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On strong cellularity type properties of Lindelöf groups

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Abstract

We prove several facts about cellularity and κ-cellularity of λ-Lindelöf groups generated by their κ-stable subspaces. For example, if a Lindelöf group G is generated by its κ-stable subspace then κ-cellularity (and hence cellularity) of G does not exceed κ. In particular, ω_1-cellularity (and hence cellularity) of a Lindelöf group does not exceed ω_1 if this group is generated by its ω_1-Lindelöf subspace which is a P-space. For any cardinal μ with $\omega < \mu \leqslant \mathfrak{c}$ a Lindelöf group G is constructed which is separable (and hence has countable cellularity) while ω-cellularity of G is equal to μ.

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We consider Tychonoff spaces (spaces, for short) and Hausdorff (and hence Tychonoff) topological groups only. The word "map" is used to denote a continuous mapping between spaces. Everywhere below λ, μ and τ are infinite cardinal numbers.

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1. Introduction

It is well known that cellularity (i.e., the Souslin number) $c(X)$ of compacta (i.e., compact Hausdorff spaces) X may be arbitrarily large but cellularity of any compact group is countable. Tkačenko [9] proved that cellularity of every σ-compact group is also countable. In particular, it is true for groups generated by their compact subspaces (for example, for the free topological groups of compacta). Since σ-compact groups are Lindelöf and Lindelöfness is the nearest property of compactness type to that of compactness, it was natural (after Tkačenko's result) to find out the behavior of cellularity in the class of Lindelöf groups.

The situation for Lindelöf groups is much more complicated than for compact ones. Generalizing Tkačenko's result on σ-compact groups, Uspenskii [12] showed that $c(G) \leqslant \omega$ for any Lindelöf Σ-group G (a space X is called *Lindelöf* Σ if it is a continuous image of a perfect preimage of a separable metrizable space). Tkačenko [10] and Uspenskii [12] proved that $c(G) \leqslant 2^\tau$ for any τ-Lindelöf group G (a space X is called τ*-Lindelöf* if every open cover of X has a subcover of cardinality $\leqslant \tau$). Tkačenko [10] constructed a Lindelöf group G with $c(G) = \omega_1$. In [3], a Lindelöf group G is presented with $c(G) = \mathfrak{c}$. The following problem arose in connection with the results cited above.

Problem 1. When is

(a) $c(G) \leqslant \omega$;
(b) $c(G) \leqslant \omega_1$,

for a Lindelöf (and for an arbitrary) group G?

A partial answer is contained in Uspenskii's paper [13] where the notion of τ-stability of a space is used. Recall that a space X is called τ*-stable* [1] if $\mathrm{nw}(Y) \leqslant \tau$ for any space Y which is a continuous image of X and has a *condensation* (i.e., a one-to-one map) onto a space of weight $\leqslant \tau$. It is completely natural to consider τ-stability in the case of topological groups because any G_τ-neighbourhood (i.e., a G_τ-set) containing the identity element of an arbitrary topological group G contains a closed subgroup H such that G/H has a condensation onto a space with a τ-locally finite base (which will have the weight $\leqslant \tau$ if G is τ-Lindelöf or $c(G) \leqslant \tau$). In [13] (see Theorems 4, 5 and Corollary 1), Uspenskii gave an upper bound for the cellularity of a λ-Lindelöf group supposing that it is generated by a μ-stable subspace. In particular, he stated that $c(G) \leqslant \omega_1$ *for any Lindelöf group G generated by its ω-stable subspace*. He also showed that the last inequality cannot be strengthened.

Sometimes it is more productive to consider more strong cardinal functions than cellularity in the theory of topological groups. Recall the corresponding definitions.

A family η of subsets of a space X is called *dense in a family* ξ of subsets of X if every element of η is contained in some element of ξ and $\mathrm{cl} \cup \eta = \mathrm{cl} \cup \xi$. Evidently, the inequality $c(X) \leqslant \tau$ is equivalent to the property: any family ξ of open sets in X has a dense in ξ subfamily of cardinality $\leqslant \tau$. This observation explains the following definition (and the term "τ-cellularity").

Recall that:

- a G_τ*-set* (= a *set of type* G_τ) in a space X is the intersection of at most τ-many open sets in X;
- a G_τ*-family* in a space X is a family of G_τ-sets in X;
- the τ*-cellularity* $\mathrm{cel}_\tau(X)$ of a space X is $\max(\omega, \min\{\lambda\colon$ any G_τ-family ξ in X has a dense in ξ subfamily of cardinality $\leqslant \lambda\})$.

It seems that the definition of $\mathrm{cel}_\tau X$ was given by M.G. Tkačenko. The case of $\mathrm{cel}_\omega X$ was considered, for example, in [13].

Evidently,

$$c(X) \leqslant \mathrm{cel}_\lambda(X) \leqslant \mathrm{cel}_\mu(X) \quad \text{for } \lambda \leqslant \mu.$$

Note that Uspenskii [13] proved the inequality $\mathrm{cel}_\omega(G) \leqslant \omega$ for Lindelöf Σ- (in particular, for σ-compact) groups and Pasynkov [6] stated the inequality $\mathrm{cel}_\tau(G) \leqslant 2^\tau$ for τ-Lindelöf groups G. These results strengthen Tkačenko's and Uspenskii's ones on $c(G)$ cited above. In this paper, a strengthening of Uspenskii's results from [13] will be obtained and some additional information will be given even for $c(G)$ of some Lindelöf groups G.

The following problems may be formulated in connection with the cited above results and Problem 1.

Problem 2. When is

(c) $\mathrm{cel}_\omega(G) \leqslant \omega$;
(d) $\mathrm{cel}_\omega(G) \leqslant \omega_1$,

for a Lindelöf (and for any) group G?

Problem 3. Is it true that $c(G) = \mathrm{cel}_\omega(G)$ for all Lindelöf groups G?

A complete answer to Problem 3 will be obtained in this paper. It will be shown that, for Lindelöf groups, the difference between $c(G)$ and $\mathrm{cel}_\omega(G)$ may be arbitrary (in possible limits).

Often the cardinal function $\tau - \mathrm{cl}(X) = \min\{\mu\colon$ for any G_τ-family ξ in X, the closure $\mathrm{cl}(\bigcup \xi)$ is a G_μ-set in $X\}$ is considered together with $\mathrm{cel}_\tau X$.

Evidently,

$$\lambda - \mathrm{cl}(X) \leqslant \mu - \mathrm{cl}(X) \quad \text{for } \lambda \leqslant \mu.$$

Note that the inequality $\omega - \mathrm{cl}(X) \leqslant \omega$ which implies, for example, the almost perfect normality of X [2], is useful in examinations of inductive dimensions [6] and is often proved together with the inequality $\mathrm{cel}_\omega(X) \leqslant \omega$. The last inequality is stated by Uspenskii in [12] for Lindelöf Σ-groups. Some upper bounds for $\omega - \mathrm{cl}(G)$ for special Lindelöf groups will be given in this paper.

The proof of the main result of Section 2 (Theorem 1) is based on considering the inverse spectrum (= inverse system) S_G defined by a topological group G, which consists of all quotient-spaces G/H_α, where H_α is a closed subgroup of G. Generally speaking, this spectrum is not continuous (i.e., for some well-ordered by inclusion family of closed subgroups $H_\alpha, \alpha \in A$, of G and for $H = \bigcap\{H_\alpha\colon \alpha \in A\}$ the space G/H may not be the limit of the inverse spectrum of the spaces G/H_α), but sometimes this problem may be taken care of (see, for example, [6,7]). In this paper, the continuity of S_G is stated in some places.

2. Cellularity and τ-cellularity of λ-Lindelöf groups for $\lambda < \tau$

Recall that $\mathrm{iw}(X) \leqslant \tau$ for a space X if X has a condensation onto a space of weight $\leqslant \tau$.

Definition 1. For cardinal numbers λ and μ, $\lambda \leqslant \mu$, a space X will be called (λ, μ)-*stable* if $\mathrm{nw}(Y) \leqslant \mu$ for any continuous image Y of X with $\mathrm{iw}(Y) \leqslant \lambda$.

Note that the (λ, λ)-stability coincides with λ-stability.

Definition 2. A topological group G will be called (*strongly*) *algebraically* (λ, μ)-*stable*, $\lambda \leqslant \mu$, if for any closed normal subgroup (for any closed subgroup) N, $\mathrm{iw}(G/N) \leqslant \lambda$ implies $\mathrm{nw}(G/N) \leqslant \mu$. A group will be called (*strongly*) *algebraically* λ-*stable* if it is (strongly) algebraically (λ, λ)-stable.

Lemma 1. *If a topological group is generated by a* (λ, μ)-*stable subspace then it is algebraically* (λ, μ)-*stable.*

Proof. Assume that a group G is generated by its (λ, μ)-stable subspace A and H is a closed normal subgroup of G such that $\mathrm{iw}(G/H) \leqslant \lambda$; let $p : G \to X = G/H$ be the canonical map. Then, for $B = pA$ we have $\mathrm{iw}(B) \leqslant \lambda$ and so $\mathrm{nw}(B) \leqslant \mu$. Since $\mathrm{nw}(B^n) \leqslant \mu$, $n = 2, 3, \ldots$, and G/H is a continuous image of a countable discrete union of powers B^n, we have $\mathrm{nw}(G/H) \leqslant \mu$. □

Recall that an inverse spectrum $S = \{X_\alpha, p_{\beta\alpha}; A\}$ is called λ-*continuous* for an infinite cardinal number λ (see, for example, [6]) if, for any monotone mapping $j : \lambda \to A$, there exists $\gamma = \sup j\lambda$ in A and $\Delta\{p_{\gamma j(\theta)}\colon \theta \in \lambda\}$ is a homeomorphism onto the limit of the inverse spectrum $\{X_{j(\theta)}, p_{j(\kappa) j(\theta)};\ \theta \in \lambda\}$.

The following proposition is a generalization of Theorem 4 from [13] (for a slightly stronger notion of λ-continuity than in [13]).

Proposition 1. *Assume that* $\lambda < \tau$ *and a* λ-*Lindelöf group* G *is generated by a subspace* X *which is* (μ, τ)-*stable for all* μ *with* $\lambda \leqslant \mu < \tau$. *Then* G *is the limit of a* λ^+-*continuous* τ-*directed inverse spectrum* $S = \{G_\alpha, p_{\beta\alpha}; A\}$ *of topological groups such that*:

(1) $\mathrm{nw}(G_\alpha) \leqslant \tau$ *for all* $\alpha \in A$;
(2) *the limit homomorphisms* $p_\alpha : G \to G_\alpha$ *are open.*

We need the following lemma to prove Proposition 1.

Lemma 2. *Suppose that G is a λ-Lindelöf group and a family $\eta = \{N_\alpha\colon \alpha \in A\}$ of closed normal subgroups of G is λ-directed (assuming also that $\alpha < \beta$ if $N_\beta \subset N_\alpha$); let $N = \bigcap \eta$. Then*

(1) *for any neighbourhood O of the identity e of G, there exists a neighbourhood U of e and $\alpha \in A$ such that $U \cdot N_\alpha \subset O \cdot N$;*
(2) *the topological group $X = G/N$ is the limit of the inverse spectrum of topological groups $S = \{X_\alpha = G/N_\alpha, p_{\beta\alpha}; A\}$, where $p_{\beta\alpha}$ is the canonical map of X_β onto X_α for $\beta > \alpha$.*

Proof. Let $p : G \to X$ and $f_\alpha : X \to X_\alpha$ be the canonical maps. Evidently, X is λ-Lindelöf.

Take a neighbourhood O of e. There exists a neighbourhood U of e such that $U^2 \subset O$. The family $\xi = \{G \setminus N_\alpha\colon \alpha \in A\} \cup \{U \cdot N\}$ is an open cover of G. Since G is λ-Lindelöf, there exists a subcover $\{N_\alpha\colon \alpha \in B\} \cup \{U \cdot N\}$ of ξ with $B \subset A$, $|B| \leqslant \lambda$. Since η is λ-centered, there exists $\beta \in A$ such that $\beta > \alpha$ for any $\alpha \in B$. Then $N_\beta \subset \bigcap\{N_\alpha\colon \alpha \in B\} = G \setminus \bigcup\{(G \setminus N_\alpha)\colon \alpha \in B\} \subset U \cdot N$. Hence $U \cdot N_\beta = U^2 \cdot N \subset O \cdot N$, so (1) is proved.

Let Y be the limit of S and $p_\alpha : Y \to X_\alpha$, $\alpha \in A$, be its projections. Since $f_\alpha = p_{\beta\alpha} \circ f_\beta$ for $\beta > \alpha$, a continuous homomorphism $f : X \to Y$ is defined so that $f_\alpha = p_\alpha \circ f$, $\alpha \in A$. Since $N = \bigcap \eta$, the map f is a monomorphism. If $y = \{y_\alpha\}_{\alpha \in A} \in Y$ then the sets $f^{-1} y_\alpha$ are closed in X and their family is λ-centered (because A is λ-directed). It follows from the λ-Lindelöfness of X that $f^{-1} y = \bigcap\{f_\alpha^{-1} y_\alpha\colon \alpha \in A\} \neq \emptyset$. Hence f is an epimorphism. Finally, (1) implies that f is open and so we may identify X and Y by means of f. □

Proof of Proposition 1. We may suppose (see [4]) that G is a subgroup of the product Π of topological groups G_i of weight $\leqslant \lambda$, $i \in I$. Let $A = \{\alpha \subset I\colon |\alpha| \leqslant \tau\}$; pr_α be the restriction to G of the projection of the product Π onto the subproduct $\Pi_\alpha = \prod\{G_i\colon i \in \alpha\}$, $N_\alpha = \mathrm{pr}_\alpha^{-1} e_\alpha$ (where e_α is the identity of the group Π_α), $G_\alpha = G/N_\alpha$, π_α be the canonical map of G onto G_α, $\alpha \in A$; $p_{\beta\alpha}$ be the canonical map of G_β onto G_α for $\beta > \alpha$ (i.e., for $\beta \supset \alpha$), $\alpha, \beta \in A$. Evidently, *we have a τ-directed inverse spectrum of topological groups $S = \{G_\alpha, p_{\beta\alpha}; A\}$*. Let H be the limit of S and $p_\alpha : H \to G_\alpha$ be the limit projections, $\alpha \in A$. Evidently, for every α, the group G_α has a continuous monomorphism to the group Π_α.

Let us prove (1).

Let $|\alpha| = \nu$. Take the case when $\lambda \leqslant \nu < \tau$. Then $w(\Pi_\alpha) \leqslant \nu < \tau$ and, by Lemma 1, $\mathrm{nw}(G_\alpha) \leqslant \tau$. Now let $\nu = \tau$. Take an injective and (strongly) monotone mapping $j : \tau \to A$ such that $j(\theta) < \alpha$ for all $\theta \in \tau$ and $\alpha = \bigcup\{j(\theta)\colon \theta \in \tau\} = \sup\{j(\theta)\colon \theta \in \tau\}$. Then the family of all $N_{j(\theta)}$, $\theta \in \tau$, is λ-directed and $N_\alpha = \bigcap\{N_{j(\theta)}\colon \theta \in \tau\}$. Since $w(\Pi_{j(\theta)}) < \tau$, by Lemma 1, $\mathrm{nw}(G_{j(\theta)}) \leqslant \tau$, $\theta \in \tau$. Hence, by Lemma 2, $\mathrm{nw}(G_\alpha) \leqslant \tau$. We have proved (1).

Let us prove that S is λ^+-continuous.

Take a monotone mapping $j : \lambda^+ \to A$. Since $\lambda^+ \leqslant \tau$, we have $\delta = \bigcup j(\lambda^+) \in A$ and $\delta = \sup j(\lambda^+)$. Let $S_\delta = \{G_{j(\theta)}, p_{j(\kappa)j(\theta)}; \theta \in \lambda^+\}$. If $\delta \in j(\lambda^+)$ then, evidently, the group G_δ is the limit of S_δ. If $\delta \notin j(\lambda^+)$ then $|\delta| > \lambda$. Hence the set $B = \{\beta \subset \delta\colon |\beta| \leqslant \lambda\}$ is

λ-directed and $N_\delta = \bigcap\{N_\beta\colon \beta \in B\}$. By Lemma 2, G_δ is the limit of the inverse spectrum $\Sigma_\delta = \{G_\beta, p_{\gamma\beta}; B\}$ and its limit projections coincide with the homomorphisms $p_{\delta\beta}$, $\beta \in B$. The homomorphisms $p_{\delta j(\theta)}$ define a continuous homomorphism i_δ of G_δ to the limit H_δ of S_δ such that $p_{\delta j(\theta)} = p_{j(\theta)} \circ i_\delta$, where $p_{j(\theta)}$ is the limit projection of the spectrum S_δ, $\theta \in \lambda^+$. The surjectivity of all $p_{\delta j(\theta)}$ implies the density of $i_\delta G_\delta$ in H_δ. Since, for any $\beta \in B$, there exists the minimal number $\theta(\beta) \in \lambda^+$ such that $\beta \subset j(\theta(\beta))$, the continuous homomorphism $k_\beta = p_{j(\theta(\beta))\beta} \circ p_{j(\theta(\beta))} : H_\delta \to G_\beta$ is defined. Since the inequality $\gamma > \beta$, for $\gamma, \beta \in B$, implies the inequality $\theta(\gamma) \geqslant \theta(\beta)$, we convince ourselves that

$$\begin{aligned} k_\beta &= p_{j(\theta(\beta))\beta} \circ p_{j(\theta(\beta))} = p_{j(\theta(\beta))\beta} \circ p_{j(\theta(\gamma))j(\theta(\beta))} \circ p_{j(\theta(\gamma))} \\ &= p_{\gamma\beta} \circ p_{j(\theta(\gamma))\gamma} \circ p_{j(\theta(\gamma))} = p_{\gamma\beta} \circ k_\gamma. \end{aligned}$$

Hence a continuous homomorphism $k : H_\delta \to G_\delta$ is defined such that $p_{\delta\beta} \circ k = k_\beta$ for any $\beta \in B$. But since

$$p_{\delta\beta} \circ k \circ i_\delta = k_\beta \circ i_\delta = p_{j(\theta(\beta))\beta} \circ p_{j(\theta(\beta))} \circ i_\delta = p_{j(\theta(\beta))\beta} \circ p_{\delta j(\theta(\beta))} = p_{\delta\beta}$$

for any $\beta \in B$, we conclude that $k \circ i_\delta$ is the identity map of G_δ. The Hausdorffness of H_δ and the density of $i_\delta G_\delta$ in H_δ imply that i_δ and k are mutually inverse homeomorphisms. We have proved that S is λ^+-continuous.

Since $\pi_\alpha = p_{\beta\alpha} \circ \pi_\beta$ for any $\alpha, \beta \in A$, $\alpha < \beta$, a continuous homomorphism $\pi : G \to H$ is defined such that $\pi_\alpha = p_\alpha \circ \pi$, $\alpha \in A$. The epimorphness of all π_α implies that πG is dense in H. Since G is a subspace of Π, we conclude that π is a topological embedding. Finally, for any point $h \in H$, the family $h^* = \{\pi_\alpha^{-1} p_\alpha h\colon \alpha \in A\}$ is closed in G and is τ-(and so λ-)centered. It follows from the λ-Lindelöfness of G that $\bigcap h^* \neq \emptyset$. Since $\bigcap\{p_\alpha^{-1} p_\alpha h\colon \alpha \in A\} = \{h\}$, we have $h \in \pi G$. Hence π is an isomorphism. The openness of π_α, the surjectivity of π and the relation $\pi_\alpha = p_\alpha \circ \pi$ give us the openness of p_α. □

Theorem 1. *Suppose that $\lambda < \tau$ and a λ-Lindelöf group G is generated by a subspace X which is (μ, τ)-stable for all μ with $\lambda \leqslant \mu < \tau$. Then*

$$c(X) \leqslant \mathrm{cel}_\omega(G) \leqslant \mathrm{cel}_\tau(G) \leqslant \tau \quad \textit{and} \quad \omega - \mathrm{cl}(G) \leqslant \tau - \mathrm{cl}(G) \leqslant \tau.$$

Moreover, for any G_τ-family ξ in G, there exist a normal closed subgroup N of G and a closed (and G_τ-) family η in $H = G/N$ such that $\mathrm{nw}(H) \leqslant \tau$, $|\eta| \leqslant \tau$ and the family $p^{-1}\eta$ (where p is the canonical map of G onto H) is dense in ξ.

Proof. Let ξ be a G_τ-family in G.

Take an inverse spectrum $S = \{G_\alpha, p_{\beta\alpha}; A\}$ with the properties such as in Proposition 1.

Since S is τ-directed, we may find a family ζ in G such that its elements are contained in elements of ξ; for every $F \in \zeta$ there exist $\alpha = \alpha(F) \in A$ and a closed set $\Phi = \Phi(F)$ in G_α such that $F = p_\alpha^{-1}\Phi$; $\bigcup \zeta = \bigcup \xi$. Suppose that there is no dense in ζ subsystem of cardinality $\leqslant \tau$. Then, for any ordinal number $\theta < \tau^+$, there exist $\alpha(\theta) \in A$, a closed (and G_τ-) set Φ_θ in $G_{\alpha(\theta)}$ and a point $x_\theta \in F_\theta = p_{\alpha(\theta)}^{-1}\Phi_\theta \in \zeta$ such that

$$x_\theta \in F_\theta \setminus \bigl(\mathrm{cl} \cup \{F_\kappa\colon \kappa < \theta\}\bigr), \quad \theta < \tau^+. \tag{1}$$

Let $C_\theta = \{x_\kappa\colon \kappa < \theta\}$, $2 \leqslant \theta < \tau^+$; $C = \{x_\theta\colon \theta < \tau^+\}$.

The τ-directedness of S allows us to suppose that

$$\alpha(\kappa) \leqslant \alpha(\theta) \quad \text{for } \kappa < \theta < \tau^+. \tag{2}$$

Since $\mathrm{nw}(G_\alpha) \leqslant \tau$ for all $\alpha \in A$, there exists $\theta(1) < \tau^+$ such that $p_{\alpha(1)}C_{\theta(1)}$ is dense in $p_{\alpha(1)}C$. Let $\delta(1) = \alpha(\theta(1))$. It is possible, by means of transfinite induction and using the τ-directedness of S and τ^+, to choose $\theta(k) < \tau^+$ and $\delta(k) \in A$, $k < \lambda^+$, so that

$$\theta(k) < \theta(l) \quad \text{if } k < l < \lambda^+, \qquad \delta(k) = \alpha\big(\theta(k)\big) \quad \text{and}$$
$$\mathrm{cl}\, p_{\delta(k)}C_{\theta(k+1)} = \mathrm{cl}\, p_{\delta(k)}C, \quad k < \lambda^+. \tag{3}$$

Since $\lambda^+ \leqslant \tau$, we have $\theta(\infty) = \sup\{\theta(k)\colon k < \lambda^+\} < \tau^+$. It follows from (3) and (2) that the mapping $\delta : \lambda^+ \to A$ is monotone. Hence, there exists $\delta(\infty) = \sup\{\delta(k)\colon k < \lambda^+\}$ in A. The set $p_{\delta(\infty)}C_{\theta(\infty)}$ is dense in $p_{\delta(\infty)}C$ because S is λ^+-continuous. Hence $p_{\delta(\infty)}x_{\theta(\infty)} \in \mathrm{cl}\, p_{\delta(\infty)}C_{\theta(\infty)}$. It follows from the openness of $p_{\delta(\infty)}$ that $x_{\theta(\infty)} \in \mathrm{cl}\, p^{-1}_{\delta(\infty)} p_{\delta(\infty)}C_{\theta(\infty)}$. But (see (2)) $\delta(\infty) \geqslant \alpha(\kappa)$ for all $\kappa < \theta(\infty)$. Hence

$$x_{\theta(\infty)} \in \mathrm{cl}\, p^{-1}_{\delta(\infty)} p_{\delta(\infty)}C_{\theta(\infty)} \subset \mathrm{cl} \bigcup \big\{ p^{-1}_{\alpha(\kappa)} p_{\alpha(\kappa)} x_\kappa\colon \kappa < \theta(\infty) \big\}$$
$$\subset \mathrm{cl} \bigcup \big\{ F_\kappa\colon \kappa < \theta(\infty) \big\}.$$

But, by (1), this is impossible. Thus there exists a dense subfamily ζ' of ζ of cardinality $\leqslant \tau$.

Since S is τ-directed, there exist an index $\alpha \in A$ and a closed family η in $H = G_\alpha$ such that $\zeta' = p_\alpha^{-1}\eta$. It follows from the openness of p_α that $\mathrm{cl} \bigcup \xi = \mathrm{cl} \bigcup \zeta = \mathrm{cl} \bigcup \zeta' = p_\alpha^{-1}\, \mathrm{cl} \bigcup \eta$. Since $\mathrm{nw}(H) \leqslant \tau$, we conclude that $\mathrm{cl} \bigcup \eta$ and $\mathrm{cl} \bigcup \xi$ are G_τ-sets in H and in G respectively. Evidently, the required normal subgroup N of G is $p_\alpha^{-1} e_\alpha$, where e_α is the identity of the group H. □

Corollary 1. *If a Lindelöf group G is generated by its (ω, ω_1)-stable (in particular, ω- or ω_1-stable) subspace then*

$$c(G) \leqslant \mathrm{cel}_\omega(G) \leqslant \mathrm{cel}_{\omega_1}(G) \leqslant \omega_1 \quad \textit{and} \quad \omega - \mathrm{cl}(G) \leqslant \omega_1 - \mathrm{cl}(G) \leqslant \omega_1.$$

This corollary is a generalization and a strengthening of Corollary 2 in [13].

Corollary 2. *If a Lindelöf group G is generated by a μ-stable subspace, $\mu > \omega$, then*

$$c(G) \leqslant \mathrm{cel}_\omega(G) \leqslant \mathrm{cel}_\mu(G) \leqslant \mu \quad \textit{and} \quad \omega - \mathrm{cl}(G) \leqslant \mu - \mathrm{cl}(G) \leqslant \mu.$$

Recall that a space X is a P-space if every G_δ-subset of X is open in X. Also recall that, for a discrete space X and an infinite cardinal number τ, the one-point τ-Lindelöfication $L_\tau X$ is the disjoint union of X and a point l with the topology consisting of all subsets of X and all sets of type $\{l\} \cup (X \setminus L)$, where $|L| \leqslant \tau$. Evidently, $L_\tau X$ is a τ-Lindelöf P-space. Since every first countable continuous image of any ω_1-Lindelöf P-space consists of not greater than ω_1 points, any ω_1-Lindelöf P-space is (ω, ω_1)-stable. Thus we have the following.

Corollary 3. *If a Lindelöf group G is generated by a ω_1-Lindelöf (in particular, Lindelöf) subspace which is a P-space then*

$$c(G) \leqslant \mathrm{cel}_\omega(G) \leqslant \mathrm{cel}_{\omega_1}(G) \leqslant \omega_1 \quad \textit{and} \quad \omega - \mathrm{cl}(G) \leqslant \omega_1 - \mathrm{cl}(G) \leqslant \omega_1.$$

This corollary is a generalization and a strengthening of Corollary 3 in [13], Theorem 3.8 in [5] and Corollary 4.14 in [11].

3. Lindelöf groups with countable cellularity and arbitrary (possible) ω-cellularity

Theorem 2. *For any μ, $\omega < \mu \leqslant \mathfrak{c}$, there exists a Lindelöf separable group F_μ with $\mathrm{nw}(F_\mu) = \mu$, $\mathrm{iw}(F_\mu) = \omega$ and $(c(F_\mu) = \omega <)\, \mathrm{cel}_\omega(F_\mu) = \mu$ which is (ω, μ)-stable but is not (ω, λ)-stable for $\lambda < \mu$.*

Proof. Let R^2 be the set of all points of the space $\mathbb{R}^2$ and $\mathcal{T}_e$ be the topology of $\mathbb{R}^2$.

Following Przimusin'ski [8, Corollary 4], represent $\mathbb{R}^2$ as the union of two disjoint sets T_1 and T_2 so that, for any $n \in \omega$ and any closed in $(\mathbb{R}^2)^n$ set F, the relation $F \cap (T_i)^n = \emptyset$ for $i = 1$ or $i = 2$ implies the relation $|F|_n \leqslant \omega$ (i.e., there exists a countable set $A \subset \mathbb{R}^2$ such that $F \subset \bigcup\{(\mathbb{R}^2)_1 \times \cdots \times (\mathbb{R}^2)_{i-1} \times A \times (\mathbb{R}^2)_{i+1} \times \cdots \times (\mathbb{R}^2)_n \colon i = 1, \ldots, n\}$).

Below $\mathcal{T}_e^n$ denotes the topology of $(\mathbb{R}^2)^n$. Put $R_i = \{(x, y) \in T_i \colon y = 0\}$, $i = 1, 2$. We may suppose that $|R_2| = \mathfrak{c}$. Let $\mathcal{T}$ be a new topology on R^2 a base of which is $\mathcal{T}_e \cup (\{\{(a, 0)\} \cup \{(x, y) \in R^2 \colon (x - a)^2 + (y - \varepsilon)^2 < \varepsilon^2\} \colon (a, 0) \in R_2, \ \varepsilon > 0\})$. Put $P = (R^2, \mathcal{T})$.

For any μ, $\omega < \mu \leqslant \mathfrak{c}$, fix a subset A_μ of R_2 of cardinality μ and let P_μ be the set $R^2 \setminus (R_2 \setminus A_\mu)$ with the topology induced by $\mathcal{T}$ on it. Evidently, the discrete topology is induced on A_μ in P_μ. Hence $\mathrm{nw}(P_\mu) \geqslant \mu$. Since the space $P_\mu \setminus A_\mu$ has a countable base and $|A_\mu| = \mu$, we have $\mathrm{nw}(P_\mu) \leqslant \mu$. Thus $\mathrm{nw}(P_\mu) = \mu$. It follows from this that $\mathrm{nw}((P_\mu)^n) = \mu$ for any $n \in \omega$. Since the free topological group $F_\mu = F(P_\mu)$ of the space P_μ is a continuous image of a countable discrete union of all finite powers of the space P_μ and F_μ contains P_μ as a subspace, we have $\mathrm{nw}(F_\mu) = \mu$.

Evidently, the space P_μ is separable. Hence all of its finite powers and the group F_μ are also separable and so $c(F_\mu) = \omega$.

We now show that the space F_μ is (ω, μ)-stable but not (ω, λ)-stable for $\lambda < \mu$. The first assertion follows from the fact that the network weight of all continuous images of F_μ is not greater than μ. The second one follows from the inequality $\mathrm{nw}(F_\mu) \geqslant \mu$ and from the fact that F_μ has a condensation onto a separable metrizable space. Indeed, P_μ has a condensation onto a subset Q_μ of $\mathbb{R}^2$. Hence F_μ has a continuous isomorphism onto the free topological group G_μ of Q_μ. The existence of a countable base in Q_μ implies the existence of a countable network in G_μ. This allows to condense G_μ onto a separable metrizable space. Hence $\mathrm{iw}(F_\mu) = \omega$.

We next prove that F_μ is Lindelöf. It is sufficient for this to prove that all finite powers of P_μ are Lindelöf (the method of the proof is similar to one in [8]).

First, we shall prove that P_μ itself is Lindelöf. Let η be its open cover. Topologies $\mathcal{T}_e$ and $\mathcal{T}$ coincide on T_1. Hence there exists a countable subfamily ζ of η covering T_1. The set $\bigcup \zeta$ contains a neighbourhood O of T_1 in the subspace $R^2 \setminus (R_2 \setminus A_\mu)$ of the space $\mathbb{R}^2$. It follows from this (by the choice of the sets T_i) that $P_\mu \setminus O$ is countable. This allows to choose a countable subcover of η.

Suppose that all powers $(P_\mu)^m$, $1 \leqslant m < n$, are Lindelöf. We show that $X = (P_\mu)^n = \prod\{(P_\mu)_k\colon k = 1, \ldots, n\}$ is also Lindelöf. Let η be an open cover of X. As above, there exists a countable subfamily of η covering some neighbourhood O of $(T_1)^n$ in the subspace $(R^2 \setminus (R_2 \setminus A_\mu))^n$ of the space $(\mathbb{R}^2)^n$. By the choice of the sets T_i, there exists a countable set C in P_μ such that $(P_\mu)^n \setminus O$ is contained in the union of subproducts Π_k, $k = 1, \ldots, n$, of the product X such that the factor of Π_k is contained in $(P_\mu)_k$ is C and other factors coincide with the corresponding factors of X. Thus $X \setminus O$ is contained in the countable union of subsets homeomorphic to $(P_\mu)^{n-1}$ which are Lindelöf, by the inductive assumption. It follows from this that η has a countable subcover. We have proved that all finite powers of P_μ and (so) F_μ are Lindelöf.

Finally we prove the equality $\mathrm{cel}_\omega(F_\mu) = \mu$. The inequality $\mathrm{cel}_\omega(F_\mu) \leqslant \mu$ follows from the relation $\mathrm{nw}(F_\mu) \leqslant \mu$. All one-point sets in F_μ are of type G_δ because F_μ may be condensed onto a (separable) metrizable space. Since P_μ is a subspace of F_μ and contains a discrete (in itself) subspace A_μ of cardinality μ, the G_δ-family of all one-point subsets of A_μ has no dense subfamily of cardinality $< \mu$. □

References

[1] A.V. Arhangelskii, Factorization theorems and spaces of functions: Stability and monolithicity, Dokl. Akad. Nauk SSSR 265 (5) (1982) 1039–1043 (in Russian), also: Soviet Math. Dokl. 26 (1982) 177–181 (in English).

[2] A.Ch. Chigogidze, On a generalization of perfectly normal spaces, Topology Appl. 13 (1982) 15–20.

[3] P.M. Gartside, E.A. Reznichenko, O.V. Sipacheva, Mal'tsev and retral spaces, Topology Appl. 80 (1997) 115–129.

[4] I.I. Guran, On topological groups close to finally compact groups, Dokl. Akad. Nauk SSSR 256 (6) (1981) 1305–1307 (in Russian), also: Soviet Math. Dokl. 23 (1981) 173–175 (in English).

[5] G.L. Itzkowitz, V.V. Tkačuk, Any pseudo-pG-compact group G is fine, New Zealand J. Math. 32 (2003) 127–138.

[6] B.A. Pasynkov, On inverse systems and cardinal functions of topological spaces, Topology Appl. 54 (1993) 97–110.

[7] B.A. Pasynkov, On relative cellularity of Lindelöf subspaces of topological groups, Topology Appl. 57 (1994) 249–258.

[8] T. Przimusin'ski, On the notion of n-cardinality, Proc. Amer. Math. Soc. 69 (2) (1978) 333–338.

[9] M.G. Tkačenko, On the Souslin property in free topological groups over compacta, Mat. Zametki 34 (1983) 601–607 (in Russian).

[10] M.G. Tkačenko, On topologies of free groups, Czechoslovak Math. J. 34 (109) (1984) 541–551.

[11] M.G. Tkačenko, $\mathbb{R}$-factorizable groups and subgroups of Lindelöf P-groups, Topology Appl. 136 (2004) 135–167.

[12] V.V. Uspenskii, A topological group generated by Lindelöf Σ-space has the Souslin property, Dokl. Akad. Nauk SSSR 265 (4) (1982) 823–826 (in Russian), also: Soviet Math. Dokl. 26 (1982) 166–169 (in English).

[13] V.V. Uspenskii, On continuous images of Lindelöf topological groups, Dokl. Akad. Nauk SSSR 285 (4) (1985) 824–827 (in Russian).