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Orthonormal bases and quasi-splitting subspaces in pre-Hilbert spaces

D. Buhagiar [a,*], E. Chetcuti [b], H. Weber [c]

[a] Department of Mathematics, Faculty of Science, University of Malta, Msida MSD.06, Malta
[b] Department of Mathematics, Junior College, University of Malta, Msida MSD.06, Malta
[c] Dipartimento di matematica e informatica, Universitá degli Studi di Udine, 1-33100 Udine, Italy

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ABSTRACT

Let S be a pre-Hilbert space. We study quasi-splitting subspaces of S and compare the class of such subspaces, denoted by $E_q(S)$, with that of splitting subspaces $E(S)$. In [D. Buhagiar, E. Chetcuti, Quasi splitting subspaces in a pre-Hilbert space, Math. Nachr. 280 (5–6) (2007) 479–484] it is proved that if S has a non-zero finite codimension in its completion, then $E_q(S) \neq E(S)$. In the present paper it is shown that if S has a total orthonormal system, then $E_q(S) = E(S)$ implies completeness of S. In view of this result, it is natural to study the problem of the existence of a total orthonormal system in a pre-Hilbert space. In particular, it is proved that if every algebraic complement of S in its completion is separable, then S has a total orthonormal system.

1. Introduction

In what follows S is a real or complex pre-Hilbert space (= inner product space) and H is its completion, i.e. a Hilbert space containing S as a dense subspace. For any subset $A \subseteq S$ we write $\overline{A}$ to denote the closure of A in H and $A^{\perp_S}$ the orthogonal complement of A in S, i.e. $A^{\perp_S} = \{x \in S \mid \langle x, a\rangle = 0,\ \forall a \in A\}$. Let us recall that the orthogonal (Hilbert) dimension $\dim S$ is the cardinality of any maximal orthonormal system of S. For a vector space K we denote by $\mathrm{d}(K)$ the linear (Hamel) dimension of K.

In the Hilbert space model for quantum mechanics, the events of a quantum system can be identified with projections on a Hilbert space or, equivalently, a collection of closed subspaces of a Hilbert space [2,5,9,11,13]. Two classes of closed subspaces of S that can naturally replace the lattice of projections in a Hilbert space are those of *orthogonally closed*, and *splitting* subspaces. We recall that a subspace M of S is orthogonally closed if $M^{\perp_S\perp_S} = M$, and is splitting if $S = M \oplus M^{\perp_S}$. It is not difficult to check that every splitting subspace is orthogonally closed. By the classical Amemiya–Araki–Piron Theorem, equality between these two classes holds if and only if S is complete [1,10] (see also [5,9]). When endowed with the partial ordering of set-theoretical inclusion $\subseteq$ and orthocomplementation $\perp_S$, the set of orthogonally closed subspaces $F(S)$ and the set of splitting subspaces $E(S)$ carry an algebraic structure with orthocomplementation. It is not difficult to check that $E(S) \subseteq F(S)$. In general, the algebraic structures of these two orthoposets are different; $F(S)$ is a complete lattice whereas $E(S)$ is an orthomodular poset and the following three statements are equivalent:

(i) $F(S)$ is orthomodular;
(ii) $E(S)$ is a complete lattice;
(iii) S is complete.

* Corresponding author.
E-mail addresses: david.buhagiar@um.edu.mt (D. Buhagiar), emanuel.chetcuti@um.edu.mt (E. Chetcuti), weber@dimi.uniud.it (H. Weber).

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Chapter 4 of the book of A. Dvurečenskij [5] and Chapter 4 of the book of J. Hamhalter [9] serve as a very good introduction on the subject.

The class $E_q(S)$ of *quasi-splitting* subspaces of S was introduced in [4] as an intermediate between $E(S)$ and $F(S)$. A subspace M of S is quasi-splitting if it is closed in S and $M \oplus M^{\perp_S}$ is a dense subspace of S. Equivalently, a closed subspace M of S is quasi-splitting if $\overline{M^{\perp_S}} = M^{\perp_H}$ [4, Proposition 2.2].

If S is complete, then $E(S) = E_q(S) = F(S)$. The inclusions

$$E(S) \subseteq E_q(S) \subseteq F(S)$$

hold though, in general, they are proper. Motivated by the Amemiya–Araki–Piron Theorem, the authors of [4] conjectured that: $E_q(S) = E(S)$ *if and only if* S *is a Hilbert space* and also have settled this in the affirmative for the case when $\mathrm{d}(H/S)$ is finite. As will be seen further on, this question is closely related to the problem of characterizing those pre-Hilbert spaces that admit an orthonormal basis, i.e. an orthonormal system (ONS) that is total. It is known that in a Hilbert space every maximal orthonormal system (MONS) is an orthonormal basis (ONB). This fact distinguishes Hilbert spaces completely [6–8]. Moreover, it is possible to exhibit pre-Hilbert spaces admitting no ONB. Indeed, it can happen that $\dim S \neq \dim H$ [3,6].

The main result of Section 2 says that if S has an ONB, then $E_q(S) = E(S)$ if and only if S is complete. (In particular, this means that $E(S) \neq E_q(S)$ when S is an incomplete separable pre-Hilbert space.) In Section 3 of the paper we investigate when a pre-Hilbert space has an ONB. It is shown that when every linear complement of S in H is separable, then S has an ONB. This means, for example, that S admits an ONB when $\mathrm{d}(H/S) \leqslant \aleph_0$. (In particular, all hyperplanes have an ONB.) The relation between $\dim S$, $\dim H$ and $\mathrm{d}(S)$ is also studied in Section 3.

2. Pre-Hilbert spaces in which every quasi-splitting subspace is splitting

In this section it is proved that if S is a pre-Hilbert space with an ONB, then $E_q(S) = E(S)$ if and only if S is complete. This result is first proved for the case when S is separable. (In such a case S always has an ONB; see for example [3, V.24] or [12].) The proof is divided in shorter lemmas. The main lemma, which is also referred to in Section 3, is Lemma 1. We denote the set $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In the proof of the following lemma we use the result that $M \cap S$ is dense in M whenever M is a closed subspace of H with finite $\dim M^{\perp_H}$ ([5, Theorem 4.1.2], [9, Lemma 4.2.3]).

Lemma 1. *Let $(z_n)_{n\in\mathbb{N}_0}$ be a sequence of vectors in H such that $\|z_0\| = 1$ and $z_0 \perp z_1$. There is a double sequence $(y_{mn})_{m,n\in\mathbb{N}_0,\, n\leqslant m}$ in S and a sequence $(y_n)_{n\in\mathbb{N}_0}$ in H such that $y_0 = z_0$ and $y_1 = z_1$, with the following properties*:

(i) $\|y_{mn} - y_n\| \leqslant 1/2^m$ *for* $m, n \in \mathbb{N}_0$ *with* $n \leqslant m$;
(ii) $\langle y_{mn}, y_{pq}\rangle = \langle y_n, y_q\rangle = 0$ *for* $m, n, p, q \in \mathbb{N}_0$ *with* $q < n$, $q \leqslant p$, $n \leqslant m$;
(iii) $y_n - z_n \in \operatorname{span} Y_n$ *where*

$$Y_n = \{y_k \mid 0 \leqslant k < n\} \cup \{y_{ij} \mid 0 \leqslant j \leqslant i < n\}.$$

Proof. By induction on $n \in \mathbb{N}_0$ we define vectors $y_n, y_{n0}, \ldots, y_{nn}$ such that condition (i)–(iii) are satisfied by the vectors of Y_{n+1}. Set $y_0 = z_0$ and $y_{00} = 0$. In the nth induction step we proceed as follows. Define y_n as the orthoprojection of z_n in $Y_n^{\perp_H}$. Condition (iii) is satisfied since $y_n \in z_n + \operatorname{span} Y_n$.

Let us further observe that for $n = 1$ we have $Y_1 = \{0, z_0\}$ and therefore $y_1 = z_1$.

We now construct inductively the vectors y_{nl} $(l = 0, \ldots, n)$ such that $y_{nl} \perp A_l$, where A_l is the set

$$\{y_j \mid 0 \leqslant j \leqslant n,\ \ j \neq l\} \cup \{y_{ij} \mid 0 \leqslant j \leqslant i < n,\ \ j \neq l\} \cup \{y_{nj} \mid 0 \leqslant j < l\}.$$

Suppose that $y_{nl'}$ $(l' < l)$ are constructed. Since $y_l \perp A_l$, by the above stated result we can find $y_{nl} \in S$ such that $y_{nl} \perp A_l$ and $\|y_{nl} - y_l\| \leqslant 1/2^n$. It is clear then that conditions (i) and (ii) are satisfied by the vectors of Y_n. □

Lemma 2. *With the assumption and notations of Lemma* 1, *let $M := \{y_{m0} \mid m \in \mathbb{N}_0\}^{\perp_S\perp_S}$.*

(i) *If $z_0, z_1 \in H \setminus S$ and $\operatorname{span}\{z_0, z_1\} \cap S \neq \{0\}$, then $M \notin E(S)$.*
(ii) *If $u \in H$, $u \perp M$ and $u \perp M^{\perp_S}$, then $u \perp z_n$ for all $n \in \mathbb{N}_0$.*

Proof. (i) Suppose that $z_0, z_1 \notin S$, $\operatorname{span}\{z_0, z_1\} \cap S \neq \{0\}$ and $M \in E(S)$ for contradiction. Then there is a non-zero vector $h \in S$ such that

$$h = \alpha z_0 + \beta z_1 = h_1 + h_2,$$

where $h_1 \in M$ and $h_2 \in M^{\perp_S}$. Then $h_1 = \alpha z_0$ and $h_2 = \beta z_1$ because $z_0 \in \overline{M}$ and $z_1 \in \overline{M^{\perp_S}}$. This contradicts the assumption that z_0 and z_1 are not elements of S.

(ii) Given any $m \in \mathbb{N}_0$, observe that $y_{m0} \in M$ and $y_{mn} \in M^{\perp_S}$ $(n \neq 0)$, i.e. $u \perp y_{mn}$ and $u \perp y_n$ for all $0 \leqslant n \leqslant m$. In view of property (iii) of Lemma 1, $u \perp z_n$ for all $n \in \mathbb{N}_0$. □

Theorem 3. *A separable pre-Hilbert space S is complete if and only if $E_q(S) = E(S)$.*

Proof. We only need to prove sufficiency since the necessity is obvious. Let $(z_n)_{n\in\mathbb{N}}$ be total sequence in H. If we assume that S is not complete, we can choose z_1 not to be an element of S.

Let $h \in S$ such that $\langle h, z_1\rangle \neq 0$. Then $z_0' := \frac{z_1}{\|z_1\|^2} - \frac{h}{\langle h,z_1\rangle}$ is not in S since $z_1 \in H \setminus S$ and $h \in S$. Let $z_0 := \frac{z_0'}{\|z_0'\|}$. Observe that $z_0 \perp z_1$. Let y_{mn} and y_n be chosen according to Lemma 1. We show that $M := \{y_{m0} \mid m \in \mathbb{N}_0\}^{\perp_S\perp_S} \in E_q(S) \setminus E(S)$. By part (i) of Lemma 2, M is not splitting. If M is not quasi-splitting, then there exists a non-zero vector $u \in H$ such that $u \perp M$ and $u \perp M^{\perp_S}$. By part (ii) of the same lemma, it follows that $u \perp z_n$ for all $n \in \mathbb{N}_0$. This is a contradiction since $(z_n)_{n\in\mathbb{N}}$ is total in H. □

Proposition 4. *Let M, N be two subspaces of S such that $M \subseteq N$.*

(i) *$M \in E(S)$ implies $M \in E(N)$.*
(ii) *Let $N \in E(S)$. Then $M \in E(N)$ if and only if $M \in E(S)$.*
(iii) *Let $N \in E_q(S)$. Then $M \in E_q(N)$ implies $M \in E_q(S)$.*
(iv) *Let $N \in E(S)$. Then $M \in E_q(N)$ if and only if $M \in E_q(S)$.*

Proof. We leave the proofs of (i) and (ii) to the reader.

(iii) Let $u \in H$, such that $u \perp M$ and $u \perp M^{\perp_S}$. Since $M^{\perp_S} \cap N$, $N^{\perp_S} \subseteq M^{\perp_S}$, we have that $u \perp M^{\perp_S} \cap N$ and $u \perp N^{\perp_S}$. However, since $N \in E_q(S)$, $u \perp N^{\perp_S}$ implies that $u \in \overline{N}$. On the other-hand, $M \in E_q(N)$ and therefore $u \perp M$ and $u \perp M^{\perp_S} \cap N$ implies that $u = 0$, i.e. $M \in E_q(S)$.

(iv) We need to show only one direction since $E(S) \subseteq E_q(S)$ and therefore we can use (iii) to deduce that $M \in E_q(N)$ implies $M \in E_q(S)$. Let $u \in \overline{N}$ such that $u \perp M$. We show that $u \in \overline{M^{\perp_S} \cap N}$. Fix $\epsilon > 0$. Since $M \in E_q(S)$, there exists $v \in M^{\perp_S}$ such that $\|u - v\| \leqslant \epsilon$. Since $N \in E(S)$, $v = v_1 + v_2$, where $v_1 \in N$ and $v_2 \in N^{\perp_S}$. Observe that $v_1 = v - v_2 \in N \cap M^{\perp_S}$ and $u - v_1 \perp v_2$. Therefore

$$\|u - v_1\| \leqslant \big\|(u - v_1) - v_2\big\| = \|u - v\| \leqslant \epsilon$$

and this completes the proof. □

Corollary 5. *If S has an incomplete separable quasi-splitting subspace, then $E_q(S) \neq E(S)$.*

Proof. Let $M \in E_q(S)$ be incomplete and separable. If $E_q(S) = E(S)$, then by (iii) of Proposition 4 we get that $E_q(M) \subseteq E_q(S) = E(S)$. It follows from Proposition 4(i) that $E_q(M) = E(M)$ and therefore, in view of Theorem 3, that M is complete, a contradiction. □

Theorem 6. *If S has an ONB, then $E(S) = E_q(S)$ if and only if S is complete.*

Proof. Let A be an ONB of S. In what follows we use the fact that for any $B \subseteq A$ the space $\overline{\operatorname{span} B} \cap S$ is quasi-splitting. Suppose that S is not complete and let $x \in H \setminus S$. There exists a countable subset A_0 of A such that $x \in \overline{\operatorname{span} A_0}$. Since $\overline{\operatorname{span} A_0} \cap S$ is an incomplete separable quasi-splitting subspace of S, we have $E_q(S) \neq E(S)$ by Corollary 5. The converse is trivial. □

In Section 2, it is proved that if every algebraic complement of S in H is separable—particularly if $\mathrm{d}(H/S) \leqslant \aleph_0$—then S has an ONB. With this in mind, one deduces the following result which extends [4, Theorem 2.11] (where the result is proved for finite $\mathrm{d}(H/S)$).

Corollary 7. *If every algebraic complement of S in H is separable, then $E(S) \neq E_q(S)$. In particular, if $0 < \mathrm{d}(H/S) \leqslant \aleph_0$, then $E(S) \neq E_q(S)$.*

3. Orthonormal bases in pre-Hilbert spaces

The starting point of this section is the known fact that there are pre-Hilbert spaces having their orthogonal dimension not equal to that of the completion, i.e. $\dim S \neq \dim H$. Such pre-Hilbert spaces admit no ONB, i.e. no MONS of S is a MONS of H. Conditions forcing a pre-Hilbert space S to have an ONB are studied in this section.

Theorem 8. *Let M be a subspace of H containing S and A a MONS in S such that $\dim A^{\perp_M} \leqslant \aleph_0$. Then S contains an ONS that is maximal in M.*

Proof. Let $(z_n)_{n\in\mathbb{N}_0}$ be a MONS in $A^{\perp_M}$. Observe that $A\cup\{z_n \mid n\in\mathbb{N}_0\}$ is a MONS in M. We can use Lemma 1 to obtain a double sequence $(y_{mn})_{m,n\in\mathbb{N}_0,n\leqslant m}$ in S and a sequence $(y_n)_{n\in\mathbb{N}_0}$ in H with properties (i)–(iii) of same lemma. The set

$$A_0 := \{a\in A \mid \exists m,n\in\mathbb{N}_0,\ n\leqslant m,\ \text{with } y_{mn}\not\perp a\}$$

is countable. Let B be an ONB of

$$\operatorname{span}\bigl(\{y_{mn} \mid m,n\in\mathbb{N}_0,\ n\leqslant m\}\cup A_0\bigr).$$

Then $C := (A\setminus A_0)\cup B$ is an ONS of S such that $A\subseteq\overline{\operatorname{span} C}$ and $z_n\in\overline{\operatorname{span} C}$ for all $n\in\mathbb{N}_0$, i.e. C is maximal in M. □

Corollary 9. *If every algebraic complement of S in H is separable, then S has an ONB. In particular, if $\mathrm{d}(H/S)\leqslant\aleph_0$, then S has an ONB.*

Proof. Let A be a MONS in S. Since $A^{\perp_H}$ is contained in an algebraic complement of S, $A^{\perp_H}$ is separable and therefore $\dim A^{\perp_M}\leqslant\aleph_0$. We can now apply Theorem 8 and deduce that S contains an ONS that is maximal in H; i.e. an ONB of H. □

Example 12 below shows that in Corollary 9 the assumption that *every* algebraic complement of S in H is separable cannot be weakened to the assumption that S has *one* algebraic complement in H which is separable.

In what follows, for a closed subspace M of a Hilbert space H, we denote by P_M the orthogonal projection of H onto M.

Lemma 10. *Let U be a subspace of a Hilbert space V and A, B subsets of U such that $U=\operatorname{span}(A\cup B)$. Let further $V_1=\overline{\operatorname{span} A}$ and $V_2=V_1^{\perp_V}$. Then the following two conditions are equivalent:*

(1) *A is an ONS, $P_{V_2}B$ is total in V_2, P_{V_1} is one-to-one when restricted to B and $A\cup P_{V_1}B$ is a set of linearly independent vectors;*
(2) *A is a MONS of U, $\overline{U}=V$ and $A\cup B$ is a set of linearly independent vectors.*

Proof. (1) ⇒ (2). To show that U is dense in V one only has to note that since $V_1\subseteq\overline{U}$ and $B\subseteq U$, it follows that $P_{V_2}B\subseteq\overline{U}$, which in turn implies that $V_2\subseteq\overline{U}$ since $P_{V_2}B$ is total in V_2.

We now show that A is a MONS in U. Assume that $u\in U$ with $u\perp A$. Say

$$u=\sum_{i=1}^{n}\lambda_i a_i+\sum_{j=1}^{m}\mu_j b_j,$$

where $a_i\in A$, $b_j\in B$ and λ_i,μ_j are scalars. Then $P_{V_1}u=0$ since $u\perp A$, so that

$$\sum_{j=1}^{m}\mu_j P_{V_1}(b_j)\in\operatorname{span}\{a_i \mid 1\leqslant i\leqslant n\}.$$

Since P_{V_1} is one-to-one when restricted to B and $A\cup P_{V_1}B$ consists of linearly independent vectors, it follows that $\mu_j=0$ for all $1\leqslant j\leqslant m$, i.e. $u=\sum_{i=1}^{n}\lambda_i a_i$. Consequently, $u=0$ because by assumption $u\perp A$.

(2) ⇒ (1). To see that $P_{V_2}B$ is total in V_2, one should observe that

$$V_1\oplus V_2=V=\overline{U}\subseteq\overline{V_1\oplus\operatorname{span}(P_{V_2}B)}=V_1\oplus\overline{\operatorname{span}(P_{V_2}B)},$$

hence $V_2=\overline{\operatorname{span}(P_{V_2}B)}$. We now show that P_{V_1} is one-to-one when restricted to B, and that the set $A\cup P_{V_1}B$ consists of linearly independent vectors. Consider a linear combination $\sum_{i=1}^{n}\lambda_i a_i+\sum_{j=1}^{m}\mu_j P_{V_1}(b_j)=0$. Then

$$\sum_{i=1}^{n}\lambda_i a_i+\sum_{j=1}^{m}\mu_j b_j=\sum_{i=1}^{n}\lambda_i a_i+\sum_{j=1}^{m}\mu_j P_{V_1}(b_j)+\sum_{j=1}^{m}\mu_j P_{V_2}(b_j)=P_{V_2}\left(\sum_{j=1}^{m}\mu_j b_j\right)\in U\cap V_2=\{0\},$$

where the last equality follows from the fact that A is a MONS of U. Consequently, $\alpha_i=\mu_j=0$ for every $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$ since $A\cup B$ consists of linearly independent vectors. □

Remark 11. Observe that if any (and hence both) of the conditions of Lemma 10 are satisfied, then the following statements hold:

(i) $V_2\cap U=\{0\}$;
(ii) If $V_1=\operatorname{span}(A\cup P_{V_1}B)$, then $V=U+V_2$, and therefore V_2 is an algebraic complement of U;
(iii) If $V_2=\operatorname{span}(P_{V_2}B)$, then $V=U+V_1$.

Example 12. Given two cardinals κ,τ satisfying $\aleph_0\leqslant\kappa<\tau\leqslant\kappa^{\aleph_0}$, let V_1, V_2 be two Hilbert spaces with $\dim V_1=\kappa$, $\dim V_2=\tau$ and let $V=V_1\oplus V_2$. Then V contains a dense subspace U satisfying:

(i) $\dim U = \kappa$ and $\dim \overline{U} = \tau$;
(ii) V_2 is a non-separable algebraic complement of U;
(iii) V_1 contains an algebraic complement of U. In particular, if $\kappa = \aleph_0$, then U has a separable algebraic complement.

Proof. First observe that

$$\kappa^{\aleph_0} \leqslant \tau^{\aleph_0} \leqslant \left(\kappa^{\aleph_0}\right)^{\aleph_0} = \kappa^{\aleph_0},$$

hence $\kappa^{\aleph_0} = \tau^{\aleph_0}$. Therefore $\mathrm{d}(V_1) = \kappa^{\aleph_0} = \tau^{\aleph_0} = \mathrm{d}(V_2)$.

Let A be an ONB of V_1 and C be a Hamel basis of V_1 containing A. Let further D be a Hamel basis of V_2. Since $|D| = |C \setminus A|$, there is a bijection g from D onto $C \setminus A$. Define

$$B := \left\{ g(d) + d \;\middle|\; d \in D \right\}.$$

From Lemma 10 ((1) $\Rightarrow$ (2)) and Remark 11 it is clear that the subspace $U := \operatorname{span}(A \cup B)$ has the desired properties. □

Let us note that in the above example we have a pre-Hilbert space whose dimension is strictly less than the dimension of its completion and therefore cannot contain an ONB. We now modify the above example to construct a pre-Hilbert space that has no ONB, but its dimension agrees with the dimension of its completion.

Example 13. Let V and U be as in Example 12 and $V_0 := V \oplus H_0$ be the direct sum of V and a Hilbert space H_0 with $\dim H_0 = \tau$. Then $U_0 := U \oplus H_0$ is a dense subspace of V_0 not containing an ONB, and $\dim U_0 = \dim V_0$.

Proof. Evidently $\dim U_0 = \dim V_0 = \tau$. We show that U_0 does not have an ONB. First observe that if E is a total subset of U_0, then $P_V E \subseteq U$ and $P_V E$ is total in V. This means that $|P_V E| \geqslant \dim V = \tau$.

On the other hand, if E is an ONS contained in U_0, then $\dim U \geqslant |P_V E|$. To see this, observe that if A is a MONS of U, then for any $a \in A$, the set $E_a := \{e \in E \mid e \not\perp a\}$ is countable. Moreover, since A is a MONS in U,

$$P_V\left(\bigcup_{a \in A} E_a\right) = P_V E,$$

and therefore,

$$|P_V E| = \left|P_V\left(\bigcup_{a \in A} E_a\right)\right| \leqslant \left|\bigcup_{a \in A} E_a\right| \leqslant |A| \cdot \aleph_0 = \dim U = \kappa. \qquad \square$$

How much bigger than $\dim S$ can $\dim H$ be? Upper bounds for $\dim H$ in terms of $\dim S$ and $\mathrm{d}(S)$ are given in the next theorem.

Theorem 14. *Let S be a pre-Hilbert space and H be its completion. Then*

$$\dim H \leqslant \mathrm{d}(S) \leqslant (\dim S)^{\aleph_0}.$$

Proof. We may assume that H has infinite dimension. Let A be a MONS in S and B a subset of S such that $A \cup B$ is a Hamel basis of S. Define $H_1 := \overline{\operatorname{span} A}$ and let C be an ONB of $H_2 := H_1^{\perp_H}$. We first show that

$$|C| \leqslant \begin{cases} \mathrm{d}(S), \\ (\dim S)^{\aleph_0}. \end{cases}$$

For any $b \in B$, the set

$$C_b := \{c \in C \mid c \not\perp P_{H_2} b\}$$

is countable. By Lemma 10 we know also that $P_{H_2} B$ is total in H_2. Hence, for every $c \in C$ there exists $b \in B$ such that $c \not\perp P_{H_2} b$, i.e. $C = \bigcup_{b \in B} C_b$. Hence

$$|C| \leqslant |B| \cdot \aleph_0 \leqslant \mathrm{d}(S) \cdot \aleph_0 = \mathrm{d}(S).$$

For the second inequality, observe that by Lemma 10, P_{H_1} is one-to-one when restricted to B and $P_{H_1} B$ consists of linearly independent vectors. Hence

$$|C| \leqslant |B| \cdot \aleph_0 = |P_{H_1} B| \cdot \aleph_0 \leqslant \mathrm{d}(H_1) = (\dim H_1)^{\aleph_0} = (\dim S)^{\aleph_0}.$$

Consequently,

$$\dim H \leqslant \dim S + |C| \leqslant \begin{cases} \dim S + \mathrm{d}(S) = \mathrm{d}(S), \\ \dim S + (\dim S)^{\aleph_0} = (\dim S)^{\aleph_0}, \end{cases}$$

and therefore,

$$\mathrm{d}(S) \leqslant \mathrm{d}(H) = (\dim H)^{\aleph_0} \leqslant \left((\dim S)^{\aleph_0}\right)^{\aleph_0} = (\dim S)^{\aleph_0}. \qquad \square$$

Remark 15. In view of Theorem 14 it is natural to ask whether there exists a Hilbert space V having a dense subspace U with $\dim V = \tau$, $\dim U = \aleph$ and $\mathrm{d}(U) = \lambda$, where $\aleph, \tau, \lambda$ are cardinal numbers satisfying $\aleph_0 \leqslant \aleph \leqslant \tau \leqslant \lambda \leqslant \aleph^{\aleph_0}$. An easy modification of Example 12 shows that the answer is in the affirmative. Indeed, choose V_1, V_2, V, A, C as in the proof of Example 12 and let D be a total, linearly independent subset of V_2 with cardinality λ. Since $|D| = \lambda \leqslant \aleph^{\aleph_0} = |C \setminus A|$, there is an injection g from D into $C \setminus A$. Define $B := \{g(d) + d \mid d \in D\}$ and $U := \mathrm{span}(A \cup B)$. Then U and V have the desired properties.

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