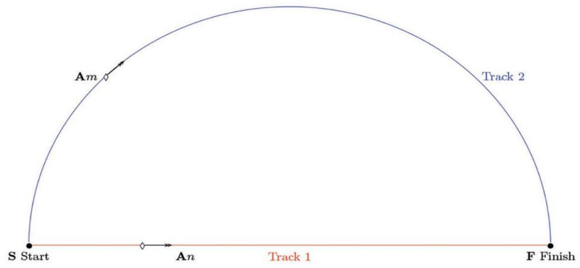

The ants Am and An are let go simultaneously at the starting point S to walk steadily with their respective constant speeds, along Track 1 and Track 2, respectively, to reach the finish point F.

In mathematics, real numbers are everywhere; but are they so real?

Let me propose the following hypothetical game. The host has several point objects that can move with constant speeds; I like to imagine these as identical little ants, labelled say A1, A2, A3, and so on. It is known that the speed with which A2 walks is double that of A1, the speed of A3 is triple the speed of A1, and, in general, the speed of ant number k equals k times the speed of A1. Consider the two tracks labelled by Track 1 and Track 2 in the figure shown, joining the Starting point S to the Finish point F. Track 1 is straight, and Track 2 is semi-circular. We can suppose that the length of the paths is enormously bigger than the size of the ants. The game proceeds as follows: One by one, every player is asked to choose two ants, and release them simultaneously from S so that they reach F through the two separate paths, walking steadily with their respective constant speed. The host takes record of the time lag between the arrival of the two ants. The winner is the player whose pair of ants arrive at F with the smallest time difference.

Is there a choice that secures a win? Can a suitable pair of ants be chosen to guarantee a simultaneous arrival?

The reader must have realised that all the game is about is to try to find a 'best' approximation to the number $\pi/2$ with numbers of the form m/n, with m and n being whole numbers. The answer to the second question is a straight no! The answer to the first depends on whether we are ready to 'embrace the infinite as a real'. With a finite number of ants, a simple calculation easily yields the best choice, that is, the closest approximation to the ratio of the lengths of the tracks. Finiteness, however, limits the precision of our describing of the ratio $\pi/2$. If the host has an infinite number of ants, there is no such closest approximation; given any pair, one can always find another one which gives a smaller time lag, and yet another which is even better, and so on.

In this case, there is no limit on how small one can make the time lag; in the limit, it gets equal to zero! In other words, an infinite number of ants is required if we truly are to describe the finite(!) length of the semicircle up to an indeterminate level of accuracy.

Emanuel Chetcuti is an associate professor at the Department of Mathematics within the Faculty of Science of the University of Malta. In 2006, he was awarded the Birkhoff-von Neumann prize by the International Quantum Structures Association.

Sound bites

• Gleason's Theorem has a fundamental role in the Hilbert space model for quantum mechanics. The heart of the theorem lies in the treatment of the three-dimensional case: Imagine a sphere in three dimensions. Any number of points on the sphere are said to be in orthogonal position when the line segments joining any two of the points to the centre of the sphere are perpendicular. The maximum number of points on the sphere that can be in orthogonal position is three. The theorem describes all possible ways that one can assign a value, ranging between 0 and 1, to every point on the sphere, so that the values assigned to every maximal number of points that are in orthogonal position always add up to one.

• Now, imagine a solid ball. Any number of points of the ball are said to be in standard position if the distance between any two points equals the radius of the ball. The maximum number of points that can be in standard position is four. In a 2015 paper (Equilateral weights on the unit ball of Rn, Real Analysis Exchange, 40(1), 37-51), E. Chetcuti (the author of this page) and J. Muscat (also a member of the Department of Mathematics at the University of Malta) proved that there is only one way of assigning a value to every point in the ball so that the sum of the values assigned to every maximal number of points that are in standard position is 1. It can only be done in the trivial way of assigning the value ¼ to every point.

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Did you know?

• The language of calculus, that appears everywhere in modern science and technology, could not have been developed without the assumption of the continuity of the continuum (the real line), i.e. without the presupposition of the existence of real numbers such as pi, the square root of 2 and many others.

• A real number is an infinite sequence of whole numbers. So, do real numbers exist? Generally, mathematicians do not really query the existence of infinite sets and prove the reals' existence from axioms.

• In mathematics, there is an infinity of infinites. For example, there are more points on the continuum than there are counting numbers, and beyond the continuum lies an interminable progression of evermore enormous, indeed all endless, entities.

For more trivia see: www.um.edu.mt/think