INTRINSIC TOPOLOGIES ON ORDERED STRUCTURES – APPLICATIONS

Kevin Abela

Supervisor: Prof. Emanuel Chetcuti

A thesis submitted in partial fulfillment
of the requirements for the degree of
Ph.D. in Mathematics

Department Of Mathematics
University Of Malta

December 2019 - December 2025

L-Università ta' Malta

FACULTY/INSTITUTE/CENTRE/SCHOOL Science

DECLARATION OF AUTHENTICITY FOR DOCTORAL STUDENTS

(a) Authenticity of Thesis/Dissertation

I hereby declare that I am the legitimate author of this Thesis/Dissertation and that it is my original work.

No portion of this work has been submitted in support of an application for another degree or qualification of this or any other university or institution of higher education.

I hold the University of Malta harmless against any third party claims with regard to copyright violation, breach of confidentiality, defamation and any other third party right infringement.

(b) Research Code of Practice and Ethics Review Procedure

I declare that I have abided by the University's Research Ethics Review Procedures. Research Ethics & Data Protection form code NA.

☑ As a Ph.D. student, as per Regulation 66 of the Doctor of Philosophy Regulations, I accept that my thesis be made publicly available on the University of Malta Institutional Repository.

□ As a Doctor of Sacred Theology student, as per Regulation 17 (3) of the Doctor of Sacred Theology Regulations, I accept that my thesis be made publicly available on the University of Malta Institutional Repository.

□ As a Doctor of Music student, as per Regulation 26 (2) of the Doctor of Music Regulations, I accept that my dissertation be made publicly available on the University of Malta Institutional Repository.

□ As a Professional Doctorate student, as per Regulation 55 of the Professional Doctorate Regulations, I accept that my dissertation be made publicly available on the University of Malta Institutional Repository.

Page intentionally left blank

To Rakel and Dimitria

Acknowledgements

Foremost, I would like to express my deepest gratitude to my supervisor, Prof. Emanuel Chetcuti. He has been pivotal to my academic journey that extends beyond my PhD studies. His invaluable guidance and unwavering support throughout the course of my research enabled me to grow both mathematically and personally.

I would also like to thank Prof. Hans Weber for his contributions to our joint work and for introducing me to lattice uniformities. I am also grateful to the faculty and staff of the Department of Mathematics at the University of Malta, especially Prof. David Buhagiar, Prof. Joseph Muscat, and Dr. Beatriz Zamora. The weekly seminar provided an ideal environment to present and discuss my research, as well as an opportunity to explore different subjects and perspectives.

A very special thanks goes to Mr. Ramon Vella for his guidance during the early days of my mathematical studies. I will always cherish our discussions, which extended far beyond curricular mathematics.

I am deeply grateful to my parents, Andrew and Antoinette, for their unfailing support and continuous encouragement throughout my years of study.

Finally, and most importantly, I would like to thank my wife, Michela, and my daughters, Rakel and Dimitria. Their love, patience, and understanding made this research possible.

Abstract

This thesis is divided into two main parts. The first three chapters focus on the study of order and unbounded order convergence. We examine various well-established definitions of order convergence, that have emerged over time, each associated with a corresponding topology. These notions are compared in detail, with particular attention to the conditions under which they coincide or differ. Furthermore, in the context of a semi-finite measure space, we investigate the relationship between the topologies on L^∞ arising from the duality (L^∞, L^1), and we compare these to the order topology. Notably, we establish a condition under which the Mackey topology is strictly weaker than the order topology.

Compared to order convergence, unbounded order convergence is relatively new and is generally studied on Riesz spaces. In this thesis, we explore it in the broader context of lattices. Our results show that, similar to the order topology, the unbounded order topology is independent of the definition of order convergence. In addition, we extend key properties known to hold in Riesz spaces to lattices. We prove that order continuity of unbounded order convergence is equivalent to the lattice being infinitely distributive. Moreover, we show that the O-closure and $\mathfrak{u}$O-closure of a sublattice coincide and form a sublattice. Furthermore, we show that the $\mathfrak{u}$O-adherence of an ideal is an O-closed ideal. We also examine the MacNeille completion of a sublattice Y relative to that of a lattice L, identifying two conditions under which the completion of Y embeds regularly in that of L.

The last chapter is dedicated to the study of lattice uniformities. It is known that for a locally solid Riesz space (X, τ) there exists a locally solid linear topology $\mathfrak{u}\tau$ on X such that unbounded τ-convergence coincides with $\mathfrak{u}\tau$-convergence. This topology is the weakest locally solid linear topology that agrees with τ on all order bounded subsets. Thus, for a uniform lattice $(L,\mathcal{U})$, we introduce the weakest lattice

uniformity $\mathcal{U}^*$ on L that coincides with $\mathcal{U}$ on each order bounded subset of L. We see that if $\mathcal{U}$ is the uniformity induced by the topology of a locally solid Riesz space (X,τ), then the $\mathcal{U}^*$-topology coincides with $\mathfrak{u}\tau$. We also provide answers to several questions posed in [44, 37].

Publications

This thesis is based on research carried out between 2019 and 2025. The first three chapters draw upon results presented in [1, 2], while the final chapter builds on the work in [3].

Introduction

Order convergence in partially ordered sets has long been studied. One finds different definitions for order convergence that all reduce for complete lattices to the requirement that $\limsup = \liminf$. Each notion of order convergence gives rise to a topology, defined by taking as closed sets those subsets of the poset such that no net in them order converges to a point outside of the set. This topology is called the order topology and it is intrinsic to the poset since it is the finest topology which preserves order convergence. The simplest example is the set of real numbers. The order topology on the real line is the standard topology and if a sequence converges with respect to the order topology, it is also order convergent. Caution is required, since in general nets that converge with respect to the order topology need not be order convergent. Order convergence is not topological in general. For example, the order topology on a Riesz space is topological if and only if it is finite-dimensional (see, for example, [44, Theorem 18.36]). The question of characterizing those posets for which order convergence is topological has received considerable attention. Our work builds on earlier results in [12, 14, 15, 16, 17], in which the order topology associated with a partially ordered structure arising naturally in the realm of functional analysis is studied and related to the functional-analytic properties of the structure.

A systematic treatment of the order topology on various structures associated with a von Neumann algebra was carried out in [17]. In that case, the order topology comes from the standard operator order defined on the self-adjoint operators acting on a Hilbert space; namely, for self-adjoint operators T and S, we have $T \leq S$ if the operator $S-T$ has a positive spectrum. It was shown that the order topology on the self-adjoint part of a σ-finite von Neumann algebra, albeit far from being a linear topology in general, on bounded parts, coincides with the strong operator topology. The proof is based on the Noncommutative Egoroff Theorem and rests heavily on the assumption of σ-finiteness. Moreover, finite σ-finite von Neumann algebras were characterized in terms of the order-topological properties of the associated lattice of projections. In [2], for semi-finite measure spaces, we give a complete picture between the topologies arising from the duality (L^∞, L^1) and the order topology. Furthermore, we provide a condition under which the Mackey topology is strictly coarser than the order topology.

The concept of unbounded order convergence (also referred to as $\mathfrak{u}$O-convergence) was first introduced by Hidegorô Nakano under the term individual convergence [38, 39]. The now commonly used term unbounded order convergence was later coined by Ralph DeMarr [21]. In sequence spaces ℓ^p $(1 \leq p \leq \infty)$, unbounded order convergence coincides with pointwise convergence. Furthermore, in L^p spaces $(1 \leq p < \infty)$, it is equivalent to almost everywhere convergence.

A pivotal advancement in the theory of unbounded order convergence ($\mathfrak{u}$O-convergence) was made by Gao, Xanthos, and Troitsky [28], who demonstrated that $\mathfrak{u}$O-convergence passes freely to and from regular sublattices. In particular, they established that for a sublattice, the notions of $\mathfrak{u}$O-closure and order closure (O-closure) coincide. This fundamental connection between unbounded and classical order structures has spurred further investigation. In subsequent work [27], the relationship between $\mathfrak{u}$O-closure and O-closure of sublattices was further studied.

Bilokopytov and Troitsky extended the study of unbounded order convergence by analyzing the concept of $\mathfrak{u}$O-convergence in spaces of continuous functions, including $C(X), C_b(X), C_0(X)$ and $C^\infty(X)$, where X is a completely regular Hausdorff topological space. In this context, they provided a detailed characterization of $\mathfrak{u}$O-convergence in $C(X)$, showing, notably, that a sequence $\mathfrak{u}$O-converges if and only if it converges pointwise on a co-meagre subset [9].

Our work on unbounded order convergence builds upon the aforementioned developments. In [3], we demonstrate that the unbounded order topology is independent of the definition of order convergence. In [1], we investigate $\mathfrak{u}$O-convergence within the setting of infinitely distributive lattices. We establish that in a distributive lattice, $\mathfrak{u}$O-convergence is order continuous if and only if the lattice is infinitely distributive. Additionally, we show that similarly to the Riesz spaces case, both the $\mathfrak{u}$O-closure and the O-closure of a sublattice are themselves sublattices, and, moreover, these two closure notions coincide.

Furthermore, we explore the MacNeille completion of a sublattice within the MacNeille completion of the ambient lattice. In particular, we identify two sufficient conditions under which the MacNeille completion of the sublattice embeds regularly in that of the larger lattice.

Uniform lattices were introduced in [47, 49] as a generalization of topological Boolean rings—that is, Boolean rings endowed with an FN-topology—and locally solid Riesz spaces, or more generally, locally solid ℓ-groups. A uniform lattice is defined as a lattice equipped with a lattice uniformity, meaning a uniformity under which the lattice operations $\vee$ and $\wedge$ are uniformly continuous.

In particular, the uniformity induced by a group topology τ on a Boolean ring is a lattice uniformity if and only if τ is an FN-topology. Similarly, the right, left, or two-sided uniformity induced by a group topology τ on an ℓ-group is a lattice uniformity if and only if τ is locally solid (see [47, Propositions 1.1.7 and 1.1.8]).

In recent years, several authors have investigated the notion of unbounded τ-convergence in a locally solid Riesz space (X, τ). This concept generalizes earlier work on unbounded convergence, which was initially studied in the setting where τ is the norm topology on a Banach lattice. A broader treatment was developed in [37], where τ is taken to be any locally solid linear topology on a Riesz space.

In particular, Taylor [37, Theorem 2.3] established that there exists a locally solid linear topology $\mathfrak{u}\tau$ on X such that unbounded τ-convergence coincides with $\mathfrak{u}\tau$-convergence. Furthermore, as shown in [44, Proposition 2.19], the topology $\mathfrak{u}\tau$ is the weakest locally solid linear topology that agrees with τ on all order bounded subsets of X.

In our work [3], we consider uniform lattices $(L,\mathcal{U})$ and introduce the weakest lattice uniformity $\mathcal{U}^*$ on L that coincides with $\mathcal{U}$ on all order bounded subsets. This construction parallels the development of the topology $\mathfrak{u}\tau$ in the setting of locally solid Riesz spaces. In particular, we show that when the lattice uniformity $\mathcal{U}$ is induced by a locally solid topology τ on a Riesz space, the topology induced by $\mathcal{U}^*$ coincides exactly with $\mathfrak{u}\tau$. Furthermore, we answer several questions posed in [44, 37].

Finally, an important application of order-theoretic and lattice structures arises in economics and finance. The work on the GRAS (Generalised RAS) algorithm [4] and Social Accounting Matrices (SAMs) [5] demonstrates how matrix balancing and block decomposition can maintain structural consistency in economic and financial data while adjusting for incomplete or updated observations. Moreover, lattice-theoretic approaches underpin the representation of preferences, aggregation of economic quantities, and decision-making in infinite-dimensional commodity or state spaces [20, 8]. The block decomposition techniques used in [5] further allow for efficient computation and analysis by partitioning complex economic systems into submatrices. Such connections illustrate that the abstract mathematical structures we study can directly inform practical methods for modeling, aggregating, and analyzing economic and financial systems.

Page intentionally left blank

Table of Contents

1 Basic Concepts

In this section, we review several well-known basic definitions of partially ordered sets, lattices and Riesz spaces. In particular, we review the construction of the MacNeille completion.

1.1 Partially Ordered Sets and Lattices

Definition 1.1.1. *Let P be a set. A partial order (or order) on P is a relation $\leq$ on P such that, if $a, b, c \in P$, then*

(i) *$a \leq a$ (reflexivity),*

(ii) *$a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry),*

(iii) *$a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity).*

A pair $(P, \leq)$ is called a *partially ordered set* (or poset). Sometimes we write $b \geq a$ instead of $a \leq b$, and write $a < b$ when $a \leq b$ and $a \neq b$. A *total order* is a partial order in which every two elements are comparable. A set endowed with a total order is said to be *totally ordered*. An element $1 \in P$ is called a maximum element (or greatest element) and $0 \in P$ is called a minimum element (or least element) if $0 \leq x \leq 1$ for every $x \in P$. Whenever maximum and minimum elements exist, they are unique.

Definition 1.1.2. *Let P be a poset and $B \subseteq P$.*

(i) *An element $a \in P$ is an upper bound of B if $b \leq a$ for every $b \in B$. Similarly, $a \in P$ is a lower bound of B if $a \leq b$ for every $b \in B$.*

(ii) *If $a \in P$ is an upper bound of B and $a \leq c$ for every upper bound $c \in P$ of B, then a is the supremum (or least upper bound) of B in P. This is denoted by $\sup^P B$ or $\bigvee^P B$. Similarly, if a is a lower bound of B and $c \leq a$ for every lower bound $c \in P$ of B, then a is the infimum (or greatest lower bound) of B in P. This* *is denoted by* $\inf^P B$ *or* $\bigwedge^P B$. *When there is no ambiguity, we simply write* $\sup B$ *or* $\bigvee B$ *for the supremum and* $\inf B$ *or* $\bigwedge B$ *for the infimum.*

(iii) *For* $a, b \in P$, *the set* $\{x \in P : a \leq x \leq b\}$ *is an order interval, denoted by* $[a, b]$.

(iv) *If there exist* $a, b \in P$ *such that* $B \subseteq [a, b]$, *then* B *is order bounded.*

(v) B *is order convex if, whenever* $a, b \in B$ *with* $a \leq b$, *we have* $[a, b] \subseteq B$.

In the following example, we note that a subset of a partially ordered set might not have a supremum nor an infimum, and when they exist they are unique.

Example 1. *Consider the subset* $B := \{x \in \mathbb{Q} : 2 < x^2 < 3\}$ *of* $\mathbb{Q}$, *viewed also as a subset of* $\mathbb{R}$. *Then* B *has neither an infimum nor a supremum in* $\mathbb{Q}$. *Indeed,* $\sqrt{2}$ *and* $\sqrt{3}$ *exist in* $\mathbb{R}$ *but not in* $\mathbb{Q}$. *On the other hand, in* $\mathbb{R}$ *we have* $\bigvee^{\mathbb{R}} B = \sqrt{3}$ *and* $\bigwedge^{\mathbb{R}} B = \sqrt{2}$. *Thus, the existence of suprema and infima depends on the ambient poset.*

Proposition 1.1.3. *If* P *is a poset with* $B \subseteq P_0 \subseteq P$ *and* $\bigwedge^P B$ *exists and is an element of* P_0, *then* $\bigwedge^{P_0} B = \bigwedge^P B$. *A dual statement holds for suprema.*

Proof. Let $a := \bigwedge^P B$. Then B is bounded below in P_0 by a. Let $b \in P_0$ be another lower bound of B. Since $b \in P$, we have $b \leq a$, hence $a = \bigwedge^{P_0} B$. □

Proposition 1.1.4. *Let* A *and* B *be subsets of a poset* P.

(i) *If* $\bigvee B$ *or* $\bigwedge B$ *exist, they are unique.*

(ii) *Suppose* $A \subseteq B$. *If both* A *and* B *have suprema in* P, *then* $\bigvee A \leq \bigvee B$. *Dually, if both* A *and* B *have infima in* P, *then* $\bigwedge B \leq \bigwedge A$.

Definition 1.1.5. *Let P_1 and P_2 be posets. A map $\varphi : P_1 \to P_2$ is said to be:*

(i) *order-preserving (or isotone) if $x \leq y$ in P_1 implies $\varphi(x) \leq \varphi(y)$ in P_2;*

(ii) *an order-embedding (written $\varphi : P_1 \hookrightarrow P_2$) if $x \leq y$ in P_1 if and only if $\varphi(x) \leq \varphi(y)$ in P_2;*

(iii) *an order-isomorphism if it is an order-embedding which maps P_1 onto P_2.*

Definition 1.1.6. *A poset in which every finite subset has an infimum and a supremum is called a lattice. A subset S of a lattice L is said to be a sublattice if $a \vee^L b \in S$ and $a \wedge^L b \in S$ for all $a, b \in S$.*

Definition 1.1.7. *A sublattice S of L is said to be regular if, for every subset $A \subseteq S$, whenever $\bigwedge^S A$ exists in S, we have $\bigwedge^S A = \bigwedge^L A$, and dually for suprema.*

Lemma 1.1.8. *Let L be a lattice. Then for all $a, b, c \in L$:*

(i) *$(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (associative laws);*

(ii) *$a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ (commutative laws);*

(iii) *$a \vee a = a$ and $a \wedge a = a$ (idempotency laws);*

(iv) *$a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ (absorption laws).*

Lemma 1.1.9. *Let L be a lattice. Then for all $a, b, c, d \in L$:*

(i) *If $a \leq b$, then $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$;*

(ii) *If $a \leq b$ and $c \leq d$, then $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$;*

(iii) *$a \leq b$ if and only if $a \vee b = b$ (equivalently, $a \wedge b = a$).*

Proof. We shall only prove statements that involve $\wedge$; dual statements for $\vee$ are analogous. (i) First note that $a \wedge c \leq a \leq b$ and $a \wedge c \leq c$. Then, $a \wedge c$ is a lower bound of $\{b, c\}$, hence $a \wedge c \leq b \wedge c$.

(ii) Using (i) we have that $a \wedge c \leq b \wedge c$ and $b \wedge c \leq b \wedge d$. By transitivity it follows that $a \wedge c \leq b \wedge d$.

(iii) If $a \leq b$, then a is a lower bound of $\{a, b\}$, hence $a \leq a \wedge b$. Since $a \wedge b \leq a$, we conclude $a \wedge b = a$. Conversely, if $a \wedge b = a$ then a is a lower bound of $\{a, b\}$, as required. □

Definition 1.1.10. *Let L_1 and L_2 be lattices. A map $\varphi : L_1 \to L_2$ is said to be:*

(i) a *lattice homomorphism* if φ is both *join-preserving* and *meet-preserving*, i.e., for every $a, b \in L_1$, $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ and $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$;

(ii) a *lattice embedding* if φ is an injective lattice homomorphism;

(iii) a *lattice isomorphism* if φ is a bijective lattice homomorphism.

Proposition 1.1.11 ([19], Proposition 2.19, p. 44)**.** *Let L_1 and L_2 be lattices and $f : L_1 \to L_2$ be a map.*

(i) *The following are equivalent:*

  (a) *f is order-preserving;*

  (b) *$f(a) \vee f(b) \leq f(a \vee b)$ for every $a, b \in L_1$;*

  (c) *$f(a \wedge b) \leq f(a) \wedge f(b)$ for every $a, b \in L_1$.*

  In particular, if f is a lattice homomorphism, then f is order-preserving.

(ii) *f is a lattice isomorphism if and only if it is an order - isomorphism.*

Definition 1.1.12. *A poset Γ is said to be directed (resp. filtered), if for every $\gamma_1, \gamma_2 \in \Gamma$ there exists $\gamma \in \Gamma$ such that $\gamma_1 \leq \gamma$ and $\gamma_2 \leq \gamma$ (resp. $\gamma \leq \gamma_1$ and $\gamma \leq \gamma_2$).*

Definition 1.1.13. *A net in a set X is a function f from a directed set Γ into X. We write $f(\gamma)$ as x_γ and the net by $(x_\gamma)_{\gamma \in \Gamma}$.*

Definition 1.1.14. *Let P be a poset and let κ be an infinite cardinal.*

(i) *P is called κ-order-complete if every non-empty κ-subset (i.e., subset of cardinality at most κ) of P has a supremum and an infimum. P is said to be order-complete if it is κ-order-complete for every infinite cardinal κ.*

(ii) *P is called κ-Dedekind-complete if every non-empty κ-subset that is bounded above has a supremum and every non-empty κ-subset that is bounded below has an infimum. P is said to be Dedekind-complete if it is κ-Dedekind-complete for every infinite cardinal κ.*

(iii) *P is called* order separable *if every non-empty subset B of P with a supremum contains a countable subset with the same supremum and, dually, every non-empty subset of P having an infimum contains a countable subset with the same infimum.*

(iv) *P is called* monotone order separable *if every directed subset admitting a supremum contains an increasing sequence with the same supremum and, dually, every filtered subset of P with an infimum contains a decreasing sequence with the same infimum.*

Clearly, every order-complete poset is Dedekind complete and every order separable poset is monotone order separable. We write *σ-Dedekind complete* when the cardinal κ is $\aleph_0$.

Example 2. *For a set X, consider the power set $\mathcal{P}(X)$ ordered by inclusion. For $A, B \in \mathcal{P}(X)$, $A \cap B$ is the infimum and $A \cup B$ is the supremum. Furthermore, $\mathcal{P}(X)$ is an order-complete lattice in which, for any collection $\{A_\alpha : \alpha \in \mathscr{A}\}$, the supremum $\bigvee_{\alpha \in \mathscr{A}} A_\alpha$ is $\bigcup_{\alpha \in \mathscr{A}} A_\alpha$, and the infimum $\bigwedge_{\alpha \in \mathscr{A}} A_\alpha$ is $\bigcap_{\alpha \in \mathscr{A}} A_\alpha$.*

In this thesis, the collection 2^X will denote the space of closed subsets of a topological space X.

Example 3 (Hyperspace)**.** *For a topological space X, let 2^X denote the space of closed subsets of X ordered by inclusion. For $A, B \in 2^X$, $A \cap B$ is the infimum and $A \cup B$ is the supremum. For a collection $\{A_\alpha :$*

$\alpha \in \mathscr{A}\}$, the supremum is $\overline{\bigcup_{\alpha \in \mathscr{A}} A_\alpha}$ and the infimum is $\bigcap_{\alpha \in \mathscr{A}} A_\alpha$. Thus, 2^X is an order-complete lattice.

Example 4 ([29], Example 3.6)**.** *Let $A := \{(x, 0) : x \in [0, 1)\}$ and $B := \{(0, y) : y \in [0, 1)\}$. Then $P := A \cup B \cup \{(1, 1)\}$ under coordinate-wise order is order-complete.*

The next proposition shows that for a poset P to be Dedekind complete, it suffices to show that every non-empty subset which is bounded above has a supremum.

Proposition 1.1.15. *A poset P is Dedekind complete if and only if every non-empty subset of P that is bounded above in P has a supremum in P.*

Proof. One direction follows by definition. Conversely, we need to show that every non-empty subset of P that is bounded below in P has an infimum in P. Let $A \subseteq P$ be bounded below. Let B be the set of lower bounds of A. Then B is non-empty and bounded above by every element of A. By hypothesis $\bigvee B$ exists in P, and $\bigvee B = \bigwedge A$. □

Theorem 1.1.16. *A lattice L is order separable if and only if it is monotone order separable.*

Proof. First assume that L is order separable and $(x_\gamma)_{\gamma \in \Gamma}$ is an increasing net with $\bigvee_{\gamma \in \Gamma} x_\gamma = x$. Then there exists a countable subset $B \subseteq \{x_\gamma : \gamma \in \Gamma\}$ satisfying $\bigvee_{y \in B} y = x$. Let $B := \{x_{\gamma_n} : n \in \mathbb{N}\}$. Define $\alpha_1 := \gamma_1$. For α_2 choose $\xi \in \Gamma$ such that $\alpha_1 \leq \xi$ and $\gamma_2 \leq \xi$. Let $\alpha_2 := \xi$. Repeating this process, we get an increasing sequence $(x_{\alpha_n})_{n \in \mathbb{N}}$ which is bounded above by x. Finally we show that $\bigvee_{n \in \mathbb{N}} x_{\alpha_n} = x$. If $x_{\alpha_n} \leq b$ for every $n \in \mathbb{N}$, then $x_{\gamma_n} \leq b$ for every $n \in \mathbb{N}$, concluding that $x \leq b$ as required.

Conversely assume L is monotone order separable. Let $A \subseteq L$ with $\sup A = x$ and $\Gamma := \{B \subseteq A : |B| < \omega\}$ ordered by inclusion. It is easy to see that Γ is directed. For each $B \in \Gamma$ let $x_B := \bigvee_{b \in B} b$ to get an increasing net $(x_B)_{B \in \Gamma}$ with supremum x. By the monotone order separability of L, there exists $\{A_n : n \in \mathbb{N}\} \subseteq \Gamma$ such that $(x_{A_n})_{n \in \mathbb{N}}$ is increasing with supremum x. Let $C := \bigcup_{n=1}^{\infty} A_n$. Then, $C \subseteq A$ is countable and $\bigvee C = x$, as required. □

Proposition 1.1.17. *Let P and Q be order isomorphic posets. If Q is order-complete, then P is order-complete as well.*

Proof. Consider the order isomorphism $\varphi : P \to Q$. Let $A \subseteq P$. Then $\varphi(A) = \{\varphi(a) : a \in A\}$ has a supremum $\varphi(p)$ in Q. Therefore p is the supremum of A in P. Indeed, since φ is an order isomorphism, we have $a \leq p$ for every $a \in A$. Moreover, if $b \in P$ is such that $a \leq b$ for every $a \in A$, then $\varphi(b)$ is an upper bound of $\varphi(A)$. Hence, $\varphi(p) \leq \varphi(b)$, which implies $p \leq b$. □

Proposition 1.1.17 also holds for other properties, such as κ-Dedekind completeness, order separability, and monotone order separability. The mapping $\varphi : P \to Q$ *preserves suprema and infima* if for every $A \subseteq P$, whenever $x_0 = \bigvee A$ exists in P, then $\bigvee\{\varphi(x) : x \in A\}$ exists in Q and

$$\varphi(x_0) = \bigvee\{\varphi(x) : x \in A\}.$$

Likewise for infima.

Given a subset A of a poset P, the set of upper bounds of A, denoted by A^+, is given by

$$A^+ := \{x \in P : a \leq x \text{ for every } a \in A\}.$$

Similarly, given a subset B of P, the set of lower bounds of B, denoted by B^-, is defined as

$$B^- := \{x \in P : x \leq b \text{ for every } b \in B\}.$$

For a poset P, if $A = \varnothing$ then $A^+ = P$. The converse also holds if P does not have a minimum. One also has that $A^+ = \varnothing$ if and only if A is not bounded above.

Similarly, for a subset B of a poset P, if $B = \varnothing$ then $B^- = P$. The converse also holds if P does not have a maximum. One also has that $B^- = \varnothing$ if and only if B is not bounded below.

For every $A \subseteq P$, we write A^{+-} for $(A^+)^-$. A subset satisfying $A = A^{+-}$ is called an *l-cut* in P. The collection of l-cuts, ordered by inclusion, is non-empty since P is an l-cut.

Lemma 1.1.18. *Let P be a poset and $A, B \subseteq P$. Then:*

(i) $A \subseteq A^{+-}$ *and* $A \subseteq A^{-+}$.

(ii) *If* $A \subseteq B$, *then* $B^+ \subseteq A^+$ *and* $B^- \subseteq A^-$.

(iii) $A^+ = A^{+-+}$ *and* $A^- = A^{-+-}$.

Proof. (i) Let $a \in A$. Then for every $b \in A^+$, $a \leq b$, showing that $a \in A^{+-}$. Similarly, one can show that $A \subseteq A^{-+}$.

(ii) Assume $A \subseteq B$ and $x \in B^+$, then $b \leq x$ for every $b \in B$. From $A \subseteq B$, it follows that $a \leq x$ for every $a \in A$, hence $x \in A^+$. Similarly one can show that $B^- \subseteq A^-$.

(iii) From (i) and (ii), it follows that $A^{+-+} \subseteq A^+$. To see the reverse inclusion, take $a \in A^+$. Then $b \leq a$ for every $b \in A^{+-}$, so that $a \in A^{+-+}$. That $A^- = A^{-+-}$ can be shown in a similar way. □

Lemma 1.1.19. *For any subset A of a poset P, the set A^{+-} is an l-cut, and A^{+-} is the smallest l-cut containing A. In particular, the set $(\leftarrow, x]$ is an l-cut.*

Proof. That A^{+-} is an l-cut follows from Lemma 1.1.18 (iii). Moreover, if B is another l-cut containing A, from Lemma 1.1.18 (ii) it follows that $A^{+-} \subseteq B^{+-} = B$. Therefore, A^{+-} is the smallest l-cut containing A. To see that $(\leftarrow, x]$ is an l-cut, it suffices to note that $\{x\}^{+-} = (\leftarrow, x]$. □

Proposition 1.1.20. *Let P be a poset and $\{A_\alpha : \alpha \in \mathscr{A}\}$ a family of subsets of P. Then $\bigcap_{\alpha\in\mathscr{A}} A_\alpha^{+-} = \left(\bigcup_{\alpha\in\mathscr{A}} A_\alpha^{+}\right)^{-}$.*

Proof. First note that $A_\alpha^+ \subseteq \bigcup_{\alpha\in\mathscr{A}} A_\alpha^+$. By Lemma 1.1.18 (ii),

$$(\bigcup_{\alpha\in\mathscr{A}} A_\alpha^+)^- \subseteq \bigcap_{\alpha\in\mathscr{A}} A_\alpha^{+-}.$$

To see the reverse inclusion, take $x \in \bigcap_{\alpha\in\mathscr{A}} A_\alpha^{+-}$. Then for every $\alpha \in \mathscr{A}$ and every $y \in A_\alpha^+$, we have $x \leq y$. In particular, $x \leq y$ for every $y \in \bigcup_{\alpha\in\mathscr{A}} A_\alpha^+$, i.e., $x \in (\bigcup_{\alpha\in\mathscr{A}} A_\alpha^+)^-$. Therefore, the reverse inclusion holds. □

Definition 1.1.21. *A subset A of a poset P is said to be join-dense in P if, for every $x \in P$, there exists $B \subseteq A$ such that $\bigvee B = x$.*

Definition 1.1.22. *A subset A of a poset P is said to be meet-dense in P if, for every $x \in P$, there exists $B \subseteq A$ such that $\bigwedge B = x$.*

Theorem 1.1.23 (MacNeille Completion)**.** *Given a non-empty poset P containing no smallest or largest element, there exists an order-complete lattice $DM(P)$ and an order embedding $\varphi : P \to DM(P)$, where $x \mapsto (\leftarrow, x]$. More precisely, $DM(P)$ is the set of all l-cuts of P, partially ordered by inclusion. In addition, P is join-dense and meet-dense in $DM(P)$. Equivalently, every l-cut A of P satisfies*

$$\Big(\bigcup_{(\leftarrow,x]\subseteq A} (\leftarrow, x]\Big)^{+-} = A = \bigcap_{A\subseteq(\leftarrow,x]} (\leftarrow, x].$$

Proof. First we show that the set of all l-cuts of P is order-complete. Let $(A_\alpha)_{\alpha\in\mathscr{A}}$ be an arbitrary subset of $DM(P)$. Let $A := (\bigcup_{\alpha\in\mathscr{A}} A_\alpha)^{+-}$. By Lemma 1.1.19, A is an l-cut, and moreover the smallest l-cut containing $(\bigcup_{\alpha\in\mathscr{A}} A_\alpha)$. If B is another l-cut containing A_α for every $\alpha \in \mathscr{A}$, it follows that $\bigcup_{\alpha\in\mathscr{A}} A_\alpha \subseteq B$. Thus by Lemma 1.1.18 (i), we get $A \subseteq B$. For the infimum, consider $C := \bigcap_{\alpha\in\mathscr{A}} A_\alpha$. The set C is an l-cut by noting that $C \subseteq A_\alpha$ for every $\alpha \in \mathscr{A}$ and

$C^{+-} \subseteq A_\alpha$ by Lemma 1.1.18 (i). Then $C^{+-} \subseteq C$ and, by Lemma 1.1.18 (i) it follows that $C = C^{+-}$. If D is another l-cut satisfying $D \subseteq A_\alpha$ for every $\alpha \in \mathscr{A}$, it follows that $D \subseteq C$. Thus C is the greatest lower bound, i.e., the infimum.

By Lemma 1.1.19, we observe that φ is an injective mapping of P into $DM(P)$. Next, we show that φ preserves suprema and infima. Let $E \subseteq P$, and let $x_0 := \sup E$. Using the above, the set $\{(\leftarrow, x] : x \in E\}$ has supremum $A = (\bigcup_{x\in E}(\leftarrow, x])^{+-}$. It suffices to show that $A = (\leftarrow, x_0]$; equivalently, that $A^+ = [x_0, \rightarrow)$. Clearly $z \in A^+$ if and only if $x \leq z$ for every $x \in E$ if and only if $x_0 \leq z$ if and only if $z \in [x_0, \rightarrow)$.

Next we show that the mapping φ preserves infima. Let $E \subseteq P$ with $x_0 := \inf E$. We know that $B := \bigcap_{x\in E}(\leftarrow, x]$ is the infimum of $\{(\leftarrow, x] : x \in E\}$, thus it suffices to show that $B = (\leftarrow, x_0]$. Indeed,

$$z \in B \iff z \leq x \quad \forall x \in E \iff z \leq x_0 \iff z \in (\leftarrow, x_0].$$

Finally, we show that every $A \in DM(P)$ satisfies

$$A = \sup\{(\leftarrow, x] : (\leftarrow, x] \subseteq A\} = \inf\{(\leftarrow, x] : A \subseteq (\leftarrow, x]\}.$$

The claim is trivial if $A = \emptyset$. Assume that $A \neq \emptyset$. Since A is an l-cut, we have $x \in A \iff (\leftarrow, x] \subseteq A$. Then,

$$A = \bigcup_{x\in A}(\leftarrow, x] = \bigcup_{(\leftarrow,x]\subseteq A}(\leftarrow, x] = \Big(\bigcup_{(\leftarrow,x]\subseteq A}(\leftarrow, x]\Big)^{+-}$$

Furthermore, $A \subseteq (\leftarrow, x]$ if and only if x is an upper bound of A. Then,

$$A = A^{+-} = \bigcap_{x\in A^+}(\leftarrow, x] = \bigcap_{A\subseteq(\leftarrow,x]}(\leftarrow, x]$$

$\square$

Theorem 1.1.24 ([19], Theorem 7.41, p. 167)**.** *Let P be a partially ordered set and let $\varphi : P \hookrightarrow DM(P)$, where $x \mapsto (\leftarrow, x]$, be the order-embedding of P into its MacNeille completion. Then,*

(i) *$\varphi(P)$ is both join-dense and meet-dense in $DM(P)$.*

(ii) *If L is an order-complete lattice and P is a subset of L which is both join-dense and meet-dense in L. Then L and $DM(P)$ are order-isomorphic via an order-isomorphism that agrees with φ on P.*

From Theorem 1.1.24 it follows that up to order isomorphism, the MacNeille completion is unique.

Example 5 ([35], p. 190)**.** *Let $P = \{x_1, x_2\}$, where x_1 and x_2 are incomparable. For $A \subseteq P$, the set A^{+-} is one of $\emptyset, \{x_1\}, \{x_2\}$, or P. Thus, $DM(P) = \{\emptyset, \{x_1\}, \{x_2\}, P\}$, which is isomorphic to the four-element Boolean lattice.*

The example above is trivial in nature; however, it highlights an important aspect of the order-completion of a poset. If we remove the element P from $DM(P)$, the order-completion may no longer be a lattice. On the other hand, if the poset P is a lattice, and we delete the smallest and largest elements from $DM(P)$, the remaining set is still a Dedekind complete lattice. For a poset P, the $DM(P)$ completion of P without its greatest and smallest elements will be denoted by P^δ.

Lemma 1.1.25 ([35], p. 190)**.** *Let L be a lattice without smallest and largest elements. Then L can be embedded as a sublattice in a Dedekind complete lattice.*

Proof. Let L be a lattice without smallest and largest elements. The set L^δ is Dedekind complete since $DM(L)$ is Dedekind complete by construction; removing the trivial bounds $\emptyset$ and L does not affect the existence of suprema and infima of non-empty bounded subsets. To see that L^δ is a lattice, take l-cuts A, B in $DM(L)$ such that $A \neq L$ and $B \neq L$. By the meet-density of L in $DM(L)$, there exist $x_1, x_2 \in L$ satisfying $A \subseteq (\leftarrow, x_1]$ and $B \subseteq (\leftarrow, x_2]$. Again, using the fact that L does not have a largest element,

$$A \vee^{L^\delta} B \subseteq (\leftarrow, x_1] \vee^{L^\delta} (\leftarrow, x_2] = (\leftarrow, x_1 \vee x_2] \neq L.$$

Thus $A \vee^{L^\delta} B \in L^\delta$. The argument for infimum follows by duality. □

Definition 1.1.26. *A lattice L is called a distributive lattice if*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all $a, b, c \in L$.

Proposition 1.1.27 ([19], Lemma 4.3, p. 85)**.** *Let L be a lattice. The following are equivalent:*

(i) *L is distributive;*

(ii) *$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.*

Proof. Assume that L is distributive. Then, by distributivity

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)\\ &= a \vee ((a \vee b) \wedge c)\\ &= a \vee (a \wedge c) \vee (b \wedge c)\\ &= a \vee (b \wedge c).\end{aligned}$$

Conversely, assume that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for every $a, b, c \in L$. Then, by the assumed identity,

$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= ((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c)\\ &= a \wedge ((a \wedge b) \vee c)\\ &= a \wedge (a \vee c) \wedge (b \vee c)\\ &= a \wedge (b \vee c).\end{aligned}$$

□

For a lattice L and fixed $a, b \in L$, define the maps $f_{(a,b)}, g_{(a,b)} : L \to L$ by

$$f_{(a,b)}(x) := (x \wedge b) \vee a, \quad g_{(a,b)}(x) := (x \vee a) \wedge b,$$

for all $x \in L$.

Proposition 1.1.28 ([3], Proposition 3.10)**.** (a) *For a lattice L, the following statements are equivalent:*

(i) *L is distributive.*

(ii) *For all $a, b \in L$, $f_{(a,b)}$ is a lattice homomorphism from L to L.*

(iii) *For all $a, b \in L$, $g_{(a,b)}$ is a lattice homomorphism from L to L.*

(b) *If L is distributive, then for all $a, b \in L$, $f_{(a,b)} = f_{(a,a\vee b)} = g_{(a,a\vee b)}$ and $g_{(a,b)} = g_{(a\wedge b,b)} = f_{(a\wedge b,b)}$.*

Proof. (i) $\Rightarrow$ (ii) is obvious. Furthermore, (ii) $\Rightarrow$ (iii) follows by noting that

$$\begin{aligned} g_{(a,b)}(x \vee y) &= (x \vee y \vee a) \wedge b \\ &= ((x \vee a) \vee (y \vee a)) \wedge b \\ &= (((x \vee a) \vee (y \vee a)) \wedge b) \vee (a \wedge b) \\ &= f_{(a\wedge b,b)}((x \vee a) \vee (y \vee a)) \\ &= f_{(a\wedge b,b)}(x \vee a) \vee f_{(a\wedge b,b)}(y \vee a) \\ &= g_{(a,b)}(x) \vee g_{(a,b)}(y). \end{aligned}$$

Similarly, one can show that $g_{(a,b)}(x \wedge y) = g_{(a,b)}(x) \wedge g_{(a,b)}(y)$. Finally, to show that (iii) $\Rightarrow$ (i), observe that for every $x, a, b \in L$, $g_{(a\wedge x,b)}(x \vee a) = g_{(a\wedge x,b)}(x) \vee g_{(a\wedge x,b)}(a)$ which is exactly the distributive law. Part (b) follows directly from the distributive property of L. □

Definition 1.1.29. *A lattice L is said to satisfy the join-infinite distributive law if for $x \in L$ and $\{x_\alpha : \alpha \in \mathscr{A}\} \subseteq L$ such that $\bigvee_{\alpha \in \mathscr{A}} x_\alpha$ exists in L, one has*

$$x \wedge \bigvee_{\alpha \in \mathscr{A}} x_\alpha = \bigvee_{\alpha \in \mathscr{A}} (x \wedge x_\alpha).$$

Definition 1.1.30. *A lattice L is said to satisfy the meet-infinite distributive law if for $x \in L$ and $\{x_\alpha : \alpha \in \mathscr{A}\} \subseteq L$ such that $\bigwedge_{\alpha \in \mathscr{A}} x_\alpha$ exists in L, one has*

$$x \vee \bigwedge_{\alpha \in \mathscr{A}} x_\alpha = \bigwedge_{\alpha \in \mathscr{A}} (x \vee x_\alpha).$$

Definition 1.1.31. *A lattice L is infinitely distributive if it satisfies both the join-infinite distributive law and the meet-infinite distributive law.*

The inequalities $x \wedge \bigvee_{\alpha \in \mathscr{A}} x_\alpha \geq \bigvee_{\alpha \in \mathscr{A}}(x \wedge x_\alpha)$ and $x \vee \bigwedge_{\alpha \in \mathscr{A}} x_\alpha \leq \bigwedge_{\alpha \in \mathscr{A}}(x \vee x_\alpha)$ hold in every lattice. If a lattice satisfies the join or meet-infinite distributive laws, then it is distributive by Proposition 1.1.27. However, not every distributive lattice satisfies the join- or meet-infinite distributive laws. The next examples exhibit distributive lattices that do not satisfy the join-infinite distributive law.

Example 6. *Let L be the collection $2^{\mathbb{R}}$ ordered by inclusion. Clearly this is an order-complete distributive lattice. For every $n \in \mathbb{N}$, consider the increasing sequence $X_n := [2^{-n}, \infty)$. Then $(\sup_{n \in \mathbb{N}} X_n) \wedge \{0\} = \overline{\bigcup_{n \in \mathbb{N}} X_n} \cap \{0\} = [0, \infty) \cap \{0\} = \{0\}$ but $\sup_{n \in \mathbb{N}}(X_n \wedge \{0\}) = \emptyset$.*

Example 7. *Let $\mathbb{Z}^+ \cup \{0\}$ be ordered by $a \leq b$ if a divides b. This is an order-complete distributive lattice where 1 and 0 are the minimal and maximal elements, respectively. However, it is not join-infinite distributive. Let $A := \{p : p \text{ is odd prime}\}$. Then, $2 \wedge (\bigvee_{p \in A} p) = 2$, but $\bigvee_{p \in A}(2 \wedge p) = 1$.*

We conclude this section with a brief mention of ideals and filters. For a detailed treatment, including their role in Stone's representation theorem for Boolean algebras, see [19].

Definition 1.1.32. *Let P be a poset and $Q \subseteq P$.*

(i) *Q is a down-set if, whenever $x \in Q, y \in P$ and $y \leq x$, then $y \in Q$.*

(ii) *Dually, Q is an up-set if, whenever $x \in Q, y \in P$ and $x \leq y$, then $y \in Q$.*

Definition 1.1.33. *Let L be a lattice. A non-empty subset $\mathcal{I}$ is called an order ideal if it is a down-set satisfying $a \vee b \in \mathcal{I}$ for every $a, b \in \mathcal{I}$. It is said to be proper if $\mathcal{I} \neq L$.*

Proposition 1.1.34. *Let L be a lattice. Every order ideal $\mathcal{I}$ is a regular sublattice of L.*

Proof. Let $A \subseteq \mathcal{I}$ and $a := \bigvee^{\mathcal{I}} A$ exists. Let b be an upper-bound of A in L. Then $a \wedge b$ is an upper-bound of A in $\mathcal{I}$, concluding that $a = \bigvee^{L} A$. □

The following example shows that the converse of Proposition 1.1.34 is not necessarily true.

Example 8. *Consider the lattice $\mathbb{R}^2$ ordered coordinate-wise, i.e., $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. Take $S := \{(x, 0) : x \in \mathbb{R}\}$, then S is a regular sublattice but not an order ideal in $\mathbb{R}^2$ because it is not a down-set.*

The set of order ideals will be denoted by $\mathscr{I}(L)$. An order ideal is said to be *prime* if $a, b \in L$ and $a \wedge b \in \mathcal{I}$, then $a \in \mathcal{I}$ or $b \in \mathcal{I}$. The set of prime order ideals is denoted by $\mathscr{I}_p(L)$. When there is no ambiguity, the set of order ideals and prime order ideals will be denoted by $\mathscr{I}$ and $\mathscr{I}_p$, respectively.

Lemma 1.1.35. *Let L be a lattice. Then the intersection of two order ideals is also an order ideal.*

However, the following example shows that the intersection of two prime order ideals is not necessarily a prime order ideal.

Example 9. *Let $\mathcal{P}(X)$ be the power set of a set X ordered by inclusion. Let x and y be distinct elements of X and consider the prime order ideals $\mathcal{I}_x = \{A \subseteq X : x \notin A\}$ and $\mathcal{I}_y = \{A \subseteq X : y \notin A\}$. It should be clear that $\mathcal{I}_x \cap \mathcal{I}_y$ is not a prime order ideal.*

Definition 1.1.36. *Let L be a lattice. A non-empty subset $\mathcal{F}$ is called a filter if*

(i) *$a, b \in \mathcal{F}$ implies $a \wedge b \in \mathcal{F}$,*

(ii) *$a \in L$, $b \in \mathcal{F}$ and $b \leq a$ implies $a \in \mathcal{F}$.*

It is said to be proper if $\mathcal{F} \neq L$.

A filter is said to be prime if, whenever $a, b \in L$ and $a \vee b \in \mathcal{F}$, one has $a \in \mathcal{F}$ or $b \in \mathcal{F}$. The set of prime filters is denoted by $\mathscr{F}_p(L)$. When the lattice is clear from the context, we simply write $\mathscr{F}$ and $\mathscr{F}_p$ for the set of filters and prime filters, respectively.

Lemma 1.1.37. *A subset $\mathcal{I}$ of a lattice L is a proper prime order ideal if and only if $L \setminus \mathcal{I}$ is a prime filter.*

Definition 1.1.38. *Let L be a lattice, and let $\mathcal{I}$ be a proper order ideal. $\mathcal{I}$ is said to be a maximal order ideal if the only order ideal strictly containing $\mathcal{I}$ is L. Equivalently, $\mathcal{I}$ is a maximal order ideal if and only if it is a maximal element of $\mathscr{I} \setminus \{L\}$ ordered under inclusion. Dually, one defines a maximal filter, which is also known as an ultrafilter.*

Theorem 1.1.39 ([19], Lemma 10.11)**.** *Let L be a distributive lattice. Then every maximal order ideal in L is prime, and every ultrafilter is a prime filter.*

1.2 Riesz Spaces

In this section, we present an overview of Riesz spaces and establish several well-known properties.

Definition 1.2.1. *An ordered vector space is a real vector space $\mathscr{V}$ endowed with a partial order that is compatible with the vector space structure of $\mathscr{V}$; i.e.,*

(i) $x \leq y$ *implies* $x + z \leq y + z$ *for all* $x, y, z \in \mathscr{V}$*;*

(ii) $x \geq 0$ *implies* $\alpha x \geq 0$ *for all* $x \in \mathscr{V}$ *and all* $\alpha \geq 0$ *in* $\mathbb{R}$*.*

Proposition 1.2.2. *Let $\mathscr{V}$ be an ordered vector space. Then the following properties hold:*
(i) If $x \leq y$, then $\alpha x \leq \alpha y$ for all $x, y \in \mathscr{V}$ and all $\alpha \geq 0$ in $\mathbb{R}$.
(ii) If $x \leq y$, then $\alpha y \leq \alpha x$ for all $x, y \in \mathscr{V}$ and all $\alpha < 0$ in $\mathbb{R}$.
(iii) If $w \leq x$ and $y \leq z$, then $w + y \leq x + z$ for all $w, x, y, z \in \mathscr{V}$.

Proof. (i) Observe that $y - x \geq 0$ since $x \leq y$. By the definition of ordered vector space, $0 \leq \alpha(y - x) = \alpha y - \alpha x$, hence $\alpha x \leq \alpha y$.

(ii) From (i) and the fact that $-\alpha \geq 0$, we obtain $-\alpha x \leq -\alpha y$. It follows that $\alpha y \leq \alpha x$.

(iii) Observe that $w + y \leq x + y$ by the definition of an ordered vector space. Moreover, by the fact that $y \leq z$, it follows that $x + y \leq x + z$. By the transitivity of $\leq$, we obtain that $w + y \leq x + z$. $\square$

We now turn to Riesz spaces (also called vector lattices).

Definition 1.2.3. *A Riesz space V is an ordered vector space that is also a lattice, in which the lattice operations are compatible with the vector space structure.*

For an element $x \in V$, the positive part of x is defined by $x^+ := x \vee 0$, and the negative part of x is defined by $x^- := (-x) \vee 0$. The absolute value of x is defined by $|x| := x \vee (-x)$. An element $x \in V$ is said to be *positive* if $x \geq 0$. The set of positive elements of a Riesz space V is denoted by V_+ , equivalently $V_+ := \{u \in V : u \geq 0\}$. This set is called the *positive cone.*

Proposition 1.2.4. *Let V be a Riesz space. Then, for all $x, y, z \in V$ and all scalars $\alpha \geq 0$, the following hold:*
(i) $x \vee y = -((-x) \wedge (-y))$ and $x \wedge y = -((-x) \vee (-y))$;
(ii) $x + y = (x \wedge y) + (x \vee y)$;
(iii) $x + (y \vee z) = (x + y) \vee (x + z)$ and $x + (y \wedge z) = (x + y) \wedge (x + z)$;
(iv) $\alpha(x \vee y) = \alpha x \vee \alpha y$ and $\alpha(x \wedge y) = \alpha x \wedge \alpha y$.

Proof. (i) First note that $x \wedge y \leq x$ and $x \wedge y \leq y$. It follows that $-y \leq -(x \wedge y)$ and $-x \leq -(x \wedge y)$. Thus $(-x) \vee (-y) \leq -(x \wedge y)$. Now let z be another upper bound of $-x$ and $-y$, then $-z \leq x$ and $-z \leq y$, hence $-z \leq x \wedge y$. This implies $-(x \wedge y) \leq z$. Therefore $-(x \wedge y)$ is the supremum of $-x$ and $-y$. The second identity is proved analogously.

(ii) From $x \wedge y \leq y$ it follows that $y - x \wedge y \geq 0$. Adding x to both sides yields $x + y - (x \wedge y) \geq x$. Similarly, from $x \wedge y \leq x$ we obtain $x + y - (x \wedge y) \geq y$. It follows that $x + y - (x \wedge y) \geq x \vee y$. Equivalently, $(x \wedge y) + (x \vee y) \leq x + y$. On the other hand, from the fact that $x \leq x \vee y$, subtracting $x \vee y$ yields $x - (x \vee y) \leq 0$ and adding y to both sides yields $x + y - (x \vee y) \leq y$. Similarly, from $y \leq x \vee y$, we obtain $x + y - (x \vee y) \leq x$. Then $x + y - (x \vee y) \leq x \wedge y$. Combining both inequalities, we conclude that $(x \wedge y) + (x \vee y) = x + y$.

(iii) First note that $x + (y \wedge z) \leq x + y$ and $x + (y \wedge z) \leq x + z$. Hence, $x + (y \wedge z) \leq (x + y) \wedge (x + z)$. Next observe that $y = (-x) + (x + y) \geq -x + (x + y) \wedge (x + z)$, and similarly, $z = (-x) + (x + z) \geq -x + (x + z) \wedge (x + y)$. Thus, $-x + (x + z) \wedge (x + y) \leq y \wedge z$. Combining both inequalities, we conclude that $x + (y \wedge z) = (x + y) \wedge (x + z)$. The other equality is proved analogously.

(iv) For $\alpha = 0$, the result is trivial. For $\alpha > 0$ we observe that $\alpha x \wedge \alpha y \geq \alpha(x \wedge y)$. Next let z be any lower bound of αx and αy. Then, $\frac{z}{\alpha} \leq x$ and $\frac{z}{\alpha} \leq y$, so $\frac{z}{\alpha} \leq x \wedge y$, concluding that $z \leq \alpha(x \wedge y)$. The dual statement follows analogously. $\square$

Theorem 1.2.5 (Riesz Decomposition Inequality)**.** *Let $x, x_1, \ldots, x_n$ be positive elements of a Riesz space V. Then:*

$$x \wedge \Big(\sum_{i=1}^{n} x_i\Big) \leq \sum_{i=1}^{n}(x \wedge x_i).$$

Proposition 1.2.6. *Let V be a Riesz space, $A, B \subseteq V$, $x \in V$, and $\alpha \geq 0$. Then:*

(i) $\sup(x + A) = x + \sup A$, *whenever one of them exists.*

(ii) $\sup(-A) = -\inf(A)$, *whenever one of them exists.*

(iii) $\sup(A + B) = \sup(A) + \sup(B)$, *whenever the right hand side exists.*

(iv) $\sup(\alpha A) = \alpha \sup(A)$, *whenever the right hand side exists.*

Proof. (i) Let $y := \sup(x+A)$. Then for every $a \in A$ we have $a+x \leq y$, hence $a \leq y-x$. Now let w be another upper bound of A. Then $a \leq w$ for all $a \in A$, so $a + x \leq x + w$ for all $a \in A$. Therefore $y \leq w + x$, which implies $y - x \leq w$. Hence $y - x = \sup(A)$, or equivalently $\sup(x + A) = x + \sup(A)$.

(ii) Let $y := \sup(-A)$. Then for every $a \in A$ we have $-a \leq y$. Hence, multiplying both sides by -1, we get $-y \leq a$ for all $a \in A$. Let x be any lower bound of A. Then $x \leq a$ and multiplying both sides by -1 gives $-a \leq -x$ for all $a \in A$. Thus $-x$ is an upper bound of $-A$, hence $y \leq -x$ and multiplying by -1 yields $x \leq -y$, as required.

(iii) Let $x := \sup(A)$ and $y := \sup(B)$. Then for every $a \in A$ and $b \in B$, we have that $a + b \leq x + y$. Let z be another upper bound of $A + B$. Then for all $a \in A$ and $b \in B$, we have $a + b \leq z$. Hence, for each fixed $b \in B$, the element $z - b$ is an upper bound of A, so $x \leq z - b$. Thus, $x + b \leq z$ for every $b \in B$ so that $b \leq z - x$ for every $b \in B$. Consequently, $y \leq z - x$ which implies that $x + y \leq z$.

(iv) For $\alpha = 0$, the result is obvious. Let $\alpha > 0$ then the right hand side exists if and only if $t = \sup(A)$ exists. Let $x = \alpha t$, then $\alpha a \leq x$, so x is an upper bound of αA. Let y be another upper bound of αA, then $\alpha a \leq y$ for every $a \in A$. This implies that $a \leq \alpha^{-1} y$ for every $a \in A$, concluding that $t \leq \alpha^{-1} y$. Then $x = \alpha t \leq y$, as required. □

Theorem 1.2.7. *Let V be a Riesz space and $x \in V$. Then,*

(i) $x = x^+ - x^-$.

(ii) $|x| = x^+ + x^-$.

(iii) $x^+ \wedge x^- = 0$.

Proof. (i) From Proposition 1.2.4 (ii) and Proposition 1.2.6 (ii) it follows that

$$x + 0 = x \vee 0 + x \wedge 0 = x \vee 0 - (-x \vee 0) = x^+ - x^-.$$

(ii) Using Proposition 1.2.4 (iii), (iv) and (i) it follows that

$$\begin{aligned} |x| = x \vee -x &= -x + (2x \vee 0) \\ &= -x + 2x^+ \\ &= 2x^+ - x \\ &= 2x^+ - (x^+ - x^-) \\ &= x^+ + x^-. \end{aligned}$$

(iii) Observe that

$$\begin{aligned} x^+ \wedge x^- &= (x^+ - x^-) \wedge 0 + x^- \\ &= x \wedge 0 + x^- \\ &= -((-x) \vee 0) + x^- \\ &= -x^- + x^- \\ &= 0. \end{aligned}$$

□

Theorem 1.2.8. *Let $x, y \in V$, where V is a Riesz space. Then:*

(i) $x = (x-y)^+ + (x \wedge y)$;

(ii) $x \vee y = \frac{1}{2}(x + y + |x-y|)$ *and* $x \wedge y = \frac{1}{2}(x + y - |x-y|)$;

(iii) $|x-y| = (x \vee y) - (x \wedge y)$;

(iv) $|x| \vee |y| = \frac{1}{2}(|x+y| + |x-y|)$;

(v) $|x| \wedge |y| = \frac{1}{2}\big(\big||x+y| - |x-y|\big|\big)$;

(vi) $|x+y| \vee |x-y| = |x| + |y|$.

Proof. (i) From Proposition 1.2.4 (ii), $x + y = x \vee y + x \wedge y$, so

$$x = (x \vee y) + (x \wedge y) - y = (x-y) \vee 0 + (x \wedge y) = (x-y)^+ + (x \wedge y).$$

(ii) We shall prove that $x \wedge y = \frac{1}{2}(x+y-|x-y|)$, the other identity similarly follows.

$$\begin{aligned} x + y - |x-y| &= x + y - ((x-y) \vee (y-x)) \\ &= x + y + (y-x) \wedge (x-y) \\ &= (x + y + y - x) \wedge (x + y + x - y) \\ &= 2y \wedge 2x = 2(x \wedge y). \end{aligned}$$

(iii) Add and subtract the identities in (ii).

(iv) Since $|x+y| = (x+y) \vee (-x-y)$, we obtain

$$\begin{aligned} |x+y| + |x-y| &= ((x+y) \vee (-x-y)) + |x-y| \\ &= ((x + y + |x-y|) \vee (-x - y + |x-y|)) \\ &= 2((x \vee y) \vee (-x \vee -y)) \\ &= 2((x \vee -x) \vee (y \vee -y)) \\ &= 2(|x| \vee |y|). \end{aligned}$$

(v) Using (ii) and (iv) twice, we obtain

$$\big||x+y| - |x-y|\big| = 2(|x+y| \vee |x-y|) - (|x+y| + |x-y|)$$

$$
\begin{aligned}
&= 2(|x| + |y|) - 2(|x| \vee |y|) \\
&= 2((|x| + |y| - (|x| \vee |y|))) \\
&= 2((|x| + |y| + (-|x| \wedge -|y|))) \\
&= 2((|x| + |y| - |x|) \wedge (|x| + |y| - |y|)) \\
&= 2(|y| \wedge |x|).
\end{aligned}
$$

(vi) We use (iii) and (v) to get

$$
\begin{aligned}
|x+y| \vee |x-y| &= \big||x+y| - |x-y|\big| + |x+y| \wedge |x-y| \\
&= 2(|x| \wedge |y|) + \big||x| - |y|\big| \\
&= 2(|x| \wedge |y|) + (|x| \vee |y| - |x| \wedge |y|) \\
&= |x| \wedge |y| + |x| \vee |y| \\
&= |x| + |y|.
\end{aligned}
$$

□

Theorem 1.2.9 ([10], p. 296)**.** *For all x, y, z in a Riesz space V, we have*

$$|x \vee z - y \vee z| + |x \wedge z - y \wedge z| = |x - y|.$$

Corollary 1.2.10 (Birkhoff's Inequalities)**.** *For all x, y, z in a Riesz space V, we have*

$$|x \vee z - y \vee z| \leq |x - y| \quad \textit{and} \quad |x \wedge z - y \wedge z| \leq |x - y|.$$

Definition 1.2.11. *Let V be a Riesz space.*

(i) *A vector subspace W is called a Riesz subspace if it is a sublattice of V.*

(ii) *A Riesz subspace W of V is called regular if it is a regular sublattice of V.*

(iii) *V is called Archimedean if, for $x, y \in V^+$, the condition $nx \leq y$ for all $n \in \mathbb{N}$ implies $x = 0$.*

(iv) *A Riesz subspace W of V is called order dense in V if, for every $0 < x \in V$ there exists $w \in W$ such that $0 < w \leq x$.*

(v) *A Riesz subspace W of V is called majorizing in V if, for every $x \in V$ there exists $w \in W$ such that $x \leq w$.*

(vi) *A subset $A \subseteq V$ is solid if, whenever $w \in A$ and $u \in V$ satisfy $|u| \leq |w|$, then $u \in A$.*

(vii) *A solid vector subspace $U \subseteq V$ is called a Riesz order ideal.*

Lemma 1.2.12. *Every Riesz subspace W of a Dedekind complete Riesz space V is Archimedean. In particular, every Dedekind complete Riesz space is Archimedean.*

Proof. Let $x, y \in W_+$ such that $nx \leq y$ for every $n \in \mathbb{N}$. By Dedekind completeness, let $z := \bigvee_{n\in\mathbb{N}}^{V} nx$ in V. Then, $nx = (n+1)x - x \leq z - x$ for every $n \in \mathbb{N}$, concluding that $x \leq 0$, i.e., $x = 0$. ☐

Theorem 1.2.13. *Let W be a Riesz order ideal in a Riesz space V. Then, W is a Riesz subspace of V.*

Proof. From the definition of a Riesz order ideal, we have that $|w| \in W$ for every $w \in W$. In particular, for any $u, v \in W$, it follows that $|u - v| \in W$. By Theorem 1.2.8 (ii) , this implies that $u \vee v, u \wedge v \in W$. Hence, W is a sublattice of V. Since W is already a vector subspace, it follows that W is a Riesz subspace of V. ☐

Proposition 1.2.14 ([25], Proposition 353J, p. 239)**.** *Let W be a Riesz order ideal in a Riesz space V.*

(i) *If V is Dedekind complete (resp. σ-Dedekind complete) then so is W;*

(ii) *If V is order separable then W is also order separable.*

Remark 1.2.15. *It is well known that if V is a vector space and W is a vector subspace of V, then one can consider the quotient space $V/W = \{v^\bullet : v \in V\}$ where $v^\bullet := v + W$. The space V/W becomes a* *vector space when considering the operations* $(u+v)^\bullet = u^\bullet + v^\bullet$ *and* $(\alpha v)^\bullet = \alpha v^\bullet$ *for all* $u, v \in V$ *and* $\alpha \in \mathbb{R}$*. Moreover, if* V *is a partially ordered set, one can define the relation* $\preceq$ *on* V/W *by defining* $v_1^\bullet \preceq v_2^\bullet$ *when there exists* $w_1, w_2 \in W$ *such that* $v_1 + w_1 \leq v_2 + w_2$ *in* V*. When there is no risk of ambiguity, we shall also write* $\leq$ *instead of* $\preceq$*.*

Theorem 1.2.16. *Let* W *be a Riesz order ideal of a Riesz space* V*. Then the quotient space* V/W *with the induced order, is a Riesz space.*

Proof. We start by showing that $\leq$ is a partial order. It is easy to see that it is reflexive. To see that it is antisymmetric, assume that $v_1^\bullet \leq v_2^\bullet$ and $v_2^\bullet \leq v_1^\bullet$. Then there exists $w_1, w_2 \in W$ such that $v_1 - v_2 \leq w_1$ and $v_2 - v_1 \leq w_2$. Hence, $|v_2 - v_1 + w_1| \leq |w_1 + w_2|$. Since W is a Riesz order ideal, $v_2 - v_1 + w_1 \in W$ and so $v_2 - v_1 \in W$, thus, $v_1^\bullet = v_2^\bullet$. To see that it is transitive, suppose $v_1^\bullet \leq v_2^\bullet$ and $v_2^\bullet \leq v_3^\bullet$. Then there exist $w_1, w_2 \in W$ such that $v_1 - v_2 \leq w_1$ and $v_2 - v_3 \leq w_2$. Adding these inequalities yields $v_1 - v_3 \leq w_1 + w_2$, so $v_1^\bullet \leq v_3^\bullet$. Thus, V/W is a partially ordered set.

Next we show that this partial order is compatible with the vector space structure of V/W. Assume $v_1^\bullet \leq v_2^\bullet$ and $v_3^\bullet \in V/W$. Then there exists $w \in W$ such that $v_1 - v_2 \leq w$. It follows that $(v_1 + v_3) - (v_3 + v_2) \leq w$, and therefore $(v_1 + v_3)^\bullet \leq (v_3 + v_2)^\bullet$. Furthermore, suppose $0^\bullet \leq v^\bullet$ and let $\alpha \in \mathbb{R}$ with $\alpha \geq 0$. Then, $-v \leq w$ for some $w \in W$, which implies $-\alpha v \leq \alpha w$. Hence $0^\bullet \leq (\alpha v)^\bullet$. Thus V/W is an ordered vector space. Finally, if $v_1^\bullet \leq v_2^\bullet$ and $\alpha \in \mathbb{R}$ with $\alpha \geq 0$, it is easy to see that $(\alpha v_1)^\bullet \leq (\alpha v_2)^\bullet$.

We are left to show that V/W is a lattice, where $v_1^\bullet \vee v_2^\bullet := (v_1 \vee v_2)^\bullet$ and $v_1^\bullet \wedge v_2^\bullet := (v_1 \wedge v_2)^\bullet$. Clearly $v_1^\bullet \leq (v_1 \vee v_2)^\bullet$ and $v_2^\bullet \leq (v_1 \vee v_2)^\bullet$. If $v^\bullet$ is another upper bound of $v_1^\bullet$ and $v_2^\bullet$, there exist $w_1, w_2 \in W$ such that $v_1 - v \leq w_1$ and $v_2 - v \leq w_2$. By Proposition 1.2.4 (iii), $(v_1 - v) \vee (v_2 - v) = (v_1 \vee v_2) - v \leq w_1 \vee w_2 \in W$, hence $v_1^\bullet \vee v_2^\bullet = (v_1 \vee v_2)^\bullet$. The statement for infimum follows dually. $\square$

The Riesz space in Theorem 1.2.16 is called the *quotient Riesz space* of V by W.

Definition 1.2.17. *A linear operator $T : V \to U$ between Riesz spaces U and V is said to be*

(i) *a Riesz homomorphism if $T(x \vee y) = T(x) \vee T(y)$ for all $x, y \in V$.*

(ii) *Positive if $T(x) \geq 0$ whenever $x \geq 0$.*

The following three propositions are easy to show.

Proposition 1.2.18. *Let $T : V \to U$ be a linear operator between Riesz spaces U and V. The following statements are equivalent:*

(i) *T is a Riesz homomorphism.*

(ii) *$T(x) \vee T(y) = 0_U$ whenever $x, y \in V$ satisfy $x \vee y = 0_V$.*

(iii) *$T(x \wedge y) = T(x) \wedge T(y)$ for all $x, y \in V$.*

(iv) *$T(x) \wedge T(y) = 0_U$ whenever $x, y \in V$ satisfy $x \wedge y = 0_V$.*

Proposition 1.2.19. *Let V and W be Riesz spaces, and $T : V \to W$ be a Riesz homomorphism. Then:*

(i) *T is positive;*

(ii) *$T(x^+) = (T(x))^+$, $T(x^-) = (T(x))^-$, and $T(|x|) = |T(x)|$ for all $x \in V$;*

(iii) *$\ker(T)$ is a Riesz order ideal in V.*

Definition 1.2.20. *A Riesz homomorphism between Riesz spaces V and W is called a σ-Riesz homomorphism if $T(v) = \bigvee^W_{n \in \mathbb{N}} T(v_n)$ whenever $v = \bigvee^V_{n \in \mathbb{N}} v_n$.*

Remark 1.2.21. *If W is a σ-order-complete Riesz order ideal in V, then the quotient map $T : V \to V/W$ is a Riesz homomorphism. Moreover, since $Ker(T) = W$, it follows that T is also a σ-Riesz-homomorphism.*

Theorem 1.2.22. *Let W be a σ-order-complete Riesz order ideal of a Riesz space V. If V is σ-Dedekind-complete, then so is V/W.*

Proof. Let $B = \{b_n^\bullet : n \in \mathbb{N}\}$ be a subset of V/W bounded above by $b^\bullet$. For each $n \in \mathbb{N}$, there exists $w_n \in W$ such that $b_n - b \le w_n$. Hence the set $\{b_n - w_n : n \in \mathbb{N}\}$ is bounded above in V and therefore has a supremum $v \in V$. By Remark 1.2.21, the quotient map is a σ-Riesz homomorphism, and thus $v^\bullet = \left(\bigvee_{n\in\mathbb{N}}^{V}(b_n - w_n)\right)^\bullet = \bigvee_{n\in\mathbb{N}}^{V/W} b_n^\bullet$. □

Theorem 1.2.23. *Let V be a Riesz space, W a Riesz order ideal in V, and U a linear subspace of W. Then:*

(i) *U is a Riesz order ideal in W if and only if U is a Riesz order ideal in V;*

(ii) *If U is a Riesz order ideal in V, then W/U is an order ideal in V/U.*

Proof. (i) If U is a Riesz order ideal in V, then it is trivially a Riesz order ideal in W. Conversely, assume that U is a Riesz order ideal in W. Suppose $v \in V$ and $u \in U$ satisfy $|v| \le |u|$. Since $U \subseteq W$ and W is a Riesz order ideal in V, it follows that $v \in W$. Finally, because U is a Riesz order ideal in W, it follows that $v \in U$.

(ii) Let $v^\bullet \in V/U$ and $w^\bullet \in W/U$ satisfy $|v^\bullet| \le |w^\bullet|$. Then $(|v|)^\bullet \le (|w|)^\bullet$, which means that $|v| - |w| \le u$ for some $u \in U$. As W is a Riesz order ideal in V and $|v| \le u + |w|$ with $u \in W$, it follows that $v \in W$. Therefore $v^\bullet \in W/U$, and hence W/U is a Riesz order ideal in V/U. □

1.3 The space L^0

Definition 1.3.1. *Let X be a set. A collection Σ of subsets of X is called a* σ-algebra on X *if:*

(i) *$X \in \Sigma$;*

(ii) *$X \setminus E \in \Sigma$ for every $E \in \Sigma$;*

(iii) $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$ *for every sequence* $(E_n)_{n\in\mathbb{N}}$ *in* Σ.

Elements of a σ*-algebra are called measurable subsets of* X*, and the pair* (X, Σ) *is called a measurable space.*

Definition 1.3.2. *A map* $f : X \to Y$ *from a measurable space* (X, Σ) *into a topological space* $(Y, \mathscr{T})$ *is said to be* measurable *if* $f^{-1}(V) \in \Sigma$ *for every* $V \in \mathscr{T}$.

Let (X, Σ) be a measurable space. The set of measurable functions $f : X \to \mathbb{R}$ will be denoted by $\mathscr{L}^0(X, \Sigma)$, or simply by $\mathscr{L}^0$. For $f, g \in \mathscr{L}^0$, define a relation $\leq$ on $\mathscr{L}^0$ by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. Clearly, $\leq$ is a partial order on $\mathscr{L}^0$, which we shall refer to it as the *pointwise partial order* on $\mathscr{L}^0$.

Theorem 1.3.3. *Equipped with pointwise addition, multiplication, and scalar multiplication, together with the pointwise partial order, the space* $\mathscr{L}^0$ *is a* σ*-Dedekind complete Riesz space.*

Definition 1.3.4. *Let* (X, Σ) *be a measurable space. A* measure *on* X *is a function* $\mu : \Sigma \to [0, \infty]$ *such that:*

(i) $\mu(\emptyset) = 0$;

(ii) *for every sequence* $(E_n)_{n\in\mathbb{N}}$ *of pairwise disjoint sets in* Σ, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

In this case, the triple (X, Σ, μ) *is called a measure space. When the measure* μ *is understood from the context, we denote by* Σ^f *the collection* $\{E \in \Sigma : 0 < \mu(E) < \infty\}$. *i.e., the sets of positive finite measure.*

Definition 1.3.5. *Let* (X, Σ, μ) *be a measure space. Then* $N \subseteq X$ *is called negligible if there is a set* $E \in \Sigma$ *such that* $N \subseteq E$ *and* $\mu(E) = 0$. *A set* $A \subseteq X$ *is called conegligible if the complement* $X \setminus A$ *is negligible.*

If (X, Σ, μ) is a measure space and $P(x)$ is a statement concerning elements x of a subset $A \subseteq X$, we say that $P(x)$ *holds almost everywhere on* A if $P(x)$ is true for all $x \in A$ except possibly on a negligible subset of A. In particular, if $A = X$, we simply say that $P(x)$ *holds almost everywhere*, often abbreviated as "$P(x)$ a.e.". The set of real-valued measurable functions that vanish almost everywhere on X, i.e $\{f \in \mathscr{L}^0 : f(x) = 0 \text{ a.e.}\}$ is denoted by $\mathcal{N}(X, \Sigma, \mu)$ or simply $\mathcal{N}$.

Proposition 1.3.6. *Let (X, Σ, μ) be a measurable space, $\mathcal{N}$ is a σ-order-complete Riesz order ideal of $\mathscr{L}^0$.*

For a measure space (X, Σ, μ), the quotient space $\mathscr{L}^0/\mathcal{N} = \{f^\bullet : f \in \mathscr{L}^0\}$ is denoted by $L^0(X, \Sigma, \mu)$, or simply L^0. Clearly, $f^\bullet \leq g^\bullet$ in L^0 if and only if $f(x) \leq g(x)$ a.e.

Theorem 1.3.7 ([23], 241G(a), p. 137)**.** *L^0 is an Archimedean, σ-Dedekind complete Riesz space.*

Proof. That L^0 is a σ-Dedekind complete Riesz space follows by Theorems 1.3.6, 1.3.3, and 1.2.22. That it is Archimedean follows by Lemma 1.2.12. □

Definition 1.3.8. *Let (X, Σ, μ) be a measure space and let $\mathcal{E} \subseteq \Sigma$. A set $H \in \Sigma$ is called an essential supremum of $\mathcal{E}$ if:*

(i) *$E \setminus H$ is negligible for every $E \in \mathcal{E}$; and*

(ii) *whenever $G \in \Sigma$ such that $E \setminus G$ is negligible for every $E \in \mathcal{E}$, it follows that $H \setminus G$ is negligible.*

Definition 1.3.9. *A measure space (X, Σ, μ) is said to be:*

(i) *complete if every negligible subset of X is measurable;*

(ii) *totally finite if $\mu(X) < \infty$;*

(iii) *σ-finite if there exists a sequence $(E_n)_{n\in\mathbb{N}}$ in Σ^f that covers X, that is $X = \bigcup_{n\in\mathbb{N}} E_n$;*

(iv) *semi-finite if for every $E \in \Sigma$ with $\mu(E) > 0$ there exists an $F \in \Sigma^f$ such that $F \subseteq E$;*

(v) *localisable if it is semi-finite and every subcollection of Σ has an essential supremum.*

The following theorem provides two conditions under which L^0 is Dedekind complete and order separable, respectively.

Theorem 1.3.10 ([23], Theorem 241G(b), Exercise 241Y(d), p. 140)**.** *Let (X, Σ, μ) be a semi-finite measure space. Then,*

(i) *$\mathscr{L}^0$ is Dedekind complete if and only if (X, Σ, μ) is localisable;*

(ii) *L^0 is order separable if and only if (X, Σ, μ) is σ-finite.*

Corollary 1.3.11. *Let (X, Σ, μ) be a semi-finite measure space. Then, L^0 is Dedekind complete if and only if (X, Σ, μ) is localisable.*

1.4 Integrability

In this section, we briefly discuss integrable functions. For a detailed exposition of this topic, see [18, 24].

Definition 1.4.1. *Let (X, Σ, μ) be a measure space. An element $f \in \mathscr{L}^0$ is said to be* integrable *if $\int |f|\, d\mu < \infty$.*

Remark 1.4.2. *Let $p \geq 1$ be a real number. Denote by $\mathscr{L}^p$ the set of measurable p^{th}-power integrable real functions; that is, $\mathscr{L}^p$ is the set of $f \in \mathscr{L}^0$ such that $\int |f|^p\, d\mu < \infty$.*

Define the following function on $\mathscr{L}^p$ for $1 \leq p < \infty$:

$$\rho_p : \ \mathscr{L}^p \longrightarrow [0, \infty) : f \longmapsto \left(\int |f|^p\, d\mu \right)^{\frac{1}{p}}.$$

Then ρ_p is non-negative and $\rho_p(f) = 0$ if and only if $f \in \mathcal{N}$. Moreover, if $f \in \mathscr{L}^p$ and $\alpha \in \mathbb{R}$, then $(\rho_p(\alpha f))^p = |\alpha|^p \int |f|^p\, d\mu$, so that $\rho_p(\alpha f) = |\alpha| \rho_p(f)$. Finally, for any $f, g \in \mathscr{L}^1$,

$$\rho_1(f+g) = \int |f+g|\, d\mu \leq \int |f| + |g|\, d\mu = \rho_1(f) + \rho_1(g).$$

Theorem 1.4.3 (*Cauchy-Minkowski Inequality*)**.** *Let $1 \leq p < \infty$ and $f, g \in \mathscr{L}^p$. Then $f + g \in \mathscr{L}^p$ and $\rho_p(f+g) \leq \rho_p(f) + \rho_p(g)$.*

If $p > 1$ is a real number, then there exists a $q = \frac{p}{(p-1)} > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The numbers p and q are called *conjugate exponents.*

Theorem 1.4.4 (*Hölder's Inequality*)**.** *Let $1 < p, q < \infty$ be conjugate exponents. If $f \in \mathscr{L}^p$ and $g \in \mathscr{L}^q$, then fg is integrable and $\rho_1(fg) \leq \rho_p(f)\rho_q(g)$.*

Theorem 1.4.5. *Let (X, Σ, μ) be a measure space and let $1 \leq p < \infty$. Then $\mathscr{L}^p$ is a Riesz order ideal of $\mathscr{L}^0$ containing $\mathcal{N}$.*

As the space $\mathscr{L}^0$ is σ-Dedekind complete, it follows from Theorem 1.4.5 and Proposition 1.2.14 that $\mathscr{L}^p$ is also σ-Dedekind complete.

Following remark 1.2.15, we consider the quotient space $\mathscr{L}^p/\mathcal{N} := \{f^\bullet : f \in \mathscr{L}^p\}$ denoted by $L^p(X, \Sigma, \mu)$ or simply L^p, where $f^\bullet := \{g \in \mathscr{L}^p : f - g \in \mathcal{N}\}$. We define a relation $\leq$ on L^p by setting $f^\bullet \leq g^\bullet$ whenever there exist $h_1, h_2 \in \mathcal{N}$ such that $f + h_1 \leq g + h_2$ in $\mathscr{L}^p$.

Theorem 1.4.6. *Let (X, Σ, μ) be a measure space and $1 \leq p < \infty$, then L^p is a σ-Dedekind-complete Riesz space.*

Proof. From Theorem 1.4.5, $\mathscr{L}^p$ is a Riesz order ideal in $\mathscr{L}^0$ that contains $\mathcal{N}$. By Theorem 1.3.6, $\mathcal{N}$ is a Riesz order ideal in $\mathscr{L}^0$, so it follows by Theorem 1.2.22 (ii) that L^p is a Riesz order ideal in L^0. Finally, by Theorem 1.3.7 together with Proposition 1.2.14, we conclude that L^p is a σ-Dedekind-complete. $\square$

In fact more can be said for the space L^p.

Theorem 1.4.7 ([23], 244L & 244Y (b), p. 167 & p. 170)**.** *Let (X, Σ, μ) be a measure space and $1 \leq p < \infty$. Then L^p is order-separable and Dedekind complete.*

Definition 1.4.8. *Let $\mathscr{V}$ be a vector space over the scalar field $\mathbb{C}$. A function $p : \mathscr{V} \to [0, \infty)$ is called a seminorm on $\mathscr{V}$ if, for all $x, y \in \mathscr{V}$ and $a \in \mathbb{R}$,*

(i) *$p(x + y) \leq p(x) + p(y)$;*

(ii) *$p(ax) = |a|p(x)$.*

A norm is a seminorm satisfying $p(x) = 0$ implies that $x = 0$.

From Remark 1.4.2 and Theorem 1.4.3, the map ρ_p is a seminorm on $\mathscr{L}^p$. The following theorem shows that ρ_p defines a norm on L^p for $1 \leq p < \infty$.

Theorem 1.4.9 ([23], Theorem 244F, p. 162)**.** *Let (X, Σ, μ) be a measure space and let $1 \leq p < \infty$. Then the seminorm ρ_p induces a norm $\|\cdot\|_p$ on the quotient space L^p, defined by $\|f^\bullet\|_p = \rho_p(f)$, for all $f \in \mathscr{L}^p$.*

Theorem 1.4.10 ([23], Theorem 244G, p. 163)**.** $(L^p, \|\cdot\|_p)$ *is a Banach space.*

1.5 The Space L^∞ of Essentially Bounded Functions

In the previous section, we discussed the spaces L^p of integrable functions for $1 \leq p < \infty$. In this section, we consider the case when $p = \infty$.

Definition 1.5.1. *Let (X, Σ, μ) be a measure space. A measurable function $f : X \to \mathbb{R}$ is said to be essentially bounded if there exists $M > 0$ such that $\mu(x \in X : |f(x)| > M) = 0$. Equivalently $|f| \leq M$ a.e. The set of all such functions is denoted by $\mathscr{L}^\infty(X, \Sigma, \mu)$, or simply $\mathscr{L}^\infty$.*

Proposition 1.5.2. *Let (X, Σ, μ) be a measure space. If $f, g \in \mathscr{L}^\infty$ and $c \in \mathbb{R}$, then $f + g, cf$ and fg are elements of $\mathscr{L}^\infty$.*

Proof. Let $f, g \in \mathscr{L}^\infty$. Then there exist $M_1, M_2 \in \mathbb{R}$ such that $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ a.e. Hence,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2 \quad \text{a.e.}$$

Also,

$$|cf(x)| = |c||f(x)| \leq |c|M_1 \quad \text{a.e.}$$

Concluding that $f + g \in \mathscr{L}^\infty$ and $cf \in \mathscr{L}^\infty$. Finally, $|f(x)g(x)| \leq |f(x)||g(x)| \leq M_1 M_2$ a.e, which shows $fg \in \mathscr{L}^\infty$. □

Proposition 1.5.3. *The space $\mathscr{L}^\infty$ is a Riesz order ideal in $\mathscr{L}^0$. Moreover, $\mathscr{L}^\infty$ is σ-Dedekind-complete and, if (X, Σ, μ) is localisable, then $\mathscr{L}^\infty$ is Dedekind complete.*

Proof. From Proposition 1.5.2, $\mathscr{L}^\infty$ is a vector subspace of $\mathscr{L}^0$. It is easy to see that $\mathscr{L}^\infty$ is a Riesz order ideal in $\mathscr{L}^0$ containing $\mathcal{N}$. For

σ-order completeness (respectively, Dedekind completeness), the result follows by Proposition 1.2.14 (i) and Theorem 1.3.3 (respectively, Theorem 1.3.10 (i)). $\square$

For every $f \in \mathscr{L}^\infty$, the *essential supremum* of f is defined as the infimum of the set $\{M \in \mathbb{R} : |f(x)| \leq M \text{ a.e.}\}$. We now define

$$\rho_\infty : \mathscr{L}^\infty \to [0, \infty) : f \mapsto \operatorname{ess\,sup} |f|.$$

Proposition 1.5.4 ([23], Proposition 243D, p. 153)**.** *Let $f, g \in \mathscr{L}^\infty$ then*

(i) *If $f(x) = g(x)$ a.e, then* $\operatorname{ess\,sup} |f| = \operatorname{ess\,sup} |g|$*;*

(ii) $|f| \leq \operatorname{ess\,sup} |f|$ *a.e;*

(iii) $\operatorname{ess\,sup} |fg| \leq (\operatorname{ess\,sup} |f|)(\operatorname{ess\,sup} |g|)$*;*

(iv) $\operatorname{ess\,sup} |f + g| \leq \operatorname{ess\,sup} |f| + \operatorname{ess\,sup} |g|$.

Proposition 1.5.5. *Let (X, Σ, μ) be a measure space. Then:*

(i) *ρ_∞ is a seminorm on $\mathscr{L}^\infty$ satisfying $\rho_\infty(fg) \leq \rho_\infty(f)\rho_\infty(g)$ for all $f, g \in \mathscr{L}^\infty$.*

(ii) *The seminorm ρ_∞ induces a norm on the quotient space $\mathscr{L}^\infty/\mathcal{N}$ given by $\|f^\bullet\|_\infty = \rho_\infty(f)$.*

Proof. (i) Clearly ρ_∞ satisfies $\rho_\infty(f+g) \leq \rho_\infty(f)+\rho_\infty(g)$ and $\rho_\infty(fg) \leq \rho_\infty(f)\rho_\infty(g)$ by Proposition 1.5.4 (iii) and (iv). Next let $a \in \mathbb{R}$. For $a = 0$, the result is immediate. Suppose $a \neq 0$. Then,

$$\begin{aligned}
\rho_\infty(af) = \text{ess sup}|af| &= \inf\{M \in \mathbb{R} : |af(x)| \leq M \text{ a.e}\} \\
&= \inf\{M \in \mathbb{R} : |a||f(x)| \leq M \text{ a.e}\} \\
&= \inf\left\{M \in \mathbb{R} : |f(x)| \leq \frac{M}{|a|} \text{ a.e}\right\} \\
&= |a| \inf\left\{\frac{M}{|a|} \in \mathbb{R} : |f(x)| \leq \frac{M}{|a|} \text{ a.e}\right\} \\
&= |a|\rho_\infty(f)
\end{aligned}$$

(ii) We start by showing that $\|\cdot\|_\infty$ is well-defined. Assume that $f^\bullet = g^\bullet$. Then $|f(x)| = |g(x)|$ a.e, and by proposition 1.5.4 (i), it follows that $\rho_\infty(f) = \rho_\infty(g)$, or equivalently $\|f^\bullet\|_\infty = \|g^\bullet\|_\infty$. Now if $\|f^\bullet\|_\infty = 0$, then ess sup$|f| = 0$, i.e $|f(x)| = 0$ a.e, thus $f \in \mathcal{N}$. The other properties of a norm follow because ρ_∞ is a seminorm. □

Let L^∞ be the space $\mathscr{L}^\infty/\mathcal{N}$.

Theorem 1.5.6. *The space L^∞ is a Riesz space.*

Proof. By Proposition 1.5.3, we know that $\mathscr{L}^\infty$ is a Riesz order ideal in $\mathscr{L}^0$ and that it contains $\mathcal{N}$. Therefore, by Theorem 1.2.23 (ii), L^∞ is a Riesz order ideal in L^0. Hence, by Theorem 1.2.13, L^∞ is a Riesz space. □

Theorem 1.5.7 ([23], Theorem 243E, p. 154)**.** *The space $(L^\infty, \|\cdot\|_\infty)$ is a Banach space.*

Theorem 1.5.8. *The space L^∞ is Dedekind σ-complete. Moreover, if (X, Σ, μ) is localisable, then L^∞ is Dedekind complete.*

Proof. By Theorem 1.5.6, L^∞ is a Riesz order ideal in L^0. Moreover, Theorem 1.3.7 shows that L^0 is σ-Dedekind complete. Hence, by Theorem 1.2.14, it follows that L^∞ is also σ-Dedekind complete. Finally, if (X, Σ, μ) is localisable, then L^∞ is Dedekind complete by Corollary 1.3.11 together with Theorem 1.2.14. □

Theorem 1.5.9 ([45], Proposition 3.3.9)**.** *The space L^∞ is order separable if and only if (X, Σ, μ) is σ-finite.*

Proof. Suppose first that L^∞ is order separable, and define $\mathscr{F} := \{F \in \Sigma : 0 < \mu(F) < \infty\}$. The net $\{\chi_F^\bullet : F \in \mathscr{F}\}$ is increasing with supremum $\chi_X^\bullet$. By order separability, there exists a countable family $\{F_i : i \in \mathbb{N}\}$ such that $\bigvee_{i\in\mathbb{N}} \chi_{F_i}^\bullet = \chi_X^\bullet$. Let $F := \bigcup_{i=1}^\infty F_i$. Then $\chi_{F_i}^\bullet \leq \chi_F^\bullet$ for every $i \in \mathbb{N}$, and hence $\mu(X \setminus F) = 0$. Therefore, (X, Σ, μ) is σ-finite.

Conversely, if (X, Σ, μ) is σ-finite, then L^0 is order separable by Theorem 1.3.10 (ii), and L^∞ is a Riesz order ideal in L^0. By Theorem 1.2.14 (ii), it follows that L^∞ is order separable. □

1.6 The Embedding of L^∞ into $\mathscr{B}(L^p)$ for $1 \leq p < \infty$.

Proposition 1.6.1. *Let $g \in \mathscr{L}^\infty$ and $f \in \mathscr{L}^p$ with $(1 \leq p \leq \infty)$. Then $fg \in \mathscr{L}^p$ and $\rho_p(fg) \leq \rho_\infty(g)\rho_p(f)$.*

Proof. If $p = \infty$ and $f, g \in \mathscr{L}^\infty$, the result follows by Proposition 1.5.4 (iii). For $1 \leq p < \infty$, since $|g(x)| \leq \rho_\infty(g)$ a.e., we have

$$\rho_p(fg)^p = \int |f|^p |g|^p \; d\mu \leq \rho_\infty(g)^p \int |f|^p \; d\mu = \rho_\infty(g)^p \rho_p(f)^p < \infty.$$

□

A linear operator is a mapping $T : \mathscr{V} \to \mathscr{G}$ between two vector spaces $\mathscr{V}$ and $\mathscr{G}$ over the same field $\mathbb{R}$ satisfying $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for all $v_1, v_2 \in \mathscr{V}$. If $\mathscr{V}$ and $\mathscr{W}$ are normed spaces with norms $\|\cdot\|_{\mathscr{V}}$ and $\|\cdot\|_{\mathscr{W}}$, respectively, then T is called a *bounded linear operator* if there exists $C \geq 0$ such that $\|T(v)\|_{\mathscr{G}} \leq C \|v\|_{\mathscr{V}}$ for all $v \in \mathscr{V}$. The set of bounded linear operators from $\mathscr{V}$ into $\mathscr{G}$ is denoted by $\mathscr{B}(\mathscr{V}, \mathscr{G})$; when $\mathscr{V} = \mathscr{G}$ we simply write $\mathscr{B}(\mathscr{V})$. For any $T \in \mathscr{B}(\mathscr{V}, \mathscr{G})$, the *operator norm* of T is given by

$$\|T\| := \inf\{C \geq 0 : \|T(v)\|_{\mathscr{G}} \leq C \|v\|_{\mathscr{V}} \text{ for every } v \in \mathscr{V}\}.$$

For two linear operators $S, T : \mathscr{V} \to \mathscr{G}$ and a scalar $\alpha \in \mathbb{R}$, we define operators $S + T : \mathscr{V} \to \mathscr{G}$ and $\alpha T : \mathscr{V} \to \mathscr{G}$ by $(S+T)(v) := S(v) + T(v)$ and $(\alpha T)(v) := \alpha(T(v))$ for all $v \in \mathscr{V}$.

Lemma 1.6.2. *For any $g \in \mathscr{L}^\infty$ and $1 \leq p \leq \infty$ the mapping $M_g^p : L^p \to L^p : f^\bullet \mapsto (fg)^\bullet$ is a bounded linear operator with $\|M_g^p\|_p \leq \|g^\bullet\|_\infty$.*

Proof. Let $1 \leq p \leq \infty$, $f_1, f_2 \in \mathscr{L}^p$, and $\alpha, \beta \in \mathbb{R}$. To see that M_g^p is well defined, note that if $f_1^\bullet = f_2^\bullet$, then $(f_1 g)(x) = (f_2 g)(x)$ almost everywhere. Next, M_g^p is linear since

$$\begin{aligned} M_g^p((\alpha f_1 + \beta f_2)^\bullet) &= ((\alpha f_1 + \beta f_2)g)^\bullet \\ &= ((\alpha(f_1 g) + \beta(f_2 g))^\bullet \\ &= \alpha(f_1 g)^\bullet + \beta(f_2 g)^\bullet \\ &= \alpha M_g^p(f_1^\bullet) + \beta M_g^p(f_2^\bullet). \end{aligned}$$

Finally, for any $f \in \mathscr{L}^p$, $\|M_g^p(f^\bullet)\|_p = \|(fg)^\bullet\|_p = \rho_p(fg) \leq \rho_\infty(g)\rho_p(f)$. Thus M_g^p is bounded, and $\|M_g^p\|_p \leq \rho_\infty(g) = \|g^\bullet\|_\infty$. □

The following theorem shows that L^∞ can be embedded in $\mathscr{B}(L^p)$.

Theorem 1.6.3 ([45], Theorem 3.4.3)**.** *For any $1 \leq p \leq \infty$, the map $T_p : L^\infty \longrightarrow \mathscr{B}(L^p) : g^\bullet \mapsto M_g^p$ is a bounded linear operator. If $p = \infty$, then T_p is an isometry. When $p \neq \infty$, the following are equivalent:*

(i) *(X, Σ, μ) is semi-finite;*

(ii) *T_p is an isometry;*

(iii) *T_p is injective.*

Definition 1.6.4. *A linear functional is a linear operator $\lambda : \mathscr{V} \to \mathbb{R}$. A family of linear functionals $\mathscr{F}$ is said to be separating if, for every $x \in \mathscr{V}$ with $x \neq 0$, there exists some $\ell \in \mathscr{F}$ such that $\ell(x) \neq 0$.*

Remark 1.6.5. *The integral defines a linear functional I on L^1 by $I(f^\bullet) = \int f \, d\mu$. Clearly, I is well-defined and satisfies:*

(i) *$I(f^\bullet) \geq 0$ for all $f^\bullet > 0^\bullet$;*

(ii) *If $f^\bullet \geq 0^\bullet$ and $I(f^\bullet) = 0$, then $f^\bullet = 0^\bullet$;*

(iii) *I is bounded with $\|I\| \leq 1$, and $\|I\| = 1$ whenever there exists $F \in \Sigma$ with $0 < \mu(F) < \infty$. Indeed, if we take $k := \frac{\chi_F}{\mu(F)}$, then $|I(k^\bullet)| = 1 = \|k\|_1$.*

Let $g \in \mathscr{L}^\infty$ and $f^\bullet \in L^1$. Define

$$(I \circ M_g^1)(f^\bullet) = I(M_g^1(f^\bullet)) = I((fg)^\bullet) = \int fg\, d\mu.$$

Theorem 1.6.6. *Every $g \in \mathscr{L}^\infty$ induces a bounded linear functional $I \circ M_g^1$ on L^1 with $\left\|I \circ M_g^1\right\|_1 \leq \|g^\bullet\|_\infty$.*

Proof. By remark 1.6.5, we have $\|I\|_1 \leq 1$. By Lemma 1.6.2, $M_g^1 \in \mathscr{B}(L^1)$ and $\left\|M_g^1\right\| \leq \|g^\bullet\|_\infty$. Therefore the composition $I \circ M_g^1$ satisfies

$$\left\|I \circ M_g^1\right\| \leq \|I\| \left\|M_g^1\right\| \leq 1 \cdot \|g^\bullet\|_\infty = \|g^\bullet\|_\infty .$$

□

We end this section with the following theorem, which shows how L^∞ is viewed as a subspace of the space of linear functionals on L^1, denoted by $(L^1)^*$.

Theorem 1.6.7 ([23], Theorem 243G, p. 155)**.** *The mapping $T_1^* : L^\infty \to (L^1)^* : g^\bullet \mapsto I \circ M_g^1$ is a bounded linear operator. Moreover:*

(i) *T_1^* is an isometry if and only if (X, Σ, μ) is semi-finite;*

(ii) *T_1^* is a surjective isomorphism if and only if (X, Σ, μ) is localisable.*

1.7 A note on nets

Recall that a net is a generalization of a sequence. In this section, we give a brief overview of nets. For further details, the reader can consult [40].

Definition 1.7.1. *Let Γ be a directed set and $\Gamma_0 \subseteq \Gamma$. Then Γ_0 is cofinal in Γ if, for every $\gamma \in \Gamma$, there exists $\gamma_0 \in \Gamma_0$ such that $\gamma \leq \gamma_0$. A mapping $g : \Omega \to \Gamma$ between two directed sets is called final if its range $g(\Omega)$ is cofinal in Γ.*

Definition 1.7.2. *Let $f : \Gamma \to X$ be a net in X. If Ω is a directed set and $g : \Omega \to \Gamma$ is a final function such that $g(\gamma_1) \leq g(\gamma_2)$ whenever $\gamma_1 \leq \gamma_2$ in Ω, then the composite function $f \circ g : \Omega \to \Gamma$ is called a subnet of f. The subnet of $(x_\alpha)_{\alpha\in\Gamma}$ is denoted by $(x_{\gamma_\alpha})_{\alpha\in\Omega}$.*

For a net $(x_\gamma)_{\gamma\in\Gamma}$, the set $E_\Gamma(\gamma_0) := \{x_\gamma : \gamma \geq \gamma_0\}$ is said to be a residual of $(x_\gamma)_{\gamma\in\Gamma}$. Let $A \subseteq X$. A net $(x_\gamma)_{\gamma\in\Gamma}$ is said to be *eventually* in A if there exists $\gamma_0 \in \Gamma$ such that $E_\Gamma(\gamma_0) \subseteq A$. On the other hand, it is said to be cofinal in A if for every $\gamma \in \Gamma$ there exists $\gamma_0 \geq \gamma$ with $x_{\gamma_0} \in A$.

If X is a poset, a net is called increasing (respectively, decreasing) if $x_\gamma \leq x_\beta$(respectively, $x_\beta \leq x_\gamma$) for every $\gamma \leq \beta$. It is said to be *diverse* if, whenever $\gamma \neq \beta$, the elements x_γ and x_β are incomparable.

Although this thesis primarily works with nets, it is well known that there is a one to one correspondence between nets and filters. In what follows, we review only the essential aspects. We begin by recalling the definition of a filter, but rather than considering an arbitrary lattice, we restrict our attention to the complete lattice $\mathcal{P}(X)$ ordered by inclusion.

Definition 1.7.3. *A filter $\mathcal{F}$ on a set X is a non-empty collection of subsets of a set X such that:*

(i) *$\emptyset \notin \mathcal{F}$;*

(ii) *if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$;*

(iii) *if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.*

Definition 1.7.4. *If a filter $\mathcal{F}$ on a set X is maximal with respect to inclusion among collections satisfying* (i)–(iii)*, that is, if there is no proper extension of $\mathcal{F}$ that is also a filter, then $\mathcal{F}$ is called a maximal filter. Such a maximal filter is also referred to as an ultrafilter.*

Theorem 1.7.5. *Let $\mathcal{H}$ be a collection of subsets of X satisfying the finite intersection property, i.e., for all finitely many elements $F_1, ..., F_k$*

of $\mathcal{H}$, $\cap_{i=1}^{k} F_i \neq \emptyset$. Then there exists a maximal filter $\mathcal{F}$ on X such that $\mathcal{H} \subseteq \mathcal{F}$.

Proof. Consider

$$\mathscr{A} := \{\mathcal{H} \subseteq \mathcal{G} : \mathcal{G} \text{ is a collection of sets with the f.i.p}\}.$$

We partial order $\mathscr{A}$ by set inclusion. Clearly, $\mathscr{A}$ is partially ordered and every totally ordered subset has a supremum. By Zorn's lemma, there exists a maximal element $\mathcal{F} \in \mathscr{A}$. We now show that $\mathcal{F}$ is a filter. First, $\emptyset \notin \mathcal{F}$ by the finite intersection property. Next, if $A, B \in \mathcal{F}$, then $\mathcal{F} \cup \{A \cap B\} \in \mathscr{A}$ also has the finite intersection property, hence maximality of $\mathcal{F}$ forces $A \cap B \in \mathcal{F}$. Finally, if $A \in \mathcal{F}$ and $A \subseteq B$, then $\mathcal{F} \cup \{B\}$ has the finite intersection property, so again by maximality $B \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a maximal filter on X containing $\mathcal{H}$. □

Recall that a filter $\mathcal{F}$ on X is said to be a *prime filter* if, whenever $A, B \subseteq X$ with $A \cup B \in \mathcal{F}$, it follows that $A \in \mathcal{F}$ or $B \in \mathcal{F}$. The next theorem collects several important properties of maximal filters.

Theorem 1.7.6 ([19], Theorem 10.14, p. 234)**.** *Let $\mathcal{F}$ be a filter on X. Then the following are equivalent:*

(i) *$\mathcal{F}$ is a maximal filter;*

(ii) *$\mathcal{F}$ is a prime filter;*

(iii) *for every $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$;*

(iv) *for each $B \subseteq X$, if $A \cap B \neq \emptyset$ for all $A \in \mathcal{F}$, then $B \in \mathcal{F}$;*

(v) *given pairwise disjoint sets $A_1, A_2, \dots, A_n$ with $A_1 \cup A_2 \cup \dots \cup A_n = X$, there exists a unique j such that $A_j \in \mathcal{F}$.*

Theorem 1.7.7. *Let $(x_\gamma)_{\gamma \in \Gamma}$ be a net in a set X, and let $\mathscr{A}$ be a family of subsets of X satisfying:*

(i) *the net $(x_\gamma)_{\gamma \in \Gamma}$ is cofinal in every $A \in \mathscr{A}$;*

(ii) *for every $A_1, A_2 \in \mathscr{A}$ there exists $A \in \mathscr{A}$ such that $A \subseteq A_1 \cap A_2$.*

Then there exists a subnet of $(x_\gamma)_{\gamma\in\Gamma}$ which is eventually in A for every $A \in \mathscr{A}$.

Proof. First note that $\mathscr{A}$ is a filtered set when ordered by inclusion. Define $\Omega := \{(\gamma, A) : \gamma \in \Gamma,\ A \in \mathscr{A},\ x_\gamma \in A\}$. Order Ω by setting $(\gamma_1, A_1) \preceq (\gamma_2, A_2)$ whenever $\gamma_1 \leq \gamma_2$ and $A_2 \subseteq A_1$. To see that Ω is directed, let $(\gamma_1, A_1), (\gamma_2, A_2) \in \Omega$. By assumption (ii), there exist $A \in \mathscr{A}$ with $A \subseteq A_1 \cap A_2$. Moreover, there is $\gamma_0 \in \Gamma$ with $\gamma_1 \leq \gamma_0$ and $\gamma_2 \leq \gamma_0$. Since $(x_\gamma)_{\gamma\in\Gamma}$ is cofinal in A, there exists $\beta \in \Gamma$ satisfying $\gamma_0 \leq \beta$ and $x_\beta \in A$. Then $(\beta, A) \in \Omega$, and we have $(\gamma_1, A_1) \preceq (\beta, A)$ and $(\gamma_2, A_2) \preceq (\beta, A)$. Define $g : \Omega \to \Gamma : g(\gamma, A) \mapsto \gamma$. Clearly, $g(\Omega)$ is cofinal in Γ, so $(x_{g(\gamma,A)})_{(\gamma,A)\in\Omega}$ is the required subnet. Indeed, let $A \in \mathscr{A}$ and $\gamma_0 \in \Gamma$ such that $x_{\gamma_0} \in A$. If $(\gamma_0, A) \preceq (\beta, B)$ we have that $B \subseteq A$ and $\gamma_0 \leq \beta$. Then, $x_\beta \in B \subseteq A$, i.e $x_{g(\beta,B)} \in A$ for every $(\gamma_0, A) \preceq (\beta, B)$. $\square$

Theorem 1.7.8. *For a net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P, there exists a family $\mathcal{F}$ of subsets of P satisfying*

(i) *$(x_\gamma)_{\gamma\in\Gamma}$ is cofinal in every $C \in \mathcal{F}$;*

(ii) *if $C_1, C_2 \in \mathcal{F}$, then $C_1 \cap C_2 \in \mathcal{F}$;*

(iii) *for every $A \subseteq P$, either $A \in \mathcal{F}$ or $P \setminus A \in \mathcal{F}$.*

Proof. Let $\mathcal{F}' = \{A \subseteq P : E_\Gamma(\gamma_0) \subseteq A \text{ for some } \gamma_0 \in \Gamma\}$. It is clear that $\mathcal{F}'$ satisfies the finite intersection property, and hence, by Theorem 1.7.5, it can be extended to a maximal filter $\mathcal{F}$. For every $C \in \mathcal{F}$ and every $\gamma_0 \in \Gamma$, we have $C \cap E_\Gamma(\gamma_0) \neq \emptyset$. Hence there exists $\gamma \geq \gamma_0$ with $x_\gamma \in C$. This shows that $\mathcal{F}$ satisfies (i). Finally, note that (ii) follows by the definition of a filter, and (iii) follows by Theorem 1.7.6 (iii). $\square$

Definition 1.7.9. *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P is said to be universal if, for every $A \subseteq P$, it is eventually in A or in $P \setminus A$.*

Theorem 1.7.10 ([34], p. 81)**.** *Every net has a universal subnet.*

Proof. The proof follows from Theorems 1.7.8 and 1.7.7 □

Recall the well known result in analysis, the monotone subsequence theorem, which states that every sequence in $\mathbb{R}$ has a monotone subsequence. The next result is an analogue of the monotone subsequence theorem for nets.

Proposition 1.7.11 ([30], Theorem 1)**.** *Every universal net in a partially ordered set P has a monotone or diverse subnet.*

2 Different modes of order convergence

2.1 Order Convergence

Recall that for a net $(x_\gamma)_{\gamma\in\Gamma}$, the residual $\{x_\gamma : \gamma \geq \gamma_0\}$ is denoted by $E_\Gamma(\gamma_0)$.

Definition 2.1.1. *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P is said to* O_1-converge *to x if there exist an increasing net $(y_\gamma)_{\gamma\in\Gamma}$ and a decreasing net $(z_\gamma)_{\gamma\in\Gamma}$ in P such that eventually $y_\gamma \leq x_\gamma \leq z_\gamma$, and $\bigvee_{\gamma\in\Gamma} y_\gamma = x = \bigwedge_{\gamma\in\Gamma} z_\gamma$. This will be written as $x_\gamma \xrightarrow{\mathrm{O}_1} x$.*

Definition 2.1.2. *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P is said to* O_2-converge *to x if there exist a directed subset $M \subseteq P$, and a filtered subset $N \subseteq P$, such that $\sup M = \inf N = x$, and for every $(m,n) \in M \times N$ there exists $\gamma_0 \in \Gamma$ such that $E_\Gamma(\gamma_0) \subseteq [m,n]$. This will be written as $x_\gamma \xrightarrow{\mathrm{O}_2} x$.*

Definition 2.1.3. *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P is said to* O_3-converge *to x if there exist two subsets $M, N \subseteq P$ such that $\sup M = \inf N = x$, and for every $(m,n) \in M\times N$ there exists $\gamma_0 \in \Gamma$ with $E_\Gamma(\gamma_0) \subseteq [m,n]$. This will be written as $x_\gamma \xrightarrow{\mathrm{O}_3} x$.*

It is clear that for a net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P,

$$x_\gamma \xrightarrow{\mathrm{O}_1} x \;\Rightarrow\; x_\gamma \xrightarrow{\mathrm{O}_2} x \;\Rightarrow\; x_\gamma \xrightarrow{\mathrm{O}_3} x \qquad (x \in P).$$

[1]The convergences defined above are not necessarily equivalent. The following examples illustrate this fact. For another example showing that O_2-convergence does not imply O_1-convergence see [36].

Example 10 (O_2-convergence does not imply O_1-convergence on Boolean algebras)**.** *Let X be an uncountable set and let $\mathscr{A}$ be the algebra of all finite and cofinite subsets of X (ordered by inclusion). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements of X and set $A_n := \{x_n\}$. Then the sequence $(A_n)_{n\in\mathbb{N}}$ O_2-converges to $\emptyset$ in $\mathscr{A}$, since the filtered family of all cofinite subsets of X has infimum $\emptyset$. However, it is not O_1-convergent, because any O_1-convergent sequence in $\mathscr{A}$ is eventually constant.*

The following example, credited to D. Fremlin [43, Ex. 2, pg. 140], shows that the notions of O_1- and O_2-convergence differ, even in Riesz spaces.

Example 11 (O_2-convergence does not imply O_1-convergence)**.** *Let X be an uncountable discrete space and $X_0 := X \cup \{\infty\}$ the one-point compactification of X. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in X_0 with all terms distinct. Then the corresponding sequence of characteristic functions $f_n := \chi_{\{x_n\}}$ lies in the Banach lattice $C(X_0, \mathbb{R})$. The sequence $(f_n)_{n\in\mathbb{N}}$ is O_2-convergent to 0, but is not O_1-convergent.*

Example 12 (O_3-convergence does not imply O_2-convergence)**.** *Let $A := \{a_n : n \in \mathbb{N}\}$ and $B := \{b_n : n \in \mathbb{N}\}$ be two countable sets such that $A \cap B = \varnothing$, and let $P := A \cup B \cup \{0, 1\}$ equipped with the partial order defined by:*

$$\mathbf{0} \le x \le \mathbf{1}\ (\forall x \in P) \qquad \textit{and} \qquad x \le y \iff x = a_n,\, y = b_m\ (\exists n \le m),$$

for all $x, y \in A \cup B$. Note that any infinite directed (or filtered) subsets of P must contain $\mathbf{1}$ (resp. $\mathbf{0}$). On the other hand, $\sup A = \mathbf{1}$. Thus, the sequence $(b_n)_{n\in\mathbb{N}}$ is O_3-convergent to $\mathbf{1}$ but not O_2-convergent.

[1]For every $i \in \{1, 2, 3\}$, when sequences are considered instead of nets, we refer to this as sequential O_i-convergence.

Proposition 2.1.4. *Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a poset P and $i \in \{1,2,3\}$. For any $x, z \in P$:*

(i) *If $(x_\gamma)_{\gamma\in\Gamma}$ is increasing with supremum x, then $x_\gamma \xrightarrow{\mathrm{O}_i} x$.*

(ii) *If $x_\gamma \xrightarrow{\mathrm{O}_i} x$ and $x_\gamma \leq z$ for all $\gamma \in \Gamma$, then $x \leq z$.*

(iii) *Every* O_i-convergent *net is eventually order bounded.*

Proof. (i) It suffices to prove the result for $i = 1$. If $(x_\gamma)_{\gamma\in\Gamma}$ is increasing with supremum x, in the definition of O_1-convergence, let $y_\gamma = x_\gamma$ and $z_\gamma = x$ for every $\gamma \in \Gamma$.

(ii) Since O_1-convergence implies O_2-convergence, which in turn implies O_3-convergence, it suffices to prove the claim for $i = 3$. Assume $x_\gamma \xrightarrow{\mathrm{O}_3} x$. Then there exist $M, N \subseteq P$ with $\sup M = x = \inf N$, and for every $(m,n) \in M \times N$ there is some $\gamma(m,n) \in \Gamma$ such that $E_\Gamma(\gamma(m,n)) \subseteq [m,n]$. If $x_\gamma \leq z$ for every $\gamma \in \Gamma$, it follows that $m \leq z$ for every $m \in M$. Since $\sup M = x$, it follows that $x \leq z$.

(iii) As in (ii), it suffices to prove the claim for $i = 3$. Recall that a net $(x_\gamma)_{\gamma\in\Gamma}$ is eventually order bounded if there exists $a, b \in P$ and $\gamma_0 \in \Gamma$ such that $E_\Gamma(\gamma_0) \subseteq [a,b]$. Now suppose $x_\gamma \xrightarrow{\mathrm{O}_3} x$. Then there exist subsets $M, N \subseteq P$ with $\sup M = x = \inf N$, and for every $(m,n) \in M \times N$ there is some $\gamma(m,n) \in \Gamma$ such that $E_\Gamma(\gamma(m,n)) \subseteq [m,n]$. Thus, choosing any $m \in M$ and $n \in N$ we get that the net is eventually contained in $[m,n]$, hence eventually order bounded. □

Lemma 2.1.5. *For $i \in \{1,2,3\}$, every O_i-converging net $(x_\gamma)_{\gamma\in\Gamma}$ has a unique O_i-limit. Moreover, every subnet of an O_i-convergent net $(x_\gamma)_{\gamma\in\Gamma}$ also O_i-converges to the same limit.*

Proof. To show uniqueness of the limit, it suffices to prove the assertion for O_3-convergence, since O_1- and O_2-convergence imply O_3-convergence. Suppose $(x_\gamma)_{\gamma\in\Gamma}$ O_3-converges to both x and y. Then there exist subsets M_x, M_y, N_x and N_y satisfying the definition of O_3-convergence. For any $m \in M_x$ and $n \in N_y$ we have that $m \leq n$ and so $x \leq y$. By symmetry, one can show that $y \leq x$, and thus $x = y$. This proves uniqueness of the limit.

To show the second part of the lemma, we shall prove it separately for for O_1-, O_2- and O_3-convergence, respectively.

If $(x_\gamma)_{\gamma\in\Gamma}$ O_1-converges to x, there exist an increasing net $(y_\gamma)_{\gamma\in\Gamma}$ and a decreasing net $(z_\gamma)_{\gamma\in\Gamma}$ such that eventually $y_\gamma \leq x_\gamma \leq z_\gamma$, and $\bigvee_{\gamma\in\Gamma} y_\gamma = x = \bigwedge_{\gamma\in\Gamma} z_\gamma$. For any subnet $(x_{\gamma_\omega})_{\omega\in\Gamma'}$, the net $(y_{\gamma_\omega})_{\omega\in\Gamma'}$ is increasing, $(z_{\gamma_\omega})_{\omega\in\Gamma'}$ is decreasing, $\bigvee_{\omega\in\Gamma'} y_{\gamma_\omega} = x = \bigwedge_{\omega\in\Gamma'} z_{\gamma_\omega}$ and eventually $y_{\gamma_\omega} \leq x_{\gamma_\omega} \leq z_{\gamma_\omega}$, which shows that the subnet O_1-converges to x.

If $(x_\gamma)_{\gamma\in\Gamma}$ O_2-converges to x, there exists a directed set M and a filtered set N such that $\bigvee M = x = \bigwedge N$, and for every $(m,n) \in M \times N$ the net $(x_\gamma)_{\gamma\in\Gamma}$ is in $[m,n]$. For any subnet $(x_{\gamma_\omega})_{\omega\in\Gamma'}$, the cofinality of Γ' in Γ ensures that for each $(m,n) \in M \times N$ there exists $\omega_0 \in \Gamma'$ with $\gamma_{\omega_0} \geq \gamma(m,n)$. Consequently, $E_{\Gamma'}(\omega_0) \subseteq E_\Gamma(\gamma(m,n)) \subseteq [m,n]$, which shows that the subnet O_2-converges to x.

The proof for O_3-convergence is similar to the the one of O_2-convergence. □

In example 12, we showed that in a poset, O_3-convergence does not necessarily imply O_2-convergence. However, if one considers a lattice instead of a poset, then O_2- and O_3-convergence coincide.

Theorem 2.1.6. *If L is a lattice, then* O_2*-convergence and* O_3*-convergence are equivalent.*

Proof. It suffices to show that O_3-convergence implies O_2-convergence. Assume that $(x_\gamma)_{\gamma\in\Gamma}$ O_3-converges to x. Then there exist subsets $M, N \subseteq L$ such that $\sup M = x = \inf N$, and for every $m \in M$ and $n \in N$ there exists $\gamma(m,n) \in \Gamma$ with $E_\Gamma(\gamma(m,n)) \subseteq [m,n]$. Define $M' := \{\bigvee A : A \subseteq M \text{ and } |A| < \omega\}$ and $N' := \{\bigwedge B : B \subseteq N \text{ and } |B| < \omega\}$. Since L is a lattice, M' is directed and N' is filtered and, $\sup M' = x = \inf N'$. Moreover, the net $(x_\gamma)_{\gamma\in\Gamma}$ is eventually in the interval $\left[a', b'\right]$ for any $a' \in M'$ and $b' \in N'$. Thus $(x_\gamma)_{\gamma\in\Gamma}$ O_2-converges to x. $\square$

Matthews and Anderson further showed that O_1-convergence and O_2-convergence coincide when the poset P is monotone order separable. The following argument provides a simplification of their proof from [36, Theorem 1].

Theorem 2.1.7 ([2], Theorem 4)**.** *If P is a monotone order separable poset, O_2-convergence implies O_1-convergence.*

Proof. Suppose $(x_\gamma)_{\gamma\in\Gamma}$ O_2-converges to $x \in P$. Then there exist a directed subset $M \subseteq P$ and a filtered subset $N \subseteq P$ such that $\sup M = \inf N = x$, and for every $(m,n) \in M \times N$ there exists $\gamma(m,n)$ with $m \le x_\gamma \le n$ for all $\gamma \ge \gamma(m,n)$. Since P is monotone order separable, there exists an increasing sequence $(m_i)_{i\in\mathbb{N}}$ in M, and a decreasing sequence $(n_i)_{i\in\mathbb{N}}$ in N, such that $\sup_{i\in\mathbb{N}} m_i = \inf_{i\in\mathbb{N}} n_i = x$. Define $\gamma_1 := \gamma(m_1, n_1)$. For $i \ge 2$ define inductively γ_i by choosing an arbitrary $\gamma_i \in \Gamma$ satisfying $\gamma_i \ge \gamma_{i-1}$ and $\gamma_i \ge \gamma(m_i, n_i)$. Let $\Gamma_0 := \{\gamma : \gamma \ngeq \gamma_1\}$ and for $i \ge 1$ let $\Gamma_i := \{\gamma : \gamma \ge \gamma_i \text{ and } \gamma \ngeq \gamma_{i+1}\}$. If $\Gamma \setminus \bigcup_{i=0}^{\infty} \Gamma_i \ne \emptyset$, then $(x_\gamma)_{\gamma\in\Gamma}$ must be eventually constant equal to x, and therefore the assertion follows. Suppose that $\{\Gamma_i : i \ge 0\}$ is a partition of Γ and $\{\gamma : \gamma \ge \gamma_i\} = \bigcup_{k \ge i} \Gamma_k$. Put m_0 and n_0 equal to any element of P and define the nets $(m_\gamma)_{\gamma\in\Gamma}$ and $(n_\gamma)_{\gamma\in\Gamma}$ by setting $m_\gamma = m_i$ and $n_\gamma = n_i$ if $\gamma \in \Gamma_i$. Then, $(m_\gamma)_{\gamma\ge\gamma_1}$ is increasing with supremum x, $(n_\gamma)_{\gamma\ge\gamma_1}$ is decreasing with infimum x and $m_\gamma \le x_\gamma \le n_\gamma$ for every $\gamma \ge \gamma_1$, i.e., eventually, $m_\gamma \le x_\gamma \le n_\gamma$. $\square$

For two nets $(x_\gamma)_{\gamma\in\Gamma}$ and $(y_\omega)_{\omega\in\Omega}$, and for $\diamond \in \{\vee, \wedge\}$, we define the net $(x_\gamma \diamond y_\omega)_{(\gamma,\omega)\in\Gamma\times\Omega}$ where $\Gamma \times \Omega$ is endowed with the product order, that is, $(\gamma_1, \omega_1) \le (\gamma_2, \omega_2)$ if and only if $\gamma_1 \le \gamma_2$ and $\omega_1 \le \omega_2$.

Proposition 2.1.8. *Let L be an infinitely distributive lattice, let $i \in \{1, 2\}$, and let $\diamond \in \{\vee, \wedge\}$. If $(x_\gamma)_{\gamma\in\Gamma} \xrightarrow{\mathrm{O}_i} x$ and $(y_\omega)_{\omega\in\Omega} \xrightarrow{\mathrm{O}_i} y$, then* *the net*

$$(x_\gamma \diamond y_\omega)_{(\gamma,\omega)\in\Gamma\times\Omega}$$

O_i*-converges to* $x \diamond y$.

Proof. We prove the result for $i = 2$ and $\vee$. The cases $i = 1$ and $\wedge$ follow similarly. There exist directed subsets M_x, M_y, and filtered subsets N_x, N_y, such that for $(m_x, n_x) \in M_x \times N_x$, and $(m_y, n_y) \in M_y \times N_y$, one can find $\gamma(m_x, n_x)$, $\omega(m_y, n_y)$ satisfying $E_\Gamma(\gamma(m_x, n_x)) \subseteq [m_x, n_x]$ and $E_\Omega(\omega(m_y, n_y)) \subseteq [m_y, n_y]$. Define $M := M_x \vee M_y = \{m_x \vee m_y : m_x \in M_x, m_y \in M_y\}$ and $N := N_x \vee N_y = \{n_x \vee n_y : n_x \in N_x, n_y \in N_y\}$. Then M is directed, N is filtered, and $\bigvee M = x \vee y$, and $\bigwedge N = x \vee y$, where the last equality follows from infinite distributivity. Finally, for $m_x \vee n_y \in M$ and $n_x \vee n_y \in N$, it is easy to see that $x_\gamma \vee y_\omega \in [m_x \vee m_y\,,\, n_x \vee n_y]$ for $(\gamma, \omega) \geq (\gamma(m_x, n_x), \omega(m_y, n_y))$. □

It is worth noting that in Proposition 2.1.8, infinite distributivity is essential; the conclusion might not hold under the weaker assumption of distributivity.

Example 13. *Let* L *be the complete lattice* $2^{\mathbb{R}}$ *and* $i \in \{1, 2\}$. *When endowed with set inclusion,* L *forms a bounded distributive lattice. For* $n \in \mathbb{N}$ *let* $X_n := [2^{-n}, \infty)$ *and* $X := [0, \infty)$. *Then* $(X_n)_{n\in\mathbb{N}}$ *is increasing and* $\bigvee^L X_n = X$. *In particular,* $X_n \xrightarrow{\mathrm{O}_i} X$. *On the other hand, let* $B := (-\infty, 0]$. *Then* $(X_n \wedge B)_{n\in\mathbb{N}}$ *does not* O_i*-converge to* $X \wedge B$. *Indeed,* $X_n \wedge B = \emptyset$ *for every* $n \in \mathbb{N}$, *however,* $X \wedge B = \{0\}$.

Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a partially ordered set P. Define:

$$\limsup_{\gamma\in\Gamma} x_\gamma = \bigwedge_{\gamma\in\Gamma} \bigvee_{\gamma'\geq\gamma} x_{\gamma'} \text{ and } \liminf_{\gamma\in\Gamma} x_\gamma = \bigvee_{\gamma\in\Gamma} \bigwedge_{\gamma'\geq\gamma} x_{\gamma'}$$

as the limit superior and limit inferior, respectively, of $(x_\gamma)_{\gamma\in\Gamma}$ provided that the corresponding infima and suprema exist in P.

Proposition 2.1.9. *Let P be a Dedekind complete poset. A net $(x_\gamma)_{\gamma\in\Gamma}$ in P, O_1-converges to x if and only if there exists $\gamma_0 \in \Gamma$ such that* $\limsup_{\gamma\geq\gamma_0} x_\gamma = \liminf_{\gamma\geq\gamma_0} x_\gamma = x$ *in P.*

Proof. Assume that for a net $(x_\gamma)_{\gamma\in\Gamma}$, eventually $\liminf x_\gamma = x = \limsup x_\gamma$. Then there exists $\gamma_0 \in \Gamma$ satisfying $\bigvee_{\gamma\geq\gamma_0}\bigwedge_{\beta\geq\gamma} x_\beta = x = \bigwedge_{\gamma\geq\gamma_0}\bigvee_{\beta\geq\gamma} x_\beta$. Take $y_\gamma := \bigwedge_{\beta\geq\gamma} x_\beta$ and $z_\gamma := \bigvee_{\beta\geq\gamma} x_\beta$ for every $\gamma \geq \gamma_0$. Clearly, $(y_\gamma)_{\gamma\geq\gamma_0}$ is increasing with supremum x and $(z_\gamma)_{\gamma\geq\gamma_0}$ is decreasing with infimum x. Furthermore, it is clear that for every $\gamma \geq \gamma_0$,

$$y_\gamma \leq x_\gamma \leq z_\gamma$$

concluding that $(x_\gamma)_{\gamma\in\Gamma}$ O_1-converges to x.

Conversely, we prove that if $(x_\gamma)_{\gamma\in\Gamma}$ O_1-converges to x in P, eventually $\limsup x_\gamma = x$. If $(x_\gamma)_{\gamma\in\Gamma}$ O_1-converges to x in P, there exist an increasing net $(y_\gamma)_{\gamma\in\Gamma}$, a decreasing net $(z_\gamma)_{\gamma\in\Gamma}$ and $\gamma_0 \in \Gamma$ such that $y_\gamma \leq x_\gamma \leq z_\gamma$ for every $\gamma \geq \gamma_0$ and, $\sup_\gamma y_\gamma = x = \inf_\gamma z_\gamma$. Therefore for every $\gamma \geq \gamma_0$, $a_\gamma := \bigvee_{\beta\geq\gamma} x_\beta$ exists in P by Dedekind completion. Clearly $(a_\gamma)_{\gamma\geq\gamma_0}$ is decreasing and every a_γ is an upper bound of $(y_\gamma)_{\gamma\in\Gamma}$. Indeed let $\gamma \in \Gamma$, choose $\beta \in \Gamma$ with $\gamma \leq \beta$ and $\gamma_0 \leq \beta$, then

$$y_\gamma \leq y_\beta \leq a_\beta.$$

This implies that eventually $x \leq \limsup x_\gamma$ because $x \leq a_\gamma$ for every $\gamma \geq \gamma_0$. Finally if $x' \leq \limsup_{\gamma\geq\gamma_0} x_\gamma$ we have that $x' \leq a_\gamma$ for every $\gamma \geq \gamma_0$. For any $\gamma \in \Gamma$ choose $\beta \in \Gamma$ such that $\beta \geq \gamma_0$ and $\beta \geq \gamma$. Then, $x' \leq a_\beta \leq z_\beta \leq z_\gamma$ concluding that $x' \leq x$, i.e. eventually $\limsup x_\gamma = x$. That eventually $\liminf x_\gamma = x$ follows similarly. □

Proposition 2.1.10. *Let P be a Dedekind complete poset. If a net $(x_\gamma)_{\gamma\in\Gamma}$ O_3-converges to x then it also O_1-converges to x. Thus, the three notions of order convergence coincide.*

Proof. Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net O_3-converging to x. Then there exist subsets $M, N \subseteq P$ with $\sup M = \inf N = x$, and for every $(m,n) \in M \times N$ there exists $\gamma(m,n) \in \Gamma$ with $E_\Gamma(\gamma(m,n)) \subseteq [m,n]$. Fix $(m_0,n_0) \in M \times N$. By Dedekind completeness of P, both $a_\gamma := \inf_{\beta\geq\gamma} x_\beta$ and $b_\gamma := \sup_{\beta\geq\gamma} x_\beta$ exist in P for every $\gamma \geq \gamma(m_0,n_0)$. For any $(m,n) \in M \times N$, choose $\gamma_0 \in \Gamma$ satisfying $\gamma_0 \geq \gamma(m_0,n_0)$ and $\gamma_0 \geq \gamma(m,n)$. Then $m \leq a_\beta \leq b_\beta \leq n$ for $\beta \geq \gamma_0$. Hence, $x \leq \liminf_{\gamma\geq\gamma(m_0,n_0)} x_\gamma \leq \limsup_{\gamma\geq\gamma(m_0,n_0)} x_\gamma \leq x$, and the result follows by Proposition 2.1.9. □

In light of Proposition 2.1.10, when working with order convergence on Dedekind complete posets, we write O-convergence instead of O_i-convergence. In examples 10 and 11, we saw that O_2-convergence and O_1-convergence are not equivalent. The next result shows, however, that every O_2-convergent net has a subnet O_1-converging to the same limit.

Theorem 2.1.11 ([3], Corollary 2.5 (ii))**.** *Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a poset P that O_2-converges to x. Then there exists a subnet of $(x_\gamma)_{\gamma\in\Gamma}$ that O_1-converges to x.*

Proof. Assume that $(x_\gamma)_{\gamma\in\Gamma}$ is a net in P that O_2-converges to x. Then there exist a directed set $M \subseteq P$ and a filtered set $N \subseteq P$ such that $\sup M = \inf N = x$, and for every $(m,n) \in M \times N$ there exists $\gamma(m,n) \in \Gamma$ with $E_\Gamma(\gamma(m,n)) \subseteq [m,n]$. Define $\Omega := \{(m,n) : m \in M \text{ and } n \in N\}$. For $(m_1,n_1),(m_2,n_2) \in \Omega$, set $(m_1,n_1) \leq_\Omega (m_2,n_2)$ if $m_1 \leq m_2$ and $n_2 \leq n_1$. Then $(\Omega, \leq_\Omega)$ is directed. Now define

$$\Lambda := \{((m,n),\gamma) \in \Omega \times \Gamma : \gamma(m,n) \leq \gamma\}$$

with order given by $((m_1,n_1),\gamma_1) \leq ((m_2,n_2),\gamma_2)$ if $(m_1,n_1) \leq_\Omega (m_2,n_2)$ and $\gamma_1 \leq_\Gamma \gamma_2$.

We first check that Λ is directed. Indeed, for every $((m_1,n_1),\gamma_1),((m_2,n_2),\gamma_2) \in \Lambda$, there exists $(m_3,n_3) \in \Omega$ such that (m_1,n_1),

$(m_2, n_2) \leq_\Omega (m_3, n_3)$. Since Γ is directed, there exists $\gamma_3 \in \Gamma$ with $\gamma_1, \gamma_2, \gamma(m_3, n_3) \leq_\Gamma \gamma_3$. Then, $((m_3, n_3), \gamma_3)$ is the required element in Λ. Define the order-preserving map $\varphi : \Lambda \to \Gamma$ by $\varphi((m, n), \gamma) = \gamma$. Then the net $(x_{\varphi(\alpha)})_{\alpha\in\Lambda}$ is a subnet of $(x_\gamma)_{\gamma\in\Gamma}$. Finally, for each $\alpha = ((m, n), \gamma) \in \Lambda$ set $m_{\varphi(\alpha)} = m$ and $n_{\varphi(\alpha)} = n$. Then $(m_{\varphi(\alpha)})_{\alpha\in\Lambda}$ is increasing with supremum x, $(n_{\varphi(\alpha)})_{\alpha\in\Lambda}$ is decreasing with infimum x, and for all $\alpha \in \Lambda$,

$$m_{\varphi(\alpha)} \leq x_{\varphi(\alpha)} \leq n_{\varphi(\alpha)}.$$

Thus, $(x_{\varphi(\alpha)})_{\alpha\in\Lambda}$ O_1-converges to x. □

Recall that for a subset $A \subseteq P$, we define the set $A^+ := \{x \in P : a \leq x \text{ for every } a \in A\}$ and dually, $A^- := \{x \in P : x \leq a \text{ for every } a \in A\}$. For a net $(x_\gamma)_{\gamma\in\Gamma}$, set

$$P_\Gamma := \bigcup_{\gamma\in\Gamma} [E_\Gamma(\gamma)]^- \text{ and } Q_\Gamma := \bigcup_{\gamma\in\Gamma} [E_\Gamma(\gamma)]^+.$$

Lemma 2.1.12 ([51], Remark 1)**.** *Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a poset P, and let $M, N \subseteq P$. Then $M \subseteq P_\Gamma$ and $N \subseteq Q_\Gamma$ if and only if, for every $m \in M$ and $n \in N$, there exists $\gamma_0 \in \Gamma$ such that $E_\Gamma(\gamma_0) \subseteq [m, n]$.*

Proof. Assume that for every $(m, n) \in M \times N$ there exists $\gamma_0 \in \Gamma$ such that $E_\Gamma(\gamma_0) \subseteq [m, n]$. Then clearly $m \in [E_\Gamma(\gamma_0)]^- \subseteq P_\Gamma$ and $n \in [E_\Gamma(\gamma_0)]^+ \subseteq Q_\Gamma$. Hence, $M \subseteq P_\Gamma$ and $N \subseteq Q_\Gamma$.

Conversely, suppose $M \subseteq P_\Gamma$ and $N \subseteq Q_\Gamma$. Then for each $m \in M$ and $n \in N$, there exists $\gamma_1, \gamma_2 \in \Gamma$ such that $m \in [E_\Gamma(\gamma_1)]^-$ and $n \in [E_\Gamma(\gamma_2)]^+$. Choose, $\gamma \in \Gamma$ with $\gamma_1 \leq \gamma$ and $\gamma_2 \leq \gamma$. Then, $E_\Gamma(\gamma) \subseteq [m, n]$, as required. □

The following proposition was partially established in [51, Theorem 1]. An equivalent formulation in terms of l-cuts was later given in [2, Proposition 2].

Proposition 2.1.13 ([3, 51]). *Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a poset P, and let $x \in P$. Then the following are equivalent:*

(i) $x_\gamma \xrightarrow{\mathrm{O}_3} x$;

(ii) $\sup P_\Gamma = x = \inf Q_\Gamma$;

(iii) $\bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{+-} = (\leftarrow, x]$ *and* $\bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{-+} = [x, \rightarrow)$.

Proof. (i) $\Rightarrow$ (ii) Assume that $x_\gamma \xrightarrow{\mathrm{O}_3} x$. Let $M, N \subseteq P$ satisfy $\sup M = \inf N = x$ and, for every $(m, n) \in M \times N$, there exists $\gamma(m, n)$ such that $E_\Gamma(\gamma(m, n)) \subseteq [m, n]$. By Lemma 2.1.12, $M \subseteq P_\Gamma$ and $N \subseteq Q_\Gamma$, so it suffices to show that x is an upper-bound of P_Γ and x is an lower-bound of Q_Γ.

Fix $a \in P_\Gamma$ and $b \in N$. Choose $\gamma(a), \gamma(b) \in \Gamma$ such that $a \leq x_\gamma$ for all $\gamma(a) \leq \gamma$ and $x_\gamma \leq b$ for all $\gamma(b) \leq \gamma$. Let $\gamma_0 \in \Gamma$ satisfy $\gamma(a), \gamma(b) \leq \gamma_0$. Then $E_\Gamma(\gamma_0) \subseteq [a, b]$. It then follows that $a \leq x$. A similar argument shows that $\inf Q_\Gamma = x$.

(ii) $\Rightarrow$ (iii) Assume that $\sup P_\Gamma = x = \inf Q_\Gamma$. Since $E_\Gamma(\gamma)^+ \subseteq Q_\Gamma$, by Proposition 1.1.18 it follows that $(\leftarrow, x] \subseteq E_\Gamma(\gamma)^{+-}$ for every $\gamma \in \Gamma$. For the reverse inclusion, let $z \in \bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{+-}$. Then $z \leq a$ for every $a \in E_\Gamma(\gamma)^+$ and $\gamma \in \Gamma$. Hence, $z \in (\leftarrow, x]$. The other equality follows by a symmetric argument.

(iii) $\Rightarrow$ (i) Take $M := P_\Gamma$ and $N := Q_\Gamma$ in the definition of O_3-convergence. If $\bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{+-} = (\leftarrow, x]$ and $\bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{-+} = [x, \rightarrow)$ then it follows immediately that $\sup M = x = \inf N$. The result therefore follows by Lemma 2.1.12. ☐

Recall that, by Theorem 1.1.23, every poset P can be embedded into an order-complete poset $DM(P)$. Birkhoff introduced an alternative notion of order convergence, defined via the embedding of the poset in its MacNeille completion [11].

Definition 2.1.14. *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P is said to* O^{DM}-converge *to $x \in P$ if the net $\big((\leftarrow, x_\gamma]\big)_{\gamma\in\Gamma}$* O-*converges to $(\leftarrow, x]$ in $DM(P)$.*

However, in [11, pg. 244], Birkhoff does not clearly distinguish between different modes of convergence. In particular, he states that O_1-convergence is equivalent to O^{DM}-convergence. The following theorem shows that this claim is incorrect. Moreover, in [2, Theorem 3], it is proved that O^{DM}-convergence coincides with O_3-convergence.

Theorem 2.1.15 ([2], Theorem 3)**.** *Let P be a poset. Then:*

(i) O_3*-convergence is equivalent to* O^{DM}*-convergence.*

(ii) *If, in addition, P is a lattice, then* O_2*-,* O_3*-, and* O^{DM}*-convergence are equivalent.*

Proof. Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a poset P, and let $x \in P$.
(i) Observe that $E_\Gamma(\gamma)^+ = \left(\bigcup_{\beta\geq\gamma}(\leftarrow, x_\beta]\right)^+$.

Hence

$$(\leftarrow, x] = \bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{+-} = \bigcap_{\gamma\in\Gamma}\left(\bigcup_{\beta\succeq\gamma}(\leftarrow, x_\beta]\right)^{+-} = \limsup_\gamma(\leftarrow, x_\gamma].$$

From Proposition 1.1.20, we have $\bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{-+} = \left(\bigcup_{\gamma\in\Gamma} E_\Gamma(\gamma)^-\right)^+$. Therefore,

$$(\leftarrow, x] = \left(\bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{-+}\right)^- = \left(\bigcup_{\gamma\in\Gamma} E_\Gamma(\gamma)^-\right)^{+-} = \liminf_\gamma(\leftarrow, x_\gamma].$$

Then by Proposition 2.1.13 (iii) we have,

$$\begin{aligned} x_\gamma \xrightarrow{\mathrm{O}_3} x &\iff \bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{+-} = (\leftarrow, x] = \big(\bigcap_{\gamma\in\Gamma} E_\Gamma(\gamma)^{-+}\big)^- \\ &\iff \liminf_\gamma(\leftarrow, x_\gamma] = (\leftarrow, x] = \limsup_\gamma(\leftarrow, x_\gamma] \\ &\iff x_\gamma \xrightarrow{\mathrm{O}^{\mathrm{DM}}} x. \end{aligned}$$

(ii) This follows directly from (i) together with Theorem 2.1.6. □

Corollary 2.1.16. *Let L be a lattice and $(x_\gamma)_{\gamma\in\Gamma}$ a net in L. Then $x_\gamma \xrightarrow{\mathrm{O}_2} x$ in L if and only if $x_\gamma \xrightarrow{\mathrm{O}^{DM}} x$ in $DM(L)$.*

2.2 Order Topology

Each notion of order convergence gives rise to a corresponding topology, called the *order topology*. In this section, we introduce these topologies and compare them. We start by recalling the definition of convergence of a net $(x_\gamma)_{\gamma\in\Gamma}$ in a topological space.

Definition 2.2.1. *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a topological space $(X, \mathscr{T})$ is said to converge to $x \in X$ with respect to $\mathscr{T}$ if, for every neighbourhood U of x, there exists $\gamma_0 \in \Gamma$ such that $E_\Gamma(\gamma_0) \subseteq U$.*

For a subset F of a poset P, define

$$F_1^{\mathrm{O}_i} := \{x \in P : \text{there exists a net in } F \text{ that } \mathrm{O}_i\text{-converges to } x\}\,.$$

The set $F_1^{\mathrm{O}_i}$ is called the 1-O_i-adherence of F. More generally, we define the λ-O_i-adherence $F_\lambda^{\mathrm{O}_i}$ recursively as follows.

Set $F_0^{\mathrm{O}_i} := F$, and for every ordinal $\lambda > 0$ define the λ-O-adherence $F_\lambda^{\mathrm{O}_i}$ by:

$$F_\lambda^{\mathrm{O}_i} := \left(\bigcup_{\beta<\lambda} F_\beta^{\mathrm{O}_i}\right)_1^{\mathrm{O}_i}.$$

Definition 2.2.2. *For $i \in \{1, 2, 3\}$, a subset F of a partially ordered set P is called O_i-closed (respectively, sequentially O_i-closed) if the O_i-limit of every O_i-convergent net (respectively, sequence) in F is contained in F. Equivalently, F is O_i-closed if $F = F_1^{\mathrm{O}_i}$.*

Proposition 2.2.3. *Let P be a poset and let $i \in \{1, 2, 3\}$. Let $\mathcal{C}$ denote the collection of all O_i-closed subsets of P. Then:*

(i) *$P \in \mathcal{C}$ and $\emptyset \in \mathcal{C}$;*

(ii) *If $F_1, F_2 \in \mathcal{C}$, then $F_1 \cup F_2 \in \mathcal{C}$;*

(iii) *If $(F_\eta)_{\eta\in\mathscr{A}} \subseteq \mathcal{C}$, then $\bigcap_{\eta\in\mathscr{A}} F_\eta \in \mathcal{C}$.*

Proof. (i) Trivially, $P \in \mathcal{C}$ and $\emptyset \in \mathcal{C}$.

(ii) Let $F_1, F_2 \in \mathcal{C}$ and $(x_\gamma)_{\gamma\in\Gamma} \subseteq F_1 \cup F_2$ be a net that O_i-converges to x. Then either F_1 or F_2 contains a subnet of $(x_\gamma)_{\gamma\in\Gamma}$. By Proposition 2.1.5, this subnet O_i-converges to x, and hence $x \in F_1 \cup F_2$.

(iii) Let $(x_\gamma)_{\gamma\in\Gamma} \subseteq \bigcap_{\eta\in\mathscr{A}} F_\eta$ be a net that O_i-converges to x. Since $(x_\gamma)_{\gamma\in\Gamma} \subseteq F_\eta$ for every $\eta \in \mathscr{A}$, it follows that $x \in \bigcap_{\eta\in\mathscr{A}} F_\eta$. Hence $\bigcap_{\eta\in\mathscr{A}} F_\eta \in \mathcal{C}$.

□

Definition 2.2.4 (Order Topology)**.** *For $i \in \{1,2,3\}$, the order topology on a poset P, denoted by $\tau_{\mathrm{O}_i}(P)$, is the topology on P whose closed sets are precisely the O_i-closed subsets of P.*

Definition 2.2.5 (Sequential Order Topology)**.** *For $i \in \{1,2,3\}$, the sequential order topology on a poset P, denoted by $\tau^s_{\mathrm{O}_i}(P)$, is the topology on P whose closed sets are precisely the sequentially O_i-closed subsets of P.*

The O_i-closure of $F \subseteq P$ is the smallest O_i-closed subset of P containing F; equivalently, it is the topological closure of F with respect to the order topology. Note that, in general, this set is strictly larger than $F_1^{\mathrm{O}_i}$.

Remark 2.2.6. *We note that, for every $i \in \{1,2,3\}$, $\tau_{\mathrm{O}_i} \subseteq \tau^s_{\mathrm{O}_i}$. Furthermore, since O_1-convergence implies O_2-convergence and O_2-convergence implies O_3-convergence, we have*

$$\tau_{\mathrm{O}_3} \subseteq \tau_{\mathrm{O}_2} \subseteq \tau_{\mathrm{O}_1} \textit{ and } \tau^s_{\mathrm{O}_3} \subseteq \tau^s_{\mathrm{O}_2} \subseteq \tau^s_{\mathrm{O}_1}.$$

In Proposition 2.1.10 we observed that the three notions of order convergence coincide when P is Dedekind complete. In this case, it is customary to denote the order topology by $\tau_{\mathrm{O}}(P)$, or simply by τ_{O} when there is no risk of confusion.

Let P_0 be a subset of a poset P. Since P_0 itself is a poset, we can endow it with order topology $\tau_{\mathrm{O}_i}(P_0)$, as well as with the subspace topology $\tau_{\mathrm{O}_i}(P)\restriction_{P_0}$ induced by $\tau_{\mathrm{O}_i}(P)$. In general, these two topologies are not comparable.

Example 14 ($\tau_{\mathrm{O}}(P_0) \nsubseteq \tau_{\mathrm{O}}(P)\restriction_{P_0}$)**.** *Consider the Dedekind complete lattices $P := [0,1]$ and the subset $P_0 := \{\frac{1}{2} - \frac{1}{n} : n \in \mathbb{N}, n \geq 2\} \cup \{1\}$ endowed with the natural order. It is easy to see that $\tau_{\mathrm{O}}(P)\restriction_{P_0}$ is the discrete topology. Let $F := \{\frac{1}{2} - \frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$. The order closure of F in P_0 is P_0. However, the order closure of F in P is $F \cup \{\frac{1}{2}\}$. Thus, $(F \cup \{\frac{1}{2}\}) \cap P_0 = F$, which shows that $\tau_{\mathrm{O}}(P_0) \nsubseteq \tau_{\mathrm{O}}(P)\restriction_{P_0}$.*

Example 15 ($\tau_{\mathrm{O}}(P)\restriction_{P_0} \nsubseteq \tau_{\mathrm{O}}(P_0)$)**.** *Let $P := \mathcal{P}(\mathbb{N})$ and $P_0 := \{A \in P : |A| < \infty\}$, ordered by inclusion. Clearly both P and P_0 are Dedekind complete lattices. Moreover, every increasing and decreasing net in P_0 satisfying the definition of* O_1*-convergence is eventually constant. Consider $F := \{\{n\} : n \in \mathbb{N}\}$. The set F is order closed in $\tau_O(P_0)$, since no net in F has an order limit outside F in P_0. On the other hand, the sequence $(\{n\})_{n\in\mathbb{N}}$* O_1*-converges to $\varnothing$ in P. Thus, $\tau_{\mathrm{O}}(P)\restriction_{P_0} \nsubseteq \tau_{\mathrm{O}}(P_0)$.*

In examples 10 and 11, we showed that O_1-convergence and O_2-convergence differ on boolean algebras and Riesz spaces (in particular, lattices). Nevertheless, the following theorem demonstrates that they generate the same order topology.

Theorem 2.2.7 ([2], Theorem 5)**.** *Let P be a poset.*

(i) $\tau_{\mathrm{O}_1} = \tau_{\mathrm{O}_2}$.

(ii) *If P is a lattice then $\tau_{\mathrm{O}_1} = \tau_{\mathrm{O}_2} = \tau_{\mathrm{O}_3}$.*

Proof. (i) We need to show that every O_1-closed subset F of P is O_2-closed. Let F be an O_1-closed subset of P. Suppose that $(x_\gamma)_{\gamma\in\Gamma}$ is a net in F that O_2-converges to $x \in P$. By Theorem 2.1.11, there exists a subnet $(x_\gamma)_{\gamma\in\Gamma'}$ that O_1-converges to x. Since F is O_1-closed,

it follows that $x \in F$. Hence F is O_2-closed. This completes the proof of (i).
(ii) This follows by (i) and Theorem 2.1.15. $\square$

As in the case of Dedekind complete partially ordered sets when working with lattices, we denote the order topology by τ_{O}.

Theorem 2.2.8 ([2], Theorem 5(ii))**.** *Let P be a poset and let $DM(P)$ denote its MacNeille completion. Then $\tau_{\mathrm{O}_3}(P) \supseteq \tau_{\mathrm{O}}(DM(P)) \restriction_P$.*

Proof. Let $F \subseteq P$ be closed with respect to $\tau_{\mathrm{O}}(DM(P)) \restriction_P$, and let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in F that O_3-converges to some $x \in P$. By (i) of Theorem 2.1.15, this net is also O^{DM}-convergent to x in $DM(P)$. Since F is closed with respect to $\tau_{\mathrm{O}}(DM(P)) \restriction_P$, it follows that $x \in F$. Therefore, F is O_3-closed. $\square$

The inclusion of Theorem 2.2.8 may be proper. The following example by Olejček (see [41]) was used in [2] to highlight this fact.

Example 16 ([2])**.** *Let $\mathbb{Z}' := \mathbb{Z}\setminus\{0\}$. Define $L_1 := \bigcup_{i\in\mathbb{Z}'}\{a(i), b(i)\} \cup \{0\}$, and endow L_1 with partial order $\leq$ specified as follows:*

- *$a(i) \geq a(j) \geq 0$ and $b(i) \geq b(j) \geq 0$ for every $i, j \in \mathbb{Z}'$ satisfying $1 \leq i \leq j$,*
- *$a(i) \leq a(j) \leq 0$ and $b(i) \leq b(j) \leq 0$ for every $i, j \in \mathbb{Z}'$ satisfying $-1 \geq i \geq j$,*
- *$a(i) \leq b(i)$ if $i \geq 1$ and $a(i) \geq b(i)$ if $i \leq -1$.*

For every $k \in \mathbb{N}$ let L_k be a copy of L_1. Endow $L' := \bigcup_{k\in\mathbb{N}} L_k \cup \{e\}$ with the partial order $\preceq$ induced by:[2]

$$x \preceq y \iff (\exists k \in \mathbb{N}) \begin{cases} (x, y \in L_k,\ x \leq y) \text{ or} \\ (x = a_k(-1),\ y = a_{k+1}(-1)) \text{ or} \\ (x = a_{k+1}(1),\ y = a_k(1)) \text{ or} \\ (x = a_k(-1),\ y = e) \text{ or} \\ (x = e,\ y = a_k(1)). \end{cases}$$

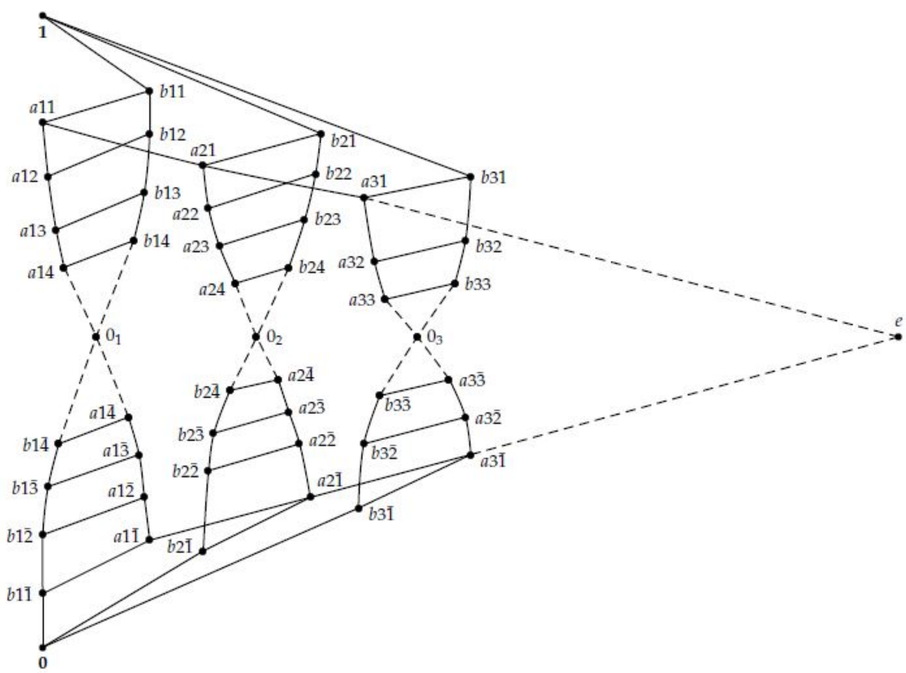

Figure 1: A poset P satisfying $\tau_{\mathrm{O}_3}(P) \supset \tau_{\mathrm{O}}(DM(P))|_P$.

One can verify that $L := L' \cup \{\mathbf{0}, \mathbf{1}\}$ is a complete lattice. Figure 1 shows the Hasse diagram of L. Define $B := \{b_k(i) : k \in \mathbb{N},\ i \in \mathbb{Z}'\}$ and

[2]For typographic reasons we write aki (resp. bki) instead of $a_k(i)$ (resp. $b_k(i)$) and $\bar{i}$ instead of $-i$.

$O := \{0_k : k \in \mathbb{N}\}$. Then, $L \setminus O$ is a lattice and its MacNeille completion can be identified with L. To prove the claim, it suffices to show that $\tau_{\mathrm{O}}(L \setminus O) \neq \tau_{\mathrm{O}}(L)|_{L\setminus O}$, equivalently that $\tau_{\mathrm{O}_1}(L \setminus O) \neq \tau_{\mathrm{O}}(L)|_{L\setminus O}$. To this end, one needs to observe that in $L\setminus O$ every (nontrivial) monotonic net has to either be eventually in L_k (and therefore converges to 0_k) for some k or be eventually in one of the sets $\{a_k(1) : k \in \mathbb{N}\}$ or $\{a_k(-1) : k \in \mathbb{N}\}$. Hence, the set B is closed w.r.t. $\tau_{\mathrm{O}_1}(L \setminus O)$. On the other-hand, every subset $F \subset L$ that is closed with respect to $\tau_{\mathrm{O}}(L)$ and that contains B must necessarily contain O – and therefore e – since $(0_k)_{k\in\mathbb{N}}$ O-converges to e. This means that B is not closed w.r.t. $\tau_{\mathrm{O}}(L)|_{L\setminus O}$.

Definition 2.2.9. *Let L be a lattice. A topology $\mathscr{T}$ on L is said to be order-continuous if every net (in particular, sequence) in L that* O_2*-converges to some $x \in L$ also converges to x with respect to $\mathscr{T}$; that is $x_\gamma \xrightarrow{\mathrm{O}_2} x \implies x_\gamma \xrightarrow{\mathscr{T}} x$.*

Theorem 2.2.10. *Let L be a lattice. Then the order topology τ_{O} is the finest order-continuous topology on L.*

Proof. We first show that τ_{O} is order-continuous. Suppose, for contradiction, that there exists and open set U containing x and a net $(x_\gamma)_{\gamma\in\Gamma}$ that O_2-converges to x but is not eventually in U. Then there exists a subnet $(x_\gamma)_{\gamma\in\Gamma'}$ contained in $L \setminus U$. By Theorem 2.1.5, this net also O_2-converges to x, which implies $x \in L \setminus U$, a contradiction.

Next we show that τ_{O} is the finest order-continuous topology. Let $\mathscr{T}$ be another order continuous topology, and let F be a closed set in $\mathscr{T}$. Suppose $(x_\gamma)_{\gamma\in\Gamma}$ is a net in F that O_2-converges to $x \in L$. Assume, for contradiction, that $x \notin F$. By hypothesis, the net converges to x with respect to $\mathscr{T}$, which implies that it is eventually in $L \setminus F$, a contradiction. □

By Theorem 2.2.10, every order-convergent net in a lattice converges to the same limit with respect to the order topology. The following example shows that the converse is not necessarily true.

Example 17. *Consider the Dedekind complete space ℓ^∞, ordered by point-wise partial order. For each $n \in \mathbb{N}$, let e_n denote the sequence that is 1 at the n^{th} term and on all other terms is 0. For every $k \in \mathbb{N}$, consider the sequence $(ke_n)_{n\in\mathbb{N}}$ in ℓ^∞.*

We note that the sequence $(ke_n)_{n\in\mathbb{N}}$ O-converges to the zero sequence, denoted by $\mathbf{0}$. Indeed, let b_n be the sequence that is 0 on the first n-1 terms and 1 thereafter. Then $\bigwedge_{n\in\mathbb{N}} kb_n = \mathbf{0}$, and we have $ke_m \leq kb_n$ for every $m \geq n$. By Theorem 2.2.10, it follows that $(ke_n)_{n\in\mathbb{N}}$ converges to $\mathbf{0}$ with respect to τ_{O}. Hence, for every open neighbourhood U of $\mathbf{0}$, there exists $N(k,U) \in \mathbb{N}$ such that $E_{\mathbb{N}}(N(k,U)) \subseteq U$. Define

$$\Gamma := \{(k,U) : k \in \mathbb{N} \text{ and } U \text{ a } \tau_{\mathrm{O}}\text{-open neighbourhood of } \mathbf{0}\}$$

Order Γ by $(k_1,U_1) \leq (k_2,U_2)$ whenever $k_1 \leq k_2$ and $U_2 \subseteq U_1$. Then, Γ is a directed set. Define $\varphi : \Gamma \to \ell_\infty : (k,U) \mapsto ke_{N(k,U)}$. Clearly, $\varphi((k,U)) \in U$. Moreover, the net $(\varphi(k,U))_{(k,U)\in\Gamma}$ τ_{O} - converges to $\mathbf{0}$. Indeed, let V be a τ_{O}-neighbourhood of $\mathbf{0}$, and consider $(1,V) \in \Gamma$. For any $(1,V) \leq (k,U)$, we have $U \subseteq V$ and hence $\varphi(k,U) \in V$ for every $(1,V) \leq (k,U)$. However, this net is not eventually order bounded, and therefore it does not O-converge to $\mathbf{0}$.

We conclude this section by noting that the order topology need not be Hausdorff. In particular, τ_{O} is not Hausdorff on Boolean algebras and Riesz spaces.

Recall that for a topological space $(X, \mathscr{T})$, the interior of $A \subseteq X$, denoted by $\mathrm{int}(A)$, is given by $X \setminus \overline{X \setminus A}$. A subset $A \subseteq X$ is called regular open if it coincides with the interior of its closure, i.e. $A = \mathrm{int}(\overline{A})$. Let $\mathcal{R}$ be the collection of regular open sets:

$$\mathcal{R} := \{A \subseteq X : A = \mathrm{int}(\overline{A})\}.$$

Then $\mathcal{R}$ is a lattice, where the order is inclusion, i.e., $A \leq B$ if and only if $A \subseteq B$, and the operations are given by $A \vee B = X \setminus \overline{X \setminus \overline{A \cup B}}$ and $A \wedge B = A \cap B$ for all $A, B \in \mathcal{R}$. The complement of $A \in \mathcal{R}$, denoted by A', is $X \setminus \overline{A}$. Then, $(\mathcal{R}, \subseteq, ')$ is a complete Boolean algebra.

Example 18 (τ_{O} is not Hausdorff for Boolean algebras)**.** *Let $X = [0, 1]$ with the natural topology, and consider the lattice $\mathcal{R}$ of regular open sets. For every $p \in X$, consider the intervals $[0, p)$ and $(p, 1]$. We show that $\emptyset$ and X cannot be separated by open neighbourhoods of $\tau_O(\mathcal{R})$. Consider the countable base $\{[0, p) : p \in \mathbb{Q} \cap (0, 1]\} \cup \{(p, 1] : p \in \mathbb{Q} \cap [0, 1)\} \cup \{(p, q) : p, q \in \mathbb{Q} \cap [0, 1]\}$ for the natural topology on $[0, 1]$, and enumerate it as $(U_n)_{n\in\mathbb{N}}$. Each U_n is an upper-bound of a decreasing sequence $(A^n_m)_{m\in\mathbb{N}} \subseteq \mathcal{R}$ with infimum $\emptyset$. Since $\tau_O(\mathcal{R})$ preserves order convergence, every open neighbourhood $\mathscr{U} \in \tau_O(\mathcal{R})$ of $\emptyset$ contains a residual of $(A^n_m)_{m\in\mathbb{N}}$. Taking $n = 1$, choose $B_1 = A^1_m \in \mathscr{U}$. Then, consider the decreasing sequence $(A^2_m \vee A^1_m)_{m\in\mathbb{N}}$ which decreases to A^1_m. Again, there exists $B_2 = A^2_k \vee A^1_m \in \mathscr{U}$. Continuing inductively, we obtain an increasing sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathscr{U}$. Let $B = \bigcup_{n\in\mathbb{N}} B_n$. Then B is dense in X, and the only regular open set containing B is X. Thus, $\bigvee_{n\in\mathbb{N}} B_n = X$, so $X \in \overline{\mathscr{U}}$.*

Example 19 (τ_{O} is not Hausdorff for Riesz spaces)**.** *We use a similar argument to the Example 18 to show that the order topology is not Hausdorff on Riesz spaces. Let $X = C[0, 1]$, the space of continuous functions on $[0, 1]$. Denote by $\mathbf{1}$ the constant 1 function and by $\mathbf{0}$ the constant function equal to 0. Let U and V be neighbourhoods of $\mathbf{0}$ and $\mathbf{1}$, respectively. Let $(a_n)_{n\in\mathbb{N}}$ be an enumeration of the rational numbers in $(0, 1)$. For each $n, k \in \mathbb{N}$, consider the interval $I_{n,k} := [a_n - \frac{1}{k}, a_n + \frac{1}{k}] \cap [0, 1]$. Define a continuous function $f_{n,k}$ on $[0, 1]$ by letting $f_{n,k}(a_n) = 1$, extending linearly on $[a_n - \frac{1}{k}, a_n + \frac{1}{k}]$, and letting $f_{n,k}(x) = 0$ for every $x \in [0, 1] \setminus I_{n,k}$. Observe that for fixed $n \in \mathbb{N}$, the sequence $(f_{n,k})_{k\in\mathbb{N}}$ order converges to $\mathbf{0}$, and hence converges to $\mathbf{0}$ with respect to τ_O. Thus, for $n = 1$, we may choose $k_1 \in \mathbb{N}$ such that $g_1 := f_{1,k_1} \in U$. Next, note that the sequence $(f_{2,k} \vee g_1)_{k\in\mathbb{N}}$ order converges to g_1, so there exists $k_2 \in \mathbb{N}$ such that $g_2 := f_{2,k_2} \vee$*

$f_{1,k_1} \in U$. Proceeding inductively, we construct an increasing sequence $(g_n)_{n\in\mathbb{N}}$ in U with pointwise supremum $\mathbf{1}$, and by Theorem 2.2.10, it also converges to $\mathbf{1}$ in τ_O. Therefore, (g_n) is eventually in V, which shows that $U \cap V \neq \varnothing$.

2.3 Unbounded Order Convergence

Definition 2.3.1. *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a lattice L is said to unbounded order converge* (uO-converge) *to $x \in L$ if for every $s, t \in L$ and $s \leq t$, $(x_\gamma \wedge t) \vee s \xrightarrow{\mathrm{O_2}} (x \wedge t) \vee s$. In this case we write $x_\gamma \xrightarrow{\mathrm{uO}} x$.*

Lemma 2.3.2. *If a net uO-converges, then its uO-limit is unique.*

Proof. Take $s := x$ and $t := x \vee y$ in the definition of uO-convergence. Then, by applying the same reasoning as in Lemma 2.1.5, it follows that if a net uO-converges to both x and y, then $x = y$. Hence, the uO-limit is unique. □

In the following lemma we observe that in a distributive lattice the condition $s \leq t$ in the definition of unbounded order convergence is redundant.

Lemma 2.3.3. *Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a distributive lattice L. Then $x_\gamma \xrightarrow{\mathrm{uO}} x$ iff $(x_\gamma \vee s) \wedge t \xrightarrow{\mathrm{O_2}} (x \vee s) \wedge t$ for all $s, t \in L$.*

Proof. The proof follows directly by Proposition 1.1.28. □

Proposition 2.3.4. *Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in a distributive lattice L.*

(i) *If $x_\gamma \xrightarrow{\mathrm{uO}} x$ and eventually $x_\gamma \leq u$ (respectively, $x_\gamma \geq u$) for some $u \in L$, then $x \leq u$ (respectively, $x \geq u$).*

(ii) *If $(x_\gamma)_{\gamma\in\Gamma}$ is monotone, the following implication holds:*

$$x_\gamma \xrightarrow{\mathrm{uO}} x \quad \Longrightarrow \quad \begin{cases} x_\gamma \uparrow x \text{ (if the net is increasing)}, \\ x_\gamma \downarrow x \text{ (if the net is decreasing)}. \end{cases}$$

Proof. (i) Assume that $x_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} x$, and that $x_\gamma \leq u$ for every $\gamma \geq \gamma_0$. Define $a := u \wedge x$ and $b := u \vee x$. Then by definition of $\mathfrak{u}$O-convergence, we have $(x_\gamma \wedge b) \vee a \xrightarrow{\mathrm{O}_2} (x \wedge b) \vee a$. Since $x = (x \wedge b) \vee a \leq (u \wedge b) \vee a = u$, we conclude that $x \leq u$. The dual statement is proved analogously.

(ii) Assume that $(x_\gamma)_{\gamma\in\Gamma}$ is increasing and that $x_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} x$. By part (i), it suffices to show that x is an upper bound for the net. Fix $\gamma_0 \in \Gamma$. By $\mathfrak{u}$O-convergence, we have

$$(x_\gamma \wedge x_{\gamma_0}) \vee x \xrightarrow{\mathrm{O}_2} (x \wedge x_{\gamma_0}) \vee x = x.$$

Hence there exist subsets $M, N \subseteq L$ with $\bigvee M = x = \bigwedge N$, such that for each $(m, n) \in M \times N$, there is $\gamma(m, n) \in \Gamma$ satisfying $(x_\gamma \wedge x_{\gamma_0}) \vee x \in [m, n]$, for all $\gamma \geq \gamma(m, n)$. Fix $(m, n) \in M \times N$, and take $\gamma' \in \Gamma$ satisfying $\gamma' \geq \gamma(m, n)$ and $\gamma' \geq \gamma_0$. Then

$$x_{\gamma_0} \vee x = \big(x_{\gamma'} \wedge x_{\gamma_0}\big) \vee x \in [m, n] \qquad \forall (m, n) \in M \times N.$$

This implies that $x_{\gamma_0} \vee x = x$, and therefore $x_{\gamma_0} \leq x$. As γ_0 was arbitrary, it follows x is an upper bound for $(x_\gamma)_{\gamma\in\Gamma}$. □

Using Example 6, we observe that the assumption of infinite distributivity in Proposition 2.3.4 is essential; without it, the conclusion does not necessarily hold.

Example 20. *Let L be the complete lattice $2^{\mathbb{R}}$ of closed subsets of $\mathbb{R}$. For each $n \in \mathbb{N}$, define $X_n := [2^{-n}, \infty)$ and $X := [0, \infty)$. Then $(X_n)_{n\in\mathbb{N}}$ is increasing and $\bigvee^L X_n = X$, that is, $X_n \uparrow X$ in L. In particular, $X_n \xrightarrow{\mathrm{O}_2} X$. On the other hand, $(X_n)_{n\in\mathbb{N}}$ does not $\mathfrak{u}$O-converge to X. Indeed, let $A := (-\infty, -1]$ and $B := (-\infty, 0]$. Then for every $n \in \mathbb{N}$, we have $(X_n \wedge B) \vee A = A$, so that $(X_n \wedge B) \vee A \xrightarrow{\mathrm{O}_2} A$. However, $(X \wedge B) \vee A = \{0\} \cup A$, showing that $\mathfrak{u}$O-convergence fails.*

Proposition 2.3.5. *Let L be an infinitely distributive lattice. If*

$(x_\gamma)_{\gamma\in\Gamma} \xrightarrow{\mathfrak{u}\mathrm{O}} x$ and $(y_\omega)_{\omega\in\Omega} \xrightarrow{\mathfrak{u}\mathrm{O}} y$, then the following hold:

$$\big((x_\gamma \vee y_\omega) : (\gamma, \omega) \in \Gamma \times \Omega\big)$$

$\mathfrak{u}$O-converges to $x \vee y$, and similarly,

$$\big((x_\gamma \wedge y_\omega) : (\gamma, \omega) \in \Gamma \times \Omega\big)$$

$\mathfrak{u}$O-converges to $x \wedge y$.

Proof. For any $s, t \in L$, we have

$$(x_\gamma \wedge t) \vee s \xrightarrow{\mathrm{O}_2} (x \wedge t) \vee s,$$
$$(y_\omega \wedge t) \vee s \xrightarrow{\mathrm{O}_2} (y \wedge t) \vee s.$$

Therefore, by Proposition 2.1.8,

$$((x_\gamma \wedge t) \vee s) \vee ((y_\omega \wedge t) \vee s) \xrightarrow{\mathrm{O}_2} ((x \wedge t) \vee s) \vee ((y \wedge t) \vee s),$$

and

$$((x_\gamma \wedge t) \vee s) \wedge ((y_\omega \wedge t) \vee s) \xrightarrow{\mathrm{O}_2} ((x \wedge t) \vee s) \wedge ((y \wedge t) \vee s).$$

By distributivity,

$$((x_\gamma \vee y_\omega) \wedge t) \vee s \xrightarrow{\mathrm{O}_2} ((x \vee y) \wedge t) \vee s,$$

and

$$((x_\gamma \wedge y_\omega) \wedge t) \vee s \xrightarrow{\mathrm{O}_2} (x \wedge y \wedge t) \vee s.$$

□

Theorem 2.3.6. *Let L be a distributive lattice and $(x_\gamma)_{\gamma\in\Gamma}$ a net in L. Then L is infinitely distributive if and only if $x_\gamma \xrightarrow{\mathrm{O}_2} x \implies x_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} x$.*

Proof. If L is infinitely distributive, that O_2-convergence implies $\mathfrak{u}$O-convergence follows by Proposition 2.1.8. Conversely, assume that if a net $(x_\gamma)_{\gamma\in\Gamma}$ O_2-converges to x, then it $\mathfrak{u}$O-converges to x. Let $A \subseteq L$ and $y \in L$. We want to show that if $\bigvee A = u$ (resp. $\bigwedge A = v$) then $\bigvee\{a \wedge y : a \in A\} = u \wedge y$ (resp. $\bigwedge\{a \vee y : a \in A\} = v \vee y$). Let $\mathcal{A} := \{B \subseteq A : |B| < \omega\}$ and for every $B \in \mathcal{A}$ let $a_B = \bigvee B$, to get an increasing net $\{a_B : B \in \mathcal{A}\}$ in L with supremum u. Then $(a_B)_{B\in\mathcal{A}}$ is O_2-convergent to u, and by hypothesis, $(a_B)_{B\in\mathcal{A}}$ $\mathfrak{u}$O-converges to u.

Take $t = y$ and $s = a_0 \wedge y$, where $a_0 \in A$ is arbitrary. Then

$$(a_B \wedge y) \vee (a_0 \wedge y) \xrightarrow{\mathrm{O}_2} (u \wedge y) \vee (a_0 \wedge y) = u \wedge y.$$

Observe that the net

$$\{(a_B \wedge y) \vee (a_0 \wedge y) : B \in \mathcal{A}\}$$

is increasing, and so by Proposition 2.3.4,

$$\bigvee\{(a_B \wedge y) \vee (a_0 \wedge y) : B \in \mathcal{A}\} = u \wedge y.$$

Since $(a_B \wedge y) \geq (a_0 \wedge y)$ when $a_0 \in B$, it follows that

$$\bigvee\{(a_B \wedge y) \vee (a_0 \wedge y) : B \in \mathcal{A}\} = \bigvee\{(a_B \wedge y) : B \in \mathcal{A}\}.$$

Finally, if $B = \{a_1, \ldots, a_n\}$, then $a_B \wedge y = \bigvee_{i\leq n}(a_i \wedge y)$, because L is distributive. Thus

$$\bigvee\{a \wedge y : a \in A\} = \bigvee\{(a_B \wedge y) : B \in \mathcal{A}\} = u \wedge y.$$

A similar argument can be made to prove the dual statement. $\square$

Proposition 2.3.7. *Let L be an infinitely distributive lattice. If $(x_\gamma)_{\gamma\in\Gamma}$ is an order bounded net, then $x_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} x$ if and only if $x_\gamma \xrightarrow{\mathrm{O}_2} x$.*

Proof. The implication $x_\gamma \xrightarrow{\mathrm{O}_2} x$ implies $x_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} x$ follows directly from Theorem 2.3.6. Conversely, suppose there exist $a, b \in L$ such that $x_\gamma \in [a, b]$ for every $\gamma \in \Gamma$, and assume $x_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} x$. For all α, $\beta, \gamma \in \Gamma$,

$(x_\alpha \wedge x_\gamma) \vee x_\beta \in [a,b]$, and $(x \wedge x_\gamma) \vee x_\beta \in [a,b]$. Repeating this argument we get that $x \in [a,b]$. Hence, $x_\gamma = (x_\gamma \wedge b) \vee a \xrightarrow{\mathrm{O}_2} (x \wedge b) \vee a = x$, as required. $\square$

Although we have so far discussed $\mathfrak{u}$O-convergence in the setting of lattices, the concept is most commonly studied in the narrower context of Riesz spaces. In what follows, we present several results concerning $\mathfrak{u}$O-convergence in Riesz spaces.

Proposition 2.3.8 ([42], Proposition 7.2)**.** *For a net $(x_\gamma)_{\gamma \in \Gamma}$ and an element x of a Riesz space V, the following statements are equivalent:*

(i) $|x_\gamma - x| \wedge u \xrightarrow{\mathrm{O}_2} 0$ *for every* $u \in V_+$*;*

(ii) $(x_\gamma \wedge t) \vee s \xrightarrow{\mathrm{O}_2} (x \wedge t) \vee s$ *for every* $s, t \in V$*.*

Proof. Assume that $(x_\gamma)_{\gamma \in \Gamma}$ is a net and $x \in V$ such that $|x_\gamma - x| \wedge u \xrightarrow{\mathrm{O}_2} 0$ for every $u \in V_+$. Take $s, t \in V$. Then $(t-s)^+ \geq 0$, and

$$|(x_\gamma \wedge t) \vee s - (x \wedge t) \vee s| \leq |x_\gamma - x| \wedge (t-s)^+ \xrightarrow{\mathrm{O}_2} 0.$$

Conversely, let $u > 0$ and choose $s = x$, $t = x + u$, then $|(x_\gamma \wedge (x+u)) \vee x - (x \wedge (x+u)) \vee x| \xrightarrow{\mathrm{O}_2} 0$. Then,

$$\begin{aligned} |(x_\gamma \wedge (x+u)) \vee x - x| \xrightarrow{\mathrm{O}_2} 0 &\implies |((x_\gamma - x) \wedge u) \vee 0| \xrightarrow{\mathrm{O}_2} 0 \\ &\implies ((x_\gamma - x) \vee 0) \wedge u \xrightarrow{\mathrm{O}_2} 0 \\ &\implies (x_\gamma - x)^+ \wedge u \xrightarrow{\mathrm{O}_2} 0. \end{aligned} \tag{1.1}$$

On the other hand, take $s = x - u$ and $t = x$. Then $|((x_\gamma \wedge x) \vee (x-u)) - ((x \wedge x) \vee (x-u))| \xrightarrow{\mathrm{O}_2} 0$ which gives

$$\begin{aligned} |((x_\gamma \wedge x) \vee (x-u)) - x| \xrightarrow{\mathrm{O}_2} 0 &\implies |((x_\gamma - x) \wedge 0) \vee -u| \xrightarrow{\mathrm{O}_2} 0 \\ &\implies |-[((x - x_\gamma) \vee 0) \wedge u]| \xrightarrow{\mathrm{O}_2} 0 \\ &\implies |((x - x_\gamma) \vee 0) \wedge u| \xrightarrow{\mathrm{O}_2} 0 \\ &\implies |(x_\gamma - x)^- \wedge u| \xrightarrow{\mathrm{O}_2} 0 \end{aligned}$$

$$\implies (x_\gamma - x)^- \wedge u \xrightarrow{\mathrm{O}_2} 0. \qquad (1.2)$$

Combining, (1.1) and (1.2), we obtain $|(x_\gamma - x)| \wedge u \xrightarrow{\mathrm{O}_2} 0$ as desired. □

In the literature of $\mathfrak{u}$O-convergence in Riesz spaces, property (i) of Proposition 2.3.8 often appears in the definition of $\mathfrak{u}$O-convergence. Moreover, Proposition 2.3.8 shows that this definition coincides with our definition of $\mathfrak{u}$O-convergence on lattices.

The following example shows that in ℓ_p spaces with $1 \leq p \leq \infty$, unbounded order convergence coincides with pointwise convergence.

Example 21. $\mathfrak{u}$O*-convergence coincides with pointwise convergence in* ℓ^p *for* $1 \leq p \leq \infty$ *and in* c_0*. We prove the result for* ℓ^∞*; the statement for the other spaces can be shown similarly. Let* $(x_\gamma)_{\gamma\in\Gamma}$ *be a net converging pointwise to* x*, and let* $u \in \ell^\infty$ *with* $u > 0$*. For every* $m \in \mathbb{N}$*, define*

$$y_m = \left(\min\left\{u_1, \tfrac{1}{m}\right\}, \dots, \min\left\{u_m, \tfrac{1}{m}\right\}, u_{m+1}, u_{m+2}, \dots\right),$$

which yields a decreasing sequence $(y_n)_{n\in\mathbb{N}}$*. Clearly,* $\mathbf{0} = \inf\{y_n : n \in \mathbb{N}\}$*. For each* $i \in \mathbb{N}$*, since* $(x_\gamma)_{\gamma\in\Gamma}$ *converges pointwise to* x*, there exists* $\gamma_0(i) \in \Gamma$ *such that*

$$|x_\gamma^i - x^i| \leq \frac{1}{i}$$

for all $\gamma \geq \gamma_0(i)$*. Fix* $n \in \mathbb{N}$*, and for all* $i \in \{1, 2, 3, ..., n\}$*, let* $\gamma_0 \geq \gamma_0(i)$*. Then for* $\gamma \geq \gamma_0$*,* $|x_\gamma - x| \wedge u \leq y_n$ *which shows that* $|x_\gamma - x| \wedge u \xrightarrow{\mathrm{O}_2} \mathbf{0}$*.*

Conversely, assume that $(x_\gamma)_{\gamma\in\Gamma}$ $\mathfrak{u}$O*-converges to* x*. Fix* $i \in \mathbb{N}$ *and let* $u^{(i)} = (\delta_{n,i})_{n\in\mathbb{N}}$*, the sequence with* 1 *in the* i*-th coordinate and* 0 *elsewhere. Then*

$$|x_\gamma - x| \wedge u^{(i)} \xrightarrow{O_2} 0.$$

Hence there exists a decreasing net $(y_\omega)_{\omega\in\Omega}$ *with infimum* $\mathbf{0}$ *and for every* $\omega \in \Omega$ *there exists an* $\gamma(\omega) \in \Gamma$ *such that,*

$$|x_\gamma - x| \wedge u^{(i)} \leq y_\omega, \quad \textit{for all } \gamma \geq \gamma(\omega).$$

Equivalently, $|x^i_\gamma - x^i| \wedge 1 \leq y^i_\omega$ *for* $\gamma \geq \gamma(\omega)$*. As* i *was arbitrary, we conclude that* $(x_\gamma)_{\gamma\in\Gamma}$ *converges pointwise to* x*.*

For a given measure space (X, Σ, μ), consider the Riesz space L^0. This space consists of real-valued measurable functions on X, where functions that agree almost everywhere are identified. The order is defined by

$$f^\bullet \leq g^\bullet \quad \Longleftrightarrow \quad f \leq g \text{ a.e.},$$

where $f^\bullet$ denotes the equivalence class of f.

Proposition 2.3.9 ([28], Proposition 3.1)**.** *For a sequence* $(f^\bullet_n)_{n\in\mathbb{N}}$ *in* L^0 *and* $f^\bullet \in L^0$*, the following are equivalent:*

(i) $f^\bullet_n \xrightarrow{\mathfrak{u}\mathrm{O}} f^\bullet$*;*

(ii) $f^\bullet_n \longrightarrow f^\bullet$ *almost everywhere;*

(iii) $f^\bullet_n \xrightarrow{\mathrm{O}_2} f^\bullet$*.*

In view of example 21 and Proposition 2.3.9, it is natural that $\mathfrak{u}$O-convergence is generally studied in the setting of Riesz spaces. The following theorem explains why O_2-convergence is generally preferred over O_1-convergence in the definition of $\mathfrak{u}$O-convergence.

Recall from Corollary 2.1.16, that on lattices, O_2-convergence passes freely to its MacNeille completion. The following theorem is an analogue, showing that $\mathfrak{u}$O-convergence passes freely between a Riesz space and a regular Riesz subspace.

Theorem 2.3.10 ([28], Theorem 2.5)**.** *Let* W *be a Riesz subspace of a Riesz space* V*. The following statements are equivalent.*

(i) *W is regular;*

(ii) *For any net $(y_\gamma)_{\gamma\in\Gamma}$ in W and $y \in W$, $y_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} y$ in W implies $y_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} y$ in V;*

(iii) *For any net $(y_\gamma)_{\gamma\in\Gamma}$ in W and $y \in W$, $y_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} y$ in W iff $y_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} y$ in V.*

If V is an Archimedean Riesz space, there is another completion, namely the universal completion. It is known that V is a regular Riesz subspace of the universal completion. In particular, $\mathfrak{u}$O-convergence passes freely between V and the universal completion. For details on the construction of the universal completion V^u of an Archimedean Riesz space, see [7, p. 187].

2.4 The unbounded order topology

In example 11, it was shown that on Riesz spaces, O_1-convergence and O_2-convergence are not equivalent. Thus, in the definition of $\mathfrak{u}$O-convergence, one may consider another type of unbounded order convergence by considering O_1-convergence instead of O_2-convergence.

Definition 2.4.1. *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a lattice L is said to be* $\mathfrak{u}\mathrm{O}_1$-convergent *to $x \in L$ if $(x_\gamma \wedge t) \vee s \xrightarrow{\mathrm{O}_1} (x \wedge t) \vee s$ for every $s, t \in L$ with $s \leq t$. In this case we write $x_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}_1} x$.*

The following theorem generalizes the argument of Theorem 2.1.11 to the setting of $\mathfrak{u}\mathrm{O}_1$-convergence and $\mathfrak{u}$O-convergence.

Theorem 2.4.2 ([3], Theorem 2.4)**.** *Let L be a lattice and let $(x_\gamma)_{\gamma\in\Gamma}$ be a net $\mathfrak{u}$O-convergent to x. Then there exists a subnet $(x_{\gamma_\omega})_{\omega\in\Omega}$ that $\mathfrak{u}\mathrm{O}_1$-converges to x.*

Proof. Let $\mathscr{F} := \{f_{(s,t)} : s, t \in L \text{ and } s \leq t\}$, where $f_{(s,t)}(x) := (x \wedge t) \vee s$. Assume that $x_\gamma \xrightarrow{\mathfrak{u}\mathrm{O}} x$. Then, for every $f_{(s,t)} \in \mathscr{F}$, we have $f_{(s,t)}(x_\gamma) \xrightarrow{\mathrm{O}_2} f_{(s,t)}(x)$. Hence, there exist sets $M_{f(s,t)}$ and $N_{f(s,t)}$ with

$\sup M_{f(s,t)} = f_{(s,t)}(x) = \inf N_{f(s,t)}$, such that the net $(f_{(s,t)}(x_\gamma))_{\gamma\in\Gamma}$ is eventually in $[m, n]$ for every $(m, n) \in M_{(s,t)} \times N_{(s,t)}$.

Let $\mathscr{F}_0 = \{F \subseteq \mathscr{F} : |F| < \omega\}$ denote the collection of finite sets of $\mathscr{F}$. Assume $F_0 \in \mathscr{F}_0$ with $|F_0| = n$, and write $F_0 = \{f_{(s_1,t_1)}, \dots, f_{(s_n,t_n)}\}$. Define

$$M_{F_0} := \Big\{\{m_{f_{(s_i,t_i)}}, \dots, m_{f_{(s_n,t_n)}}\} : m_{f_{(s_i,t_i)}} \in M_{f_{(s_i,t_i)}} \text{ for } i = 1, \dots, n\Big\}$$

and

$$N_{F_0} := \Big\{\{n_{f_{(s_i,t_i)}}, \dots, n_{f_{(s_n,t_n)}}\} : n_{f_{(s_i,t_i)}} \in N_{f_{(s_i,t_i)}} \text{ for } i = 1, \dots, n\Big\}.$$

Let $\Omega = \big\{(F_0, A, B) : F_0 \in \mathscr{F}_0, A \in M_{F_0} \text{ and } B \in N_{F_0}\big\}$. On Ω we define a relation $\preceq$ by declaring $(F_1, A_1, B_1) \preceq (F_2, A_2, B_2)$ if and only if

(i) $F_1 \subseteq F_2$, and

(ii) for every $f_{(s,t)} \in F_1$ with corresponding elements $m^1_{f_{(s,t)}} \in A_1$, $m^2_{f_{(s,t)}} \in A_2$, $n^1_{f_{(s,t)}} \in B_1$, and $n^2_{f_{(s,t)}} \in B_2$, we have

$$m^1_{f_{(s,t)}} \le m^2_{f_{(s,t)}} \quad \text{and} \quad n^2_{f_{(s,t)}} \le n^1_{f_{(s,t)}}.$$

We first show that $(\Omega, \preceq)$ is a partially ordered set.

(i) Reflexivity follows by definition.

(ii) (Antisymmetry) : Suppose $(F_1, A_1, B_1) \preceq (F_2, A_2, B_2)$ and $(F_2, A_2, B_2) \preceq (F_1, A_1, B_1)$. Then $F_1 = F_2$, and for each $f_{(s,t)} \in F_1$ we have $m^1_{f_{(s,t)}} \le m^2_{f_{(s,t)}}$ and $m^2_{f_{(s,t)}} \le m^1_{f_{(s,t)}}$, hence $m^1_{f_{(s,t)}} = m^2_{f_{(s,t)}}$ by the antisymmetry of $\le$. Thus $A_1 = A_2$. The same argument shows $B_1 = B_2$, so $(F_1, A_1, B_1) = (F_2, A_2, B_2)$.

(iii) (Transitivity) : If $(F_1, A_1, B_1) \preceq (F_2, A_2, B_2)$ and $(F_2, A_2, B_2) \preceq (F_3, A_3, B_3)$. Then $F_1 \subseteq F_2 \subseteq F_3$, $m^1_{f_{(s,t)}} \le m^2_{f_{(s,t)}} \le m^3_{f_{(s,t)}}$ and $n^3_{f_{(s,t)}} \le n^2_{f_{(s,t)}} \le n^1_{f_{(s,t)}}$ for every $f_{(s,t)} \in F_1$. Then $(F_1, A_1, B_1) \preceq (F_3, A_3, B_3)$ as required.

Next we show that the partially ordered set $(\Omega, \preceq)$ is directed. Let $(F_1, A_1, B_1), (F_2, A_2, B_2) \in \Omega$. We need to find $(F_3, A_3, B_3) \in \Omega$ such that

$$(F_1, A_1, B_1) \preceq (F_3, A_3, B_3) \quad \text{and} \quad (F_2, A_2, B_2) \preceq (F_3, A_3, B_3).$$

If $F_1 \cap F_2 = \emptyset$, set

$$(F_3, A_3, B_3) := (F_1 \cup F_2, A_1 \cup A_2, B_1 \cup B_2).$$

If $F_1 \cap F_2 \neq \emptyset$, take $f_{(s,t)} \in F_1 \cap F_2$. For $i \in \{1, 2\}$ and $m^i_{f_{(s,t)}} \in A_i$, since $M_{f_{(s,t)}}$ is directed, there exists $m^3_{f_{(s,t)}} \in M_{f_{(s,t)}}$ with $m^i_{f_{(s,t)}} \leq m^3_{f_{(s,t)}}$ for $i = 1, 2$. Define A_3 by taking these upper bounds for all common $f_{(s,t)} \in F_1 \cap F_2$, and leaving the elements unchanged for $f_{(s,t)} \in F_1 \setminus F_2$ or $f_{(s,t)} \in F_2 \setminus F_1$. A similar construction, using that each $N_{f_{(s,t)}}$ is filtered, produces B_3.

Thus

$$(F_3, A_3, B_3) := (F_1 \cup F_2, A_3, B_3)$$

satisfies $(F_1, A_1, B_1) \preceq (F_3, A_3, B_3)$ and $(F_2, A_2, B_2) \preceq (F_3, A_3, B_3)$. Hence $(\Omega, \preceq)$ is directed.

Consider the product $\Omega \times \Gamma$ with order defined by $((F_1, A_1, B_1), \gamma_1) \ll ((F_2, A_2, B_2), \gamma_2)$ if and only if $(F_1, A_1, B_1) \preceq (F_2, A_2, B_2)$ and $\gamma_1 \leq \gamma_2$. Since both Ω and Γ are directed sets, their product $\Omega \times \Gamma$ is directed as well.

For every $(F, A, B) \in \Omega$ and each $f_{(s,t)} \in F$ with $m_{f_{(s,t)}} \in A$ and $n_{f_{(s,t)}} \in B$, there exists $\gamma_{(s,t)} \in \Gamma$ such that $f_{(s,t)}(x_\gamma) \in [m_{f_{(s,t)}}, n_{f_{(s,t)}}]$ for all $\gamma \geq \gamma_{(s,t)}$. Since F is finite, we can choose $\gamma(F, A, B) \in \Gamma$ such that $\gamma_{(s,t)} \leq \gamma(F, A, B)$ for every $f_{(s,t)} \in F$.

Let $\Lambda := \{((F, A, B), \gamma) : (F, A, B) \in \Omega, \gamma \in \Gamma \text{ and } \gamma(F, A, B) \leq \gamma\}$. Then Λ is cofinal in $\Omega \times \Gamma$. Indeed, let $((F, A, B), \gamma) \in \Omega \times \Gamma$. Choose $\beta \in \Gamma$ such that $\gamma(F, A, B) \leq \beta$ and $\gamma \leq \beta$. Then $((F, A, B), \beta) \in \Lambda$ and $((F, A, B), \gamma) \ll ((F, A, B), \beta)$.

Now consider the map $\varphi : \Lambda \to \Gamma$ where $((F, A, B), \gamma) \mapsto \gamma$. Clearly φ is isotone: if $((F_1, A_1, B_1), \gamma_1) \ll ((F_2, A_2, B_2), \gamma_2)$, then $\gamma_1 \leq \gamma_2$, hence $\varphi((F_1, A_1, B_1), \gamma_1) \leq \varphi((F_2, A_2, B_2), \gamma_2)$. To see that φ is a final map, take $\lambda \in \Gamma$ and $((F, A, B), \gamma) \in \Lambda$. Choose $\beta \in \Gamma$ such that $\gamma \leq \beta$ and $\lambda \leq \beta$. Then, $((F, A, B), \beta) \in \Lambda$ and $\varphi(((F, A, B), \beta)) \geq \lambda$. Thus, φ is a final map.

Consider the subnet $(x_{\varphi((F,A,B),\gamma)})_{((F,A,B),\gamma)\in\Lambda}$ of $(x_\gamma)_{\gamma\in\Gamma}$. We show that this subnet is $\mathfrak{u}\mathrm{O}_1$-convergent to x. Fix $f_{(s,t)} \in \mathscr{F}$ and define

$$\Lambda_{f_{(s,t)}} := \{((F_0, A, B), \gamma) : f_{(s,t)} \in F_0\}.$$

For every $((F_0, A, B), \gamma) \in \Lambda_{f_{(s,t)}}$, set $m_{((F_0,A,B),\gamma))} = m_{f_{(s,t)}}$ and $n_{((F_0,A,B),\gamma))} = n_{f_{(s,t)}}$. By construction, the net $(m_{((F_0,A,B),\gamma)})_{((F,A,B),\gamma)\in\Lambda_{f_{(s,t)}}}$ is increasing and

$$\sup\{m_{((F_0,A,B),\gamma)} : ((F_0, A, B), \gamma) \in \Lambda_{f_{(s,t)}}\} = f_{(s,t)}(x).$$

Similarly, the net $(n_{((F_0,A,B),\gamma)})_{((F,A,B),\gamma)\in\Lambda_{f_{(s,t)}}}$ is decreasing and

$$\inf\{n_{((F_0,A,B),\gamma)} : ((F_0, A, B), \gamma) \in \Lambda_{f_{(s,t)}}\} = f_{(s,t)}(x).$$

Now observe that

$$m_{((F_0,A,B),\gamma)} \leq f_{(s,t)}(x_{\varphi((F_1,A_1,B_1,\alpha))}) \leq n_{((F_0,A,B),\gamma)}$$

for every $((F_0, A, B), \gamma) \ll ((F_1, A_1, B_1), \alpha)$ Finally, note that for $((F_0, A_0, B_0), \lambda_0) \in \Lambda_{f_{(s,t)}}$, we have that

$$E_{\Lambda_{f_{(s,t)}}}\big(((F_0, A, B), \gamma)\big) = E_\Lambda\big(((F_0, A, B), \gamma)\big),$$

which shows that the subnet $\big(x_{\varphi((F,A,B),\gamma)}\big)_{((F,A,B),\gamma)\in\Lambda}$ is indeed $\mathfrak{u}\mathrm{O}_1$-convergent to x. □

Recall that for O_1-, O_2- and O_3-convergence, we defined the order-adherence and the order topologies $\tau_{\mathrm{O}_1}, \tau_{\mathrm{O}_2}$ and τ_{O_3}, respectively. For $\mathfrak{u}\mathrm{O}$- and $\mathfrak{u}\mathrm{O}_1$-convergence, we can similarly define the respective *unbounded order adherence* ($\mathfrak{u}\mathrm{O}$-adherence). The unbounded order adherence is denoted by $X_\lambda^{\mathfrak{u}\mathrm{O}_i}$ for every ordinal $\lambda > 0$ and defined by recursion in the same way that we defined the λ-O-adherence X_λ^{O}, i.e.

$$X_\lambda^{\mathfrak{u}\mathrm{O}_i} = \left(\bigcup_{\beta<\lambda} X_\beta^{\mathfrak{u}\mathrm{O}_i} \right)_1^{\mathfrak{u}\mathrm{O}_i}.$$

Definition 2.4.3. *A subset F of a lattice L is called $\mathfrak{u}\mathrm{O}_1$-closed if the $\mathfrak{u}\mathrm{O}_1$-limit of every net in F belongs to F. Equivalently, F is $\mathfrak{u}\mathrm{O}_1$-closed if $F = F_1^{\mathfrak{u}\mathrm{O}_1}$.*

Definition 2.4.4. *A subset F of a lattice L is called $\mathfrak{u}\mathrm{O}$-closed if the $\mathfrak{u}\mathrm{O}$-limit of every net in F belongs to F. Equivalently, F is $\mathfrak{u}\mathrm{O}$-closed if $F = F_1^{\mathfrak{u}\mathrm{O}}$.*

Definition 2.4.5. *The topology $\tau_{\mathfrak{u}\mathrm{O}_1}$ on L is the topology whose closed sets are precisely the $\mathfrak{u}\mathrm{O}_1$-closed subsets of P.*

Definition 2.4.6 (Unbounded Order Topology)**.** *The unbounded order topology on L, denoted by $\tau_{\mathfrak{u}\mathrm{O}}$, is the topology whose closed sets are precisely the $\mathfrak{u}\mathrm{O}$-closed subsets of L.*

In Theorem 2.2.7 we saw that on a poset P, $\tau_{\mathrm{O}_1} = \tau_{\mathrm{O}_2}$. A natural question that arises, posed in [44, Question 18.50], is whether the same equality holds for $\tau_{\mathfrak{u}\mathrm{O}_1}$ and $\tau_{\mathfrak{u}\mathrm{O}}$.

Theorem 2.4.7 ([3], Proposition 2.8)**.** *Let L be a lattice. Then $\tau_{\mathfrak{u}\mathrm{O}_1} = \tau_{\mathfrak{u}\mathrm{O}}$.*

Proof. The inclusion $\tau_{\mathfrak{u}\mathrm{O}} \subseteq \tau_{\mathfrak{u}\mathrm{O}_1}$ is immediate. The reverse inclusion $\tau_{\mathfrak{u}\mathrm{O}_1} \subseteq \tau_{\mathfrak{u}\mathrm{O}}$ follows from Theorem 2.4.2. □

2.4.1 The O_2-closure and $\mathfrak{u}\mathrm{O}$-closure of sublattices and ideals

From the definition of unbounded order convergence, it is evident that it is closely related to O_2-convergence. In the setting of Riesz spaces, the O-adherence and $\mathfrak{u}$O-adherence have been extensively studied on sublattices [27]. In this section we extend the discussion to λ-O-adherence and λ-$\mathfrak{u}$O-adherence of sublattices and ideals in infinitely distributive lattices.

In Theorem 2.2.7 and Proposition 2.4.7, it was shown that when working with lattices, the order and unbounded order topologies that arise from different modes of convergence are equivalent. Thus in this section, we will simply write λ-O-adherence and λ-$\mathfrak{u}$O-adherence.

Lemma 2.4.8. *Let L be an infinitely distributive lattice and $X \subseteq L$. Then,*

(i) $X_1^{\mathrm{O}} \subseteq X_1^{\mathfrak{u}\mathrm{O}}$*;*

(ii) *The cuts $[a, \rightarrow)$ and $(\leftarrow, a]$ are* O*-closed and* $\mathfrak{u}$O*-closed;*

(iii) *There exist an ordinal λ such that $X_\lambda^{\mathrm{O}} = X_{\lambda+1}^{\mathrm{O}}$.*

Proof. (i) This follows easily from Theorem 2.3.6.

(ii) We give the proof for $[a, \rightarrow)$; the case of $(\leftarrow, a]$ is similar. Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in $[a, \rightarrow)$ that O_2-converges to x. Then there exist a directed set M and a filtered set N such that $\sup M = x = \inf N$ and for every $(m, n) \in M \times N$, the net $(x_\gamma)_{\gamma\in\Gamma}$ is eventually in $[m, n]$. Then $N \subseteq [a, \rightarrow)$, concluding that $x \in [a, \rightarrow)$. By the distributivity of L and Proposition 2.3.4 (i), it follows that both $[a, \rightarrow)$ and $(\leftarrow, a]$ are $\mathfrak{u}$O-closed.

(iii) To see this, it is enough to observe that $X_\kappa^{\mathrm{O}} = X_{\kappa+1}^{\mathrm{O}}$ when κ is equal to any cardinality greater than the cardinality of L. The same holds for $\mathfrak{u}$O-adherence. If we let

$$\alpha := \min\{\lambda \geq 0 : X_\lambda^{\mathrm{O}} = X_{\lambda+1}^{\mathrm{O}}\}$$

$$\beta := \min\{\lambda \geq 0 : X_\lambda^{\mathfrak{u}\mathrm{O}} = X_{\lambda+1}^{\mathfrak{u}\mathrm{O}}\},$$

then X_α^{O} coincides with the O-closure of X and $X_\beta^{\mathfrak{u}\mathrm{O}}$ with the $\mathfrak{u}$O-closure of X. If L is infinitely distributive, then $X_\alpha^{\mathrm{O}} \subseteq X_\beta^{\mathfrak{u}\mathrm{O}}$ (by Theorem 2.3.6). $\square$

Proposition 2.4.9. *Let Y be a sublattice of an infinitely distributive lattice L. For every ordinal $\lambda \geq 0$, each of the sets Y_λ^{O} and $Y_\lambda^{\mathfrak{u}\mathrm{O}}$ is a sublattice of L.*

Proof. We prove only the case of O-adherence; the $\mathfrak{u}$O-adherence case follows by the same argument (applying Proposition 2.3.5 instead of Proposition 2.1.8). We prove this result by transfinite induction. The assertion is trivially satisfied when $\lambda = 0$.

For $x,\ y \in Y_\lambda^{\mathrm{O}}$ there exist two nets $(x_\sigma)_{\sigma\in\Sigma}$ and $(y_\omega)_{\omega\in\Omega}$ in $\bigcup_{\beta<\lambda} Y_\beta^{\mathrm{O}}$ such that $x_\sigma \xrightarrow{\mathrm{O}_2} x$ and $y_\omega \xrightarrow{\mathrm{O}_2} y$. By the induction hypothesis, each Y_β^{O} $(0 \leq \beta < \lambda)$ is a sublattice of L, and hence so is $\bigcup_{\beta<\lambda} Y_\beta^{\mathrm{O}}$. Hence, $(x_\sigma \vee y_\omega)_{(\sigma,\omega)\in\Sigma\times\Omega}$ is a net in $\bigcup_{\beta<\lambda} Y_\beta^{\mathrm{O}}$. Proposition 2.1.8 implies $x_\sigma \vee y_\omega \xrightarrow{\mathrm{O}_2} x \vee y$ and so $x \vee y \in Y_\lambda^{\mathrm{O}}$. Similarly, $x \wedge y \in Y_\lambda^{\mathrm{O}}$. $\square$

Note that in Proposition 2.4.9, infinite distributivity cannot be weakened to distributivity.

Example 22. *Consider the following subsets of $2^{\mathbb{R}}$.*

$$\begin{aligned}
\mathscr{C}_- &= \{(-\infty, a] : a \leq 0\}\\
\mathscr{C}'_- &= \{(-\infty, a] : a < 0\}\\
\mathscr{C}_+ &= \{[a, +\infty) : a \geq 0\}\\
\mathscr{C}'_+ &= \{[a, +\infty) : a > 0\}
\end{aligned}$$

The ring L of subsets of $\mathbb{R}$ generated by $\mathscr{C}_- \cup \mathscr{C}_+$ consists of all subsets of $\mathbb{R}$ that have one of the following types: $\emptyset$, $(-\infty, -a]$, $[b, +\infty)$, $(-\infty, -a] \cup [b, +\infty)$, $\{0\}$, where $a, b \geq 0$. This forms a distributive *lattice. The subring Y generated by $\mathscr{C}'_- \cup \mathscr{C}'_+$ consists of all subsets of $\mathbb{R}$ that have one of the following types: $\emptyset$, $(-\infty,-a]$, $[b,+\infty)$, $(-\infty,-a]\cup[b,+\infty)$, where $a,b>0$. Clearly Y is a sublattice of L. The* O*-closure $\overline{Y}$ of Y in L consists of the subsets of $\mathbb{R}$ that have one of following types: $\emptyset$, $(-\infty,-a]$, $[b,+\infty)$, $(-\infty,-a]\cup[b,+\infty)$, where $a,b\geq 0$. Observe that the infimum in $\overline{Y}$ of $(-\infty,0]$ and $[0,+\infty)$ is equal to $\emptyset$, whereas the infimum taken in L equals $\{0\}$.*

Proposition 2.4.10. *Let Y be a sublattice of an infinitely distributive lattice L. For every ordinal $\lambda\geq 0$ there exists an ordinal $f(\lambda)\geq\lambda$ such that*

$$Y_\lambda^{\mathrm{O}} \subseteq Y_\lambda^{\mathfrak{u}\mathrm{O}} \subseteq Y_{f(\lambda)}^{\mathrm{O}}. \tag{2.1}$$

Proof. The function f shall be defined recursively. Let $f(0)=1$. Clearly (2.1) is satisfied. Suppose that $\{f(\beta):\beta<\lambda\}$ have been constructed satisfying $Y_\beta^{\mathrm{O}}\subseteq Y_\beta^{\mathfrak{u}\mathrm{O}}\subseteq Y_{f(\beta)}^{\mathrm{O}}$ for every $\beta<\lambda$. The inclusion $\bigcup_{\beta<\lambda}Y_\beta^{\mathrm{O}} \subseteq \bigcup_{\beta<\lambda}Y_\beta^{\mathfrak{u}\mathrm{O}}$ follows by the induction hypothesis and therefore

$$Y_\lambda^{\mathrm{O}} = \Big(\bigcup_{\beta<\lambda} Y_\beta^{\mathrm{O}}\Big)_1^{\mathrm{O}} \subseteq \Big(\bigcup_{\beta<\lambda} Y_\beta^{\mathfrak{u}\mathrm{O}}\Big)_1^{\mathrm{O}} \subseteq \Big(\bigcup_{\beta<\lambda} Y_\beta^{\mathfrak{u}\mathrm{O}}\Big)_1^{\mathfrak{u}\mathrm{O}} = Y_\lambda^{\mathfrak{u}\mathrm{O}},$$

where the last inclusion follows by Lemma 2.4.8 (i). Let $\omega:=\sup\{f(\beta):\beta<\lambda\}$. Then

$$\bigcup_{\beta<\lambda} Y_\beta^{\mathfrak{u}\mathrm{O}} \subseteq \bigcup_{\beta<\lambda} Y_{f(\beta)}^{\mathrm{O}} \subseteq \bigcup_{\beta<\omega} Y_\beta^{\mathrm{O}}.$$

Let $x\in\big(\bigcup_{\beta<\lambda}Y_\beta^{\mathfrak{u}\mathrm{O}}\big)_1^{\mathfrak{u}\mathrm{O}}$, so there exists a net $(x_\gamma)_{\gamma\in\Gamma}$ in $\bigcup_{\beta<\lambda}Y_\beta^{\mathfrak{u}\mathrm{O}}\subseteq\bigcup_{\beta<\omega}Y_\beta^{\mathrm{O}}$ such that $x_\gamma\xrightarrow{\mathfrak{u}\mathrm{O}}x$, i.e $(x_\gamma\wedge t)\vee s\xrightarrow{\mathrm{O}_2}(x\wedge t)\vee s$ for every $s,t\in L$. For any triple γ_1, γ_2 and γ_3 in Γ, the element $(x_{\gamma_1}\wedge x_{\gamma_2})\vee x_{\gamma_3}$ lies in $\bigcup_{\beta<\omega}Y_\beta^{\mathrm{O}}$ because the latter is a sublattice by Proposition 2.4.9. This implies that $(x\wedge x_{\gamma_2})\vee x_{\gamma_3}$ belongs to $\big(\bigcup_{\beta<\omega}Y_\beta^{\mathrm{O}}\big)_1^{\mathrm{O}}=Y_\omega^{\mathrm{O}}$, for every γ_2, γ_3 in Γ. For fixed $\gamma_3\in\Gamma$ and using the $\mathfrak{u}$O-convergence of $(x_\gamma)_{\gamma\in\Gamma}$, it can be concluded that $x\vee x_\gamma\in Y_{\omega+1}^{\mathrm{O}}$ for every $\gamma\in\Gamma$. Using that $Y_{\omega+1}^{\mathrm{O}}$ is a sublattice, we note that $(x\vee x_\gamma)\wedge(x\vee x_{\gamma'})\in Y_{\omega+1}^{\mathrm{O}}$ for every $\gamma, \gamma' \in \Gamma$. Invoking once more the $\mathfrak{u}$O-convergence of $(x_\gamma)_{\gamma\in\Gamma}$ to x, it can be concluded that $x = x \wedge (x \vee x_{\gamma'})$ belongs to $Y^{\mathrm{O}}_{\omega+2}$. Define $f(\lambda) = \omega + 2$, to get the required inclusion. $\square$

Theorem 2.4.11. *The* O*-closure and the* $\mathfrak{u}$O*-closure of a sublattice of an infinitely distributive lattice L coincide, and the resulting subset is again a sublattice of L.*

Proof. Let Y be a sublattice of L and

$$\alpha := \min\{\lambda \geq 0 : Y^{\mathrm{O}}_\lambda = Y^{\mathrm{O}}_{\lambda+1}\} \quad \text{and} \quad \beta := \min\{\lambda \geq 0 : Y^{\mathfrak{u}\mathrm{O}}_\lambda = Y^{\mathfrak{u}\mathrm{O}}_{\lambda+1}\},$$

i.e. O-closure of Y equals Y^{O}_α and the $\mathfrak{u}$O-closure of Y equals $Y^{\mathfrak{u}\mathrm{O}}_\beta$. These are sublattices by Proposition 2.4.9, and $Y^{\mathrm{O}}_\alpha \subseteq Y^{\mathfrak{u}\mathrm{O}}_\beta$ by Theorem 2.3.6. By definition of α, observe that

$$Y^{\mathrm{O}}_\alpha = Y^{\mathrm{O}}_{f(\alpha)} = Y^{\mathrm{O}}_{f(f(\alpha))}$$

and therefore $Y^{\mathrm{O}}_\alpha = Y^{\mathfrak{u}\mathrm{O}}_\alpha = Y^{\mathfrak{u}\mathrm{O}}_{f(\alpha)}$, by Proposition 2.4.10. In particular, we get that $Y^{\mathrm{O}}_\alpha = Y^{\mathfrak{u}\mathrm{O}}_\alpha = Y^{\mathfrak{u}\mathrm{O}}_{\alpha+1}$, i.e. Y^{O}_α is $\mathfrak{u}$O-closed. Therefore $Y^{\mathfrak{u}\mathrm{O}}_\beta$ (the $\mathfrak{u}$O-closure of Y) is contained in Y^{O}_α. $\square$

Corollary 2.4.12. *A sublattice of an infinitely distributive lattice is* O*-closed if and only if it is* $\mathfrak{u}$O*-closed.*

The following theorem is the analogue of Proposition 2.4.10, Theorem 2.4.11 and Corollary 2.4.12 for Riesz spaces. We remark that, in the Riesz spaces setting, Proposition 2.4.10 admits a slight strengthening.

Proposition 2.4.13 ([27], Proposition 2.1)**.** *Let Y be a sublattice of a Riesz space X. Then, $Y^{\mathrm{O}}_1 \subseteq Y^{\mathfrak{u}\mathrm{O}}_1 \subseteq Y^{\mathrm{O}}_2$. Moreover,*

(i) *if Y^{O}_1 is order closed, then it is the smallest order closed sublattice of X containing Y, and $Y^{\mathrm{O}}_1 = Y^{\mathfrak{u}\mathrm{O}}_1$;*

(ii) *if Y_1^{uO} is order closed, then it is the smallest order closed sublattice of X containing Y, and $Y_1^{\mathrm{uO}} = Y_2^{\mathrm{O}}$.*

We now examine the adherence of down-sets and order ideals in lattices.

Proposition 2.4.14. *Let Q be a down-set in a lattice L. Then every element $x \in Q_1^{\mathrm{O}}$ can be expressed as the supremum of an increasing net in Q. Moreover, if L is infinitely distributive, then Q_1^{O} itself is a down-set.*

Proof. Let $x \in Q_1^{\mathrm{O}}$. Then there exists a net $(x_\gamma)_{\gamma\in\Gamma}$ in Q such that $x_\gamma \xrightarrow{\mathrm{O}_2} x$. There exist a directed set M and a filtered set N satisfying $\bigvee M = x = \bigwedge N$ and, for every $(m,n) \in M \times N$, the net $(x_\gamma)_{\gamma\in\Gamma}$ is eventually in $[m,n]$. Since Q is a down-set, we have $M \subseteq Q$. As M is directed, it can be viewed as an increasing net indexed over itself, yielding the desired conclusion.

Furthermore, let $a \in Q_1^{\mathrm{O}}$ and $x \in L$ with $x \le a$, and assume that L is infinitely distributive. By the argument above, there exists an increasing net $(a_\gamma)_{\gamma\in\Gamma} \subseteq Q$ such that $\bigvee_{\gamma\in\Gamma} a_\gamma = a$. Since Q is a down-set, $\{a_\gamma \wedge x : \gamma \in \Gamma\} \subseteq Q$, and $\bigvee\{a_\gamma \wedge x : \gamma \in \Gamma\} = a \wedge x = x$, which shows that $x \in Q_1^{\mathrm{O}}$. $\square$

Theorem 2.4.15. *Let L be an infinitely distributive lattice, and let $\mathcal{I} \subseteq L$ be an order ideal. Then $\mathcal{I}_1^{\mathrm{O}} = \mathcal{I}_1^{\mathrm{uO}}$, and both are* uO*-closed order ideals, and hence* O*-closed.*

Proof. If $\mathcal{I}$ is an order ideal, then $\mathcal{I}_1^{\mathrm{O}}$ is an order ideal by Propositions 2.1.8 and 2.4.14.

Next, we prove that $\mathcal{I}_1^{\mathrm{O}}$ is O-closed. Let $x \in \mathcal{I}_2^{\mathrm{O}}$. Then there exists an increasing net $(x_\gamma)_{\gamma\in\Gamma}$ in $\mathcal{I}_1^{\mathrm{O}}$ such that $\bigvee_{\gamma\in\Gamma} x_\gamma = x$. Define $B := \{a \in \mathcal{I} : \exists\gamma \in \Gamma \text{ such that } a \le x_\gamma\}$. Then B is non-empty, directed, and x is an upper bound of B. Moreover, if $k \in L$ satisfies $b \le k$ for all $b \in B$, then $x_\gamma \le k$ for every $\gamma \in \Gamma$, and hence $x \le k$. It follows that $x = \sup B \in \mathcal{I}_1^{\mathrm{O}}$. Consequently, $\mathcal{I}_2^{\mathrm{O}} = \mathcal{I}_1^{\mathrm{O}}$, showing that $\mathcal{I}_1^{\mathrm{O}}$ is O-closed.

Finally, Theorem 2.4.11 implies that $\mathcal{I}_1^{\mathrm{O}}$ is the $\mathfrak{u}$O-closure of $\mathcal{I}$, and therefore $\mathcal{I}_1^{\mathrm{O}} = \mathcal{I}_1^{\mathfrak{u}\mathrm{O}}$. □

2.5 Unbounded order closure of regular sublattices

In Proposition 1.1.34, we showed that every order ideal is a regular sublattice. In this section, we aim to extend the results of Theorem 2.4.15 to sublattices. We begin with a result that holds in the setting of Riesz spaces.

Theorem 2.5.1 ([27], Theorem 2.13). *Let V be a Riesz space and W a regular Riesz subspace. Then $W_1^{\mathfrak{u}\mathrm{O}} = W_1^{\mathrm{O}}$, and both are order closed.*

The proof of Theorem 2.5.1 relies on the fact that, for a Riesz space V and a regular Riesz subspace W, the Dedekind completion W^δ embeds as a Regular Riesz subspace of V^δ [28, Theorem 2.10].

When working with infinitely distributive lattices, one must exercise caution. While Riesz spaces are intrinsically infinitely distributive, and the MacNeille completion of a Riesz space preserves infinite distributivity, the MacNeille completion of an infinitely distributive lattice need not be infinitely distributive, as the following example illustrates.

Example 23. *When endowed with the pointwise partial order,*

$$L := \{(0, b) : 0 \leq b < 1\} \cup \{(1, b) : 0 \leq b < +\infty,\, b \neq 1\}$$

forms an infinitely distributive lattice. It is easy to see that

$$\begin{aligned} DM(L) &= \{(0, b) : 0 \leq b < 1\} \cup \{(1, b) : 0 \leq b \leq +\infty\}, \\ L^\delta &= \{(0, b) : 0 < b < 1\} \cup \{(1, b) : 0 \leq b < +\infty\}. \end{aligned}$$

To see that L^δ (and hence $DM(L)$) does not satisfy the join-infinite distributive law, consider the sequence $x_n = \left(0, 1-\frac{1}{n}\right)$. Then $\bigvee^{L^\delta} x_n = (1,1)$ and $\left(\bigvee^{L^\delta} x_n\right) \wedge \left(1, \frac{1}{2}\right) = (1, \frac{1}{2})$. On the other hand, $\bigvee^{L^\delta} \left(x_n \wedge (1, \frac{1}{2})\right) = \left(0, \frac{1}{2}\right)$.

Definition 2.5.2. *Let Y be a sublattice of a lattice L, and let $D \subseteq Y$. Define $D^{+_Y} := D^+ \cap Y$ and $D^{-_Y} := D^- \cap Y$.*

Lemma 2.5.3. *Let Y be a sublattice of a lattice L, and let $A, B \subseteq Y$. If $A^{+-} \subseteq B^{+-}$, then $A^{+_Y-_Y} \subseteq B^{+_Y-_Y}$.*

Proof. First, we show that $B^{+_Y} \subseteq A^+$. Indeed, $B^{+_Y} \subseteq B^+ = B^{+-+} \subseteq A^{+-+} = A^+$.

Next, we show that $A^{+-_Y} \subseteq B^{+_Y-_Y}$. Let $y \in A^{+-_Y}$. Then $y \in Y$ and $y \leq b$ for every $b \in A^+$. From $B^{+_Y} \subseteq A^+$ it is clear that $y \leq c$ for every $c \in B^{+_Y}$. Hence, $y \in B^{+_Y-_Y}$, so $A^{+-_Y} \subseteq B^{+_Y-_Y}$. Finally, from $A \subseteq A^{+-_Y} \subseteq B^{+_Y-_Y}$ we conclude that $A^{+_Y-_Y} \subseteq B^{+_Y-_Y}$. $\square$

Theorem 2.5.4. *Let L be a lattice, and let $Y \subseteq L$ a sublattice. Then the map*

$$i : DM(Y) \to DM(L) : A \mapsto A^{+-}$$

defines an order-embedding of $DM(Y)$ into $DM(L)$.

Proof. This follows by Lemma 2.5.3. $\square$

Definition 2.5.5. *Let Y be a sublattice of a lattice L.*

- *We say that Y has Property (A) if, for every $A \subseteq Y$ and every $x \in A^-$ (in L), there exists $y \in A^- \cap Y$ such that $x \leq y$.*
- *Dually, Y has Property (B) if, for every $A \subseteq Y$ and every $x \in A^+$ (in L), there exists $y \in A^+ \cap Y$ such that $x \geq y$.*

Lemma 2.5.6. *Let L be a lattice, and Y a sublattice which satisfies properties (A) and (B). Then Y is a regular sublattice of L.*

Proof. Let $A \subseteq Y$ and $b = \bigvee^Y A$. Assume $x \in A^+$. Then, by property (B), there exists $a \in A^{+_Y}$ such that $a \le x$. Since $b = \bigvee^Y A \le a$, it follows that $b \le x$ for all $x \in A^+$. The dual statement for meets follows similarly by Property (A). $\square$

In the following example, we note that the converse of Lemma 2.5.6 does not necessarily hold.

Example 24. *Let $L = \{(0,b) : 0 \le b \le 1\} \cup \{(1,b) : 0 \le b \le 1\}$ be ordered pointwise, and let $L_0 = \{(0,b) : 0 \le b < 1\} \cup \{(1,b) : 0 \le b < 1\}$. Clearly, L is complete, and L_0 is a regular sublattice of L. However, property (B) fails to hold in L_0, since there exist upper bounds in L that cannot be approximated by L_0.*

Proposition 2.5.7. *Let L be a lattice, $Y \subseteq L$, and $A \subseteq Y$.*

(i) *If Y satisfies Property (A), then $A^{-+} = A^{-_Y+}$.*

(ii) *If Y satisfies Property (B), then $A^{+-} = A^{+_Y-}$.*

Proof. We shall only show the proof of (i), as the proof for (ii) follows by a similar argument. Clearly, $A^{-+} \subseteq A^{-_Y+}$. To see the reverse inclusion let $x \in A^{-_Y+}$ and $z \in A^-$. Then by Property (A) there exists $y \in A^{-_Y}$ satisfying $z \le y$. This implies that $z \le y \le x$, so $x \in A^{-+}$, concluding that $A^{-+} \supseteq A^{-_Y+}$. $\square$

Proposition 2.5.8. *Let L be a lattice, $Y \subseteq L$ a sublattice, and $\{A_\alpha : \alpha \in \mathcal{A}\}$ a collection of sets in Y satisfying $A_\alpha^{+_Y-_Y} = A_\alpha$ for all $\alpha \in \mathcal{A}$.*

(i) *If Y satisfies Property (A), then*

$$\bigcap_{\alpha\in\mathcal{A}} A_\alpha^{+-} = \left(\bigcap_{\alpha\in\mathcal{A}} A_\alpha\right)^{+-}.$$

(ii) *If Y satisfies Property (B), then*

$$\left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha\right)^{+_Y-_Y+-} = \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha^{+-}\right)^{+-}.$$

Proof. (i) We first note that the inclusion $(\bigcap_{\alpha\in\mathcal{A}} A_\alpha)^{+-} \subseteq \bigcap_\alpha A_\alpha^{+-}$ is trivial. To see the reverse inclusion, recall from Proposition 1.1.20 that $\bigcap_{\alpha\in\mathcal{A}} A_\alpha^{+-} = (\bigcup_{\alpha\in\mathcal{A}} A_\alpha^{+})^{-}$. Furthermore,

$$\begin{aligned}
\left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha^{+}\right)^{-} &= \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha^{+}\right)^{-+-} \\
&\subseteq \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha^{+_Y}\right)^{-+-} \\
&= \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha^{+_Y}\right)^{-_Y+-} \quad \text{(By property (A))} \\
&= \left(\bigcap_{\alpha\in\mathcal{A}} A_\alpha^{+_Y-_Y}\right)^{+-} \\
&= \left(\bigcap_{\alpha\in\mathcal{A}} A_\alpha\right)^{+-}.
\end{aligned}$$

(ii) To see this equality, note that

$$\begin{aligned}
\left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha\right)^{+_Y-_Y+-} &= \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha\right)^{+_Y-_Y+_Y-} \quad \text{(By property (B))} \\
&= \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha\right)^{+_Y-} \\
&= \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha\right)^{+-} \quad \text{(By property (B))} \\
&= \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha^{+-}\right)^{+-}.
\end{aligned}$$

□

Theorem 2.5.9. *Let L be a lattice, and $Y \subseteq L$ a sublattice. Let $i : DM(Y) \to DM(L) : A \mapsto A^{+-}$ be the order-embedding described in Theorem 2.5.4.*

(i) *If Y satisfies Property (A), then i preserves arbitrary meets.*

(ii) *If Y satisfies Property (B), then i preserves arbitrary joins.*

Proof. Let $\{A_\alpha : \alpha \in \mathcal{A}\} \subseteq DM(Y)$. Then, by Proposition 2.5.8, we have:

$$\begin{aligned} i\left(\bigwedge\nolimits_{\alpha\in\mathcal{A}}^{DM(Y)} A_\alpha\right) &= \left(\bigcap_{\alpha\in\mathcal{A}} A_\alpha\right)^{+-} \\ &= \bigcap_{\alpha\in\mathcal{A}} A_\alpha^{+-} \\ &= \bigwedge\nolimits_{\alpha\in\mathcal{A}}^{DM(L)} i(A_\alpha)\,, \end{aligned}$$

if Y satisfies Property (A); and

$$\begin{aligned} i\left(\bigvee\nolimits_{\alpha\in\mathcal{A}}^{DM(Y)} A_\alpha\right) &= \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha\right)^{+_Y -_Y +-} \\ &= \left(\bigcup_{\alpha\in\mathcal{A}} A_\alpha^{+-}\right)^{+-} \\ &= \bigvee\nolimits_{\alpha\in\mathcal{A}}^{DM(L)} i(A_\alpha)\,, \end{aligned}$$

if Y satisfies Property (B). $\square$

Corollary 2.5.10. *Let L be a lattice, and let Y be a sublattice satisfying Properties (A) and (B). Then $i[DM(Y)]$ is a regular sublattice of $DM(L)$, where $i : DM(Y) \to DM(L)$ and $A \mapsto A^{+-}$ is the order-embedding described in Theorem 2.5.4.*

Proposition 2.5.11. *Let L be a lattice, and let Y be an order convex sublattice.*

(i) *If Y has a maximal element, then it satisfies Property (B);*

(ii) *If Y has a minimal element, then it satisfies Property (A).*

Proof. We prove (i); the proof of (ii) is dual. Let $a_{\max}$ be the maximal element of Y. Let $A \subseteq Y$, and let $x \in A^+$. Then, by order convexity,

$a_{\max} \wedge x \in Y$ and $a_{max} \wedge x \in A^{+_Y}$. □

Proposition 2.5.12. *Let L be an order complete lattice, and let Y be a sublattice of L. Then Y is* O*-closed if and only if Y is supremum and infimum closed.*

Proof. Assume that Y is O-closed. Let $A \subseteq L$ with $\bigvee^L A = x$. Let $\mathscr{F} = \{B \subseteq A : |B| < \omega\}$. $\mathscr{F}$ is directed with respect to inclusion. For every $B \in \mathscr{F}$ let $x_B = \bigvee^L B = \bigvee^Y B$ to get an increasing net $\{x_B : B \in \mathscr{F}\}$ in Y with $\bigvee^L_{B\in\mathscr{F}} x_B = x$, i.e. a net in Y that O$_2$-converges to x. Since Y is O-closed, $x \in Y$. The dual statement can be proved similarly.

Conversely, assume that for any subset A of Y, its supremum and infimum in L belong to Y, and let $x \in Y_1^{\mathrm{O}}$. Then there exists a net $(x_\gamma)_{\gamma\in\Gamma}$ satisfying

$$\bigvee\nolimits^L_{\gamma'\in\Gamma} \bigwedge\nolimits^L_{\gamma\geq\gamma'} x_\gamma = x = \bigwedge\nolimits^L_{\gamma'\in\Gamma} \bigvee\nolimits^L_{\gamma\geq\gamma'} x_\gamma \,.$$

By hypothesis it follows that $x \in Y$. □

Proposition 2.5.13. *Let L be an order complete lattice, and let Y be an order complete and regular sublattice of L. Then Y is* O*-closed.*

Proof. Since Y is closed under arbitrary suprema and infima in L, result follows directly by Proposition 2.5.12. □

Theorem 2.5.14. *Let L be an infinitely distributive lattice and $Y \subseteq L$ a sublattice satisfying Properties (A) and (B). Then $Y_1^{\mathfrak{u}\mathrm{O}} = Y_1^{\mathrm{O}}$, and both are simultaneously* O*-closed and* $\mathfrak{u}$O*-closed.*

Proof. Identify $DM(Y)$ with $i[DM(Y)] \subseteq DM(L)$. Corollary 2.5.10 implies that $DM(Y)$ is a complete regular sublattice of $DM(L)$. In particular, by Proposition 2.5.13, $DM(Y)$ is O-closed (and therefore $\mathfrak{u}$O-closed because $DM(L)$ has maximal and minimal elements).

Next we show that $DM(Y) \cap L = Y_1^{\mathrm{O}}$. For every $x \in Y_1^{\mathrm{O}}$ there exists a net $(x_\gamma)_{\gamma\in\Gamma}$ in Y such that $x_\gamma \xrightarrow{\mathrm{O}_2} x$ in L. From [2, Theorem 3], we have that $x_\gamma \xrightarrow{\mathrm{O}_2} x$ in $DM(L)$. Since $DM(Y)$ is O-closed in $DM(L)$, it follows that $x \in DM(Y) \cap L$. To see the converse, take $x \in DM(Y) \cap L$ and $A = \{y \in Y : y \leq x\}$. By Theorem 2.5.9, $DM(Y)$ is regular in $DM(L)$ and thus $\bigvee^{DM(L)} A = \bigvee^{DM(Y)} A = x$. Since L is regular in $DM(L)$, it follows that $x \in Y_1^{\mathrm{O}}$. This shows that $DM(Y) \cap L = Y_1^{\mathrm{O}}$. Now it is easy to see that $Y_2^{\mathrm{O}} = Y_1^{\mathrm{O}}$, and therefore Y_1^{O} is simultaneously O-closed and $\mathfrak{u}$O-closed. $\square$

3 The topologies on the function space L^∞

In this section, we discuss several topologies on the function space L^∞. These will eventually be compared to the order topology.

3.1 Topologies on L^∞.

Definition 3.1.1. *A topological vector space is a vector space $\mathscr{V}$ over $\mathbb{R}$ endowed with a topology $\mathscr{T}$, such that:*

(i) *The addition map $(x, y) \mapsto x + y$ is continuous from $\mathscr{V} \times \mathscr{V}$ into $\mathscr{V}$ (where $\mathscr{V} \times \mathscr{V}$ has the product topology).*

(ii) *The scalar multiplication map $(t, x) \mapsto tx$ is continuous from $\mathbb{R} \times \mathscr{V}$ into $\mathscr{V}$ (where $\mathbb{R}$ has its usual topology and $\mathbb{R} \times \mathscr{V}$ has the product topology).*

Definition 3.1.2. *Let X be a set. A pseudometric on X is a function $\rho : X \times X \to [0, \infty)$ satisfying*

(i) *$\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$;*

(ii) *$\rho(x, y) = \rho(y, x)$ for every $x, y \in X$;*

(iii) *$\rho(x, x) = 0$ for every $x \in X$.*

Recall that for a pseudometric ρ, it is possible that $\rho(x, y) = 0$, even when $x \neq y$. A metric on X is a pseudometric ρ with the additional property that $\rho(x, y) = 0$ implies $x = y$.

Definition 3.1.3. *A pseudometric ρ on a vector space $\mathscr{V}$ is said to be translation invariant if $\rho(u, v) = \rho(u+w, v+w)$ for all $u, v, w \in \mathscr{V}$.*

Definition 3.1.4. *Let $\mathscr{V}$ be a vector space. An F-seminorm on $\mathscr{V}$ is a function $\tau : \mathscr{V} \to [0, \infty)$ satisfying, for all $u, v \in \mathscr{V}$ and $\alpha \in \mathbb{R}$,*

(i) $\tau(u + v) \leq \tau(u) + \tau(v)$*;*

(ii) $\tau(\alpha v) \leq \tau(v)$ *whenever* $|\alpha| \leq 1$*;*

(iii) $\lim_{\alpha \to 0} \tau(\alpha v) = 0$.

Remark 3.1.5. *Let $\mathscr{V}$ be a vector space and τ be an F-seminorm on $\mathscr{V}$. Define $\rho_\tau : \mathscr{V} \times \mathscr{V} \to [0, \infty)$ by $(u, v) \mapsto \tau(u - v)$. Then ρ_τ is a translation-invariant pseudometric on $\mathscr{V}$.*

Several important topologies can be generated by families of pseudometrics. The following theorem provides the general construction of a topology from a family of pseudometrics.

Theorem 3.1.6 ([23, p. 525])**.** *Let X be a set and P a non-empty family of pseudometrics on X. Define $\mathscr{T}_P$ as the family of those subsets G of X such that for every $x \in G$ there are $\{\rho_1, \rho_2, ..., \rho_n\} \subseteq P$ and $\epsilon > 0$ satisfying*

$$U(x; \rho_1, \rho_2, ..., \rho_n; \epsilon) \subseteq G$$

where $U(x; \rho_1, \rho_2, ..., \rho_n; \epsilon) = \{y \in X : \max_{1 \leq i \leq n} \rho_i(x, y) < \epsilon\}$. $\mathscr{T}_P$ is called the topology defined by P.

Proposition 3.1.7. *Let $(X, \mathscr{T}_P)$ be a topological space, where $\mathscr{T}_P$ is defined by a non-empty family P of pseudometrics. The following statements are equivalent :*

(i) $\mathscr{T}_P$ *is* T_0.

(ii) *$\mathscr{T}_P$ is Hausdorff.*

(iii) *The collection P separates the points of X; that is for every pair of distinct points $x, y \in X$, there exists $\rho \in P$ such that $\rho(x, y) > 0$.*

Theorem 3.1.8. *Let $\mathscr{V}$ be a vector space and let T be a family of F-seminorms on $\mathscr{V}$. Define $P := \{\rho_\tau : \tau \in T\}$. Then the topology $\mathscr{T}_P$ generated by P turns $\mathscr{V}$ into a topological vector space.*

3.1.1 Topology of convergence in measure

We begin this chapter by recalling the definition of a linear map and a linear functional.

Definition 3.1.9. *Let $\mathscr{V}$ and $\mathscr{W}$ be vector spaces over $\mathbb{R}$. A map $T : \mathscr{V} \to \mathscr{W}$ is called a linear map if*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $x, y \in \mathscr{V}$ and all $\alpha, \beta \in \mathbb{R}$. A linear functional is a linear map $\lambda : \mathscr{V} \to \mathbb{R}$. A family of linear functionals $\mathscr{F}$ is said to be separating if, for every $x \in \mathscr{V}$ with $x \neq 0$, there exists $\ell \in \mathscr{F}$ such that $\ell(x) \neq 0$.

Definition 3.1.10. *Let (X, Σ, μ) be a measure space. For each $F \in \Sigma^f$, define a real valued functional $\tau_F : \mathscr{L}^0 \to [0, \infty)$ by*

$$\tau_F(f) := \int |f| \wedge \chi_F \; d\mu$$

for every $f \in \mathscr{L}^0$.

Proposition 3.1.11. *Let (X, Σ, μ) be a measure space, and let $F \in \Sigma^f$. The functional $\tau_F : \mathscr{L} \to [0, \infty)$ satisfies the following properties:*

(i) *$\tau_F(f) = 0$ if and only if $f(x) = 0$ almost everywhere on F;*

(ii) *$\tau_F(f) = \tau_F(g)$ if $f = g$ almost everywhere on F;*

(iii) $\tau_F(f\chi_A) \leq \mu(F \cap A)$ *for every* $A \in \Sigma$*;*

(iv) $\tau_F(f) \leq \tau_F(g)$ *whenever* $0 \leq f \leq g$*;*

(v) $\tau_F(f) \leq \tau_G(f)$ *if* $G \in \Sigma^f$ *such that* $F \subseteq G$*;*

(vi) $\tau_F(\alpha f) \leq |\alpha|\tau_F(f)$ *for every* $\alpha \in \mathbb{R}$ *with* $|\alpha| \geq 1$*.*

Proposition 3.1.12. *Let* (X, Σ, μ) *be a measure space and* $F \in \Sigma^f$*. Then the functional* τ_F *is an* F*-seminorm on* $\mathscr{L}^0$*.*

Let (X, Σ, μ) be a measure space. By remark 3.1.5, for each $F \in \Sigma^f$, one can define a pseudometric ρ_{τ_F} on $\mathscr{L}^0$. By Theorem 3.1.6, the family of pseudometrics $\{\rho_{\tau_F} : F \in \Sigma^f\}$ generate a topology on $\mathscr{L}^0$.

Lemma 3.1.13 ([23], Remark 245B (b). p. 173)**.** *Let* (X, Σ, μ) *be a measure space. The family of pseudometrics* $\{\rho_{\tau_F} : F \in \Sigma^f\}$ *is directed. Then* $G \subseteq \mathscr{L}^0$ *is open if and only if for every* $f \in G$ *there exists* $\delta > 0$ *and* $F \in \Sigma^f$ *such that* $U(f; \rho_F; \delta) \subseteq G$*.*

Proposition 3.1.14 ([23], 245A & 245B, p. 173)**.** *Let* (X, Σ, μ) *be a measure space.*

(i) *If* $F \in \Sigma$ *satisfies* $\mu(F) < \infty$*, then the map*

$$\begin{aligned} \tilde{\rho}_{\tau_F} :& L^0 \times L^0 \to [0, \infty) \\ & (f^\bullet, g^\bullet) \mapsto \rho_{\tau_F}(f, g) \end{aligned}$$

defines a pseudometric on L^0*.*

(ii) *The family* $\{\tilde{\rho}_{\tau_F} : F \in \Sigma \text{ and } \mu(F) < \infty\}$ *is directed and generates a topology on* L^0*, denoted by* $\mathscr{T}_\mu$*.*

(iii) *A subset* $G \subseteq L^0$ *is open with respect to* $\mathscr{T}_\mu$ *if, and only if, for every* $f^\bullet \in G$*, there exist* $\delta > 0$ *and* $F \in \Sigma^f$ *such that*

$$\{g^\bullet \in L^0 : \tau_F(f - g) < \delta\} \subseteq G.$$

(iv) *The topology* $\mathscr{T}_\mu$ *is the finest topology that makes the quotient mapping* $\mathscr{L}^0 \to L^0$ *continuous.*

Theorem 3.1.15 ([23], Theorem 245E, p. 176)**.** *Let (X, Σ, μ) be a measure space. Then the following hold:*

(i) *(X, Σ, μ) is semi-finite if and only if $\mathscr{T}_\mu$ is Hausdorff.*

(ii) *(X, Σ, μ) is localisable if and only if $\mathscr{T}_\mu$ is Hausdorff and L^0 is complete under $\mathscr{T}_\mu$.*

(iii) *(X, Σ, μ) is σ-finite if and only if $\mathscr{T}_\mu$ is metrisable.*

The topology defined above is called the *topology of convergence in measure on* L^0. Since L^∞ is a Riesz subspace of L^0, the topology of convergence in measure on L^∞ is obtained by considering $\mathscr{T}_\mu \restriction_{L^\infty}$, which we denote by τ_μ.

3.1.2 The strong operator topology on L^∞

We start this section by recalling the definition of a seminorm.

Definition 3.1.16. *Let $\mathscr{V}$ be a vector space over the scalar field $\mathbb{R}$. A sublinear functional on $\mathscr{V}$ is a function $p : \mathscr{V} \to \mathbb{R}$ such that for all $x, y \in \mathscr{V}$ and $a \geq 0$,*

(i) *$p(x + y) \leq p(x) + p(y)$;*

(ii) *$p(ax) = ap(x)$.*

Furthermore, if $p(ax) = |a|p(x)$ for all $a \in \mathbb{R}$ and $x \in \mathscr{V}$, then p is called a seminorm.

Definition 3.1.17. *A family $\mathscr{S}$ of seminorms on a vector space $\mathscr{V}$ is said to be separating if, for every non-zero $v \in \mathscr{V}$, there exists a seminorm $r \in \mathscr{S}$ such that $r(v) \neq 0$.*

Definition 3.1.18. *A subset A of a vector space $\mathscr{V}$ is said to be convex if, for every $u, v \in A$ and $\alpha \in [0, 1]$, the combination $\alpha u + (1-\alpha)v \in A$.*

Note that every seminorm p on a vector space $\mathscr{V}$ induces a pseudometric $\rho_p(x, y) := p(x - y)$ for all $x, y \in \mathscr{V}$.

Theorem 3.1.19 ([32], p. 17)**.** *Let $\mathscr{V}$ be a vector space over $\mathbb{R}$, and let $\mathscr{S}$ be a family of seminorms on $\mathscr{V}$. Define $P := \{\rho_r : r \in \mathscr{S}\}$, where $\rho_r(x, y) := r(x - y)$. Then the topology generated by P turns $\mathscr{V}$ into a locally convex topological vector space, with respect to which each seminorm in $\mathscr{S}$ is continuous. Moreover, this topology is Hausdorff if and only if $\mathscr{S}$ is separating.*

Definition 3.1.20. *The topology $\mathscr{T}_{\mathscr{S}}$ generated by a family of seminorms is called the vector space topology determined by $\mathscr{S}$.*

Theorem 3.1.21. *Let $\mathscr{V}$ and $\mathscr{G}$ be normed spaces. Then the space $\mathscr{B}(\mathscr{V}, \mathscr{G})$, equipped with the operator norm, is itself a normed space.*

Proposition 3.1.22. *Let $\mathscr{V}$ and $\mathscr{G}$ be normed spaces. For each $v \in \mathscr{V}$, define $r_v : \mathscr{B}(\mathscr{V}, \mathscr{G}) \to [0, \infty) : r_v(T) \mapsto \|T(v)\|_{\mathscr{G}}$. Then the family $\{r_v : v \in \mathscr{V}\}$ is a separating family of seminorms on $\mathscr{B}(\mathscr{V}, \mathscr{G})$.*

Definition 3.1.23. *If $\mathscr{V}$ and $\mathscr{G}$ are normed spaces, the strong-operator-topology on $\mathscr{B}(\mathscr{V}, \mathscr{G})$ is the locally convex topology generated by the family of seminorms $\{\rho_{r_v} : v \in \mathscr{V}\}$, where*

$$\rho_{r_v} := r_v(S - T) = \|Sv - Tv\|_{\mathscr{G}}$$

for all $S, T \in \mathscr{B}(\mathscr{V}, \mathscr{G})$ and $v \in \mathscr{V}$.

From Lemma 1.6.2, for every $g \in \mathscr{L}^\infty$ and $1 \leq q < \infty$, the multiplication operator $M_g^q : L^q \to L^q : f^\bullet \mapsto (fg)^\bullet$, is well-defined and bounded. If (X, Σ, μ) is semi-finite, then by Theorem 1.6.3, the map $T_q : L^\infty \to \mathscr{B}(L^q) : g^\bullet \mapsto M_g^q$ is an isometric embedding. Thus, L^∞ may be identified with a subspace of $\mathscr{B}(L^q)$.

Consider the family of pseudometrics $\{\rho_{p_h} : h^\bullet \in L^q\}$ on $L^\infty \subseteq \mathscr{B}(L^q)$, where for each $h^\bullet \in L^q$ the seminorm $p_h : L^\infty \to [0, \infty) : f^\bullet \mapsto \left\|M_f^q(h^\bullet)\right\|_q = (\int |fh|^q \, \mathrm{d}\mu)^{\frac{1}{q}}$ is defined. By Theorem 3.1.19, these seminorms induce a locally convex topology on $\mathscr{B}(L^q)$. This topology is known as the *strong operator topology.* Restricting this topology to L^∞, we obtain a locally convex topology denoted by σ_q.

3.1.3 The weak and Mackey topology

We start this section by recalling some basic definitions. Although these are usually formulated over both $\mathbb{R}$ and $\mathbb{C}$, for simplicity we restrict to the real case.

Let $\mathscr{V}$ and $\mathscr{G}$ be vector spaces over $\mathbb{R}$. A map $B : \mathscr{V} \times \mathscr{G} \to \mathbb{R}$ is called a *bilinear map* if it is linear in each variable separately; that is, for fixed $y \in \mathscr{G}$ the map $x \mapsto B(x, y)$ is linear on $\mathscr{V}$, and for each fixed $x \in \mathscr{V}$, the map $y \mapsto B(x, y)$ is linear on $\mathscr{G}$.

Definition 3.1.24. *The algebraic dual space of a vector space $\mathscr{V}$, denoted by $\mathscr{V}^*$, is the set of all linear functionals $\varphi : \mathscr{V} \to \mathbb{R}$. Equipped with pointwise addition and scalar multiplication, $\mathscr{V}^*$ is itself a vector space.*

Definition 3.1.25. *Let $\mathscr{V}$ and $\mathscr{G}$ be vector spaces over $\mathbb{R}$, and let $B : \mathscr{V} \times \mathscr{G} \to \mathbb{R}$ be a bilinear map. We say that $\mathscr{V}$ and $\mathscr{G}$ are paired, or they form a pairing with respect to B.*

The pairing (or bilinear map) separates the points of $\mathscr{V}$ if for every nonzero $x \in \mathscr{V}$, there exists $y \in \mathscr{G}$ such that $B(x, y) \neq 0$. Similarly, it separates the points of $\mathscr{G}$ if for every nonzero $y \in \mathscr{G}$, there exists $x \in \mathscr{V}$ such that $B(x, y) \neq 0$.

If the pairing separates points of both $\mathscr{V}$ and $\mathscr{G}$, we say that it is separated, or that $(\mathscr{V}, \mathscr{G})$ is a duality with respect to B.

Let $\mathscr{V}$ and $\mathscr{G}$ be paired vector spaces with respect to the bilinear map $B : \mathscr{V} \times \mathscr{G} \to \mathbb{R}$. For each $y \in \mathscr{G}$, define the map $p_y : \mathscr{V} \to \mathbb{R} : x \mapsto B(x, y)$, which is a linear functional on $\mathscr{V}$, i.e., an element of $\mathscr{V}^*$. The assignment $\Psi : \mathscr{G} \to \mathscr{V}^* : y \mapsto p_y$ is a linear map. If B is injective in the second variable, Ψ is injective, and $\mathscr{G}$ can be identified by $\Psi(\mathscr{G})$, so that $\mathscr{G}$ is viewed as a linear subspace of $\mathscr{V}^*$. Under this identification, the bilinear map corresponds to the canonical map $B' : \mathscr{V} \times \mathscr{V}^* \to \mathbb{R}$ to $\mathscr{V} \times \mathscr{G}$. Similarly, one can identify $\mathscr{V}$ as a subspace of $\mathscr{G}^*$ via the mapping $x \mapsto B(x, \cdot)$.

For any paired vector spaces $\mathscr{V}$ and $\mathscr{G}$. Using Theorem 3.1.19, we can define a locally convex topology on $\mathscr{V}$ by considering the family of seminorms $\{|p_y| : y \in \mathscr{G}\}$, where $p_y(x) = B(x, y)$ for $x \in \mathscr{V}$. We denote this locally convex topology by $\sigma(\mathscr{V}, \mathscr{G})$.

Theorem 3.1.26 ([31], Proposition 2, p. 187)**.** *Let $\mathscr{V}$ and $\mathscr{G}$ be paired vector spaces with respect to the bilinear map $B : \mathscr{V} \times \mathscr{G} \to \mathbb{R}$, and equip $\mathscr{V}$ with the $\sigma(\mathscr{V}, \mathscr{G})$-topology. Then:*

(i) *For every $y \in \mathscr{G}$, the map p_y is a continuous linear functional on $\mathscr{V}$.*

(ii) *Conversely, if f is a continuous linear functional on $\mathscr{V}$, then there exists $y \in \mathscr{G}$ such that $f(x) = p_y(x)$ for all $x \in \mathscr{V}$.*

Remark 3.1.27. *The set of continuous linear functionals on $\mathscr{V}$, denoted by $\mathscr{V}'$, is called the continuous dual space. From Theorem 3.1.26, $\mathscr{V}'$ can be identified with $\mathscr{G}/\ker(\Psi)$, where $\ker(\Psi) = \{y \in \mathscr{G} : \Psi(y) = 0\}$. Furthermore, if the pairing separates the points of $\mathscr{G}$, then $\mathscr{V}'$ can be identified with $\mathscr{G}$ when $\mathscr{V}$ is endowed with the $\sigma(\mathscr{V}, \mathscr{G})$-topology.*

Theorem 3.1.28. *Let $\mathscr{V}$ and $\mathscr{G}$ be paired vector spaces with respect to the bilinear map $B : \mathscr{V} \times \mathscr{G} \to \mathbb{R}$. If B separates the points of $\mathscr{G}$, then $\sigma(\mathscr{V}, \mathscr{G})$ is the weakest locally convex topology for which all the maps $\{p_y : y \in \mathscr{G}\}$ are continuous.*

Proof. Let $\mathscr{T}$ be a locally convex topology such that each map p_y is continuous for every $y \in \mathscr{G}$. Let $x_0 \in \mathscr{V}$, $\epsilon > 0$ and $y_1, y_2, ..., y_n$ in $\mathscr{G}$. For every y_i, there exists a neighbourhood $V_i \in \mathscr{T}$ of x_0 such that $x_0 \in V_i \subseteq V(x_0; |p_i|, \epsilon)$, where $p_i(x) = B(x, y_i)$. Let $V = \bigcap_{i=1}^{n} V_i$. Then $x_0 \in V \subseteq V(x_0; |p_1|, |p_2|, .., |p_n|; \epsilon)$, which shows that very basic neighbourhood in $\sigma(\mathscr{V}, \mathscr{G})$ contains a neighbourhood from $\mathscr{T}$. Hence, $\sigma(\mathscr{V}, \mathscr{G}) \subseteq \mathscr{T}$. $\square$

Proposition 3.1.29 ([31], Proposition 4, p. 188)**.** *Let $\mathscr{V}$ and $\mathscr{G}$ be two paired vector spaces with respect to the bilinear map $B : \mathscr{V} \times \mathscr{G} \to \mathbb{R}$*

which separates points of $\mathscr{G}$, and let $\mathscr{W}$ be a proper subspace of $\mathscr{G}$. Then the topology $\sigma(\mathscr{V},\mathscr{G})$ on $\mathscr{V}$ is strictly finer than the topology $\sigma(\mathscr{V},\mathscr{W})$.

Theorem 3.1.28 shows that if the bilienar map B separates the points of $\mathscr{G}$, then $\sigma(\mathscr{V},\mathscr{G})$ is the weakest locally convex topology on $\mathscr{V}$ for which $\mathscr{G}$ is the continuous dual of $\mathscr{V}$. This topology is called the *weak topology* on $\mathscr{V}$. The weak topology is also known as the topology of pointwise convergence, because a net $(x_\gamma)_{\gamma\in\Gamma}$ in $\mathscr{V}$ converges to x if and only if $p_y(x_\gamma) \longrightarrow p_y(x)$ for every $y \in \mathscr{G}$.

Definition 3.1.30. *Suppose that $\mathscr{V}$ and $\mathscr{G}$ form a pairing which separates the points of $\mathscr{G}$. A locally convex topology $\mathscr{T}$ on $\mathscr{V}$ is said to be compatible with the pairing if $\mathscr{G}$ coincides with the continuous dual of $\mathscr{V}$ when $\mathscr{V}$ is equipped with $\mathscr{T}$.*

In view of the definition above, the weak topology $\sigma(\mathscr{V},\mathscr{G})$ is the weakest locally convex topology compatible with the pairing of $\mathscr{V}$ and $\mathscr{G}$. We now turn our attention to the *finest topology* that is compatible with this pairing.

Definition 3.1.31. *Let A be a subset of a vector space $\mathscr{V}$ over $\mathbb{R}$.*

(i) *A is said to be absorbing if for every $x \in \mathscr{V}$, there exists $\alpha > 0$ such that $x \in \lambda A$ for all $\lambda \in \mathbb{R}$ with $\alpha \leq |\lambda|$.*

(ii) *A is said to be balanced if $\lambda A \subseteq A$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.*

Theorem 3.1.32. *In a topological vector space $\mathscr{V}$, there exists a neighbourhood base $\mathscr{N}$ of 0 satisfying:*

(i) *Every $V \in \mathscr{N}$ is absorbing and balanced.*

(ii) *For every $V \in \mathscr{N}$, there exists $U \in \mathscr{N}$ such that $U + U \subseteq V$.*

Conversely, let $\mathscr{V}$ be a vector space over $\mathbb{R}$, and let $\mathscr{N}$ be a filter basis on $\mathscr{V}$ satisfying (i)-(ii). Then there exists a unique topology on $\mathscr{V}$ for which $\mathscr{V}$ is a topological vector space and $\mathscr{N}$ is a neighbourhood base of 0.

A topology $\mathscr{T}$ on a vector space $\mathscr{V}$ is said to be a vector topology, or a topology compatible with the linear structure, if $\mathscr{V}$ becomes a topological vector space under $\mathscr{T}$.

Theorem 3.1.33 ([31], Proposition 6 & Corollary, p. 88)**.** *Let $\mathscr{V}$ be a vector space, and let $\mathscr{A}$ be a collection of absorbing, balanced, and convex subsets of $\mathscr{V}$. Let $\mathscr{N}$ be the collection of all finite intersections of sets of the form λV, where $\lambda > 0$ and $V \in \mathscr{A}$. Then there exists a unique topology on $\mathscr{V}$ for which $\mathscr{V}$ is a locally convex space and $\mathscr{N}$ forms a neighbourhood base of* 0*.*

Definition 3.1.34. *Let $\mathscr{V}$ and $\mathscr{G}$ be paired vector spaces with respect to a bilinear map $B : \mathscr{V} \times \mathscr{G} \to \mathbb{R}$. For a subset $A \subseteq \mathscr{V}$, the polar of A is the subset A° of $\mathscr{G}$ defined by*

$$A^\circ := \{y \in \mathscr{G} : |B(x,y)| \leq 1 \textit{ for all } \; x \in A\}.$$

Proposition 3.1.35 ([31], Proposition 2, p. 192)**.** *Suppose that the vector spaces $\mathscr{V}$ and $\mathscr{G}$ are paired with respect to a bilinear map $B : \mathscr{V} \times \mathscr{G} \to \mathbb{R}$. Then the polar A° of a subset $A \subseteq \mathscr{V}$ satisfies the following properties:*

(i) *If $A_1 \subseteq A_2 \subseteq \mathscr{V}$, then $A_2^\circ \subseteq A_1^\circ$.*

(ii) *If $A \subseteq \mathscr{V}$ and D is the balanced hull of A (smallest balanced set containing A), then $A^\circ = D^\circ$.*

(iii) $A \subseteq A^{\circ\circ} := (A^\circ)^\circ$.

(iv) $A^\circ = A^{\circ\circ\circ}$.

(v) *A° is a balanced, convex and $\sigma(\mathscr{G},\mathscr{V})$-closed set in $\mathscr{G}$.*

(vi) *$(\lambda A)^\circ = \frac{1}{\lambda} A^\circ$ for $\lambda \in \mathbb{R}$, $\lambda \neq 0$. In particular, A° is absorbing if and only if A is bounded in $\sigma(\mathscr{V},\mathscr{G})$.*

(vii) *If $(A_\alpha)_{\alpha \in \mathscr{A}}$ is a family of subset of $\mathscr{V}$, then*

$$\Big(\bigcup_{\alpha \in \mathscr{A}} A_\alpha\Big)^\circ = \bigcap_{\alpha \in \mathscr{A}} A_\alpha^\circ.$$

Theorem 3.1.36 ([31], Theorem of the bipolars, p. 192)**.** *Let $\mathscr{V}$ and $\mathscr{G}$ be paired vector spaces. If A is a non-empty subset of $\mathscr{V}$, then $A^{\circ\circ}$ is the balanced, convex, $\sigma(\mathscr{V},\mathscr{G})$-closed hull of A; that is, $A^{\circ\circ}$ is the smallest balanced, convex set containing A which is $\sigma(\mathscr{V},\mathscr{G})$-closed.*

Let $\mathscr{V}$ and $\mathscr{G}$ be paired vector space, and let Θ be a collection of $\sigma(\mathscr{V},\mathscr{G})$-bounded subsets of $\mathscr{V}$. From Proposition 3.1.35 (v) and (vi), the polars A° are absorbing, balanced, and convex subsets of $\mathscr{G}$. By Theorem 3.1.33, these sets can be used to define a locally convex topology on $\mathscr{G}$.

This topology is called the Θ-topology, or the *topology of uniform convergence on sets in* Θ. A neighbourhood base of 0 for the Θ-topology is formed by all finite intersections of sets of the form λA°, where $\lambda > 0$ and $A \in \Theta$. Moreover, if we define $q_A(y) = \sup_{x\in A} |B(x,y)|$ for $A \in \Theta$, then

$$\begin{aligned} y \in \lambda A^\circ &\iff y = \lambda z \text{ for some } z \in A^\circ \\ &\iff |B(x,z)| = \left| B\left(x, \frac{y}{\lambda}\right)\right| \leq 1 \text{ for every } x \in A \\ &\iff q_A(y) \leq \lambda. \end{aligned}$$

Thus, the Θ-topology is precisely the locally convex topology generated by the family of seminorms $\{q_A : A \in \Theta\}$.

Theorem 3.1.37 ([31], Mackey Arens, p. 205)**.** *Suppose that $\mathscr{V}$ and $\mathscr{G}$ form a pairing that separates the points of $\mathscr{G}$. A locally convex topology $\mathscr{T}$ on $\mathscr{V}$ is compatible with the pairing $\mathscr{V}$ and $\mathscr{G}$ if and only if $\mathscr{T}$ is a Θ-topology, where Θ is a collection of balanced, convex, $\sigma(\mathscr{G},\mathscr{V})$-compact subsets of $\mathscr{G}$ that cover $\mathscr{G}$.*

Definition 3.1.38. *Let $\mathscr{V}$ and $\mathscr{G}$ be two vector spaces forming a pairing that separates the points of $\mathscr{G}$. If Θ denotes the collection of all balanced, convex, $\sigma(\mathscr{G},\mathscr{V})$-compact subsets of $\mathscr{G}$, then the corresponding Θ-topology on $\mathscr{V}$ is called the Mackey topology and is denoted by*

$\tau(\mathscr{V},\mathscr{G})$.

Proposition 3.1.39 ([31], Proposition 4, p. 205)**.** *Suppose that $\mathscr{V}$ and $\mathscr{G}$ are two vector spaces which form a pairing that separates the points of $\mathscr{G}$. A locally convex topology $\mathscr{T}$ on $\mathscr{V}$ is compatible with the pairing of $\mathscr{V}$ and $\mathscr{G}$ if and only if $\sigma(\mathscr{V},\mathscr{G}) \subseteq \mathscr{T} \subseteq \tau(\mathscr{V},\mathscr{G})$.*

3.2 Order Topology on L^∞

Assume that (X,Σ,μ) is a semi-finite measure space and consider the bilinear map $B : L^\infty \times L^1 \to \mathbb{R} : (f^\bullet, g^\bullet) \mapsto \int fg \, d\mu$. This bilinear form induces a duality between L^∞ and L^1. Moreover, by Remark 1.6.5, every $g \in L^\infty$ defines a bounded linear functional on L^1. Hence, by Theorem 1.6.7, L^∞ embeds isometrically into $(L^1)^*$. It should be noted that if (X,Σ,μ) is localisable, then this embedding is surjective.

Using this duality, we can consider the two topologies $\sigma(L^\infty, L^1)$ and $\tau(L^\infty, L^1)$. In this setting, the Mackey topology is defined by the collection of balanced, convex and $\sigma(L^1, L^\infty)$-compact subsets of L^1, or equivalently by the collection of balanced, convex and $\sigma(L^1, L^\infty)$-relatively compact[3] subsets of L^1. We next characterize the subsets of L^1 that are relatively compact with respect to $\sigma(L^1, L^\infty)$. It is important to note that, since (X,Σ,μ) is not necessarily localisable, the embedding of L^∞ into $(L^1)^*$ is not surjective. Consequently, under semi-finite measure spaces, L^∞ may be a proper subspace of $(L^1)^*$. In particular, by Proposition 3.1.29, the weak topology $\sigma(L^1, L^\infty)$ can be strictly coarser than $\sigma(L^1, (L^1)^*)$.

Definition 3.2.1. *A set $A \subseteq L^1$ is said to be uniformly integrable if for every $\epsilon > 0$, there exists a set $E \in \Sigma^f$, and a constant $M > 0$ such that*

$$\int (|f| - M\chi_E)^+ \; d\mu \leq \epsilon$$

for every $f^\bullet \in A$, where $(\cdot)^+ = \max\{\cdot, 0\}$.

[3]A subset $A \subseteq X$ is relatively compact in X if there is a compact subset containing A.

Theorem 3.2.2 ([23], Theorem 246G, p. 187)**.** *Let (X, Σ, μ) be a measure space and let $A \subseteq L^1$. Then the following are equivalent:*

(i) *A is uniformly integrable.*

(ii) $\sup_{f^\bullet \in A} |\int_F f \ d\mu| < \infty$ *for every μ-atom $F \in \Sigma$, and for every $\epsilon > 0$ there are $E \in \Sigma^f$ and $\delta > 0$ such that $|\int_F f \ d\mu| \leq \epsilon$ whenever $f^\bullet \in A$, $F \in \Sigma$ and $\mu(E \cap F) \leq \delta$.*

(iii) $\sup_{f^\bullet \in A} |\int_F f \ d\mu| < \infty$ *for every μ-atom $F \in \Sigma$, and* $\lim_{n\to\infty} \sup_{f^\bullet \in A}(|\int_{F_n} f \ d\mu|) = 0$ *whenever $(F_n)_{n\in\mathbb{N}}$ is a disjoint sequence in Σ.*

(iv) $\sup_{f^\bullet \in A} |\int_F f \ d\mu| < \infty$ *for every μ-atom $F \in \Sigma$, and* $\lim_{n\to\infty} \sup_{f^\bullet \in A}(|\int_{F_n} f \ d\mu|) = 0$ *whenever $(F_n)_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection.*

Definition 3.2.3. *Let $\mathscr{V}$ be a vector space endowed with the weak topology $\sigma(\mathscr{V}, \mathscr{G})$, and let $A \subseteq \mathscr{V}$. Then:*

(i) *A is said to be weakly compact if it is compact with respect to the weak topology $\sigma(\mathscr{V}, \mathscr{G})$.*

(ii) *A is said to be relatively weakly compact if it is a subset of a weakly compact set.*

Theorem 3.2.4 ([23], Theorem 247C, p. 189)**.** *Let (X, Σ, μ) be any measure space and A a subset of L^1. Then A is uniformly integrable if and only if it is relatively compact with respect to the weak topology $\sigma(L^1, (L^1)^*)$.*

Theorem 3.2.5 ([23], Exercise 246Y(k), p. 192)**.** *Let (X, Σ, μ) be a measure space, and let $A \subseteq L^1$ be a non-empty set. The following are equivalent:*

(i) *A is uniformly integrable;*

(ii) *Whenever $B \subseteq L^\infty$ is a non-empty filtered subset with infimum $0^\bullet$ in L^∞, we have* $\inf_{g^\bullet \in B} \sup_{f^\bullet \in A} |\int fg \ d\mu| = 0$.

Let ca(Σ) be the space of all σ-additive $\mathbb{R}$-valued measures on Σ with the variation norm (equivalent to the sup norm). For each $f \in L^1$, define the real-valued measure $\mu_f \in \mathrm{ca}(\Sigma)$ by $\mu_f(E) := \int_E f \, \mathrm{d}\mu$ for $E \in \Sigma$. The map $\Phi : f \mapsto \mu_f$ is an isometric embedding of L^1 into ca(Σ). We denote by τ_p the product topology on $\mathrm{ca}(\Sigma) \subset \mathbb{R}^\Sigma$.

Definition 3.2.6. *A subset $A \subseteq ca(\Sigma)$ is said to be equicontinuous if, for every decreasing sequence $(A_k)_{k\in\mathbb{N}}$ in Σ with $\bigcap_{k=1}^\infty A_k = \emptyset$, we have $\lim_{k\in\mathbb{N}} \sup_{\mu\in A} |\mu(A_k)| = 0$.*

The following theorem is pivotal for the theorems that follow.

Theorem 3.2.7 ([26], Theorem 2.6)**.** *Let $A \subseteq ca(\Sigma)$ be bounded. Then the following statements are equivalent:*

(i) *Every sequence in A has a subsequence which converges in τ_p;*

(ii) *The τ_p-closure of A is compact;*

(iii) *A is equicontinuous;*

(iv) *Every countable subset of A is equicontinuous.*

In the following theorem we note that the relatively compact subsets of $\sigma(L^1, L^\infty)$ and $\sigma(L^1, (L^1)^*)$ coincide.

Theorem 3.2.8 ([2], Proposition 11)**.** *Let $A \subset L^1$. The following statements are equivalent:*

(i) *A is relatively compact with respect to the weak topology $\sigma(L^1, (L^1)^*)$;*

(ii) *A is relatively compact with respect to $\sigma(L^1, L^\infty)$;*

(iii) *$\Phi[A]$ is relatively compact in $ca(\Sigma)$ with respect to τ_p;*

(iv) *$\Phi[A]$ is equicontinuous;*

(v) *A is uniformly integrable;*

(vi) *If $(y^\bullet_\gamma)_{\gamma\in\Gamma}$ is a decreasing net in L^∞ with infimum $0^\bullet$, then*

$$\lim_\gamma \sup_{u\in A} \left| \int u y_\gamma \, d\mu \right| = 0.$$

Proof. The equivalence of (i), (v) and (vi) follows from Theorem 3.2.4 and Theorem 3.2.5. The implication (i)⇒(ii) follows by Proposition 3.1.29, and implication (ii)⇒(iii) is immediate from the embedding Φ. The implication (iii)⇒(iv) follows from Theorem 3.2.7, while (iv)⇒(v) follows by Theorem 3.2.2 (iv). □

As a consequence of Proposition 3.2.8, the relatively compact sets of L^1 coincide for the weak topologies $\sigma(L^1, L^\infty)$ and $\sigma(L^1, (L^1)^*)$.

Corollary 3.2.9. $\tau(L^\infty, L^1) = \tau((L^1)^*, L^1)|_{L^\infty}$.

From proposition 3.2.8, every $A \subset L^1$ that is relatively compact with respect to $\sigma(L^1, L^\infty)$ is also a uniformly integrable subset of L^1. It follows that the solid hull $s(A) := \{f^\bullet \in L^1 : |f| \leq |g| \text{ for some } g \in A\}$ is uniformly integrable. Moreover, the convex hull $c(s(A))$ is also uniformly integrable. Since $c(s(A))$ is balanced, we conclude that the balanced convex solid hull of a relatively $\sigma(L^1, L^\infty)$-compact subset of L^1 is also relatively $\sigma(L^1, L^\infty)$-compact.

Proposition 3.2.10. (i) *$\tau(L^\infty, L^1)$ is a locally convex, solid and order-continuous topology.*

(ii) *$\tau(L^\infty, L^1)$ is the finest Hausdorff locally convex and order- continuous topology on L^∞.*

Proof. (i) Recall that $\tau(L^\infty, L^1)$ is the topology of uniform convergence on all balanced, convex, $\sigma(L^1, L^\infty)$-compact subsets of L^1. From the preceding paragraph, it follows that $\tau(L^\infty, L^1)$ is equivalently the topology of uniform convergence on all balanced, convex, solid, relatively $\sigma(L^1, L^\infty)$-compact subsets of L^1. Consequently, the space $(L^\infty, \tau(L^\infty, L^1))$ has a 0-neighbourhood basis of balanced, convex, solid sets. In other words, $\tau(L^\infty, L^1)$ is a locally convex, solid topology. Finally, by (ii)⇒(vi) of Theorem 3.2.8, it follows that $\tau(L^\infty, L^1)$ is order-continuous.

(ii) Let $\mathscr{T}$ be a Hausdorff, locally convex, and order-continuous topology on L^∞. We first show that every linear functional φ on L^∞

that is continuous with respect to $\mathscr{T}$ is also continuous with respect to $\sigma(L^\infty, L^1)$. By our assumptions it follows that the set function $\nu : \Sigma \to \mathbb{R}$ defined by $\nu(E) := \varphi(\chi_E)$ is σ-additive. Clearly ν is absolutely continuous with respect to μ. Let us verify that ν is σ-finite with respect to μ. The assumption of semi-finiteness implies that for every $E \in \Sigma$, the directed set $\{\chi_F : F \in \Sigma^f \text{ such that } F \subset E\}$ increases to χ_E. By the order-continuity of φ, the net $\{\nu(F) : F \in \Sigma^f \text{ such that } F \subset E\}$ converges to $\nu(E)$. Next we note that every disjoint system $\mathcal{D}$ in Σ satisfying $\nu(D) \neq 0$ is countable. Indeed, assume not and consider the collection of intervals $\{[\frac{1}{n+1}, \frac{1}{n}) : n \in \mathbb{N}\} \cup [1, \infty)$, then there exists an uncountable $\mathcal{D}' \subseteq \mathcal{D}$ such that $\nu(D) \in [\frac{1}{m+1}, \frac{1}{m})$ for some $m \in \mathbb{N}$ and $D \in \mathcal{D}'$, contradicting the σ-additivity of ν. If $\mathcal{D}$ is a maximal disjoint system in Σ such that $\mu(D) < +\infty$ and $\nu(D) \neq 0$ for every $D \in \mathcal{D}$, then $\mathcal{D}$ is countable. Note that $\nu(E) = 0$ for every $E \subset X \setminus \cup\mathcal{D}$, that is, ν is σ-finite with respect to μ. The Radon-Nikodým Theorem can therefore be applied to yield a function $f^\bullet \in L^1$ satisfying $\int \chi_E f \, \mathrm{d}\mu = \nu(E)$ for every $E \in \Sigma$. But for every $g^\bullet \in L^\infty_+$, there is a sequence $(s_n^\bullet)_{n\in\mathbb{N}}$ in $S := \operatorname{span}\{\chi_E^\bullet : E \in \Sigma\}$ such that $s_n^\bullet$ increases to $g^\bullet$. This implies that $\varphi(g^\bullet) = \int gf \, \mathrm{d}\mu$ for every $g^\bullet \in L^\infty_+$ and therefore $\varphi(g^\bullet) = \int gf \, \mathrm{d}\mu$ for every $g^\bullet \in L^\infty$. In particular $\varphi(g^\bullet) = B(g^\bullet, f^\bullet)$, thus φ is continuous with respect to $\sigma(L^\infty, L^1)$ and therefore the dual of $(L^\infty, \mathscr{T})$ can be identified with a subspace G of L^1. If $K \subset G$ is compact with respect to $\sigma(G, L^\infty)$, it is also compact with respect to $\sigma(L^1, L^\infty)$, concluding that $\mathscr{T} \subset \tau(L^\infty, G) \subset \tau(L^\infty, L^1)$. □

Theorem 3.2.11. *Let (X, Σ, μ) be a semi-finite measure space.*

(i) *For every $1 \leq p < q < \infty$, one has*

$$\sigma(L^\infty, L^1) \subsetneq \sigma_p \subsetneq \sigma_q \subsetneq \tau(L^\infty, L^1) \subsetneq \tau_\infty, \text{ and}$$
$$\tau_\mu \subsetneq \sigma_p,$$

(ii) *One has $\tau(L^\infty, L^1) \subsetneq \tau_O(L^\infty) \subsetneq \tau_\infty$, and $\tau_O(L^\infty) = \tau_\infty$ if and* *only if L^∞ is finite-dimensional.*

(iii) *If (X, Σ, μ) is σ-finite, then the restrictions of τ_μ and $\tau_O(L^\infty)$ to bounded subsets of L^∞ coincide.*

Furthermore, unless L^∞ is finite-dimensional, all of the inclusions in (i) are proper.

Proof. (i) $[\sigma(L^\infty, L^1) \subseteq \sigma_1]$. Assume that $f_\gamma^\bullet \xrightarrow{\sigma_1} f^\bullet$. Then for every $h^\bullet \in L^1$ we have $\rho_{p_h}(f_\gamma^\bullet, f^\bullet) \to 0$, that is, $p_{h^\bullet}(f_\gamma^\bullet - f^\bullet) \to 0$. Equivalently, $\int |(f_\gamma - f)h| \, \mathrm{d}\mu \to 0$. Since $\left|\int fh \, \mathrm{d}\mu\right| \leq \int |fh| \, \mathrm{d}\mu$, it follows that $f_\gamma^\bullet \xrightarrow{\sigma(L^\infty, L^1)} f^\bullet$.

$[\sigma_p \subseteq \sigma_q]$. Observe that

$$\begin{aligned} f_\gamma^\bullet \xrightarrow{\sigma_q} f^\bullet &\iff \left\|M_{f_\gamma - f}(h)\right\|_q^q \to 0 \text{ for all } h^\bullet \in L^q \\ &\iff \int |f_\gamma - f|^q |h|^q \, \mathrm{d}\mu \to 0 \text{ for all } h^\bullet \in L^q \\ &\iff \int |f_\gamma - f|^q g \, \mathrm{d}\mu \to 0 \text{ for all } g^\bullet \in L^1_+, \end{aligned}$$

i.e. since $L^1 = L^1_+ - L^1_-$,

$$f_\gamma^\bullet \xrightarrow{\sigma_q} f^\bullet \text{ if and only if } |f_\gamma^\bullet - f^\bullet|^q \xrightarrow{\sigma(L^\infty, L^1)} 0^\bullet.$$

For $1 \leq p < q < \infty$, note that

$$0^\bullet \leq |f_\gamma^\bullet - f^\bullet|^p g^\bullet \leq |f_\gamma^\bullet - f^\bullet|^q g^\bullet$$

for all $g^\bullet \in L^1_+$. From this it follows that $\sigma_p \subseteq \sigma_q$.

$[\sigma_q \subseteq \tau(L^\infty, L^1)]$. By Proposition 3.2.10 (ii) it suffices to show that σ_q is order-continuous. Let $f^\bullet \in L^p$ and note that the support of $f^\bullet$ is σ-finite. Indeed, $\mu(\{x \in X : |f(x)| \geq \frac{1}{\sqrt{n}}\}) \leq \sqrt[p]{n} \int |f|^p \, \mathrm{d}\mu < \infty$. Then,

$$\{x \in X : |f(x)| \neq 0\} = \bigcup_{n=1}^\infty \left\{x \in X : |f(x)| \geq \frac{1}{\sqrt{n}}\right\}.$$

Let $A_n = \{x \in X : |f(x)| \geq \frac{1}{\sqrt{n}}\}$ such that $\mathrm{supp}(f^\bullet) := \{x \in X : |f(x)| \neq 0\} = \bigcup_{n=1}^{\infty} A_n$, as required. Now we show that if $(f_\gamma^\bullet)_{\gamma\in\Gamma}$ is a decreasing net in L^∞ with infimum $0^\bullet$ and $f^\bullet \in L^q$, then $\lim_\gamma \|f_\gamma f\|_p = 0$. From the above the support of $f^\bullet$ is σ-finite. Thus, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that $(\chi_{\mathrm{supp}(f^\bullet)} f_{\gamma_n}^\bullet)_{n\in\mathbb{N}}$ decreases to $0^\bullet$ in L^∞. It follows that the set

$$S_k := \left\{x \in \mathrm{supp}(f^\bullet) : \inf_{n\in\mathbb{N}} f_{\gamma_n}(x) \geq \frac{1}{k}\right\}$$

has measure 0 and therefore $\inf_{n\in\mathbb{N}} f_{\gamma_n}\chi_{\mathrm{supp}(f^\bullet)}(x) = 0$ μ-a.e. This implies that $\inf_{n\in\mathbb{N}} f_{\gamma_n} f(x) = 0$ μ-a.e, and therefore

$$\lim_\gamma \|f_\gamma f\|_p = \inf_{\gamma\in\Gamma} \|f_\gamma f\|_p \leq \inf_{n\in\mathbb{N}} \|f_{\gamma_n} f\|_p = 0$$

where the last equality follows by the Lebesgue dominated convergence theorem.

$[\tau_\mu \subseteq \sigma_1]$. We show that if $f_\gamma^\bullet \xrightarrow{\sigma_1} f^\bullet$ then $f_\gamma^\bullet \xrightarrow{\tau_\mu} f^\bullet$. If $F \in \Sigma$ and $\mu(F) < \infty$, then $(\chi_F)^\bullet \in L^1$. Then

$$\begin{aligned}
f_\gamma^\bullet \xrightarrow{\sigma_1} f^\bullet &\iff \rho_{p_{\chi_F^\bullet}}(f_\gamma, f) \to 0 \\
&\iff \int |f_\gamma - f|\chi_F^\bullet \,\mathrm{d}\mu \to 0 \\
&\iff \int |f_\gamma - f| \wedge \chi_F^\bullet \,\mathrm{d}\mu \leq \int |f_\gamma - f|\chi_F^\bullet \,\mathrm{d}\mu \to 0 \\
&\implies f_\gamma^\bullet \xrightarrow{\tau_\mu} f^\bullet.
\end{aligned}$$

(ii) $[\tau(L^\infty, L^1) \subseteq \tau_{\mathrm{O}}(L^\infty)]$. This follows by the fact that $\tau(L^\infty, L^1)$ is order continuous and by Theorem 2.2.10, $\tau_{\mathrm{O}}(L^\infty)$ is the strongest order continuous topology.

$[\tau_{\mathrm{O}}(L^\infty) \subseteq \tau_\infty]$. If $(f_n^\bullet)_{n\in\mathbb{N}}$ is a sequence in L^∞ satisfying $\lim_{n\to\infty} \|f_n\|_\infty = 0$, then

$$-\inf_{k\geq n} \|f_k\|_\infty \, \mathbb{1}^\bullet \leq f_n^\bullet \leq \sup_{k\geq n} \|f_k\|_\infty \, \mathbb{1}^\bullet,$$

i.e. $(f_n^\bullet)_{n\in\mathbb{N}}$ is O_1-convergent to 0.

(iii) By Theorem 3.1.15 (ii), if (X, Σ, μ) is σ-finite, τ_μ is metrizable. For every sequence $(f_n^\bullet)_{n\in\mathbb{N}}$ of measurable functions converging in measure to the function $f^\bullet$, there is a subsequence $(f_{n_k}^\bullet)_{k\in\mathbb{N}}$ that converges pointwise μ-a.e to $f^\bullet$. This implies that when (X, Σ, μ) is σ-finite, every order-closed subset of the unit ball of L^∞ is closed with respect to τ_μ.

Now we show that if L^∞ is infinite-dimensional, the inclusions in (i) are proper. As (X, Σ, μ) is semi-finite, we can find a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in Σ that satisfies $0 < \mu(A_n) < \infty$ for every n. The map $\imath : c_0 \to L^\infty : (\lambda_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^\infty \lambda_n \chi_{A_n}$ is a Banach space embedding of c_0 into L^∞. Denote by E_0 the range of $\imath$.

$[\sigma(L^\infty, L^1) \neq \sigma_1]$. As σ_1 is locally solid, it suffices to show that $\sigma(L^\infty, L^1)$ is not locally solid. Using the embedding $\imath$, the topology $\sigma(c_0, \ell^1)$ can be identified with the restriction of $\sigma(L^\infty, L^1)$ to E_0. If $\sigma(L^\infty, L^1)$ was locally solid, then $\sigma(c_0, \ell^1)$ is also locally solid. This contradicts [6, Theorem 6.9, pg. 42].

$[\sigma_p \neq \sigma_q$. From this it also follows also that $\sigma_q \neq \tau(L^\infty, L^1)$.$]$ Let $1 < \alpha < q/p$ and define $e_n := \sqrt[q]{n^\alpha}(\chi_{A_n}^\bullet)$. To prove that $\sigma_p \neq \sigma_q$, we show that $0^\bullet$ is not a limit point of $F := \{e_n : n \in \mathbb{N}\} \subset L^\infty$ with respect to σ_q but it is a limit point of the set F with respect to σ_p. The function

$$g^\bullet := \sum_{n=1}^\infty \frac{(\chi_{A_n})^\bullet}{\sqrt[q]{n^\alpha \mu(A_n)}}$$

belongs to L^q and $\|e_n g\|_q = 1$ for every $n \in \mathbb{N}$. Then $U := \{f^\bullet \in L^\infty : \|fg\|_q < 1\}$ is a σ_q-open neighborhood of $0^\bullet$ satisfying $U \cap F = \emptyset$. On the other-hand let $h^\bullet \in L^p$ and $\varepsilon > 0$ be given. Let $\lambda_n := \int_{A_n} |h|^p \,\mathrm{d}\mu$. Then $\sum_{n=1}^\infty \lambda_n < \infty$, so there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $n_k \lambda_{n_k} \leq 1$ for every $k \in \mathbb{N}$. Since $\alpha p - q < 0$, we can find sufficiently large $k \in \mathbb{N}$ satisfying $\sqrt[q]{n_k^{\alpha p - q}} < \varepsilon$. Then

$$\|e_{n_k} h\|_p^p = \int_{A_{n_k}} \left(\sqrt[q]{n_k^{\alpha}}\ |h| \right)^p \mathrm{d}\mu = \sqrt[q]{n_k^{\alpha p}}\,\lambda_{n_k} = \sqrt[q]{n_k^{\alpha p - q}}\,(n_k \lambda_{n_k}) < \varepsilon,$$

i.e. $0^\bullet$ is a limit point of F with respect to σ_p.

[$\tau(L^\infty, L^1) \neq \tau_\infty$, using (ii), it suffices to show that $\tau_{\mathrm{O}}(L^\infty) \neq \tau_\infty$]. Let $B_n := \bigcup_{k \geq n} A_k$. Note that in L^∞, $((\chi_{B_n})^\bullet)_{n\in\mathbb{N}}$ is decreasing and $\inf\{(\chi_{B_n})^\bullet : n \in \mathbb{N}\} = 0^\bullet$, i.e. $((\chi_{B_n})^\bullet)_{n\in\mathbb{N}}$ converges to $0^\bullet$ with respect to $\tau_{\mathrm{O}}(L^\infty)$. However, it fails to converge to $0^\bullet$ with respect to τ_∞ since $\|(\chi_{B_n})^\bullet\|_\infty = 1$ for every $n \in \mathbb{N}$.

[$\tau_\mu \neq \sigma_1$]. We note that if $E \in \Sigma$, then

$$\mu(E \cap (\bigcup_{n=1}^{\infty} A_n)) = \mu(\bigcup_{n=1}^{\infty} (E \cap A_n)) = \sum_{n=1}^{\infty} \mu(E \cap A_n)$$

Moreover, if $\mu(E) < \infty$, it follows that $\mu(E \cap A_n)$ converges to 0. Now consider the sequence $\left(n^2 \chi_{A_n}^\bullet\right)_{n\in\mathbb{N}}$. From the fact that

$$\rho_{\tau_E}(n^2 \chi_{A_n}^\bullet, 0^\bullet) = \int n^2 \chi_{A_n} \wedge \chi_E \ \mathrm{d}\mu \leq \int \chi_{E\cap A_n} \ \mathrm{d}\mu = \mu(E \cap A_n) \to 0,$$

the sequence $\left(n^2 \chi_{A_n}^\bullet\right)_{n\in\mathbb{N}}$ converges to $0^\bullet$ with respect to τ_μ. On the other-hand let $1 < \alpha < 2$, the function $h := \sum_{k=1}^\infty \frac{\chi_{A_k}^\bullet}{\mu(A_k) k^\alpha}$ belongs to L^1 and $\int n^2 \chi_{A_n} h \ \mathrm{d}\mu = n^{2-\alpha}$, concluding that the sequence $\left(n^2 \chi_{A_n}^\bullet\right)_{n\in\mathbb{N}}$ does not converge to $0^\bullet$ with respect to σ_1.

$\square$

We now consider a general condition under which the topologies $\tau_{\mathrm{O}}(L^\infty)$ and $\tau(L^\infty, L^1)$ differ. For $A \in \Sigma$, define $L^\infty(A) := \{f\chi_A : f \in L^\infty\}$ and $L^1(A) := \{f\chi_A : f \in L^1\}$. It is straightforward to verify that $L^\infty(A)$ and $L^1(A)$ are isomorphic (as Banach lattices) to $L^\infty(A, \Sigma_A, \mu_A)$ and $L^1(A, \Sigma_A, \mu_A)$, respectively, where $\Sigma_A = \{A \cap E : E \in \Sigma\}$, and $\mu_A = \mu \restriction_A$. In what follows, therefore do not distinguish between $L^\infty(A)$ and $L^\infty(A, \Sigma_A, \mu_A)$, nor between $L^1(A)$ and $L^1(A, \Sigma_A, \mu_A)$.

Lemma 3.2.12. *Let $A \in \Sigma$.*

(i) *The restriction of $\sigma(L^1, L^\infty)$ to $L^1(A)$ (respectively, the restriction of $\tau(L^\infty, L^1)$ to $L^\infty(A)$) coincides with the topology $\sigma(L^1(A), L^\infty(A))$ (resp. $\tau(L^\infty(A), L^1(A))$ arising from the duality $B : (L^\infty(A), L^1(A)) \to \mathbb{R} : ((f\chi_A)^\bullet, (g\chi_A)^\bullet) \mapsto \int_A fg\; d\mu$.*

(ii) *The order topology $\tau_O(L^\infty(A))$ is equal to the restriction of $\tau_O(L^\infty)$ to $L^\infty(A)$.*

Proof. (i) The claim follows immediately by comparing the families of seminorms arising from the duality

$$B_A : (L^\infty(A), L^1(A)) \to \mathbb{R}, \qquad B_A\big((f\chi_A)^\bullet, (g\chi_A)^\bullet\big) = \int_A fg\, d\mu,$$

since these seminorms are exactly the restrictions of the seminorms defining $\sigma(L^1, L^\infty)$ (resp. the seminorms defining $\tau(L^\infty, L^1)$) to $L^1(A)$ (resp. $L^\infty(A)$).

(ii) The restriction statement for the order topology holds because order limits are unique, and the map

$$L^\infty \to L^\infty(A) : f^\bullet \mapsto (f\chi_A)^\bullet,$$

is isotone, idempotent, and preserves arbitrary suprema and infima. Hence a net in $L^\infty(A)$ is order-convergent in $L^\infty(A)$ if and only if it is order-convergent in L^∞, which implies

$$\tau_{\mathrm{O}}\big(L^\infty(A)\big) = \tau_{\mathrm{O}}(L^\infty)\big|_{L^\infty(A)}.$$

□

From the above, if $\tau_{\mathrm{O}}(L^\infty) = \tau(L^\infty, L^1)$, then,

$$\tau_{\mathrm{O}}(L^{\infty}(A)) = \tau(L^{\infty}(A), L^1(A))$$

is true for every $A \in \Sigma$.

Theorem 3.2.13. *Let (X, Σ, μ) be semi-finite measure space and let $A \in \Sigma$ satisfy:*

(i) $\mu(A) \neq 0$, *and*

(ii) *A contains no μ-atoms, i.e., there is no measurable subset $B \subseteq A$ with $\mu(B) > 0$ such that every measurable $C \subseteq B$ satisfies $\mu(C) = 0$ or $\mu(C) = \mu(B)$.*

Then $\tau_O(L^{\infty})$ and $\tau(L^{\infty}, L^1)$ are not equal.

Proof. Using the fact that (X, Σ, μ) is semi-finite, we assume that $\mu(A) < \infty$. We find a set $F \subseteq L^{\infty}(A)$ that is $\tau_{\mathrm{O}}(L^{\infty}(A))$-closed but not closed with respect to $\tau(L^{\infty}(A), L^1(A))$.

Let A_{ni} $(n \in \mathbb{N}, i = 1, ..., 2^n)$ be a tree in Σ such that A is the disjoint union of $A_{1,1}$ and $A_{1,2}$, and $A_{n-1,i}$ is the disjoint union of $A_{n,2i-1}$ and $A_{n,2i}$. Set $A_n = \bigcup_{i=1}^{2^{n-1}} A_{n,2i}$ and $B_n = A_{n,1}$. We show that

$$F := \left\{ \frac{1}{m}(\chi^{\bullet}_{A_n}) + m(\chi^{\bullet}_{B_n}) : m, n \in \mathbb{N} \right\}$$

is $\tau_{\mathrm{O}}(L^{\infty}(A))$-closed but not closed w.r.t to $\tau(L^{\infty}(A), L^1(A))$.

F is $\tau_{\mathrm{O}}(L^{\infty}(A))$-closed: By Theorem 1.5.8 and Theorem 1.5.9, note that $L^{\infty}(A)$ is Dedekind complete and order separable. It suffices to show that every O_2-convergent sequence $(f_j)_{j\in\mathbb{N}}$ in F, its order limit f belongs to F. Let

$$f_j := \frac{1}{m_j}(\chi^{\bullet}_{A_{n_j}}) + m_j(\chi^{\bullet}_{B_{n_j}}).$$

Since $(f_j)_{j\in\mathbb{N}}$ is O_2-convergent, it is norm bounded. Then, there exists $m \in \mathbb{N}$ and a subsequence such that $m_j = m$ for every $j \in \mathbb{N}$. Passing to a subsequence again, we can assume that $(n_j)_{j\in\mathbb{N}}$ is constant or strictly increasing. If it is constant, we are done. Assume for contradiction that it is strictly increasing. The sequence $(m(\chi_{B_{n_j}})^\bullet)_{j\in\mathbb{N}}$, decreases to $0^\bullet$ in L^∞ and the sequence $((\chi_{A_{n_j}})^\bullet)_{j\in\mathbb{N}}$ O_2-converges to $mf^\bullet$, so $((\chi_{A_{n_j}})^\bullet)_{j\in\mathbb{N}}$ is convergent with respect to the pseudometric ρ_{τ_A}. Then,

$$\begin{aligned}\int |\chi_{A_{n_j}} - \chi_{A_{n_{j+1}}}| \wedge \chi_A \,\mathrm{d}\mu \le \int |\chi_{A_{n_j}} - mf| \wedge \chi_A \,\mathrm{d}\mu +\\ \int |mf - \chi_{A_{n_{j+1}}}| \wedge \chi_A \,\mathrm{d}\mu \to 0\end{aligned}$$

Then $\mu((A_{n_j} \setminus A_{n_{j+1}}) \cup (A_{n_{j+1}} \setminus A_{n_j})) \to 0$ as $j \to \infty$. However, as $\mu((A_{n_j} \setminus A_{n_{j+1}}) \cup (A_{n_{j+1}} \setminus A_{n_j})) = \frac{\mu(A)}{2}$ for every $n_j \neq n_{j+1}$, then we get a contradiction. Thus, F is O_2-closed.

$0^\bullet$ belongs to the $\tau(L^\infty(A), L^1(A))$-closure of F : Let $K \subseteq L^1(A)$ be absolutely convex and relatively $\sigma(L^1(A), L^\infty(A))$-compact set. It suffices to show that there exists $f^\bullet \in F$ such that $q_K(f^\bullet) = \sup_{g^\bullet\in K} |B(f^\bullet, g^\bullet)| \le 1$. Choose $m \in \mathbb{N}$ satisfying $q_K(\chi_A^\bullet) \le \frac{m}{2}$. Furthermore, as $(B_n)_{n\in\mathbb{N}}$ is decreasing to $\emptyset$ and K is uniformly integrable, then there exists $n \in \mathbb{N}$ such that $q_K(\chi_{B_n}^\bullet) \le \frac{1}{2m}$. Let $f := \frac{1}{m}\chi_{A_n}^\bullet + m\chi_{B_n}^\bullet \in F$, then

$$q_K(f^\bullet) \le \frac{1}{m} q_K(\chi_{A_n}^\bullet) + m q_K(\chi_{B_n}^\bullet) \le \frac{1}{m} q_K(\chi_A^\bullet) + m q_K(\chi_{B_n}^\bullet) \le 1.$$

□

4 Lattice Uniformities

4.1 Uniformity and the uniform topology

For a subset $U \subseteq X \times X$, define $U^{-1} := \{(y, x) : (x, y) \in U\}$. A set satisfying $U = U^{-1}$ is said to be *symmetric*. The diagonal of $X \times X$ is denoted by $\Delta(X) := \{(x, x) : x \in X\}$. For sets $U, V \subseteq X \times X$, define the composition $U \circ V := \{(x, z) : (x, y) \in U$ and $(y, z) \in V$ for some $y \in X\}$. Denote by $\mathcal{D}_X$ the family of symmetric subsets of $X \times X$ containing the diagonal $\Delta(X)$. A set $U \in \mathcal{D}_X$ is said to be an *entourage of the diagonal* .

Definition 4.1.1. *A uniformity on a set X is a non-empty subfamily $\mathcal{U}$ of $\mathcal{D}_X$ satisfying the following properties:*

(i) *For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;*

(ii) *For all $U, V \in \mathcal{U}$, one has $U \cap V \in \mathcal{U}$;*

(iii) *If $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.*

Definition 4.1.2. *A uniformity $\mathcal{U}$ is said to be separated or Hausdorff if it satisfies $\bigcap_{U \in \mathcal{U}} U = \Delta(X)$.*

Given a uniformity $\mathcal{U}$ on a set X, the pair $(X, \mathcal{U})$ is called a uniform space. If the uniformity $\mathcal{U}$ is separated, $(X, \mathcal{U})$ is called a separated or Hausdorff uniform space.

Example 25. *The usual separated uniformity on $\mathbb{R}$ consists of all subsets $U \subseteq \mathbb{R}^2$ for which there exists $r > 0$ such that $\{(x, y) : |x-y| < r\} \subseteq U$.*

Lemma 4.1.3. *Let $(X, \mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then there exists a $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$.*

Proof. Let $U \in \mathcal{U}$, by definition of a uniformity, there exists $Z \in \mathcal{U}$ such that $Z \circ Z \subseteq U$. Next, choose $V \in \mathcal{U}$ such that $V \circ V \subseteq Z$. If $(x, y) \in V \circ V \circ V$, then there exist $z, w \in X$ with $(x, z), (z, w), (w, y) \in$

V. Since $(x,w) \in V \circ V \in Z$ and $(w,y) \in V \subseteq Z$, it follows that $(x,y) \in Z \circ Z \subseteq U$. Hence, $V \circ V \circ V \subseteq U$. □

For a uniform space $(X,\mathcal{U})$, $U \in \mathcal{U}$ and $A \subseteq X$. Define $U(x) := \{y \in X : (x,y) \in U\}$ and $U(A) := \{y \in X : (x,y) \in U \text{ for some } x \in A\}$.

Theorem 4.1.4. *Let $(X,\mathcal{U})$ be a uniform space. Define $\mathscr{T}_{\mathcal{U}} := \{G \subseteq X : \text{for every } x \in G \text{ there exists a } V \in \mathcal{U} \text{ such that } V(x) \subseteq G\}$. Then $\mathscr{T}_{\mathcal{U}}$ is a topology on X. Furthermore, if $(X,\mathcal{U})$ is separated, $\mathscr{T}_{\mathcal{U}}$ is a T_1-topology on X.*

Proof. It is clear by definition of $\mathscr{T}_{\mathcal{U}}$ that $\emptyset, X \in \mathscr{T}_{\mathcal{U}}$. For $G_1, G_2 \in \mathscr{T}_{\mathcal{U}}$ and $x \in G_1 \cap G_2$, there exists $V_1, V_2 \in \mathcal{U}$ such that $V_1(x) \subseteq G_1$ and $V_2(x) \subseteq G_2$. Setting $V := V_1 \cap V_2 \in \mathcal{U}$, we have $V(x) = V_1(x) \cap V_2(x) \subseteq G_1 \cap G_2$, proving that $G_1 \cap G_2 \in \mathscr{T}_{\mathcal{U}}$. The closure under arbitrary unions is verified similarly.

Next if $\mathcal{U}$ is separated, we show that $\mathscr{T}_{\mathcal{U}}$ is T_1. It suffices to show that $X \setminus \{x\}$ is open for every $x \in X$. For distinct $x, y \in X$, there exists $U \in \mathcal{U}$ satisfying $(x,y) \notin U$. Then, $U(y) \subseteq X \setminus \{x\}$, showing that $X \setminus \{x\}$ is open. □

The topology $\mathscr{T}_{\mathcal{U}}$ constructed in Theorem 4.1.4 is called the *topology induced by the uniformity $\mathcal{U}$*.

Theorem 4.1.5 ([34], Theorem 4, p. 178)**.** *Let $(X,\mathcal{U})$ be a uniform space and $A \subseteq X$. The interior of A, denoted by A^{o}, is $A^{\mathrm{o}} := \{x \in X : U(x) \subseteq A \text{ for some } U \in \mathcal{U}\}$, taken relative to the uniform topology $\mathscr{T}_{\mathcal{U}}$.*

Proof. We note that if $G \subseteq A$ is open, then $G \subseteq A^{\mathrm{o}}$. Hence, it suffices to show that the set $\{x \in X : U(x) \subseteq A \text{ for some } U \in \mathcal{U}\}$ is open with respect to $\mathscr{T}_{\mathcal{U}}$. Let $y \in \{x \in X : U(x) \subseteq A \text{ for some } U \in \mathcal{U}\}$. Then there exists $V \in \mathcal{U}$ such that $V(y) \subseteq A$. By the definition of a uniformity, there exists $W \in \mathcal{U}$ satisfying $W \circ W \subseteq V$. Now, for any

$z \in W(y)$, we have $W(z) \subseteq V(y) \subseteq A$. Therefore, $W(y) \subseteq \{x \in X : U(x) \subseteq A$ for some $U \in \mathcal{U}\}$, showing that the set is open in $\mathscr{T}_{\mathcal{U}}$. $\square$

Corollary 4.1.6. *Let $(X,\mathcal{U})$ be a uniform space. Then for every $x \in X$ and $U \in \mathcal{U}$, the set $(U(x))^{\circ}$ is a neighbourhood of x in the uniform topology $\mathscr{T}_{\mathcal{U}}$.*

Corollary 4.1.7. *Let $(X, \mathscr{T}_{\mathcal{U}})$ be a topological space, where $\mathscr{T}_{\mathcal{U}}$ is the topology induced by the uniformity $\mathcal{U}$. Then for $A \subseteq X$, $x \in \overline{A}^{\mathscr{T}_{\mathcal{U}}}$ if and only if $A \cap U(x) \neq \emptyset$ for every $U \in \mathcal{U}$.*

Definition 4.1.8. *Let $(X,\mathcal{U})$ be a uniform space. A subfamily $\mathcal{X} \subseteq \mathcal{U}$ is called a base for the uniformity $\mathcal{U}$ if for every $U \in \mathcal{U}$, there exists $B \in \mathcal{X}$ such that $B \subseteq U$.*

Proposition 4.1.9 ([34], Theorem 2, p. 177)**.** *A non-empty family $\mathcal{X} \subseteq \mathcal{D}_X$ is a base for some uniformity $\mathcal{U}_{\mathcal{X}}$ if and only if*

(i) *For every $U \in \mathcal{X}$, there exists $V \in \mathcal{X}$ such that $V \circ V \subseteq U$;*

(ii) *For every $B_1, B_2 \in \mathcal{X}$, there exists $B_3 \in \mathcal{X}$ such that $B_3 \subseteq B_1 \cap B_2$.*

Proposition 4.1.10 ([22], Proposition 8.1.14, p. 431)**.** *Let X be a topological space with topology $\mathscr{T}$ and $\mathcal{X} \subseteq \mathcal{D}_X$ a family of entourages of the diagonal that satisfy properties (i)-(ii) of Proposition 4.1.9, and they consists of open subsets of the Cartesian product $X \times X$. If for every $x \in X$ and any neighbourhood G of x there exists a $V \in \mathcal{X}$ such that $V(x) \subseteq G$, then $\mathscr{T}_{\mathcal{U}_{\mathcal{X}}} = \mathscr{T}$.*

Definition 4.1.11. *Let $(X,\mathcal{U})$ be a uniform space. A subfamily $\mathcal{Y} \subseteq \mathcal{U}$ is called a **subbase** for the uniformity $\mathcal{U}$ if the family of finite intersections of elements of $\mathcal{Y}$ is a base for $\mathcal{U}$.*

Theorem 4.1.12. *If $(X,\mathcal{U})$ is a uniform space, then the topology $\mathscr{T}_{\mathcal{U}}$ is regular.*

Proof. Let $x \in X$ and F be a $\mathscr{T}_\mathcal{U}$-closed set such that $x \notin F$. Then there exists $U \in \mathcal{U}$ satisfying $x \in U(x) \subseteq X \setminus F$. Let $V \in \mathcal{U}$ satisfy $V \circ V \subseteq U$. Then $(V(x))^\circ$ is the required open neighbourhood of x because $(V(x))^\circ \subseteq V \circ V(x) \subseteq U(x) \subseteq X \setminus F$. □

Corollary 4.1.13 ([13], p. 174)**.** *Let $(X,\mathcal{U})$ be a uniform space. Then $\mathcal{U}$ is separated if and only if $\mathscr{T}_\mathcal{U}$ is Hausdorff.*

It should be noted that distinct uniformities may induce the same topology. For instance, in [22, Example 8.1.6, Example 8.1.7, p. 429], there are two distinct uniformities which generated the discrete topology.

4.2 Uniform Spaces and Uniform Continuity

When considering uniformities on a set X, one can consider the complete lattice of uniformities. However, one should note that the intersection and union of two uniformities is not necessarily a uniformity. In what follows, we shall describe how suprema in this lattice are constructed.

Lemma 4.2.1. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces. Then the sets $\{((x_1,y_1),(x_2,y_2)) : U \in \mathcal{U}, V \in \mathcal{V},\ (x_1,x_2) \in U$ and $(y_1,y_2) \in V\}$ form a base for the product uniformity on $X \times Y$. Furthermore, this uniformity induces on $X \times Y$ the product topology $\mathscr{T}_\mathcal{U} \times \mathscr{T}_\mathcal{V}$.*

Proof. The proof is a direct application of Proposition 4.1.9 and Proposition 4.1.10. □

From Lemma 4.2.1, given a uniformity $\mathcal{U}$ on a set X, one can generate a uniformity that induces the product topology $\mathscr{T}_\mathcal{U} \times \mathscr{T}_\mathcal{U}$ on $X \times X$.

Definition 4.2.2. *A function $f : X \to Y$ between uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is said to be uniformly continuous if for every $V \in \mathcal{V}$*

there exists $U \in \mathcal{U}$ *such that* $f_2(U) \subseteq V$*, that is,* $(x, y) \in U \implies (f(x), f(y)) \in V$*.* [4]

Let f be a function from a set X into a uniform space $(Y,\mathcal{U})$. In general, it is not true that the family of sets $\{f_2^{-1}(V) : V \in \mathcal{U}\}$ is a uniformity on X. The difficulty comes from the possibility that there exists a subset of $A \subseteq X \times X$ such that $f_2^{-1}(V) \subseteq A$ for some $V \in \mathcal{U}$, but $A \neq f_2^{-1}(W)$ for any $W \in \mathcal{U}$. However, the following lemma shows that the set $\{f_2^{-1}(V) : V \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{U}_f$ on X.

Lemma 4.2.3 ([34], p. 181)**.** *Let* $f : X \to Y$ *be a map from a set* X *into a uniform space* $(Y,\mathcal{U})$*. The family* $\{f_2^{-1}(U) : U \in \mathcal{U}\}$ *forms a base for a uniformity* $\mathcal{U}_f$ *on* X*. Furthermore,* $\mathcal{U}_f$ *is the smallest uniformity on* X *that makes* f *uniformly continuous.*

Proof. For every $U \in \mathcal{U}$ and $x \in X$, we have $(f(x), f(x)) \in U$ because $\Delta(Y) \subseteq U$. It follows that $\Delta(X) \subseteq f_2^{-1}(U)$. It is also easy to see that f_2^{-1} preserves intersections and inverses. Finally, it remains to show that for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $f_2^{-1}(V) \circ f_2^{-1}(V) \subseteq f_2^{-1}(U)$. By the definition of a uniformity, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Now, let $(x, z) \in f_2^{-1}(V) \circ f_2^{-1}(V)$. Then there exists $y \in X$ such that $(x, y), (y, z) \in f_2^{-1}(V)$. Then, $(f(x), f(y)) \in V$ and $(f(y), f(z)) \in V$. Hence, $(f(x), f(z)) \in V \circ V \subseteq U$, so that $(x, z) \in f_2^{-1}(U)$, as required. □

Remark 4.2.4. *The statement in Lemma 4.2.3 can be extended to a collection of functions* $\mathcal{F}$*, where every* $f \in \mathcal{F}$ *maps* X *into a uniform space* $(Y, \mathcal{V}_f)$*. Each* $f \in \mathcal{F}$ *induces a uniformity* $\mathcal{U}_f$ *on* X*, and the union of the family* $\{\mathcal{U}_f : f \in \mathcal{F}\}$ *forms a subbase for a uniformity* $\mathcal{U}_{\mathcal{F}}$ *on* X*.*

Lemma 4.2.5 ([13], Proposition 8, p. 180)**.** *Let* X *be a set, and let* $\{\mathcal{U}_\alpha : \alpha \in \mathscr{A}\}$ *be a collection of uniformities on* X*. Then the union* $\bigcup_{\alpha \in \mathscr{A}} \mathcal{U}_\alpha$ *forms a subbase for a uniformity on* X*. This uniformity*

[4] The function $f_2 : X \times X \to Y \times Y$ is defined by $f_2(x, y) = (f(x), f(y))$.

is the supremum of the uniformities $\{\mathcal{U}_\alpha : \alpha \in \mathscr{A}\}$ *in the lattice of uniformities on* X*.*

An analogue of Lemma 4.2.5 for infima can also be established [50, Theorem 2.2]. Consequently, the set of lattice uniformities is an order-complete lattice. In this lattice, for two uniformities $\mathcal{U}$ and $\mathcal{V}$ on X, we say that $\mathcal{U}$ is finer than $\mathcal{V}$ if the identity map $e : (X,\mathcal{U}) \to (X,\mathcal{V})$ is uniformly continuous.

Theorem 4.2.6 ([34], Theorem 11, p. 183)**.** *Let* $(X,\mathcal{U})$ *be a uniform space, and let* ρ *be a pseudometric on* X*. Then* ρ *is uniformly continuous on* $X \times X$ *with respect to the product uniformity if and only if, for every* $r > 0$*, the set* $U_r := \{(x,y) : \rho(x,y) < r\}$ *is an element of* $\mathcal{U}$*.*

Proof. We assume that $\mathbb{R}$ is endowed with the uniformity described in Example 25. We now show that $U_r \in \mathcal{U}$ for every $r > 0$ if and only if ρ is uniformly continuous with respect to the product uniformity for $X \times X$ described in Lemma 4.2.1. Suppose ρ is uniformly continuous. Then for every $r > 0$, there exists $U \in \mathcal{U}$ such that for all $(x,y),(u,v) \in U$, $|\rho(x,u) - \rho(y,v)| < r$. Fix $(u,v) = (y,y) \in U$, then $\rho(x,y) < r$, thus showing that $U \subseteq U_r$. Hence, $U_r \in \mathcal{U}$. Conversely, if $(x,u),(y,v) \in U_r$, then $\rho(x,y) \leq \rho(x,u)+\rho(u,v)+\rho(v,y)$ and $\rho(u,v) \leq \rho(u,x)+\rho(x,y)+\rho(y,v)$, so $|\rho(x,y) - \rho(u,v)| < 2r$. As $U_r \in \mathcal{U}$, it follows that ρ is uniformly continuous. □

If ρ is a pseudometric, then one can define a uniformity by letting $U_r := \{(x,y) : \rho(x,y) < r\}$. Every U_r is symmetric, and $U_r \cap U_t = U_s$ where $s = \min\{r,t\}$. It follows that the family $\{U_r : r \in \mathbb{R}$ where $r > 0\}$ forms a base for a uniformity on X, denoted by $\mathcal{U}_\rho$. This uniformity is the coarsest uniformity on X with respect to which ρ is uniformly continuous. Moreover, the topology $\mathscr{T}_{\mathcal{U}_\rho}$ coincides with the pseudometric topology, since for each $x \in X$ and $r > 0$, this set $U_r(x)$ is exactly the open ball centered at x with radius r.

The next theorem is pivotal in the theory of uniform spaces. In particular, it shows that for a uniform space $(X,\mathcal{U})$, the topology $\mathscr{T}_{\mathcal{U}}$ is completely regular.

Theorem 4.2.7 ([34], Lemma 12, p. 185)**.** *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $X \times X$ such that $U_0 = X \times X$, $\Delta(X) \subseteq U_n$ and $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. Then there exists a non-negative function d on $X \times X$ such that:*

(i) *$d(x,z) \leq d(x,y) + d(y,z)$ for all x, y and z; and*

(ii) *$U_n \subseteq \left\{(x,y) : d(x,y) < \frac{1}{2^n}\right\} \subseteq U_{n-1}$ for every $n \in \mathbb{N}$.*

Moreover, if each U_n is symmetric, then there exists a pseudometric ρ on X satisfying (ii).

Definition 4.2.8. *Let $(X,\mathcal{U})$ be a uniform space. The uniformity $\mathcal{U}$ is said to be pseudometrizable if there exists a pseudometric ρ on X such that $\mathcal{U} = \mathcal{U}_\rho$.*

Theorem 4.2.9 (Metrization Theorem)**.** *A uniform space $(X,\mathcal{U})$ is pseudometrizable if and only if the uniformity $\mathcal{U}$ has a countable base.*

In the next theorem, we show that every uniformity can be generated by a family of pseudometrics.

Theorem 4.2.10 ([34], Theorem 15, p. 188)**.** *Every uniformity on X can be generated by a family of uniformly continuous pseudometrics on $X \times X$.*

Proof. Let $\mathcal{U}$ be a uniformity on X and let P denote the family of pseudometrics on $X \times X$ that are uniformly continuous with respect to $\mathcal{U}$. By remark 4.2.4, the collection $\{\mathcal{U}_\rho : \rho \in P\}$ is a subbase for a uniformity $\mathcal{U}_P$. From Theorem 4.2.6, it follows that $\mathcal{U}_P \subseteq \mathcal{U}$. The reverse inclusion $\mathcal{U} \subseteq \mathcal{U}_P$ holds by Theorem 4.2.7. □

In Theorem 4.1.12, we have seen that the uniform topology $\mathscr{T}_{\mathcal{U}}$ is regular. The following theorem strengthens this result: it shows that

$\mathscr{T}_{\mathcal{U}}$ is completely regular. Moreover, it characterizes those topologies which are induced by a uniformity.

Theorem 4.2.11 ([34], Corollary 17, p. 188)**.** *A topology $\mathscr{T}$ on a set X is the uniform topology for some uniformity on X if and only if the topological space $(X, \mathscr{T})$ is completely regular.*

The family of pseudometrics that are uniformly continuous on $X \times X$ is known as the *gage* of a uniformity. From Theorem 4.2.10, as every uniformity can be generated by a family of pseudometrics, every fact about the uniformity can be described by its gage. To see the connection between uniformities and gage, the reader can refer to [34, p. 189].

Definition 4.2.12. *Let $(X,\mathcal{U})$ be a uniform space. A net $(x_\gamma)_{\gamma\in\Gamma}$ in X is called a Cauchy net if and only if for every $U \in \mathcal{U}$, there exists γ_0 such that $(x_\alpha, x_\beta) \in U$ for all $\alpha, \beta \geq \gamma_0$.*

Given a net $(x_\gamma)_{\gamma\in\Gamma}$ in a uniform space $(X,\mathcal{U})$, consider the double net $(x_\gamma, x_\beta)_{(\gamma,\beta)\in\Gamma\times\Gamma}$, where $\Gamma \times \Gamma$ is given the product order, i.e., $(\gamma_1, \beta_1) \leq (\gamma_2, \beta_2)$ if and only if $\gamma_1 \leq \gamma_2$ and $\beta_1 \leq \beta_2$. Definition 4.2.12 can be restated as follows: a net $(x_\gamma)_{\gamma\in\Gamma}$ in X is a Cauchy net if and only if the double net $(x_\gamma, x_\beta)_{(\gamma,\beta)\in\Gamma\times\Gamma}$ is eventually in every $U \in \mathcal{U}$.

Let P be a gage for a uniformity $\mathcal{U}$ on X. For each $\rho \in P$ and $r > 0$, define $U_{\rho,r} := \{(x, y) \in X \times X : \rho(x, y) < r\}$. Then the collection $\{U_{\rho,r} : \rho \in P, \quad r > 0\}$ forms a base for the uniformity $\mathcal{U}$. A net $(x_\gamma)_{\gamma\in\Gamma}$ in X is Cauchy if and only if for every $\rho \in P$ and $r > 0$, there exists $\gamma_0(\rho, r) \in \Gamma$ such that $(x_\gamma, x_\beta) \in U_{\rho,r}$ for every $\gamma, \beta \geq \gamma_0(\rho, r)$. Equivalently, a net $(x_\gamma)_{\gamma\in\Gamma}$ is a Cauchy net if and only if for every $\rho \in P$ and $r > 0$, there exists $\gamma_0(\rho, r) \in \Gamma$ such that $\rho(x_\gamma, x_\beta) < r$ for every $\gamma, \beta \geq \gamma_0(\rho, r)$.

Lemma 4.2.13 ([34], Lemma 20, p. 191)**.** *A net $(x_\gamma)_{\gamma\in\Gamma}$ in a uniform space $(X,\mathcal{U})$ is a Cauchy net if and only if one of the following equivalent statements holds.*

(i) *The net* $(x_\gamma, x_\beta)_{(\gamma,\beta)\in\Gamma\times\Gamma}$ *is eventually in each element of some subbase for the uniformity* $\mathcal{U}$.

(ii) *The net* $\{\rho(x_\gamma, x_\beta) : (\gamma,\beta) \in \Gamma\times\Gamma\}$ *converges to 0 for every* $\rho \in P$ *in some family of pseudometrics which generates the gage* P.

Theorem 4.2.14 ([34], Theorem 21, p. 191)**.** *Let* $(X,\mathcal{U})$ *be a uniform space. Every net which converges to a point relative to the uniform topology* $\mathscr{T}_{\mathcal{U}}$ *is a Cauchy net.*

We conclude this section with a brief discussion of complete uniform spaces and the completion of a uniform space.

Definition 4.2.15. *A uniform space* $(X,\mathcal{U})$ *is complete if and only if every Cauchy net in* X *converges to a point in* X *with respect to* $\mathscr{T}_{\mathcal{U}}$.

Theorem 4.2.16 ([34], Theorem 28, P. 196)**.** *Every uniform space is uniformly isomorphic to a dense subspace of a complete uniform space. Every separated uniform space is uniformly isomorphic to a dense subspace of a complete separated uniform space.*

From Theorem 4.2.16, the completion of a uniform space $(X,\mathcal{U})$ is given by a pair $(f,(\tilde{X},\tilde{\mathcal{U}}))$, where $(\tilde{X},\tilde{\mathcal{U}})$ is a complete uniform space, and $f : X \to \tilde{X}$ is a uniform isomorphism of X onto a dense subspace of $\tilde{X}$.

Corollary 4.2.17 ([34], p. 196)**.** *Every uniform space has a completion.*

Proposition 4.2.18 ([34], Theorem 26, p. 195)**.** *Let* f *be a function whose domain is a subset* A *of a uniform space* $(X,\mathcal{U})$ *and let* $(Y,\mathcal{V})$ *be a complete separated uniform space. If* $f : A \to Y$ *is uniformly continuous, then there exists a unique uniformly continuous extension* $\tilde{f} : \overline{A}^{\mathscr{T}_{\mathcal{U}}} \to Y$ *such that* $\tilde{f}|_A = f$.

Corollary 4.2.19 ([34], p. 196)**.** *The completion of a uniform space is unique up to a uniform isomorphism.*

4.3 Lattice Uniformities

In this section, we discuss the concept of lattice uniformities. Most of the results in this section are stated without proofs; however, the interested reader may consult [48, 47, 49] for detailed proofs and a broader treatment of the subject.

Let L be a lattice. A *lattice uniformity* on L is a uniformity that makes the lattice operations $\vee$ and $\wedge$ uniformly continuous. A uniform lattice is a lattice endowed with a lattice uniformity.

Definition 4.3.1. *For a lattice uniformity* $(L,\mathcal{U})$ *and* $V \in \mathcal{U}$*, we let* $V \vee V := \{(x_1 \vee x_2, y_1 \vee y_2) : (x_1, y_1), (x_2, y_2) \in V\}$*.* $V \wedge V$ *is defined similarly.*

Proposition 4.3.2 ([47], Proposition 1.1.2)**.** *Let* $\mathcal{U}$ *be a uniformity on a lattice* L*. Then the following statements are equivalent:*

(i) $\mathcal{U}$ *is a lattice uniformity;*

(ii) *for every* $U \in \mathcal{U}$ *there exists* $V \in \mathcal{U}$ *satisfying* $V \vee V \subseteq U$ *and* $V \wedge V \subseteq U$*;*

(iii) *for every* $U \in \mathcal{U}$ *there exists* $V \in \mathcal{U}$ *satisfying* $V \vee \Delta(L) \subseteq U$ *and* $V \wedge \Delta(L) \subseteq U$*.*

Proof. It suffices to prove that (iii) implies (ii) since it is easy that (i) $\iff$ (ii) and (ii) implies (iii) is obvious. Let $U \in \mathcal{U}$ and $V, W \in \mathcal{U}$ such that $W \circ W \subseteq U$, $V \vee \Delta(L) \subseteq W$ and $V \wedge \Delta(L) \subseteq W$. We show that $V \vee V \subseteq U$. For $(x_1, x_2), (y_1, y_2) \in V$, $(x_1 \vee y_1, x_2 \vee y_1), (x_2 \vee y_1, x_2 \vee y_2) \in V \vee \Delta(L) \subseteq W$. This concludes that $(x_1 \vee y_1, x_2 \vee y_2) \in U$, as required. $\square$

Proposition 4.3.3 ([47], Proposition 1.1.3)**.** *Let* $\mathcal{U}$ *be a lattice uniformity on* L*. Then every* $U \in \mathcal{U}$ *contains a* $V \in \mathcal{U}$ *such that for all* $(a, b) \in V$ *the rectangle* $[a \wedge b, a \vee b]^2 \subseteq V$*.*

Recall that a subset $C \subseteq L$ is said to be *order convex* if, for all $a, b \in C$ and $a \leq b$, the interval $[a, b] := \{x \in L : a \leq x \leq b\}$ is contained in C; that is, $[a, b] \subseteq C$.

Definition 4.3.4. *A topology $\mathscr{T}$ on a lattice L is called locally order convex if $\mathscr{T}$ has a base of order convex open sets.*

Next, we note that the topology induced by a lattice uniformity is a locally order convex lattice topology.

Proposition 4.3.5 ([47], Proposition 1.1.6)**.** *Let $\mathcal{U}$ be a lattice uniformity on L and $\mathscr{T}_{\mathcal{U}}$ the induced topology. Then $(L, \mathscr{T}_{\mathcal{U}})$ is a locally order convex topological lattice. Moreover, every $U \in \mathcal{U}$ contains a $V \in \mathcal{U}$ such that $V(x)$ is order convex for every $x \in L$.*

Definition 4.3.6. *A lattice uniformity $\mathcal{U}$ on a lattice L is said to be exhaustive if every monotone net in L is a Cauchy net with respect to $\mathcal{U}$. Furthermore, $\mathcal{U}$ is said to be locally exhaustive if, for every $a, b \in L$ with $a \leq b$, the restriction $\mathcal{U}\restriction_{[a,b]}$ is exhaustive.*

Definition 4.3.7. *A lattice uniformity $\mathcal{U}$ on a lattice L is order continuous if every net that O_2-converges to x, converges to x with respect to the topology $\mathscr{T}_{\mathcal{U}}$.*

Theorem 4.3.8 ([49], Proposition 6.3)**.** *Let $(L,\mathcal{U})$ be a separated uniform lattice. The following conditions are equivalent:*

(i) *$(L,\mathcal{U})$ is complete (as a uniform space) and $\mathcal{U}$ is exhaustive.*

(ii) *L is an order complete lattice and $\mathcal{U}$ is order continuous.*

Proposition 4.3.9 ([49], Proposition 6.14)**.** *Let S be a sublattice of a uniform lattice $(L,\mathcal{U})$. Then:*

(i) *The closure $\overline{S}$ of S is a sublattice of L.*

(ii) *If the uniformity $\mathcal{U}\restriction_S$ induced by $\mathcal{U}$ on S is exhaustive, then $\mathcal{U}\restriction_{\overline{S}}$ is also exhaustive.*

(iii) *If* $\mathcal{U}|_{[a,b]\cap S}$ *is exhaustive for every* $a, b \in S$ *with* $a < b$*, then* $\mathcal{U} \restriction_{[a,b]\cap \overline{S}}$ *is exhaustive for every* $a, b \in \overline{S}$ *with* $a < b$*.*

It follows that if S is a dense sublattice of a uniform lattice $(L,\mathcal{U})$, and the induced uniformity $\mathcal{U} \restriction_S$ is (locally) exhaustive, then $\mathcal{U}$ is (locally) exhaustive. Furthermore, Theorem 4.2.16 can be stated for uniform lattices. In particular, the following theorem shows that every separated uniform lattice $(L,\mathcal{U})$ admits a separated lattice uniform completion $(\tilde{L},\tilde{\mathcal{U}})$.

Proposition 4.3.10 ([47], Proposition 1.3.1)**.** *Every separated uniform lattice* $(L,\mathcal{U})$ *is a sublattice and a dense subspace of a complete (as a uniform space) separated uniform lattice* $(\tilde{L},\tilde{\mathcal{U}})$*.*

Corollary 4.3.11. *Let* $(L,\mathcal{U})$ *be a (locally) exhaustive separated uniform lattice and let* $(\tilde{L},\tilde{\mathcal{U}})$ *denote its completion. Then:* $(\tilde{L},\leq)$ *is a (Dedekind) order complete lattice and* $\tilde{\mathcal{U}}$ *is order continuous.*

Proof. This follows directly from Proposition 4.3.8, Proposition 4.3.10 and Proposition 4.3.9. □

We shall now consider a condition stronger than order continuity.

Definition 4.3.12. *A uniform lattice* $(L,\mathcal{U})$ *satisfies condition (C) if for every increasing or decreasing order bounded net* $(x_\gamma)_{\gamma\in\Gamma}$ *in* L *and every* $U \in \mathcal{U}$ *there exists a* $\gamma_0 \in \Gamma$ *and an upper or lower bound* $x \in L$ *of* $(x_\gamma)_{\gamma\in\Gamma}$*, respectively, with* $(x_\gamma, x) \in U$ *for every* $\gamma \geq \gamma_0$*.*

Proposition 4.3.13 ([49], Proposition 7.1.2)**.** (i) *If* $\mathcal{U}$ *satisfies condition (C), then* $\mathcal{U}$ *is order continuous and locally exhaustive.*

(ii) *If* L *is a Dedekind complete lattice, then* $\mathcal{U}$ *is order continuous if and only if* $\mathcal{U}$ *satisfies condition (C).*

Proof. (i) Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net O_2-converging to x and let $U \in \mathcal{U}$. Then there exist an up-directed set M and a filtered set N satisfying $\sup M = x = \inf N$ and for every $(m,n) \in M \times N$, there exists $\gamma(m,n)$

such that $x_\gamma \in [m,n]$ for every $\gamma \geq \gamma(m,n)$. Furthermore by the definition of a uniformity there exists $W \in \mathcal{U}$ such that $W \circ W \subseteq U$. By Proposition 4.3.3, there exists a $V \in \mathcal{U}$ satisfying $V \subseteq W$ and $[a,b]^2 \subseteq V$ for every $a, b \in V$ with $a \leq b$. Using condition (C), it is easy to see that for any $m \in M$ and $n \in N$, $[m,x]^2 \subseteq V$ and $[x,n]^2 \subseteq V$. Then, $[m,n]^2 \subseteq U$, concluding that $\mathcal{U}$ is order continuous. That $\mathcal{U} \restriction_{[a,b]}$ is exhaustive is easy to see.

(ii) Simply note that in this case if an increasing (decreasing) net O_2-converges to x, then x is the supremum (infimum) of the net. □

Theorem 4.3.14 ([49], Corollary 7.2.4)**.** *Let $\mathcal{U}$ and $\mathcal{V}$ be exhaustive separated lattice uniformities on a lattice L, both satisfying condition (C). Then, $\mathscr{T}_{\mathcal{U}} = \mathscr{T}_{\mathcal{V}}$.*

Recall that every uniformity can be described in terms of pseudometrics. The same is true for lattice uniformities.

Proposition 4.3.15 ([48], Corollary 1.4)**.** *Let $(L, \mathcal{U})$ be a uniform lattice and $q > 1$. Then $\mathcal{U}$ is generated by a family of pseudometrics P satisfying*

$$\rho(x \vee z, y \vee z) \leq \rho(x,y) \quad \text{and} \quad \rho(x \wedge z, y \wedge z) \leq q \cdot \rho(x,y)$$

for every $x, y, z \in L$ and $\rho \in P$.

In Proposition 4.3.15 above, $q \neq 1$ unless L is a distributive lattice.

Proposition 4.3.16 ([48], Corollary 1.6)**.** *Let $(L, \mathcal{U})$ be a uniform distributive lattice. Then $\mathcal{U}$ is generated by a family of pseudometrics P satisfying*

$$\rho(x \vee z, y \vee z) \leq \rho(x,y) \quad \text{and} \quad \rho(x \wedge z, y \wedge z) \leq \rho(x,y)$$

for all $x, y, z \in L$ and $\rho \in P$.

Remark 4.3.17. *Let $f : L \to (L,\mathcal{U})$ be a lattice homomorphism, and for each $U \in \mathcal{U}$, $U_f := \{(x, y) \in L^2 : (f(x), f(y)) \in U\}$. Then the collection $\{U_f : U \in \mathcal{U}\}$ forms a base for a uniformity $\mathcal{U}_f$ on L.*

Theorem 4.3.18. *Let $(L,\mathcal{U})$ be a uniform lattice, and let $f : L \to L$ be a lattice homomorphism. Then the uniformity $\mathcal{U}_f$ is the coarsest lattice uniformity making f uniformly continuous.*

Proof. Let $U, V \in \mathcal{U}$ such that $V \vee \Delta(L) \subseteq U$ and $V \wedge \Delta(L) \subseteq U$. From Lemma 4.3.2, we show that $V_f \vee \Delta(L) \subseteq U_f$ and $V_f \wedge \Delta(L) \subseteq U_f$. Let $(x, y) \in V_f$ and $z \in L$. Then

$$\begin{aligned}(f(x \vee z), f(y \vee z)) &= (f(x) \vee f(z), f(y) \vee f(z))\\ &= (f(x), f(y)) \vee (f(z), f(z)) \in V \vee \Delta(L) \subseteq U\end{aligned}$$

concluding that $V_f \vee \Delta(L) \subseteq U_f$. Similarly, one can show that $V_f \wedge \Delta(L) \subseteq U_f$. □

Let F be a collection of lattice homomorphisms $f : L \to L$. Then the family of lattice uniformities $\{\mathcal{U}_f : f \in F\}$ forms a subbase for a lattice uniformity $\mathcal{U}_F$.

Theorem 4.3.19. *Let $(L,\mathcal{U})$ be a uniform lattice, and let F be a collection of lattice homomorphisms. Then the uniformity $\mathcal{U}_F$, generated by the family $\{\mathcal{U}_f : f \in F\}$, is a lattice uniformity on L.*

Proof. The proof of this statement is analogous to the proof of Theorem 4.3.18. □

4.4 The lattice uniformity $\mathcal{U}^*$

In this section, we review the work presented in [3]. We assume that the lattice L is distributive. Let $(L,\mathcal{U})$ be a uniform lattice, and consider the lattice uniformity described in Theorem 4.3.19 with respect to functions of the type $f_{(a,b)}$ or $g_{(a,b)}$. By Proposition 1.1.28 (b), if L is distributive, it does not matter whether one considers lattice homomorphism $f_{(a,b)}$ or $g_{(a,b)}$. Moreover due to Proposition 1.1.28 (b), we can assume that $a \leq b$. Let $\mathcal{U}_{(a,b)}$ be the lattice uniformity generated by $f_{(a,b)}$ as described in Theorem 4.3.18 where $U_{(a,b)} := \{(x,y) \in L^2 : (f_{(a,b)}(x), f_{(a,b)}(y)) \in U\}$ for every $U \in \mathcal{U}$. The uniformity $\mathcal{U}_{(a,b)}$ is the coarsest lattice uniformity for which the function $f_{(a,b)}$ is uniformly continuous.

Lemma 4.4.1. *Let $(L,\mathcal{U})$ be a uniform lattice. Then,*

(i) $\mathcal{U}_{(a,b)} \subseteq \mathcal{U}$;

(ii) $\mathcal{U}_{(a,b)} \cap [a,b]^2 = \mathcal{U} \cap [a,b]^2$.

(iii) *If $\mathcal{V}$ is a lattice uniformity such that $\mathcal{V} \subseteq \mathcal{U}$, then $\mathcal{V}_{(a,b)} \subseteq \mathcal{U}_{(a,b)}$.*

Proof. (i) To prove that $f_{(a,b)}$ is uniformly continuous with respect to $\mathcal{U}$, let $U \in \mathcal{U}$. Then there exists $W \in \mathcal{U}$ such that $(W \vee \Delta(L)) \wedge \Delta(L) \subseteq U$. For $(x,y) \in W$, we have $(f_{(a,b)}(x), f_{(a,b)}(y)) = ((x \wedge b) \vee a, (y \wedge b) \vee a) = ((x,y) \wedge (b,b)) \vee (a,a) \in U$. Then result follows from Lemma 4.3.18.

(ii) It suffices to note that if $x \in [a,b]$, $f_{(a,b)}(x) = x$, then $U_{(a,b)} \cap [a,b]^2 = U \cap [a,b]^2$.

(iii) This follows from the definitions of $\mathcal{V}_{(a,b)}$ and $\mathcal{U}_{(a,b)}$.

□

Proposition 4.4.2 ([3], proposition 4.1)**.** *Let $a,b,c,d \in L$. Then*

$$\mathcal{U}_{(a\vee(b\wedge c),\, b\wedge d)} \subseteq (\mathcal{U}_{(a,b)})_{(c,d)}.$$

In particular, we have:

(i) *If $a \leq c \leq d \leq b$, then $\mathcal{U}_{(c,d)} \subseteq \mathcal{U}_{(a,b)}$;*

(ii) $(\mathcal{U}_{(a,b)})_{(a,b)} = \mathcal{U}_{(a,b)}$.

Proof. First note that for $x \in L$, we have

$$f_{(a,b)}(f_{(c,d)}(x)) = (((x \wedge d) \vee c) \wedge b) \vee a$$

$$
\begin{aligned}
&= (x \wedge d \wedge b) \vee (c \wedge b) \vee a \\
&= (x \wedge (d \wedge b)) \vee (a \vee (b \wedge c)) \\
&= f_{(a \vee (b \wedge c),\, b \wedge d)}(x).
\end{aligned}
$$

Thus, $f_{(a,b)} \circ f_{(c,d)} = f_{(a \vee (b \wedge c),\, b \wedge d)}$. Now, the map

$$f_{(c,d)} : \big(L, (\mathcal{U}_{(a,b)})_{(c,d)}\big) \to (L, \mathcal{U}_{(a,b)})$$

is uniformly continuous, and so is $f_{(a,b)} : (L, \mathcal{U}_{(a,b)}) \to (L, \mathcal{U})$. Hence, their composition

$$f_{(a \vee (b \wedge c),\, b \wedge d)} = f_{(a,b)} \circ f_{(c,d)} : \Big(L, \big(\mathcal{U}_{(a,b)}\big)_{(c,d)}\Big) \to (L, \mathcal{U}).$$

is uniformly continuous as well. By definition $\mathcal{U}_{(a \vee (b \wedge c),\, b \wedge d)}$ is the coarsest lattice uniformity making the map $f_{(a \vee (b \wedge c),\, b \wedge d)}$ uniformly continuous. Therefore, $\mathcal{U}_{(a \vee (b \wedge c),\, b \wedge d)} \subseteq (\mathcal{U}_{a,b})_{(c,d)}$.

(i) If $a \leq c \leq d \leq b$, then $a \vee (b \wedge c) = c$ and $b \wedge d = d$. By the argument above and Lemma 4.4.1 (i), it follows that $\mathcal{U}_{(c,d)} \subseteq \big(\mathcal{U}_{(a,b)}\big)_{(c,d)} \subseteq \mathcal{U}_{(a,b)}$.

(ii) Furthermore, by setting $a = c$ and $b = d$ in the above argument, we obtain $\mathcal{U}_{(a,b)} = (\mathcal{U}_{(a,b)})_{(a,b)}$. □

Let $J(L) := \{(a,b) \in L^2 : a \leq b\}$. For any $J \subseteq J(L)$, Theorem 4.3.19 allows us to define a lattice uniformity $\mathcal{U}_J$ whose subbase is given by $\{\mathcal{U}_{(a,b)} : (a,b) \in J\}$.

Proposition 4.4.3 ([3], Proposition 4.2)**.** *Let $J, J' \subseteq J(L)$ be non-empty.*

(i) *For every lattice uniformity $\mathcal{V}$ coarser than $\mathcal{U}$, one has $\mathcal{V}_J \subseteq \mathcal{U}_J \subseteq \mathcal{U}$.*

(ii) *If for every $(a,b) \in J$ there exists $(a',b') \in J'$ with $a' \leq a \leq b \leq b'$, then $\mathcal{U}_J \subseteq \mathcal{U}_{J'}$ and $(\mathcal{U}_J)_{J'} = (\mathcal{U}_{J'})_J = \mathcal{U}_J$.*

(iii) *$\mathcal{U}_J$ is the coarsest lattice uniformity which agrees with $\mathcal{U}$ on $[a,b]$ for all $(a,b) \in J$.*

(iv) *The topology $\mathscr{T}_{\mathcal{U}_J}$ is the coarsest topology which agrees with $\mathscr{T}_{\mathcal{U}}$ on $[a,b]$ for all $(a,b) \in J$.*

Proof. (i) By definition, $\mathcal{V}_J$ is generated by the family $\{\mathcal{V}_{(a,b)} : (a,b) \in J\}$, while $\mathcal{U}_J$ is generated by $\{\mathcal{U}_{(a,b)} : (a,b) \in J\}$. Since $\mathcal{V} \subseteq \mathcal{U}$, Lemma 4.4.1 implies that $\mathcal{V}_{(a,b)} \subseteq \mathcal{U}_{(a,b)}$ for each $(a,b) \in J$. Hence $\mathcal{V}_J \subseteq \mathcal{U}_J$. Finally, as each $\mathcal{U}_{(a,b)}$ is contained in $\mathcal{U}$, it follows that $\mathcal{U}_J \subseteq \mathcal{U}$.

(ii) That $\mathcal{U}_J \subseteq \mathcal{U}_{J'}$ follows by Proposition 4.4.2 (i) and by the fact that $\mathcal{U}_J$ is the supremum of the uniformities $\mathcal{U}_{(a,b)}$ for all $(a,b) \in J$. For the second assertion it suffices to show that $\mathcal{U}_J = (\mathcal{U}_J)_{J'}$, since $\mathcal{U}_J = (\mathcal{U}_{J'})_J$ can be shown in a similar way. Clearly, $(\mathcal{U}_J)_{J'} \subseteq \mathcal{U}_J$. From (i) and Proposition 4.4.2 (ii), $\mathcal{U}_{(a,b)} = (\mathcal{U}_{(a,b)})_{(a,b)} \subseteq (\mathcal{U}_J)_{(a,b)} \subseteq (\mathcal{U}_J)_{J'}$. Hence, $\mathcal{U}_J \subseteq (\mathcal{U}_J)_{J'}$ since $\mathcal{U}_J$ is the supremum of the uniformities $\mathcal{U}_{(a,b)}$ for $(a,b) \in J$.

(iii) We first show that $\mathcal{U}_J$ agrees with $\mathcal{U}$ on $[a,b]$ for all $(a,b) \in J$. Let $(a,b) \in J$, $U \in \mathcal{U}$ and $U_{(a,b)} := \{(x,y) \in L^2 : (f_{(a,b)}(x), f_{(a,b)}(y)) \in U\}$. Then $U_{(a,b)} \cap [a,b]^2 = U \cap [a,b]^2$ since $f_{(a,b)}(x) = x$ for $x \in [a,b]$. Therefore $\mathcal{U}_{(a,b)}$ and $\mathcal{U}$ agree on $[a,b]$. Then $\mathcal{U}_J$ and $\mathcal{U}$ agree on $[a,b]$ since $\mathcal{U}_{(a,b)} \subseteq \mathcal{U}_J \subseteq \mathcal{U}$. Finally, we show that $\mathcal{U}_J$ is the weakest uniformity agreeing with $\mathcal{U}$ on $[a,b]$ for all $(a,b) \in J$. To this end, let $\mathcal{V}$ be another lattice uniformity on L which agrees with $\mathcal{U}$ on $[a,b]$ for all $(a,b) \in J$. It suffices to show that $\mathcal{U}_{(a,b)} \subseteq \mathcal{V}$ for $(a,b) \in J$. As $\mathcal{V}$ is a lattice uniformity, the map $f_{(a,b)} : (L, \mathcal{V}) \to (L, \mathcal{V})$ is uniformly continuous. Then $f_{(a,b)} : (L, \mathcal{V}) \to (L, \mathcal{U})$ is uniformly continuous because $f_{(a,b)}(L) \subseteq [a,b]$ and $\mathcal{V}|_{[a,b]} = \mathcal{U}|_{[a,b]}$, concluding that $\mathcal{U}_{(a,b)} \subseteq \mathcal{V}$.

(iv) The argument is entirely analogous to that of (iii). For each $(a,b) \in J$, the uniformities $\mathcal{U}_{(a,b)}$ and $\mathcal{U}$ induce the same topology on $[a,b]$, hence so do $\mathcal{U}_J$ and $\mathcal{U}$. Minimality follows by the same reasoning as in (iii), replacing uniform continuity by agreement of topologies. Thus $\mathscr{T}_{\mathcal{U}_J}$ is the coarsest topology which agrees with $\mathscr{T}_{\mathcal{U}}$ on $[a,b]$ for all $(a,b) \in J$. $\square$

Let $\mathcal{U}^* = \mathcal{U}_{J(L)}$. Then Proposition 4.4.3 leads to the following corollary.

Corollary 4.4.4 ([3], Corollary 4.3)**.** *Let* $(L,\mathcal{U})$ *be a uniform lattice. Then:* (i) $(\mathcal{U}^*)^* = \mathcal{U}^*$.

(ii) $\mathcal{U}^*$ is the weakest lattice uniformity which agrees with $\mathcal{U}$ on any order bounded subset of L.

(iii) $\mathscr{T}_{\mathcal{U}^*}$-topology is the weakest lattice topology which agrees with $\mathscr{T}_{\mathcal{U}}$ on any order bounded subset of L.

In Corollary 4.4.4 (ii) we established that $\mathcal{U}^*$ is the coarsest lattice uniformity which coincides with $\mathcal{U}$ on every order bounded subset of L, and that the topology $\mathscr{T}_{\mathcal{U}^*}$ is the coarsest lattice topology which agrees with $\mathscr{T}_{\mathcal{U}}$ on such subsets. The following example shows that this property need not hold for arbitrary uniformities.

Example 26. *Let $\mathcal{U}$ be the usual uniformity on $\mathbb{R}$ induced by the absolute value. Then:*

(a) *The infimum $\mathcal{V}$ of all uniformities on $\mathbb{R}$ which agree with $\mathcal{U}$ on all bounded subsets of $\mathbb{R}$ is the trivial uniformity.*

(b) *The sets $U_n := \{(x,y) \in \mathbb{R}^2 : |x-y| \leq \frac{1}{n}$ or $x,y \geq n$ or $x,y \leq -n\}$, $n \in \mathbb{N}$, form a base for $\mathcal{U}^*$.*

Proof. (a) For $a \in \mathbb{R}$ and $n \in \mathbb{N}$, define $V_{n,a}$ to be the subset of $\mathbb{R}^2$ defined by

$$(x,y) \in V_{n,a} \iff \begin{cases} |x-y| \leq \frac{1}{n}, \text{ or} \\ |x|,|y| \geq n, \text{ or} \\ |x| \geq n, |y-a| \leq \frac{1}{n}, \text{ or} \\ |y| \geq n, |x-a| \leq \frac{1}{n}\,. \end{cases}$$

The family $\{V_{n,a} : n \in \mathbb{N}\}$ forms a base for a uniformity $\mathcal{V}_a$ on $\mathbb{R}$ which agrees with $\mathcal{U}$ on all bounded subsets of $\mathbb{R}$. Then $n \to a$ $(\mathcal{V}_a)$ and therefore $n \to a$ $(\mathcal{V})$ since $\mathcal{V} \subseteq \mathcal{V}_a$. Now let $V \in \mathcal{V}$ and $a, b \in \mathbb{R}$. Then $n \to a$ $(\mathcal{V})$ and $n \to b$ $(\mathcal{V})$, so $(a, b) \in V$. This shows $V = \mathbb{R}^2$. Hence, $\mathcal{V}$ is the trivial uniformity.

(b) This immediately follows from the definition of $\mathcal{U}^*$ as the coarsest uniformity making $f_{(a,b)} : L \to L : x \mapsto (x \wedge b) \vee a$ uniformly continuous for every $a \leq b$. $\square$

Proposition 4.4.5 ([3], Proposition 4.6)**.** *Let S be a sublattice of a uniform lattice $(L,\mathcal{U})$ and $I = J(S) := \{(a, b) \in S^2 : a \leq b\}$. Then, $\mathcal{U}_I$ is separated if and only if $\mathcal{U}$ is separated and $x = \sup_{s\in S} s \wedge x = \inf_{s\in S} s \vee x$ for every $x \in L$.*

Proof. Assume that $\mathcal{U}_I$ is separated, so we have that $\bigcap_{U\in\mathcal{U}_I} U = \Delta(L)$. Then $\mathcal{U}$ is separated because $\mathcal{U}_I \subseteq \mathcal{U}$. Now $x \wedge s \leq x$ for every $s \in S$ and we claim that $x \leq y$ for every $y \in L$ satisfying $x \wedge s \leq y$ for all $s \in S$. Fix $(a, b) \in I$. Then $(y \wedge x) \wedge b = x \wedge b$, so that $(x \wedge y \wedge b) \vee a = (x \wedge b) \vee a$. By definition of $\mathcal{U}_I$, this shows that $(x, x \wedge y) \in \bigcap_{U\in\mathcal{U}_I} U = \Delta(L)$, which implies $x \leq y$. Similarly, one shows that $\inf_{s\in S}(x \vee s) = x$.

Conversely, suppose that $\mathcal{U}$ is separated and that $x = \sup_{s\in S} s \wedge x = \inf_{s\in S} s \vee x$ for every $x \in L$. Let $(x, y) \in \bigcap_{U\in\mathcal{U}_I} U$. Then, since $\mathcal{U}$ is separated, we get that $(x \wedge b) \vee a = (y \wedge b) \vee a$ for every $(a, b) \in I$. Then,

$$(x \wedge b) \vee a = (x \wedge (a \vee b)) \vee a = (y \wedge (a \vee b)) \vee a = (y \wedge b) \vee a,$$

for arbitrary a and b in S, and therefore

$$x = \sup_{b\in S} x \wedge b = \sup_{b\in S} \inf_{a\in S} (x \wedge b) \vee a = \sup_{b\in S} \inf_{a\in S} (y \wedge b) \vee a = \sup_{b\in S} y \wedge b = y.$$

This shows that $\mathcal{U}_I$ is separated. $\square$

Corollary 4.4.6. *$\mathcal{U}$ is separated if and only if $\mathcal{U}^*$ is separated.*

The remainder of this section is devoted to the case when the uniformity $\mathcal{U}$ is induced by the topology $\mathscr{T}$ of a locally solid Riesz space. We begin by recalling some fundamental results on Riesz spaces that will be needed.

Definition 4.4.7. *A (not necessarily Hausdorff) topology τ on a Riesz space V is said to be locally solid if it is topological vector space and admits a neighbourhood base at zero consisting of solid sets. The pair (V, τ), where V is a Riesz space and τ is locally solid, is known as a locally solid Riesz space.*

Definition 4.4.8. *Let (V, τ) be a locally solid Riesz space and let $A \subseteq V$ be a Riesz order ideal. A net $(x_\gamma)_{\gamma\in\Gamma}$ in V is said to be unbounded τ-convergent to $x \in V$ with respect to A if $(|x_\gamma - x| \wedge a)_{\gamma\in\Gamma}$ τ-converges to 0 for all $a \in A$.*

It was shown in [44, 37] that unbounded τ-convergence with respect to A induces a locally solid topology on V.

Theorem 4.4.9 ([44], Theorem 2.7)**.** *Let (V, τ) be a Hausdorff locally solid Riesz space, let $\mathfrak{U}$ its 0-neighbourhood system, and let A be a Riesz order ideal of V. Then the sets $\{x \in V : |x| \wedge a \in U\}$ $(a \in A_+, U \in \mathfrak{U})$, form a 0-neighbourhood base for a Hausdorff locally solid topology on V, denoted by $\mathfrak{u}_A\tau$.*

Remark 4.4.10. *Let (V, τ) be a Hausdorff locally solid Riesz space. A net $(x_\gamma)_{\gamma\in\Gamma}$ converges to x with respect to $\mathfrak{u}_A\tau$ iff $|x_\gamma - x| \wedge a \xrightarrow{\tau} 0$ for every $a \in A_+$.*

Definition 4.4.11. *A locally solid topology τ on a Riesz space V is called unbounded if $\tau = \mathfrak{u}\tau (= \mathfrak{u}_V\tau)$.*

Proposition 4.4.12. *Let (V, τ) be a Hausdorff locally solid Riesz space and let $\mathfrak{U}$ denote the solid neighbourhood system of 0 for τ. For*

$s, t \in V$ with $s \leq t$, define

$$U_{(s,t)} := \{(x, y) \in V^2 : f_{(s,t)}(x) - f_{(s,t)}(y) \in U\}.$$

Then $\{U_{(s,t)} : U \in \mathfrak{U}\}$ forms a base for a separated lattice uniformity.

Proof. To verify that $\{U_{(s,t)} : U \in \mathfrak{U}\}$ is a base for a uniformity, it suffices to check properties (i)-(ii) in Proposition 4.1.9. Let $U, W \in \mathfrak{U}$. Clearly, $\Delta(V) \subseteq U_{(s,t)}$. Since U is locally solid and hence symmetric, it follows that $U_{(s,t)}$ is symmetric as well. Moreover, $U_{(s,t)} \cap W_{(s,t)} = (U \cap W)_{(s,t)}$. Next, given $U \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ satisfying $W + W \subseteq U$. Then it is easy to see that $W_{(s,t)} \circ W_{(s,t)} \subseteq U_{(s,t)}$. Since τ is Hausdorff, it follows that $\bigcap_{U \in \mathfrak{U}} U_{(s,t)} = \Delta(V)$. Finally, by Proposition 4.3.2, this family indeed forms a base for a separated lattice uniformity. □

Corollary 4.4.13. *Let (V, τ) be a Hausdorff locally solid Riesz space and let A be a solid Riesz subspace of V. Define $J := \{(a, b) \in V^2 : b - a \in A_+\}$, and let $\mathfrak{U}$ denote the solid neighbourhood system of 0 for τ. Then the collection $\{U_{(s,t)} : U \in \mathfrak{U}$ where $(s, t) \in J\}$ forms a subbase for the separated lattice uniformity $\mathcal{U}_J$.*

Lemma 4.4.14. *Let V be a Riesz space, $s, x, y \in V$, and $a \in V_+$. Then*

$$\begin{aligned}|f_{(s,s+a)}(x) - f_{(s,s+a)}(y)| \leq |x - y| \wedge a \; &= |f_{(y-a,y)}(x) - f_{(y-a,y)}(y)| + \\ &\quad |f_{(y,y+a)}(x) - f_{(y,y+a)}(y)|.\end{aligned}$$

Proof. We first show the inequality on the left-hand side. Since $f_{(s,s+a)}(x)$, $f_{(s,s+a)}(y) \in [s, s+a]$, we have $|f_{(s,s+a)}(x) - f_{(s,s+a)}(y)| \leq a$. Moreover, by Corollary 1.2.10, $|f_{(s,s+a)}(x) - f_{(s,s+a)}(y)| \leq |x - y|$.

To see the equality on the right-hand side, observe that

$$\begin{aligned}f_{(y,y+a)}(x) - f_{(y,y+a)}(y) &= (x \wedge (y + a)) \vee y - (y \wedge (y + a)) \vee y, \\ &= (x \wedge (y + a)) \vee y - y\end{aligned}$$

$$\begin{aligned}&=((x-y)\wedge a)\vee 0\\&=(x-y)^+\wedge a,\end{aligned}$$

and

$$\begin{aligned}f_{(y-a,y)}(x)-f_{(y-a,y)}(y)=&(x\wedge y)\vee(y-a)\ -\ y\\=&((x-y)\wedge 0)\vee(-a)\\=&(-(x-y)^-)\vee(-a)\\=&-((x-y)^-\wedge a).\end{aligned}$$

Finally, since $(x-y)^+\wedge a$ and $(x-y)^-\wedge a$ are disjoint, by Theorem 1.2.4 (ii),

$|f_{(y,y+a)}(x)-f_{(y,y+a)}(y)|+|f_{(y-a,y)}(x)-f_{(y-a,y)}(y)|=$
$((x-y)^+\wedge a)\ +\ ((x-y)^-\wedge a)=((x-y)^+\wedge a)\ \vee\ ((x-y)^-\wedge a)=$
$|x-y|\wedge a.$

$\square$

Theorem 4.4.15 ([3], Theorem 4.9)**.** *Let (V,τ) be a Hausdorff locally solid Riesz space and let A be a Riesz solid subspace of V. Let $\mathcal{U}_J$ be the separated lattice uniformity induced by $J:=\{(a,b)\in V^2 : b-a\in A_+\}$. Then the topology generated by $\mathcal{U}_J$ coincides with the unbounded topology of τ with respect to A, that is,*

$$\mathscr{T}_{\mathcal{U}_J}=\mathfrak{u}_A\tau.$$

In particular, taking $A=V$, we obtain $\mathscr{T}_{\mathcal{U}^*}=\mathfrak{u}\tau$.

Proof. Let $\mathfrak{U}$ be a 0-neighbourhood base for (V,τ) consisting of solid sets. For $s,t\in V$ with $t-s\in A_+$ and $U\in\mathfrak{U}$, define

$$U_{(s,t)}(x_0):=\{x\in V : f_{(s,t)}(x)-f_{(s,t)}(x_0)\in U\}.$$

Then $\{U_{(s,t)}(x_0) : U \in \mathfrak{U},\, t-s \in A_+\}$ is a neighbourhood subbase of x_0 with respect to $\mathscr{T}_{\mathcal{U}_J}$. From Theorem 4.4.9, the family $\{U(x_0,a) : U \in \mathfrak{U},\, a \in A_+\}$, where

$$U(x_0,a) := \{x \in V : |x-x_0| \wedge a \in U\},$$

is a neighbourhood base of x_0 with respect to $\mathfrak{u}_A\tau$.

Let $U \in \mathfrak{U}$ and $s,t \in V$ with $a = t-s \in A_+$. We start by showing that $\mathscr{T}_{\mathcal{U}_J} \subseteq \mathfrak{u}_A\tau$. Let $x \in U(x_0,a)$. Consider the set $U_{(s,t)}(x_0)$. By Lemma 4.4.14, one obtains

$$|f_{(s,t)}(x) - f_{(s,t)}(x_0)| \leq |x-x_0| \wedge a \in U,$$

i.e. $x \in U_{(s,t)}(x_0)$, which shows that $U(x_0,a) \subseteq U_{(s,t)}(x_0)$.

To see the converse $\mathfrak{u}_A\tau \subseteq \mathscr{T}_{\mathcal{U}_J}$, let $a \in A_+$ and $U \in \mathfrak{U}$. Choose $V \in \mathfrak{U}$ with $V+V \subseteq U$, and set $r = x_0-a$, $s = x_0$ and $t = x_0+a$. Then, $V_{(s,t)}(x_0) \cap V_{(r,s)}(x_0) \subseteq U(x_0,a)$. Indeed, if $x \in V_{(s,t)}(x_0) \cap V_{(r,s)}(x_0)$. By Lemma 4.4.14

$$|x-x_0|\wedge a = |f_{(r,s)}(x)-f_{(r,s)}(x_0)| + |f_{(s,t)}(x)-f_{(s,t)}(x_0)| \in V+V \subseteq U,$$

concluding $x \in U(x_0,a)$. □

In Theorem 4.4.15, the uniformity $\mathcal{U}_J$ is defined using

$$J := \{\, (a,b) \in V^2 : b-a \in A_+ \,\}.$$

It may be tempting to replace J with the smaller collection

$$I := \{\, (-a,a) : a \in A_+ \,\},$$

but this is generally not sufficient. The following example highlights this fact.

Example 27. *Let $V = \mathbb{R}^{\mathbb{N}}$ and let τ be the locally solid group topology* *induced by* $\|(x_n)\| = \sum_{n=1}^{\infty} |x_n|$. *Let* $A = \ell_\infty \subseteq V$ *be a solid Riesz subspace, and set* $I := \{(-a, a) : a \in \ell_\infty\}$. *Clearly,* $\mathscr{T}_{\mathcal{U}_I} \subseteq \mathscr{T}_{\mathcal{U}_J} = \mathrm{u}_A\tau$ *since* $\mathcal{U}_I \subseteq \mathcal{U}_J$. *Consider the sequences* $x = (1, 2, 3, ...)$, $e = (1, 1, 1, ...)$, $y_n = \frac{1}{n}e$ *and* $x_n = x + y_n$. *Let* $B := \{x_n : n \in \mathbb{N}\}$. *We show* B *is* $\mathscr{T}_{\mathcal{U}_J}$*-closed but not* $\mathscr{T}_{\mathcal{U}_I}$*-closed. Let* $a \in A_+$ *and choose* $k \in \mathbb{N}$ *such that* $e \leq a \leq ke$.

B is $\mathscr{T}_{\mathcal{U}_J}$-closed: Then,

$$|x_n - x| \wedge a = y_n.$$

As $\|y_n\| = \infty$ *it follows that* B *is* $\mathscr{T}_{\mathcal{U}_J}$*-closed.*

B is not $\mathscr{T}_{\mathcal{U}_I}$-closed: Then,

$$|(x_n \wedge a) \vee (-a) - (x \wedge a) \vee (-a)| \leq \frac{1}{n}\chi_{\{1,2,...,k\}}$$

where $\chi_{\{1,2,...,k\}}$ *is the characteristic function of* $\{1, 2, \ldots, k\}$. *Therefore,*

$$\|(x_n \wedge a) \vee (-a) - (x \wedge a) \vee (-a)\| \leq \frac{k}{n} \to 0 \ (n \to \infty),$$

showing that x *belongs to the* $\mathscr{T}_{\mathcal{U}_I}$*-closure of* B. *Hence,* B *is not closed in* $\mathscr{T}_{\mathcal{U}_I}$.

4.5 The uniformity $\mathcal{U}^*$ on dense sublattices

In this section, let L be a distributive lattice. We denote by $D(L)$ the set of all pseudometrics ρ on L such that for every $x, y, z \in L$,

$$\rho(x \vee z, y \vee z) \leq \rho(x, y) \quad \text{and} \quad \rho(x \wedge z, y \wedge z) \leq \rho(x, y).$$

Recall from Proposition 4.3.16 that every lattice uniformity $\mathcal{U}$ can be generated by a subset of $D(L)$. We denote the subset of $D(L)$ by $D_{\mathcal{U}}$. Let $J \subseteq J(L)$. For every $(a, b) \in J$ and $\rho \in D(L)$, we define a pseudometric $\rho_{(a,b)}$ on L by $\rho_{(a,b)}(x,y) := \rho(f_{(a,b)}(x),\ f_{(a,b)}(y))$, where $f_{(a,b)}(x) = (x \wedge b) \vee a$.

Proposition 4.5.1. *Let $\mathcal{U}$ be a lattice uniformity on a distributive lattice L. For $J \subseteq J(L)$, the lattice uniformity $\mathcal{U}_J$ is generated by the family of pseudometrics given by $\{\rho_{(a,b)} : \rho \in D_{\mathcal{U}}$ and $(a,b) \in J\}$.*

Lemma 4.5.2. *Let $\rho \in D(L)$.*

(i) *Let $P_n : L^n \to L$ be given inductively by $P_1(x) = x$ and*

$$P_n(x_1,\dots,x_n) := \begin{cases} P_{n-1}(x_1,\dots,x_{n-1}) \vee x_n, & \text{or} \\ P_{n-1}(x_1,\dots,x_{n-1}) \wedge x_n. \end{cases}$$

Then $\rho(P_n(x_1,\dots,x_n), P_n(y_1,\dots,y_n)) \leq \sum_{i=1}^{n} \rho(x_i,y_i)$.

(ii)

$$\begin{aligned} \rho(f_{(a,b)}(x), f_{(a,b)}(y)) \leq{}& \rho(f_{(c,d)}(x), f_{(c,d)}(y)) \\ &+ 2\rho((a,c)) + 2\rho((b,d)), \end{aligned}$$

for all $a,b,c,d,x,y \in L$

Proof. The proof of (i) is by induction. The assertion is trivially true when $n=1$. Suppose that $\rho(P_{n-1}(x_1,\dots,x_{n-1}), P_{n-1}(y_1,\dots,y_{n-1})) \leq \sum_{i=1}^{n-1} \rho(x_i,y_i)$. Let $\diamond \in \{\wedge,\vee\}$. Then,

$$\begin{aligned} &\rho(P_n(x_1,\dots,x_n), P_n(y_1,\dots,y_n)) \\ &= \rho(P_{n-1}(x_1,\dots,x_{n-1}) \diamond x_n, P_{n-1}(y_1,\dots,y_{n-1}) \diamond y_n) \\ &\leq \rho(P_{n-1}(x_1,\dots,x_{n-1}) \diamond x_n, P_{n-1}(y_1,\dots,y_{n-1}) \diamond x_n) \\ &\qquad + \rho(P_{n-1}(y_1,\dots,y_{n-1}) \diamond x_n, P_{n-1}(y_1,\dots,y_{n-1}) \diamond y_n) \\ &\leq \rho(P_{n-1}(x_1,\dots,x_{n-1}), P_{n-1}(y_1,\dots,y_{n-1})) + \rho(x_n,y_n) \\ &\leq \sum_{i=1}^{n} \rho(x_i,y_i). \end{aligned}$$

To prove (ii) we first note that

$$\rho(f_{(a,b)}(x), f_{(a,b)}(y)) \leq \rho(f_{(a,b)}(x), f_{(c,d)}(x)) + \rho(f_{(c,d)}(x), f_{(c,d)}(y)) + \rho(f_{(c,d)}(y), f_{(a,b)}(y)),$$

and then apply (i) to obtain

$$\rho(f_{(a,b)}(x), f_{(a,b)}(y)) \leq \rho(f_{(c,d)}(x), f_{(c,d)}(y)) + 2\rho(a,c) + 2\rho(b,d).$$

$\square$

Let $(L,\mathcal{U})$ be a uniform lattice and $J \subseteq J(L)$. Let $\tilde{J} := \{(a,b) : a \leq b$ and there exists $(a_\alpha, b_\alpha) \in J$ such that $a_\alpha \xrightarrow{\mathscr{T}_\mathcal{U}} a$ and $b_\alpha \xrightarrow{\mathscr{T}_\mathcal{U}} b\}$. Equivalently, $\tilde{J} = \overline{J}^{\mathscr{T}_\mathcal{U}} \cap J(L)$.

Proposition 4.5.3. *Let $\emptyset \neq J \subseteq J(L)$. Then $\mathcal{U}_J = \mathcal{U}_{\tilde{J}}$.*

Proof. The inclusion $\mathcal{U}_J \subseteq \mathcal{U}_{\tilde{J}}$ follows by Proposition 4.4.3 (ii). For the reverse inclusion, take $W \in \mathcal{U}_{\tilde{J}}$. We will show that there exists $U \in \mathcal{U}_J$ such that $U \subseteq W$. By definition, there exist finitely many pairs $(a_1, b_1). \dots (a_n, b_n) \in \tilde{J}$, elements $\rho_1 \dots, \rho_n \in D_\mathcal{U}$ and $\varepsilon_1, \dots, \varepsilon_n > 0$ such that

$$\bigcap_{i=1}^{n} \{(x,y) \in L^2 : \rho_{(a_i,b_i)}(x,y) < 5\varepsilon_i\} \subseteq W.$$

For $i = 1, \dots, n$, choose $(a_i^{'}, b_i^{'}) \in J$ with $\rho(a, a') < \varepsilon_i$ and $\rho(b, b') < \varepsilon_i$. Define

$$U_i := \{(x,y) \in L^2 : \rho_{(a_i^{'}, b_i^{'})}(x,y) < \varepsilon_i\} \in \mathcal{U}_J.$$

By Lemma 4.5.2(ii), for each i and all $(x,y) \in U_i$,

$$\rho_{(a_i,b_i)}(x,y) \leq \rho_{(a_i', b_i')}(x,y) + 2\rho_i(a_i, a_i') + 2\rho_i(b_i, b_i') < 5\varepsilon_i,$$

so that $U_i \subseteq \{(x,y) : \rho_{(a_i,b_i)}(x,y) < 5\varepsilon_i\}$. Now define $U := \bigcap_{i=1}^{n} U_i$.

Then, $U \in \mathcal{U}_J$, and

$$U \subseteq \bigcap_{i=1}^{n} \{(x,y) \in L^2 : \rho_{(a_i,b_i)}(x,y) < 5\varepsilon_i\} \subseteq W.$$

Showing that $\mathcal{U}_{\tilde{J}} \subseteq \mathcal{U}_J$. ☐

Corollary 4.5.4. *Let S be a dense sublattice of the uniform lattice $(L,\mathcal{U})$, and define $\mathcal{V} = \mathcal{U} \restriction_S$. Then $\mathcal{U}^* \restriction_S = \mathcal{V}^*$.*

Proof. Let $J := J(S) = \{(a,b) \in S^2 : a \leq b\}$. First, we note that $J(L) \subseteq \tilde{J}$. By density of S in L, let $(a,b) \in J(L)$ and let $(x_\gamma)_{\gamma\in\Gamma}$ and $(y_\beta)_{\beta\in\Omega}$ be nets in S that $\mathscr{T}_{\mathcal{U}}$-converge to a and b, respectively. Then, $(x_\gamma \wedge y_\beta, x_\gamma \vee y_\beta) \subseteq J$ and $x_\gamma \wedge y_\beta \xrightarrow{\mathscr{T}_{\mathcal{U}}} a$ and $x_\gamma \vee y_\beta \xrightarrow{\mathscr{T}_{\mathcal{U}}} b$, showing that $(a,b) \in \tilde{J}$.

From Proposition 4.5.3, $\mathcal{U}^* = \mathcal{U}_J$ and it is easy to see that $\mathcal{U}_J \restriction_S = \mathcal{V}_J$. Finally, combining everything $\mathcal{U}^*|_S = \mathcal{U}_J \restriction_S = \mathcal{V}_J = \mathcal{V}^*$. ☐

Next we will be using the following result.

Lemma 4.5.5 ([46], pg. 381)**.** *Let Y be a dense subspace of a uniform space $(X,\mathcal{U})$.*

(i) *Then $\mathcal{W} \to \mathcal{W} \restriction_Y$ defines a lattice isomorphism from the lattice of all uniformities on X coarser than $\mathcal{U}$ onto the lattice of all uniformities on Y coarser than $\mathcal{U}|_Y$.*

(ii) *Let $\mathcal{W}$ be a uniformity on X coarser than $\mathcal{U}$ and $y_0 \in Y$. Then*

$$\{\overline{W}^{\mathscr{T}_{\mathcal{U}}} : W \text{ is a neighbourhood of } y_0 \text{ in } (Y, \mathcal{W} \restriction_Y)\}$$

is a neighbourhood base of y_0 in $(X,\mathcal{W})$; where $\overline{W}^{\mathscr{T}_{\mathcal{U}}}$ denotes the closure of W in $(X,\mathcal{U})$.

Theorem 4.5.6. *Let S be a dense sublattice of $(L,\mathcal{U})$ and $\mathcal{V} = \mathcal{U} \restriction_S$.*

(i) *Then $\mathcal{U}^* = \mathcal{U}$ iff $\mathcal{V}^* = \mathcal{V}$.*

(ii) *If $\mathscr{T}_{\mathcal{U}^*} = \mathscr{T}_{\mathcal{U}}$, then $\mathscr{T}_{\mathcal{V}^*} = \mathscr{T}_{\mathcal{V}}$.*

Proof. (i) By Lemma 4.5.5 (i), we have $\mathcal{U}^* = \mathcal{U}$ iff $\mathcal{U}^* \restriction_S = \mathcal{U} \restriction_S$. Result follows as $\mathcal{U}|_S = \mathcal{V}$, and from Corollary 4.5.4, $\mathcal{U}^*|_S = \mathcal{V}^*$.

(ii) If $\mathscr{T}_{\mathcal{U}^*} = \mathscr{T}_{\mathcal{U}}$, then restricting to S gives $\mathscr{T}_{\mathcal{U}^*} \restriction_S = \mathscr{T}_{\mathcal{U}} \restriction_S$. By Corollary 4.5.4, we have that, $\mathscr{T}_{\mathcal{V}^*}$ coincides with $\mathscr{T}_{\mathcal{V}}$. $\square$

It is not known whether the converse of Theorem 4.5.6 (ii) holds. What can currently be said is that, if $\mathscr{T}_{\mathcal{V}^*} = \mathscr{T}_{\mathcal{V}}$, then every $s \in S$ has the same neighbourhood system in the topologies $\mathscr{T}_{\mathcal{U}^*}$ and $\mathscr{T}_{\mathcal{U}}$. However, if one restricts attention to topologies that are uniquely determined by their 0-neighbourhood systems, then more can be concluded. The following Proposition is particularly useful in addressing a question posed in [37, Question 3.3].

Proposition 4.5.7 ([46], pg. 381/382)**.** *Let H be a dense subgroup of a topological commutative group (G, τ). Then $\rho \to \rho \restriction_H$ defines an order isomorphism from the lattice of all group topologies on G coarser than τ onto the lattice of all group topologies on H coarser than $\tau \restriction_H$.*

The following theorem provides an answer to a question posed in [37, Question 3.3].

Theorem 4.5.8. *Let H be a dense Riesz subspace of a locally solid Riesz space (V, τ), and let $\sigma = \tau \restriction_H$. Then $\mathfrak{u}\tau = \tau$ iff $\mathfrak{u}\sigma = \sigma$.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be the uniformities induced by τ and σ, respectively. From Theorem 4.4.15, we have $\mathscr{T}_{\mathcal{U}^*} = \mathfrak{u}\tau$ and $\mathscr{T}_{\mathcal{V}^*} = \mathfrak{u}\sigma$. Furthermore, by Corollary 4.5.4, it follows that $\mathfrak{u}\tau|_H = \mathfrak{u}\sigma$.

Since $\mathfrak{u}\tau$ is coarser than τ, follows from Proposition 4.5.7 that $\mathfrak{u}\tau = \tau$ iff $\mathfrak{u}\tau|_H = \tau|_H$, i.e., iff $\mathfrak{u}\sigma = \sigma$. $\square$

4.6 The uniformity $\mathcal{U}^*$ on sublattices

In this section we study the uniformity $\mathcal{U}^*$ on sublattices of distributive lattices. We begin with a theorem that generalizes [37, Proposition 2.12].

For an infinite cardinal κ, a subset U of a topological space is said to be κ-closed if U contains the limit of every convergent net $(x_\gamma)_{\gamma\in\Gamma}$ with $|\Gamma| \leq \kappa$, that is contained in U. A set is closed if it is κ-closed for every infinite cardinal κ.

Theorem 4.6.1. *Let S be a sublattice of the uniform lattice $(L,\mathcal{U})$. Then S is (sequentially) closed with respect to $\mathscr{T}_{\mathcal{U}^*}$ if and only if it is (sequentially) closed with respect to $\mathscr{T}_{\mathcal{U}}$.*

Proof. That S is closed with respect to $\mathscr{T}_{\mathcal{U}^*}$ implies that S is closed with respect to $\mathscr{T}_{\mathcal{U}}$ is immediate, since $\mathcal{U}^* \subseteq \mathcal{U}$.

Conversely, assume S is closed with respect to $\mathscr{T}_{\mathcal{U}}$, and let $(x_\gamma)_{\gamma\in\Gamma} \subseteq S$ be a net $\mathscr{T}_{\mathcal{U}^*}$-converging to $x \in L$. Fix $s \in S$ and $z \in L$ with $s \leq z$. Then the net $((x_\gamma \vee s) \wedge z)_{\gamma\in\Gamma}$ $\mathscr{T}_{\mathcal{U}}$-converges to $(x \vee s) \wedge z$. Since S is $\mathscr{T}_{\mathcal{U}}$-closed, if $z \in S$ then $(x \vee s) \wedge z \in S$. Now, for every $\gamma_0 \in \Gamma$, we have $x_{\gamma_0} \vee s \geq s$, and hence $(x \vee s) \wedge (x_{\alpha_0} \vee s) \in S$. Applying the previous argument again, it follows that the net $((x_\gamma \vee s) \wedge (x \vee s))_{\gamma\in\Gamma}$ $\mathscr{T}_{\mathcal{U}}$-converges to $x \vee s$. Thus, for every $s \in S$, one has $x \vee s \in S$. By a similar argument, for every $s \in S$ we also have $x \wedge s \in S$. Finally, for fixed $s \in S$, since $x \vee s \in S$ we get $x = x \wedge (x \vee s) \in S$. Hence $x \in S$, and so S is $\mathscr{T}_{\mathcal{U}^*}$-closed. $\square$

Corollary 4.6.2. *Let S be a sublattice of a uniform lattice $(L,\mathcal{U})$. Then the closures of S in L with respect to $\mathscr{T}_{\mathcal{U}}$ and $\mathscr{T}_{\mathcal{U}^*}$ coincide.*

Proposition 4.6.3. *The following conditions are equivalent:*

(i) *$\mathcal{U}$ is locally exhaustive;*

(ii) *$\mathcal{U}^*$ is locally exhaustive;*

(iii) *$\mathcal{U}^*$ is exhaustive.*

Proof. By Corollary 4.4.4 (ii), the uniformities $\mathcal{U}$ and $\mathcal{U}^*$ coincide on bounded sets. Hence, (i)$\Leftrightarrow$(ii). Moreover, (iii)$\Rightarrow$(ii) is immediate.

(i)$\Rightarrow$(iii): By definition, a net $(x_\gamma)_{\gamma\in\Gamma}$ is $\mathcal{U}^*$-Cauchy if and only if, for every $(a,b) \in J(L)$, the net $(f_{(a,b)}(x_\gamma))_{\gamma\in\Gamma}$ is $\mathcal{U}$-Cauchy. Now let $(x_\gamma)_{\gamma\in\Gamma}$ be a monotone net in L and $(a,b) \in J(L)$. Then the net $(f_{(a,b)}(x_\gamma))_{\gamma\in\Gamma}$ is monotone and bounded. Hence, by (i), it is $\mathcal{U}$-Cauchy. Therefore $(x_\gamma)_{\gamma\in\Gamma}$ is $\mathcal{U}^*$-Cauchy. □

Theorem 4.6.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be separated lattice uniformities on L satisfying Condition (C). Then:*

(i) *the topologies $\mathscr{T}_{\mathcal{U}^*}$ and $\mathscr{T}_{\mathcal{V}^*}$ coincide;*

(ii) *for every sublattice S of L, $\overline{S}^{\mathcal{U}} = \overline{S}^{\mathcal{V}}$.*

Proof. (i) Since $\mathcal{U}^* \subseteq \mathcal{U}$ and $\mathcal{V}^* \subseteq \mathcal{V}$, the lattice uniformities $\mathcal{U}^*$ and $\mathcal{V}^*$ also satisfy Condition (C). By Proposition 4.3.13, they are locally exhaustive, and hence exhaustive by Proposition 4.6.3. It follows from Theorem 4.3.14 that (i) holds.

(ii) follows from (i) and Corollary 4.6.2. □

Theorem 4.6.4 (ii) generalizes the following result.

Theorem 4.6.5 ([37], Theorem 5.11)**.** *Let τ and σ be Hausdorff order continuous topologies on a vector lattice V and Y a sublattice of V. Then Y is τ-closed if and only if Y is σ-closed.*

Let $\mathcal{U}$ be a separated lattice uniformity, and let $(\tilde{L},\tilde{\mathcal{U}})$ denote the completion of $(L,\mathcal{U})$. Then $\tilde{L}$ is again distributive, and both properties of local exhaustivity and exhaustivity pass directly to this completion by Proposition 4.3.9.

Theorem 4.6.6. *Let $(L,\mathcal{U})$ be a separated uniform lattice. If $\mathcal{U}$ is exhaustive, then $\mathcal{U} = \mathcal{U}^*$.*

Proof. Let $(\tilde{L},\tilde{\mathcal{U}})$ denote the completion of $(L,\mathcal{U})$. By Corollary 4.3.11, $\tilde{L}$ is an order-complete lattice, and hence $\tilde{\mathcal{U}} = (\tilde{\mathcal{U}})^*$. On the other hand, Corollary 4.5.4 gives $(\tilde{\mathcal{U}})^* \restriction_L = \mathcal{U}^*$. Combining these identities, we obtain $\mathcal{U} = \tilde{\mathcal{U}} \restriction_L = (\tilde{\mathcal{U}})^* \restriction_L = \mathcal{U}^*$. □

Finally, we conclude this section with a generalization of [37, Lemma 9.7], and hence of [33, Corollary 4.6].

Corollary 4.6.7. *Let* $(L,\mathcal{U})$ *be a separated uniform lattice, and assume that* $\mathcal{U}$ *is locally exhaustive. Then, for every sublattice* S *of* L, $\mathcal{U}^* \restriction_S = (\mathcal{U} \restriction_S)^*$.

Proof. First we note that by Proposition 4.6.3, $\mathcal{U}^*$ is exhaustive, and therefore $\mathcal{U}^* \restriction_S$ is also exhaustive. By Theorem 4.6.6, we then have $(\mathcal{U}^* \restriction_S)^* = \mathcal{U}^* \restriction_S$. Since $\mathcal{U} \restriction_S \supseteq \mathcal{U}^* \restriction_S$, it follows that $(\mathcal{U} \restriction_S)^* \supseteq (\mathcal{U}^* \restriction_S)^*$. Moreover, $\mathcal{U}^* \restriction_S \supseteq (\mathcal{U} \restriction_S)^*$ because $\mathcal{U}^* \restriction_S$ agrees with $\mathcal{U} \restriction_S$ on order-bounded subsets of S (c.f. Proposition 4.4.3 (iii)). Combining these inclusions, we obtain

$$(\mathcal{U} \restriction_S)^* \supseteq (\mathcal{U}^* \restriction_S)^* = \mathcal{U}^* \restriction_S \supseteq (\mathcal{U} \restriction_S)^*,$$

which shows that $\mathcal{U}^* \restriction_S = (\mathcal{U} \restriction_S)^*$. □

References

[1] K. Abela and E. Chetcuti. Unbounded order convergence on infinitely distributive lattices. *Positivity*, 29(4):Paper No. 46, 2025.

[2] K. Abela, E. Chetcuti, and H. Weber. On different modes of order convergence and some applications. *Positivity*, 26(1):Paper No. 14, 22, 2022.

[3] K. Abela, E. Chetcuti, and H. Weber. Lattice uniformities inducing unbounded convergence. *J. Math. Anal. Appl.*, 523(1):Paper No. 126994, 18, 2023.

[4] K. Abela and A. Theuma. Exposition of the gras method. *Xjenza*, 9(1):14–23, 2021.

[5] K. Abela and A. Theuma. Block decomposition of price multipliers in a sam framework for malta. *Xjenza*, 10(1):36–46, 2022.

[6] C. D. Aliprantis and O. Burkinshaw. *Locally solid Riesz spaces.* Academic Press, 1978. Pure and Applied Mathematics, Vol. 76.

[7] C. D. Aliprantis and O. Burkinshaw. *Locally Solid Riesz Spaces with Applications to Economics*. American Mathematical Society, 2003.

[8] C.D. Aliprantis and K.C. Border. *Infinite Dimensional Analysis: a Hitchhiker's Guide*. Springer, 2007.

[9] E. Bilokopytov and V. G. Troitsky. Order and uo-convergence in spaces of continuous functions. *Topology Appl.*, 308:Paper No. 107999, 9, 2022.

[10] G. Birkhoff. *Lattice Theory*. American Mathematical Society Colloquium Publications, vol. 25, revised edition. American Mathematical Society, New York, N. Y., 1948.

[11] G. Birkhoff. *Lattice theory*. Third edition. American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Providence, R.I., 1967.

[12] M. Bohata. Star order and topologies on von Neumann algebras. *Mediterr. J. Math.*, 15(4):Paper No. 175, 14, 2018.

[13] N. Bourbaki. *General topology*. Springer-Verlag Berlin Heidelberg, 1995.

[14] D. Buhagiar, E. Chetcuti, and H. Weber. The order topology on the projection lattice of a Hilbert space. *Topology Appl.*, 159(9):2280–2289, 2012.

[15] D. Buhagiar, E. Chetcuti, and H. Weber. Order topology on orthocomplemented posets of linear subspaces of a pre-Hilbert space. *Ann. Mat. Pura Appl. (4)*, 200(1):211–228, 2021.

[16] E. Chetcuti and J. Hamhalter. The order topology on duals of C^*-algebras and von Neumann algebras. *Studia Math.*, 254(3):219–236, 2020.

[17] E. Chetcuti, J. Hamhalter, and H. Weber. The order topology for a von Neumann algebra. *Studia Math.*, 230(2):95–120, 2015.

[18] D. L. Cohn. *Measure theory*. Birkhäuser/Springer, New York, 2013.

[19] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, New York, second edition, 2002.

[20] G. Debreu. *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. Yale University Press, 1959.

[21] R. DeMarr. Partially ordered linear spaces and locally convex linear topological spaces. *Illinois J. Math.*, 8:601–606, 1964.

[22] R. Engelking. *General topology*. Sigma series in pure mathematics. Heldermann Verlag, 1989.

[23] D. H. Fremlin. *Measure theory. Vol. 2*. Torres Fremlin, Colchester, 2003.

[24] D. H. Fremlin. *Measure theory. Vol. 1*. Torres Fremlin, Colchester, 2004.

[25] D. H. Fremlin. *Measure theory. Vol. 3*. Torres Fremlin, Colchester, 2004.

[26] P. Gänssler. Compactness and sequential compactness in spaces of measures. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 17:124–146, 1971.

[27] N. Gao and D. H. Leung. Smallest order closed sublattices and option spanning. *Proc. Amer. Math. Soc.*, 146(2):705–716, 2018.

[28] N. Gao, V. G. Troitsky, and F. Xanthos. Uo-convergence and its applications to Cesàro means in Banach lattices. *Israel J. Math.*, 220(2):649–689, 2017.

[29] A. R. Gingras. Convergence lattices. *Rocky Mountain J. Math.*, 6(1):85–104, 1976.

[30] R. W. Hansell. Monotone subnets in partially ordered sets. *Proc. Amer. Math. Soc.*, 18:854–858, 1967.

[31] J. Horváth. *Topological vector spaces and distributions. Vol. I.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[32] R. V Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras.* Academic, 1983.

[33] M. Kandić, H. Li, and V. G. Troitsky. Unbounded norm topology beyond normed lattices. *Positivity*, 22(3):745–760, 2018.

[34] J. L. Kelley. *General topology.* Van Nostrand Reinhold, 1955.

[35] W. A. J. Luxemburg and A. C. Zaanen. *Riesz spaces.* North-Holland Pub. Co.; American Elsevier Pub. Co, 1971.

[36] J. C. Mathews and R. F. Anderson. A comparison of two modes of order convergence. *Proc. Amer. Math. Soc.*, 18:100–104, 1967.

[37] M. A. Mitchell. Unbounded topologies and $\mathfrak{u}o$-convergence in locally solid vector lattices. *J. Math. Anal. Appl.*, 472(1):981–1000, 2019.

[38] H. Nakano. Teilweise geordnete Algebra. *Jpn. J. Math.*, 17:425–511, 1941.

[39] H. Nakano. Ergodic theorems in semi-ordered linear spaces. *Ann. of Math. (2)*, 49:538–556, 1948.

[40] M. O'Brien, V. G. Troitsky, and J. H. van der Walt. Net convergence structures with applications to vector lattices. *Quaest. Math.*, 46(2):243–280, 2023.

[41] V. Olejček. The order topology on a lattice and its MacNeille completion. *Internat. J. Theoret. Phys.*, 39(3):801–803, 2000. Quantum structures '98 (Liptovský Ján).

[42] F. Papangelou. Order convergence and topological completion of commutative lattice-groups. *Math. Ann.*, 155:81–107, 1964.

[43] H. H. Schaefer. *Banach lattices and positive operators.* Springer-Verlag, New York-Heidelberg, 1974.

[44] M. A. Taylor. Unbounded convergences in vector lattices. *Master's thesis*, 2018.

[45] S. Vella Bonello. Order and convergence in function spaces. Master's thesis, University of Malta, 2015.

[46] H. Weber. Unabhängige topologien, zerlegung von ringtopologien. *Mathematische Zeitschrift*, 180:379–393, 1982.

[47] H. Weber. Uniform lattices. I. A generalization of topological Riesz spaces and topological Boolean rings. *Ann. Mat. Pura Appl. (4)*, 160:347–370 (1992), 1991.

[48] H. Weber. Metrization of uniform lattices. *Czechoslovak Math. J.*, 43(118)(2):271–280, 1993.

[49] H. Weber. Uniform lattices. II. Order continuity and exhaustivity. *Ann. Mat. Pura Appl. (4)*, 165:133–158, 1993.

[50] H. Weber. On lattices of uniformities. *Order*, 24(4):249–276, 2007.

[51] E. S. Wolk. On order-convergence. *Proc. Amer. Math. Soc.*, 12:379–384, 1961.

Index

Index of Notation

Posets and Lattices

Topologies on L^∞

Order Convergence

Riesz Spaces

$\mathscr{L}^0$ The set of measurable functions on a set X, page 27

$\mathscr{L}^\infty$ The set of essentially bounded functions on a set X, page 32

$\mathscr{V}$ Vector space, page 17

Σ^f The collection of measurable sets of positive finite measure, page 27

$\text{ess sup}|f|$ The essential supremum of f, page 33

$|x|$ Absolute value of x, page 17

V/W Quotient Riesz space of V by W, page 25

V_+ Positive elements of a Riesz space, page 18

x^+ Positive part of x, page 17

x^- Negative part of x, page 17

V Riesz space, page 17

Uniform Lattices

$(\tilde{X},\tilde{\mathcal{U}})$ Completion of a uniform space $(X,\mathcal{U})$, page 115

$\Delta(X)$ Diagonal of the product $X \times X$, page 107

$\mathcal{D}_X$ Entourages of the diagonal $\Delta(X)$, page 107

$\mathcal{U}$ Uniformity, page 107

$\mathcal{U}_\rho$ Uniformity generated by a pseudometric ρ, page 112

$\mathfrak{U}$ The neighbourhood system of 0 for a Hausdorff locally solid Riesz space, page 127

$\mathfrak{u}_A\tau$ The unbounded topology of τ with respect to the Riesz order ideal A, page 126

$\mathscr{T}_\mathcal{U}$ Topology induced by the uniformity $\mathcal{U}$, page 108

$D(L)$ The set of pseudometrics on a distributive lattice L, page 130

$U \circ V$ Composition of U and V, page 107