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Isolation of Regular Graphs and *k*-Chromatic Graphs

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Abstract. Given a set $\mathcal{F}$ of graphs, we call a copy of a graph in $\mathcal{F}$ an $\mathcal{F}$-graph. The $\mathcal{F}$-isolation number of a graph G, denoted by $\iota(G, \mathcal{F})$, is the size of a smallest set D of vertices of G such that the closed neighborhood of D intersects the vertex sets of the $\mathcal{F}$-graphs contained by G (equivalently, $G - N[D]$ contains no $\mathcal{F}$-graph). Thus, $\iota(G, \{K_1\})$ is the domination number of G. For any integer $k \geq 1$, let $\mathcal{F}_{1,k}$ be the set of regular graphs of degree at least $k-1$, let $\mathcal{F}_{2,k}$ be the set of graphs whose chromatic number is at least k, and let $\mathcal{F}_{3,k}$ be the union of $\mathcal{F}_{1,k}$ and $\mathcal{F}_{2,k}$. Thus, k-cliques are members of both $\mathcal{F}_{1,k}$ and $\mathcal{F}_{2,k}$. We prove that for each $i \in \{1, 2, 3\}$, $\frac{m+1}{\binom{k}{2}+2}$ is a best possible upper bound on $\iota(G, \mathcal{F}_{i,k})$ for connected m-edge graphs G that are not k-cliques. The bound is attained by infinitely many (non-isomorphic) graphs. The proof of the bound depends on determining the graphs attaining the bound. This appears to be a new feature in the literature on isolation. Among the result's consequences are a sharp bound of Fenech, Kaemawichanurat, and the present author on the k-clique isolation number and a sharp bound on the cycle isolation number.

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1. Introduction

For standard terminology in graph theory, we refer the reader to [25]. Most of the terminology used in this paper is defined in [1].

The set of positive integers is denoted by $\mathbb{N}$. For $n \in \{0\} \cup \mathbb{N}$, $[n]$ denotes the set $\{i \in \mathbb{N} \colon i \leq n\}$. Note that $[0]$ is the empty set $\emptyset$. Arbitrary sets and graphs are taken to be finite. For a non-empty set X, $\binom{X}{2}$ denotes the set of 2-element subsets of X.

Every graph G is taken to be *simple*, meaning that its vertex set $V(G)$ and edge set $E(G)$ satisfy $E(G) \subseteq \binom{V(G)}{2}$. An edge $\{v, w\}$ may be represented by vw. If $n = |V(G)|$, then G is called an *n-vertex graph*. If $m = |E(G)|$, then G is called an *m-edge graph*. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v in G, $N_G[v]$ denotes the closed neighborhood $N_G(v) \cup \{v\}$ of v, and $d_G(v)$ denotes the degree $|N_G(v)|$ of v. For $X \subseteq V(G)$, $N_G[X]$ denotes the closed neighborhood $\bigcup_{v \in X} N_G[v]$ of X, $G[X]$ denotes the subgraph of G induced by X, and $G - X$ denotes the graph obtained by deleting the vertices in X from G. Thus, $G[X] = (X, E(G) \cap \binom{X}{2})$ and $G - X = G[V(G) \backslash X]$. With a slight abuse of notation, for $S \subseteq E(G)$, $G - S$ denotes the graph obtained by removing the edges in S from G, that is, $G - S = (V(G), E(G) \backslash S)$. For $x \in V(G) \cup E(G)$, $G - \{x\}$ may be abbreviated to $G - x$. The subscript G may be omitted where no confusion arises; for example, $N_G(v)$ may be abbreviated to $N(v)$.

For $n \geq 1$, K_n and P_n denote the graphs $([n], \binom{[n]}{2})$ and $([n], \{\{i, i+1\} : i \in [n-1]\})$, respectively. For $n \geq 3$, C_n denotes $([n], \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}\})$. A copy of K_n is called a *complete graph* or an *n-clique*. A copy of P_n is called an *n-path* or simply a *path*. A copy of C_n is called an *n-cycle* or simply a *cycle*. A 3-cycle is a 3-clique and is also called a *triangle*. If G contains a k-clique H, then we call H a *k-clique of G*.

If $D \subseteq V(G) = N[D]$, then D is called a *dominating set of G*. The *domination number of G*, denoted by $\gamma(G)$, is the size of a smallest dominating set of G. If $\mathcal{F}$ is a set of graphs and F is a copy of a graph in $\mathcal{F}$, then we call F an *$\mathcal{F}$-graph*. A subset D of $V(G)$ is called an *$\mathcal{F}$-isolating set of G* if no subgraph of $G - N[D]$ is an $\mathcal{F}$-graph. The *$\mathcal{F}$-isolation number of G*, denoted by $\iota(G, \mathcal{F})$, is the size of a smallest $\mathcal{F}$-isolating set of G. We may abbreviate $\iota(G, \{F\})$ to $\iota(G, F)$. Clearly, D is a dominating set of G if and only if D is a $\{K_1\}$-isolating set of G. Thus, $\gamma(G) = \iota(G, K_1)$.

Caro and Hansberg introduced the isolation problem in [8]. It is a natural generalization of the classical domination problem [10,11,14–17]. Ore [22] proved that $\gamma(G) \leq n/2$ for any connected n-vertex graph G with $n \geq 2$ (see [14]). This is one of the earliest results in domination theory. While the deletion of the closed neighborhood of a dominating set produces the *null graph* (the graph with no vertices), the deletion of the closed neighborhood of a $\{K_2\}$-isolating set produces an empty graph (a graph with no edges). Caro and Hansberg [8] proved that if G is a connected n-vertex graph, then $\iota(G, K_2) \leq n/3$ unless G is a 2-clique or a 5-cycle. Solving a problem of Caro and Hansberg [8], Fenech, Kaemawichanurat, and the present author [2] proved that

$$\iota(G, K_k) \leq \frac{n}{k+1} \tag{1}$$

unless G is a k-clique or $k = 2$ and G is a 5-cycle, and that the bound is sharp. Ore's result and the Caro–Hansberg result are the cases $k = 1$ and $k = 2$, respectively. Let $\mathcal{C}$ be the set of cycles. Solving another problem of Caro and Hansberg [8], the present author [1] proved that

$$\iota(G, \mathcal{C}) \leq \frac{n}{4} \tag{2}$$

unless G is a triangle, and that the bound is sharp. Domination and isolation have been particularly investigated for maximal outerplanar graphs [4,5,7–9,12,13,18–21,23,24], mostly due to connections with Chvátal's Art Gallery Theorem [9].

Fenech, Kaemawichanurat, and the present author [3] also obtained a sharp upper bound on $\iota(G, K_k)$ in terms of the number of edges. The result is the analog of (1) given by Theorem 1. We need the following construction from [3].

Construction 1 ([3]). Consider any $m, k \in \mathbb{N}$. Let $t_k = \binom{k}{2}+2$ and $q = \lfloor \frac{m+1}{t_k} \rfloor$. Then, $m+1 = qt_k+r$ for some $r \in \{0\} \cup [t_k-1]$. Let $Q_{m,k}$ be a q-element set. If $q \geq 1$, then let $v_1, \ldots, v_q$ be the elements of $Q_{m,k}$, let $U_1, \ldots, U_q$ be k-element sets such that $U_1, \ldots, U_q, Q_{m,k}$ are pairwise disjoint, and for each $i \in [q]$, let $u_i^1, \ldots, u_i^k$ be the elements of U_i, and let $G_i = (U_i \cup \{v_i\}, \binom{U_i}{2} \cup \{u_i^1 v_i\})$. If either $q = 0$, $T = (\emptyset, \emptyset)$, and G is an r-edge tree T', or $q \geq 1$, T is a tree with $V(T) = Q_{m,k}$, T' is an r-edge tree with $V(T') \cap \bigcup_{i=1}^q V(G_i) = \{v_q\}$, and $G = (V(T') \cup \bigcup_{i=1}^q V(G_i), E(T) \cup E(T') \cup \bigcup_{i=1}^q E(G_i))$, then we say that G is an (m,k)-*special graph* with *quotient graph* T and *remainder graph* T', and for each $i \in [q]$, we call G_i and v_i a k-*clique constituent of* G and the k-*clique connection of* G_i *in* G, respectively. We say that an (m,k)-special graph is *pure* if $r = 0$. (Figure 1 in [3] is an illustration of a pure (m,k)-special graph.)

Theorem 1 ([3]). *If $k \geq 1$ and G is a connected m-edge graph that is not a k-clique, then*

$$\iota(G, K_k) \leq \frac{m+1}{\binom{k}{2}+2}. \tag{3}$$

Moreover,

(i) *Equality in (3) holds if and only if either G is a pure (m,k)-special graph or $k = 2$ and G is a 5-cycle.*

(ii) *If G is an (m,k)-special graph, then* $\iota(G, K_k) = \left\lfloor (m+1)/\left(\binom{k}{2}+2\right) \right\rfloor$.

Problem 7.4 in [8] asks for bounds on $\iota(G, \mathcal{F})$ for other interesting sets $\mathcal{F}$. Section 1 of that seminal paper makes particular mention of k-colorable graphs. We address the problem for such graphs and the problem for regular graphs together. The main result presented here generalizes Theorem 1 in these two desirable directions.

If $V(G) \neq \emptyset$ and $d(v) = r$ for each $v \in V(G)$, then G is said to be r-*regular* or simply *regular*, and r is called the *degree of* G. For $k \geq 1$, let $\mathcal{F}_{1,k}$ be the set of regular graphs whose degree is at least $k-1$.

If there exists a function $f\colon V(G) \to [k]$ such that $f(v) \neq f(w)$ for every $v, w \in V(G)$ with $vw \in E(G)$, then G is said to be k-*colorable*, and f is called a *proper* k-*coloring of* G. The *chromatic number of* G, denoted by $\chi(G)$, is the smallest non-negative integer k such that G is k-colorable. If $k = \chi(G)$, then G is said to be k-*chromatic*. For $k \geq 1$, let $\mathcal{F}_{2,k}$ be the set of graphs whose chromatic number is at least k.

Let $\mathcal{F}_{3,k}$ be the union of $\mathcal{F}_{1,k}$ and $\mathcal{F}_{2,k}$. In Sect. 2, we prove the following result.

Theorem 2. *If $\ell \in \{1,2,3\}$, $k \geq 1$, and G is a connected m-edge graph that is not a k-clique, then*

$$\iota(G, \mathcal{F}_{\ell,k}) \leq \frac{m+1}{\binom{k}{2}+2}. \quad (4)$$

Moreover,

(i) *Equality in (4) holds if and only if G is a pure (m,k)-special graph, or $k = 2$ and G is a 5-cycle, or $\ell \in \{1,3\}$, $k = 3$, and G is a 4-cycle.*

(ii) *If G is an (m,k)-special graph, then $\iota(G, \mathcal{F}_{\ell,k}) = \left\lfloor (m+1)/\left(\binom{k}{2}+2\right)\right\rfloor$.*

The proof of the bound depends on determining the graphs attaining the bound (i.e., the proof of (4) uses (i) and (ii)). This appears to be a new feature in the literature on isolation.

Since k-cliques are $(k-1)$-regular and cycles are 2-regular, Theorem 1 and the following analog of (2) are immediate consequences.

Theorem 3. *If G is a connected m-edge graph that is not a triangle, then*

$$\iota(G, \mathcal{C}) \leq \frac{m+1}{5}. \quad (5)$$

Moreover,

(i) *Equality in (5) holds if and only if G is a pure $(m,3)$-special graph or a 4-cycle.*

(ii) *If G is an $(m,3)$-special graph, then $\iota(G, \mathcal{C}) = \lfloor (m+1)/5 \rfloor$.*

2. Proof of Theorem 2

In this section, we prove Theorem 2. We start with two lemmas from [1].

Lemma 1 ([1]). *If G is a graph, $\mathcal{F}$ is a set of graphs, $X \subseteq V(G)$, and $Y \subseteq N[X]$, then*

$$\iota(G, \mathcal{F}) \leq |X| + \iota(G - Y, \mathcal{F}).$$

The set of components of a graph G will be denoted by $\mathrm{C}(G)$.

Lemma 2 ([1]). *If G is a graph and $\mathcal{F}$ is a set of graphs, then*

$$\iota(G, \mathcal{F}) = \sum_{H \in \mathrm{C(G)}} \iota(H, \mathcal{F}).$$

A basic result in graph theory is that for any connected graph G, $|E(G)| \geq |V(G)| - 1$, and equality holds if and only if G is a tree.

Proposition 1. *If G is a pure (m,k)-special graph with exactly q k-clique constituents, then $m = (\binom{k}{2}+2)q - 1$ and $\iota(G, \mathcal{F}_{\ell,k}) = q$ for each $\ell \in \{1,2,3\}$.*

Proof. Suppose that G is a pure (m,k)-special graph with exactly q k-clique constituents as in Construction 1. Then, $\{v_1, \ldots, v_q\}$ is an $\mathcal{F}_{\ell,k}$-isolating set of G, so $\iota(G, \mathcal{F}_{\ell,k}) \leq q$. The subgraphs $G[U_1], \ldots, G[U_q]$ of G are k-cliques, which are $(k-1)$-regular and k-chromatic. Thus, if D is an $\mathcal{F}_{\ell,k}$-isolating set of G, then $D \cap V(G_i) \neq \emptyset$ for each $i \in [q]$. Therefore, $\iota(G, \mathcal{F}_{\ell,k}) = q$. Now, clearly $m = (\binom{k}{2} + 1)q + |E(T)|$. Since T is a q-vertex tree, $|E(T)| = q - 1$. Thus, $m = (\binom{k}{2} + 2)q - 1$. □

If G is a copy of a graph H, then we write $G \simeq H$. For a graph G, the maximum degree of G (that is, $\max\{d(v) : v \in V(G)\}$) is denoted by $\Delta(G)$.

We will use the following classical result.

Theorem 4 (Brooks' Theorem [6]). *If G is a connected n-vertex graph, then $\chi(G) \leq \Delta(G)$ unless $G \simeq K_n$ or n is odd and $G \simeq C_n$.*

Proof of Theorem 2. Let $t_k = \binom{k}{2} + 2$ and $\mathcal{F} = \mathcal{F}_{3,k}$.

The argument in the proof of Proposition 1 yields (ii). If G is a 4-cycle, then $\iota(G, \mathcal{F}_{\ell,1}) = 2 < \frac{5}{2} = \frac{m+1}{t_1}$, $\iota(G, \mathcal{F}_{\ell,2}) = 1 < \frac{m+1}{t_2}$, $\iota(G, \mathcal{F}_{1,3}) = \iota(G, \mathcal{F}_{3,3}) = 1 = \frac{m+1}{t_3}$, $\iota(G, \mathcal{F}_{2,3}) = 0 < \frac{m+1}{t_3}$, and if $k \geq 4$, then $\iota(G, \mathcal{F}_{\ell,k}) = 0 < \frac{m+1}{t_k}$. If G is a 5-cycle, then $\iota(G, \mathcal{F}_{\ell,1}) = 2 < \frac{6}{2} = \frac{m+1}{t_1}$, $\iota(G, \mathcal{F}_{\ell,2}) = 2 = \frac{m+1}{t_2}$, $\iota(G, \mathcal{F}_{\ell,3}) = 1 < \frac{m+1}{t_3}$, and if $k \geq 4$, then $\iota(G, \mathcal{F}_{\ell,k}) = 0 < \frac{m+1}{t_k}$. Together with Proposition 1, this settles the sufficiency condition in (i).

Using induction on m, we now prove that the bound in (4) holds, and we prove that it is attained only if G and k are as in (i). For the bound, since $\iota(G, \mathcal{F})$ is an integer and $\iota(G, \mathcal{F}_{\ell,k}) \leq \iota(G, \mathcal{F})$, it suffices to prove that $\iota(G, \mathcal{F}) \leq \frac{m+1}{t_k}$. If $k \leq 2$, then a $\{K_k\}$-isolating set of G is an $\mathcal{F}_{\ell,k}$-isolating set of G, so the result is given by Theorem 1. Consider $k \geq 3$. The result is trivial if $m \leq 2$ or $\iota(G, \mathcal{F}) = 0$. Suppose that $m \geq 3$, $\iota(G, \mathcal{F}) \geq 1$, and G is not a k-clique. Let $n = |V(G)|$. Since G is connected, $m \geq n - 1$, so $n \leq m + 1$.

Case 1: *G has no k-cliques.*

Subcase 1.1: *$d(v) \leq k-2$ for some $v \in V(G)$.* Let $G' = G - v$. Let $H_1, \dots, H_r$ be the distinct components of G'.

Consider any $i \in [r]$. Let D_i be a smallest $\mathcal{F}$-isolating set of H_i. Let $H_i' = H_i - N[D_i]$, let $c_i = \chi(H_i')$, and let f_i be a proper c_i-coloring of H_i'. We have that $c_i \leq k-1$ and that H_i' contains no regular graph of degree at least $k-1$. Since G has no k-cliques, $|D_i| \leq \frac{|E(H_i)|+1}{t_k}$ by the induction hypothesis.

Let $D = \bigcup_{i=1}^{r} D_i$. Suppose that $G - N[D]$ contains a connected regular graph R of degree at least $k-1$. Since $d_R(v) \leq d_G(v) \leq k-2$, $v \notin R$. We obtain that for some $i \in [r]$, H_i' contains R, a contradiction. Thus, $G - N[D]$ contains no regular graph of degree at least $k-1$. Let $f' : V(G' - N_{G'}[D]) \to [k-1]$ such that $f'(w) = f_i(w)$ for each $i \in [r]$ and each $w \in V(H_i')$. Since $d(v) \leq k-2$, $[k-1]$ has an element j that is not in $\{f'(w) : w \in N(v) \cap V(G' - N_{G'}[D])\}$. Let $f : V(G - N[D]) \to [k-1]$ such that $f(w) = f'(w)$ for each $w \in V(G' - N_{G'}[D])$, and such that if $v \in V(G - N[D])$, then $f(v) = j$. Since f is a proper $(k-1)$-coloring of $G - N[D]$, D is an $\mathcal{F}$-isolating set of G.

Since G is connected, for each $i \in [r]$, $vw_i \in E(G)$ for some $w_i \in V(H_i)$. We have

$$m \geq \left| \bigcup_{i=1}^{r} (\{vw_i\} \cup E(H_i)) \right| \geq \sum_{i=1}^{r} t_k |D_i| = t_k |D| \geq t_k \iota(G, \mathcal{F}),$$

so $\iota(G,\mathcal{F}) < \frac{m+1}{t_k}$.

Subcase 1.2: $d(v) \geq k-1$ *for each* $v \in V(G)$. Then, $\Delta(G) \geq k-1$. Let $v \in V(G)$ with $d(v) = \Delta(G)$. Let $G' = G - N[v]$.

Suppose $d(v) = k-1$. Then, G is $(k-1)$-regular. Since G is not a k-clique, we have $n \geq k+1$, so $V(G') \neq \emptyset$. Suppose that G' contains a $(k-1)$-regular graph R'. Since G is connected, $uw \in E(G)$ for some $u \in V(R')$ and some $w \in V(G)\backslash V(R')$. We have $d_G(u) \geq d_{R'}(u) + 1 \geq k$, which contradicts $\Delta(G) = k-1$. Thus, G' contains no $(k-1)$-regular graph. Since $\Delta(G') \leq \Delta(G) = k-1$, $\chi(G') \leq k-1$ by Theorem 4. Thus, $\{v\}$ is an $\mathcal{F}$-isolating set of G. By the handshaking lemma, we have $2\,m = (k-1)n \geq (k-1)(k+1) = 2t_k + k - 5$, so $m+1 \geq t_k + \frac{k-3}{2} \geq t_k$. We have $\iota(G,\mathcal{F}) = 1 \leq \frac{m+1}{t_k}$. Suppose $\iota(G,\mathcal{F}) = \frac{m+1}{t_k}$. Then, $n = k+1$ and $k = 3$. Thus, G is a 4-cycle.

Now suppose $d(v) \geq k$. Let $s = d(v)$. Suppose $N[v] = V(G)$. By the handshaking lemma,

$$
\begin{aligned}
2m &= d(v) + \sum_{x \in N(v)} d(x) \geq d(v) + d(v)(k-1) \geq k + k(k-1) = 2t_k + k - 4 \\
&\geq 2t_k - 1,
\end{aligned}
$$

so $t_k < m+1$. Since $\{v\}$ is an $\mathcal{F}$-isolating set of G, $\iota(G,\mathcal{F}) = 1 < \frac{m+1}{t_k}$. Now suppose $N[v] \neq V(G)$. Then, $V(G') \neq \emptyset$. Let $H_0 = G[N(v)]$, and let $H_1, \ldots, H_r$ be the distinct components of G'. Since G is connected, for each $i \in [r]$, $x_i y_i \in E(G)$ for some $x_i \in N(v)$ and $y_i \in V(H_i)$. Let $A = \{xy \in E(G) \colon x \in N(v),\, y \in V(G')\}$. Then, $x_1y_1, \ldots, x_ry_r \in A$, so $|A| \geq r$. By the handshaking lemma,

$$
\begin{aligned}
2|E(H_0)| &= \sum_{x \in N(v)} d_{H_0}(x) = \sum_{x \in N(v)} |N(x)\backslash(\{v\} \cup V(G'))| \\
&= \sum_{x \in N(v)} (d(x) - 1 - |N(x) \cap V(G')|) \\
&\geq (k-2)d(v) - \sum_{x \in N(v)} |N(x) \cap V(G')| = (k-2)s - |A|,
\end{aligned}
$$

so $|E(H_0)| \geq \frac{(k-2)s-|A|}{2}$. Let $A' = \{vx \colon x \in N(v)\} \cup E(H_0) \cup A$. Then,

$$
\begin{aligned}
|A'| = s + |E(H_0)| + |A| &\geq \frac{sk + |A|}{2} = \frac{(k-1)k}{2} + \frac{(s-k+1)k + |A|}{2} \\
&= t_k + \frac{(s-k+1)k + |A| - 4}{2}.
\end{aligned}
$$

Since $E(G) = A' \cup \bigcup_{i=1}^{r} E(H_i)$ and $|A| \geq r$,

$$
m = |A'| + \sum_{i=1}^{r} |E(H_i)| \geq t_k + \frac{(s-k+1)k + r - 4}{2} + \sum_{i=1}^{r} |E(H_i)|. \qquad (6)
$$

For each $i \in [r]$, let D_i be a smallest $\mathcal{F}$-isolating set of H_i. Let $D = \{v\} \cup \bigcup_{i=1}^{r} D_i$. Then, D is an $\mathcal{F}$-isolating set of G. Since G has no k-cliques, for each $i \in [r]$, H_i is not a pure $(|E(H_i)|, k)$-special graph.

Suppose $k \neq 3$ or $H_i \not\simeq C_4$ for each $i \in [r]$. By the induction hypothesis, for each $i \in [r]$, $|D_i| < \frac{|E(H_i)|+1}{t_k}$, and hence, since $t_k|D_i| < |E(H_i)| + 1$, $t_k|D_i| \leq |E(H_i)|$. By (6),

$$m \geq t_k \left(1 + \sum_{i=1}^{r} |D_i|\right) + \frac{(s-k+1)k + r - 4}{2} \geq t_k|D| + \frac{k+r-4}{2} \geq t_k|D|$$

as $k \geq 3$ and $r \geq 1$. We have $\iota(G, \mathcal{F}) \leq |D| < \frac{m+1}{t_k}$.

Now suppose $k = 3$ and $H_j \simeq C_4$ for some $j \in [r]$. We may assume that $j = 1$. Let z_1 and z_2 be the two members of $N_{H_1}(y_1)$, and let z_3 be the member of $V(H_1)\backslash N_{H_1}[y_1]$. We have $V(H_1) = \{y_1, z_1, z_2, z_3\}$. Let $G_1 = G - \{y_1, z_1, z_2\}$. Note that $N_{G_1}(v) = N_G(v)$. If $d_{G_1}(z_3) = 0$, then the components of G_1 are $(\{z_3\}, \emptyset)$ and $G_1 - z_3$, and we take I to be $G_1 - z_3$. If $d_{G_1}(z_3) \geq 1$, then G_1 is connected, and we take I to be G_1. Since $d_I(v) = d_G(v) \geq k = 3$, I is not a 4-cycle. Since G has no k-cliques, I is not a pure $(|E(I)|, k)$-special graph. By the induction hypothesis, $\iota(I, \mathcal{F}) < \frac{|E(I)|+1}{t_k} = \frac{|E(I)|+1}{5}$. Let D_I be an $\mathcal{F}$-isolating set of I of size $\iota(I, \mathcal{F})$. Since $\{y_1, z_1, z_2\} \subset N[y_1]$, $\{y_1\} \cup D_I$ is an $\mathcal{F}$-isolating set of G. Let $J = \{x_1y_1, y_1z_1, y_1z_2, z_1z_3, z_2z_3\}$. Note that $J \subseteq E(G)\backslash E(I)$. We have

$$\iota(G, \mathcal{F}) \leq 1 + |D_I| < 1 + \frac{|E(I)|+1}{5} = \frac{|E(I)| + |J| + 1}{5} \leq \frac{m+1}{5} = \frac{m+1}{t_k}.$$

Case 2: *G has at least one k-clique.* Let F be a k-clique of G. Since G is connected and $G \not\simeq K_k$, $N[v]\backslash V(F) \neq \emptyset$ for some $v \in V(F)$. Let $u \in N[v]\backslash V(F)$. We have $V(F) \subseteq N[v]$, so $|N[v]| \geq k + 1$ and

$$|E(G[N[v]])| \geq t_k - 1. \tag{7}$$

Suppose $V(G) = N[v]$. Then, $\{v\}$ is an $\mathcal{F}$-isolating set of G, so $\iota(G, \mathcal{F}) = 1 \leq \frac{m+1}{t_k}$ by (7). If $\iota(G, \mathcal{F}) = \frac{m+1}{t_k}$, then $V(G) = V(F) \cup \{u\}$ and $E(G) = E(F) \cup \{uv\}$, so G is a pure (m, k)-special graph.

Now suppose $V(G) \neq N[v]$. Let $G' = G - N[v]$ and $n' = |V(G')|$. We have

$$n \geq n' + k + 1 \tag{8}$$

and $n' \geq 1$. Let $\mathcal{H} = \mathrm{C}(G')$. For any $H \in \mathcal{H}$ and $x \in N(v)$ such that $xy \in E(G)$ for some $y \in V(H)$, we say that H is *linked to* x. Since G is connected, for each $H \in \mathcal{H}$, H is linked to at least one member of $N(v)$, so $x_Hy_H \in E(G)$ for some $x_H \in N(v)$ and some $y_H \in V(H)$. Clearly,

$$m \geq |E(F) \cup \{uv\}| + \sum_{H \in \mathcal{H}} |E(H) \cup \{x_Hy_H\}| = t_k - 1 + \sum_{H \in \mathcal{H}} (|E(H)| + 1). \tag{9}$$

Let

$$\mathcal{H}' = \{H \in \mathcal{H} \colon H \simeq K_k\}.$$

By the induction hypothesis, for each $H \in \mathcal{H}\backslash\mathcal{H}'$, $\iota(H,\mathcal{F}) \le \frac{|E(H)|+1}{t_k}$, and equality holds only if H is a pure $(|E(H)|,k)$-special graph or $k=3$ and H is a 4-cycle.

Subcase 2.1: $\mathcal{H}'=\emptyset$. By Lemma 1 (with $X=\{v\}$ and $Y=N[v]$), Lemma 2, and (9),

$$\iota(G,\mathcal{F}) \le 1+\iota(G',\mathcal{F}) = 1+\textstyle\sum_{H\in\mathcal{H}}\iota(H,\mathcal{F}) \le 1+\sum_{H\in\mathcal{H}}\frac{|E(H)|+1}{t_k} \le \frac{m+1}{t_k}. \tag{10}$$

Suppose $\iota(G,\mathcal{F}) = \frac{m+1}{t_k}$. Then, the inequality in (9) is an equality, and $\iota(H,\mathcal{F}) = \frac{|E(H)|+1}{t_k}$ for each $H\in\mathcal{H}$.

Let $H_1,\ldots,H_r$ be the distinct members of $\mathcal{H}$. Let $I=\{i\in[r]\colon H_i$ is a pure $(|E(H_i)|,k)$-special graph$\}$. For each $i\in I$, let $G_{i,1},\ldots,G_{i,q_i}$ be the distinct k-clique constituents of H_i, and for each $j\in[q_i]$, let $v_{i,j}$ be the k-clique connection of $G_{i,j}$ in H_i, and let $u_{i,j}^1,\ldots,u_{i,j}^k$ be the vertices of $G_{i,j}-v_{i,j}$, where $u_{i,j}^1$ is the neighbor of $v_{i,j}$ in $G_{i,j}$. By Proposition 1, $|E(H_i)| = t_kq_i-1$ for each $i\in I$. If $i\in[r]\backslash I$, then, by the above, $k=3$, H_i is a 4-cycle, and $|E(H_i)|+1=t_k$. Since equality holds throughout in (9), we have

$$V(G) = V(F)\cup\{u\}\cup\bigcup_{i=1}^{r}V(H_i) \tag{11}$$

and

$$E(G) = E(F)\cup\{uv\}\cup\{x_{H_1}y_{H_1},\ldots,x_{H_r}y_{H_r}\}\cup\bigcup_{i=1}^{r}E(H_i). \tag{12}$$

Let

$$D=\{v\}\cup\{y_{H_i}\colon i\in[r]\backslash I\}\cup\left(\bigcup_{i\in I}\bigcup_{j=1}^{q_i}\{u_{i,j}^1\}\right).$$

Then, D is an $\mathcal{F}$-isolating set of G, and

$$|D| = 1+\left(\sum_{i\in[r]\backslash I}\frac{|E(H_i)|+1}{t_k}\right)+\sum_{i\in I}q_i = 1+\sum_{i=1}^{r}\frac{|E(H_i)|+1}{t_k} = \frac{m+1}{t_k}$$
$$=\iota(G,\mathcal{F}).$$

Suppose that $[r]\backslash I$ has an element j. Then, $k=3$ and H_j is a 4-cycle. Suppose $x_{H_j}=u$. Then, we have that $(D\backslash\{v,y_{H_j}\})\cup\{u\}$ is an $\mathcal{F}$-isolating set of G of size $|D|-1$, contradicting $|D|=\iota(G,\mathcal{F})$. Thus, $x_{H_j}\in V(F)$. This yields that $D\backslash\{v\}$ is an $\mathcal{F}$-isolating set of G, contradicting $|D|=\iota(G,\mathcal{F})$.

Therefore, $[r]\backslash I=\emptyset$, and hence $I=[r]$. Consider any $i\in[r]$. Suppose $x_{H_i}\in V(F)$ and $y_{H_i}=v_{i,j}$ for some $j\in[q_i]$. Then, the components of $G-N[(D\backslash\{v,u_{i,j}^1\})\cup\{v_{i,j}\}]$ are $G_{i,j}-\{v_{i,j},u_{i,j}^1\}$ and $G[\{u\}\cup V(F-x_{H_i})]$, and clearly they contain no $\mathcal{F}$-graphs (a proper $(k-1)$-coloring of $G[\{u\}\cup V(F-x_{H_i})]$ is obtained by assigning different colors to the $k-1$ vertices of $F-x_{H_i}$, and coloring u differently from v). Thus, we have that $(D\backslash\{v,u_{i,j}^1\})\cup\{v_{i,j}\}$ is an $\mathcal{F}$-isolating set of G of size $|D|-1$, a contradiction. Suppose $x_{H_i} \in V(F)$ and $y_{H_i} = u_{i,j}^l$ for some $j \in [q_i]$ and $l \in [k]$. As above, we obtain that $(D\backslash\{v, u_{i,j}^1\}) \cup \{u_{i,j}^l\}$ is an $\mathcal{F}$-isolating set of G of size $|D|-1$, a contradiction. Thus, $x_{H_i} = u$. Suppose $y_{H_i} = u_{i,j}^l$ for some $j \in [q_i]$ and $l \in [k]$. Then, we have that $(D\backslash\{v, u_{i,j}^1\}) \cup \{u\}$ is an $\mathcal{F}$-isolating set of G of size $|D|-1$, a contradiction. Thus, $y_{H_i} = v_{i,j}$ for some $j \in [q_i]$. Since this has been established for each $i \in [r]$, we have that $G[\{u\} \cup \bigcup_{i=1}^{r} \bigcup_{j=1}^{q_i} \{v_{i,j}\}]$ is a tree, and hence G is a pure (m,k)-special graph with $G[N[v]], G_{1,1}, \ldots, G_{1,q_1}, \ldots, G_{r,1}, \ldots, G_{r,q_r}$ being its k-clique constituents and u being the k-clique connection of $G[N[v]]$ in G.

Subcase 2.2: $\mathcal{H}' \neq \emptyset$. Let $H' \in \mathcal{H}'$. Then, H' is linked to some $x \in N(v)$. Let
$\mathcal{H}_1 = \{H \in \mathcal{H}' \colon H \text{ is only linked to } x\}$ and
$\mathcal{H}_2 = \{H \in \mathcal{H}\backslash\mathcal{H}' \colon H \text{ is only linked to } x\}$.
For each $H \in \mathcal{H}_2$, let D_H be a smallest $\mathcal{F}$-isolating set of H. Then, $|D_H| = \iota(H, \mathcal{F}) \leq \frac{|E(H)|+1}{t_k}$ by the induction hypothesis.

Subcase 2.2.1: *Each member of $\mathcal{H}'$ is linked to at least two members of $N(v)$.* Thus, $\mathcal{H}_1 = \emptyset$. Let

$$X = \{x\} \cup V(H') \quad \text{and} \quad G^* = G - X.$$

Then, a component G_v^* of G^* satisfies $N[v]\backslash\{x\} \subseteq V(G_v^*)$, and the members of $\mathcal{H}_2$ are the other components of G^*. Let D_v^* be a smallest $\mathcal{F}$-isolating set of G_v^*. Since H' is linked to x and to some $x' \in N(v)\backslash\{x\}$, there exist $y, y' \in V(H')$ with $xy, x'y' \in E(G)$. Let $D = D_v^* \cup \{y\} \cup \bigcup_{H \in \mathcal{H}_2} D_H$. Since $X \subseteq N[y]$, D is an $\mathcal{F}$-isolating set of G. Thus,

$$\iota(G, \mathcal{F}) \leq |D_v^*| + 1 + \sum_{H \in \mathcal{H}_2} |D_H| \leq |D_v^*| + \frac{t_k}{t_k} + \sum_{H \in \mathcal{H}_2} \frac{|E(H)|+1}{t_k}.$$

We have

$$\begin{aligned} m &\geq |E(G_v^*) \cup \{vx\}| + |E(H') \cup \{xy, x'y'\}| + \sum_{H \in \mathcal{H}_2} |E(H) \cup \{x_H y_H\}| \\ &= |E(G_v^*)| + 1 + t_k + \sum_{H \in \mathcal{H}_2} (|E(H)| + 1). \end{aligned} \tag{13}$$

If $G_v^* \not\simeq K_k$, then $|D_v^*| \leq \frac{|E(G_v^*)|+1}{t_k}$ by the induction hypothesis, so

$$\iota(G, \mathcal{F}) \leq \frac{|E(G_v^*)|+1}{t_k} + \frac{t_k}{t_k} + \sum_{H \in \mathcal{H}_2} \frac{|E(H)|+1}{t_k} < \frac{m+1}{t_k}.$$

Now suppose $G_v^* \simeq K_k$. Since $|N[v]\backslash\{x\}| \geq k$ and $N[v]\backslash\{x\} \subseteq V(G_v^*)$, $V(G_v^*) = N[v]\backslash\{x\}$. Let $Y = (X \cup V(G_v^*))\backslash\{v, x, y\}$ and $G_Y = G - \{v, x, y\}$. Then, the components of $G[Y]$ and the members of $\mathcal{H}_2$ are the components of G_Y.

Suppose that $G[Y]$ contains no $\mathcal{F}$-graph. Since $v, y \in N[x]$, $\{x\} \cup \bigcup_{H \in \mathcal{H}_2} D_H$ is an $\mathcal{F}$-isolating set of G, so $\iota(G, \mathcal{F}) \leq 1 + \sum_{H \in \mathcal{H}_2} |D_H|$. Since $G_v^* \simeq K_k$, (13) gives us $m + 1 \geq 2t_k + \sum_{H \in \mathcal{H}_2}(|E(H)| + 1) > t_k(1 + \sum_{H \in \mathcal{H}_2} |D_H|) \geq t_k \iota(G, \mathcal{F})$, so $\iota(G, \mathcal{F}) < \frac{m+1}{t_k}$.

Now suppose that $G[Y]$ contains an $\mathcal{F}$-graph F_Y. Then, $\chi(F_Y) \geq k$ or F_Y is a regular graph of degree at least $k-1$. Thus, $|N_{G[Y]}[z]| \geq k$ for some $z \in V(F_Y)$; this is given by Theorem 4 if $\chi(F_Y) \geq k$. Let $W \subseteq N_{G[Y]}[z]$ such that $z \in W$ and $|W| = k$. Let $G_1 = G_v^*$, $G_2 = H'$, $v_1 = v$, $v_2 = y$, $G_1' = G_1 - v_1$, and $G_2' = G_2 - v_2$. We have

$$N_{G[Y]}[z] \subseteq Y = V(G_1') \cup V(G_2'). \tag{14}$$

Since $|V(G_1')| = |V(G_2')| = k-1$, it follows that $|W \cap V(G_1')| \geq 1$ and $|W \cap V(G_2')| \geq 1$. By (14), $z \in V(G_j')$ for some $j \in \{1,2\}$. Let $Z = V(G_j) \cup W$. Since $z \in V(G_j)$ and $G_j \simeq K_k$,

$$Z \subseteq N[z]. \tag{15}$$

We have

$$|Z| = |V(G_j)| + |W \backslash V(G_j)| = k + |W \cap V(G_{3-j}')| \geq k+1. \tag{16}$$

Let $G_Z = G - Z$. Then, $V(G_Z) = \{x\} \cup (V(G_{3-j}) \backslash W) \cup \bigcup_{H \in \mathcal{H}_2} V(H)$. The components of $G_Z - x$ are the clique $G_Z[V(G_{3-j}) \backslash W]$ (which has less than k vertices) and the members of $\mathcal{H}_2$. Moreover, $v_{3-j} \in V(G_{3-j}) \backslash W$ (by (14)), $v_{3-j} \in N_{G_Z}(x)$, and $N_{G_Z}(x) \cap V(H) \neq \emptyset$ for each $H \in \mathcal{H}_2$ (by the definition of $\mathcal{H}_2$). Thus, G_Z is connected.

Suppose $\mathcal{H}_2 \neq \emptyset$. Then, $G_Z \not\simeq K_k$. By (15), Lemma 1, and the induction hypothesis, $\iota(G, \mathcal{F}) \leq 1 + \iota(G_Z, \mathcal{F}) \leq 1 + \frac{|E(G_Z)|+1}{t_k}$. Let $w^* \in W \cap V(G_{3-j}')$ (recall that $|W \cap V(G_{3-j}')| \geq 1$). Since $G_{3-j} \simeq K_k$, $w^*w \in E(G_{3-j})$ for each $w \in V(G_{3-j}) \backslash \{w^*\}$. We have

$$\begin{aligned} m &\geq |E(G_j)| + |\{xv_j, zw^*\}| + |\{w^*w \colon w \in V(G_{3-j}) \backslash \{w^*\}\}| + |E(G_Z)| \\ &\geq \binom{k}{2} + 1 + k + |E(G_Z)| \geq t_k + 2 + |E(G_Z)| \end{aligned}$$

as $k \geq 3$. Thus, since $\iota(G, \mathcal{F}) \leq 1 + \frac{|E(G_Z)|+1}{t_k}$, $\iota(G, \mathcal{F}) < \frac{m+1}{t_k}$.

Now suppose $\mathcal{H}_2 = \emptyset$. Then, $G^* = G_v^*$, so $V(G) = V(G_v^*) \cup \{x\} \cup V(H')$. Recall that the k-clique H' is linked to at least two members of $N(v)$. Thus,

$$n = 2k+1 \quad \text{and} \quad m \geq 2\binom{k}{2} + 3 = 2t_k - 1.$$

If $|N[w]| \geq k+2$ for some $w \in V(G)$, then $|V(G - N[w])| \leq k-1$, so $\{w\}$ is an $\mathcal{F}$-isolating set of G, and hence $\iota(G, \mathcal{F}) = 1 < \frac{m+1}{t_k}$. Suppose $|N[w]| \leq k+1$ for each $w \in V(G)$. Then, $\Delta(G) = d(v) = k$, $N[z] = Z = V(G_j) \cup \{w\}$ for some $w \in V(G_{3-j}')$ (by (15) and (16)), and $V(G - N[z]) = \{x\} \cup V(G_{3-j} - w)$. If $G - N[z]$ contains no $\mathcal{F}$-graph, then $\iota(G, \mathcal{F}) = 1 < \frac{m+1}{t_k}$. Suppose that $G - N[z]$ contains an $\mathcal{F}$-graph F'. Since $|V(G - N[z])| = k$, $G - N[z] = F' \simeq K_k$. Since $\Delta(G) = k$, we have $N(x) = \{v_j\} \cup V(G_{3-j} - w)$ and, since $z \in V(G_j)$ and $w \in N[z] \cap V(G_{3-j})$, $N[w] = \{z\} \cup V(G_{3-j})$. Thus, $V(G - N[w]) = \{x\} \cup (V(G_j) \backslash \{z\})$. Since $|V(G - N[w])| = k \geq 3$ and $N[x] \cap V(G_j') = \emptyset$, $\{w\}$ is an $\mathcal{F}$-isolating set of G, so $\iota(G, \mathcal{F}) = 1 < \frac{m+1}{t_k}$.

Subcase 2.2.2: *Some member of $\mathcal{H}'$ is linked to only one member of $N(v)$.* Thus, we may assume that H' is linked to x only. For each $H \in \mathcal{H}_1 \cup \mathcal{H}_2$,

$y_H \in N(x)$ for some $y_H \in V(H)$. By the same argument in the corresponding Subcase 2.2.2 of the proof of Theorem 1.3 in [3], we obtain that $\iota(G,\mathcal{F}) \leq \frac{m+1}{t_k}$, and that equality holds only if

$$E(G) = E(H') \cup \{xy_{H'}\} \cup \bigcup_{H\in\mathcal{H}_2'} (E(H) \cup \{xy_H\}),$$

where $\mathcal{H}_2' = \mathcal{H}_2 \cup \{G_v^*\}$ for a connected subgraph G_v^* of G with

$$y_{G_v^*} = v \in V(G_v^*) = V(G)\backslash \left(\{x\} \cup V(H') \cup \bigcup_{H\in\mathcal{H}_2} V(H)\right),$$

and $\iota(H,\mathcal{F}) = \frac{|E(H)|+1}{t_k}$ for each $H \in \mathcal{H}_2'$. Thus, if $\iota(G,\mathcal{F}) = \frac{m+1}{t_k}$, then G is as in the case $\iota(G,\mathcal{F}) = \frac{m+1}{t_k}$ in Subcase 2.1 (with H' and x now taking the role of F and u, respectively, in (11) and (12)), so G is a pure (m,k)-special graph. □

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