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# SUPERPARACOMPACT TYPE PROPERTIES

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**Abstract.** In this paper we continue the study of superparacompact and weakly superparacompact spaces. Several new characterizations of superparacompact spaces are given. We also define two new covering properties which we show to be different from the above properties. The question of invariance and inverse invariance under various maps of these four covering properties is analysed. Finally we give a Tamano type theorem with respect to CO-normality.

## 1. Preliminaries

In this paper by a space we mean a $T_1$ topological space and by a map, a continuous map of spaces.

Let $\mathcal{P}=\{P_{\alpha}:\alpha\in \mathcal{A}\}$ be a collection of subsets of a set $X$. By a *chain* from $P_{\alpha}$ to $P_{\alpha'}$ we mean a finite sequence $P_{\alpha(1)}, P_{\alpha(2)}, \ldots, P_{\alpha(k)}$ of elements of $\mathcal{P}$ such that $\alpha(1)=\alpha$, $\alpha(k)=\alpha'$ and $P_{\alpha(i)}\cap P_{\alpha(i+1)}\neq\emptyset$ for $i=1, \ldots, k-1$. The collection $\mathcal{P}$ is said to be *connected* if for every pair $P_{\alpha}, P_{\alpha'}$ of elements of $\mathcal{P}$ there exists a chain from $P_{\alpha}$ to $P_{\alpha'}$. For every collection $\mathcal{P}$ the *components* of $\mathcal{P}$ are defined as maximal connected subcollections of $\mathcal{P}$, that is connected subcollections of $\mathcal{P}$ which are not proper subsets of any connected subcollection of $\mathcal{P}$.

Remember that a collection $\mathcal{P}$ of subsets of a set $X$ is said to be star-finite (star-countable) if for every $P\in \mathcal{P}$ the collection $\{Q\in \mathcal{P}:Q\cap P\neq\emptyset\}$ is finite (countable).

The following lemma will be used below (see for example [2] or [3]).

**Lemma 1.1.** 1. *Every collection* $\mathcal{P}$ *of subsets of a set* $X$ *decomposes into the union of its components.*
2. *If* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are distinct components of* $\mathcal{P}$, *then* $(\cup\mathcal{P}_1)\bigcap(\cup\mathcal{P}_2)=\emptyset$.
3. *If* $\mathcal{P}$ *is star-countable, then each component is a countable subcollection of* $\mathcal{P}$.

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The following definition is due to B.A. Pasynkov [8].

**Definition 1.1.** A star-finite open cover of the space $X$ is said to be a *finite component cover* if all of its components are finite.

For a collection $\mathcal{P}$ of subsets of a set $X$ and an infinite ordinal number $\tau$ let $\mathcal{P}^\tau = \{\cup\mathcal{Q} : \mathcal{Q} \subset \mathcal{P}, |\mathcal{Q}| < \tau\}$. The collection $\mathcal{P}^\omega = \mathcal{P}^{\omega_0}$ is usually denoted by $\mathcal{P}^F$. Thus $\mathcal{P}^F$ is the collection of all unions of finite subcollections from $\mathcal{P}$.

## 2. Characterizations of Superparacompactness

We begin by the definition of superparacompactness, which is due to B.A. Pasynkov [8].

**Definition 2.1.** A space $X$ is said to be *superparacompact* if every open cover of the space $X$ has an open finite component refinement.

The following result is known [10].

**Proposition 2.1.** *A Tychonoff space $X$ is superparacompact if and only if for every compact set $B \subset \beta X \setminus X$ there exists an open finite component cover $\mathcal{W}$ of the space $X$ such that $B \cap (\bigcup[\mathcal{W}]_{\beta X}) = \emptyset$.*

In the above, by $[\mathcal{W}]_{\beta X}$ we mean $\{[W]_{\beta X} : W \in \mathcal{W}\}$, where $[W]_{\beta X}$ is the closure of $W$ in the Čech-Stone compactification $\beta X$. In [1] it was shown that in the hypothesis of Proposition 2.1 any compactification $bX$ will do instead of $\beta X$.

We begin by giving some new characterizations of superparacompactness.

**Theorem 2.2.** *For every space $X$ the following conditions are equivalent:*

1. *The space $X$ is superparacompact;*
2. *For every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a disjont open refinement (i.e. an open refinement of order 0);*
3. *For every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a $\sigma$-discrete clopen refinement;*
4. *For every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a star-finite clopen refinement;*
5. *For every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a locally finite clopen refinement;*
6. *For every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a $\sigma$-locally finite clopen refinement;*

7. *For every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a closure preserving clopen refinement;*
8. *For every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a $\sigma$-closure preserving clopen refinement.*

**Proof.** We start with (1) $\Longrightarrow$ (2). Let the space $X$ be superparacompact and let $\mathcal{U}$ be an open cover of $X$. Then by definition, the cover $\mathcal{U}$ has a finite component open refinement $\mathcal{V}$. It is not difficult to see that (2) follows after applying Lemma 1.1 to the cover $\mathcal{V}$.

The implications (2) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (5), (5) $\Longrightarrow$ (7) and (7) $\Longrightarrow$ (8) are evident.

We now prove the implications (8) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (2). Let the space $X$ satisfy condition (8) and let $\mathcal{U}$ be an open cover of $X$. Then the cover $\mathcal{U}^F$ has a $\sigma$-closure preserving clopen refinement $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$, where each $\mathcal{F}_n$ is a closure preserving clopen collection. Let $\mathcal{F}_n = \{F_\alpha : \alpha < \gamma_n\}$ be a well-ordering of $\mathcal{F}_n$ for every $n \in \boldsymbol{N}$. For every $F_\alpha \in \mathcal{F}_n$ let $W_\alpha = F_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta$ and for every $n \in \boldsymbol{N}$ let $\mathcal{W} = \{W_\alpha : F_\alpha \in \mathcal{F}_n\}$. It is not difficult to see that each $\mathcal{W}_n$ is a discrete clopen collection and so $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$ is a $\sigma$-discrete clopen refinement of $\mathcal{U}^F$. Thus condition (3) is satisfied. Let $P_n = \bigcup \mathcal{W}_n = \bigcup\{W_\alpha : W_\alpha \in \mathcal{W}_n\}$ for every $n \in \boldsymbol{N}$. Then $P_n$ is a clopen set and therefore $Q_n = \bigcup_{k=1}^{n} P_k$ is also clopen for every $n \in \boldsymbol{N}$. Let $\mathcal{Q}_1 = \mathcal{W}_1$ and $\mathcal{Q}_{n+1} = \{W \setminus Q_n : W \in \mathcal{W}_{n+1}\}$ for every $n \in \boldsymbol{N}$. The collection $\mathcal{Q} = \bigcup_{n=1}^{\infty} \mathcal{Q}_n$ is a disjoint open refinement of $\mathcal{U}^F$ and thus (2) is satisfied.

Next we show the implication (2) $\Longrightarrow$ (1). Let the space $X$ satisfy condition (2) and let $\mathcal{U}$ be an open cover of $X$. Then the cover $\mathcal{U}^F$ has a disjoint open refinement $\mathcal{V}$. For every $V \in \mathcal{V}$ choose an element $G(V) \in \mathcal{U}^F$ such that $V \subset G(V)$. By definition one can choose a finite collection $\mathcal{U}(V) = \{U_n : U_n \in \mathcal{U}, n = 1, \dots, k_V\}$ such that $G(V) = \bigcup \mathcal{U}(V)$. Let $\mathcal{W}(V) = V \wedge \mathcal{U}(V) = \{V \cap U_n : U_n \in \mathcal{U}(V)\}$. The cover $\mathcal{W} = \bigcup\{\mathcal{W}(V) : V \in \mathcal{V}\}$ is easily seen to be a finite component open refinement of the cover $\mathcal{U}$.

Finally, the implications (5) $\Longrightarrow$ (6) $\Longrightarrow$ (8) are evident and so the theorem is proved. □

**Remark 2.1.** The equivalence of (1) and (2) in Theorem 2.2 was obtained in [1].

**Remark 2.2.** In Theorem 2.2, the phrase "*For every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a*" in items (2) to (8) can be changed to "*Every directed open cover $\mathcal{U}$ of the space $X$ has a*" and thus obtain seven more characterizations.

We also have the following characterizations with respect to well-monotone open covers, where a cover $\mathcal{U}$ is said to be *well-monotone* if the subset relation $\subset$ is a well-order on $\mathcal{U}$.

**Theorem 2.3.** *For every space $X$ the following conditions are equivalent:*

1. *The space $X$ is superparacompact;*
2. *Every well-monotone open cover of $X$ has a finite component open refinement;*
3. *Every well-monotone open cover of $X$ has a (precise) open disjoint refinement;*
4. *Every well-monotone open cover of $X$ has a closure preserving clopen refinement.*

**Proof.** The implications (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) are evident while the implication (2) $\Longrightarrow$ (3) follows from the definition of a well-monotone cover and the implication (4) $\Longrightarrow$ (3) follows from the fact that a closure preserving clopen collection can be modified to a clopen disjoint collection. We are thus left with the implication (3) $\Longrightarrow$ (1).

If the space $X$ is not superparacompact, then there is a smallest cardinal number $\mu$ such that there exists an open cover $\mathcal{U}$ of $X$ with $\mathcal{U}^F$ having no open disjoint refinement and $|\mathcal{U}| = \mu$. By this choice of $\mathcal{U}$ we know that whenever $\mathcal{W}$ is an open cover of $X$ with $|\mathcal{W}| < |\mathcal{U}|$, then $\mathcal{W}^F$ has an open disjoint refinement. Let $\mathcal{U} = \{U_\alpha, \alpha < \mu\}$ and for every $\alpha < \mu$ let $V_\alpha = \bigcup_{\beta \leq \alpha} U_\beta$. The collection $\mathcal{V} = \{V_\alpha : \alpha < \mu\}$ is a well-monotone open cover of $X$ and so, by (3), it has a precise open disjoint refinement $\mathcal{W} = \{W_\alpha : \alpha < \mu\}$. One can assume that if $W_\alpha \neq \emptyset$ then $W_\alpha \neq W_\beta$ whenever $\alpha \neq \beta$, $\alpha, \beta < \mu$.

For each $\alpha < \mu$ let $F_\alpha = X \setminus \bigcup\{W_\gamma : \gamma > \alpha\}$. Then the collection $\mathcal{F}_\alpha = \{\{X \setminus F_\alpha\} \cup \{U_\beta : \beta \leq \alpha\}\}$ is an open cover of $X$ of cardinality less than $\mu$ and so $\mathcal{F}^F$ has an open disjoint refinement $\mathcal{A}_\alpha$. Let $\mathcal{B}_\alpha = \{A \cap F_\alpha : A \in \mathcal{A}_\alpha, A \cap F_\alpha \neq \emptyset\}$. Then the collection $\mathcal{B}_\alpha$ is a clopen disjoint partial refinement of $\{U_\beta : \beta \leq \alpha\}^F$.

Finally, let $\mathcal{P}_\alpha = \{W_\alpha \cap B : B \in \mathcal{B}_\alpha\}$. This collection is a disjoint clopen collection and thus $\mathcal{H} = \bigcup_{\alpha < \mu} \mathcal{P}_\alpha$ is a clopen disjoint collection which partially refines $\mathcal{U}^F$. We now show that in fact $\mathcal{H}$ is a cover of $X$. Let $x$ be an arbitrary point of $X$. Then there exists a unique $\alpha < \mu$ such that $x \in W_\alpha$ and so $x \in X \setminus \bigcup\{W_\gamma : \gamma > \alpha\}$. There exists a $B \in \mathcal{B}_\alpha$ such that $x \in B$ which implies that $x \in W_\alpha \cap B \in \mathcal{P}_\alpha$. Therefore, the collection $\mathcal{H}$ is a refinement of $\mathcal{U}^F$ which contradicts our main assumption. □

**Remark 2.3.** Conditions analogous to those given in Theorem 2.2 and Theorem 2.3 were used in [5] and [7] to give characterizations of metacompact and paracompact spaces.

We end this section by giving another characterization for Tychonoff superparacompact spaces in terms of the Stone-Čech compactification $\beta X$.

**Theorem 2.4.** *A Tychonoff space $X$ is superparacompact if and only if, for any compact set $F \subset \beta X \setminus X$, the sets $X \times F$ and $\Delta = \{(x,x) : x \in X\} \subset X \times \beta X$ have disjoint clopen neighbourhoods in $X \times \beta X$.*

**Proof.** Let $X$ be a Tychonoff superparacompact space and let $F \subset \beta X \setminus X$ be a compact set. There exists a disjoint open cover $\mathcal{U}$ of $X$ such that $F \cap (\bigcup[\mathcal{U}]_{\beta X}) = \emptyset$. Since for every $U \in \mathcal{U}$ we have that $[U]_{\beta X}$ is clopen in $\beta X$ we get that $\bigcup\{U \times [U]_{\beta X} : U \in \mathcal{U}\}$ is a clopen neighbourhood of $\Delta$ in $X \times \beta X$.

Conversely, let $X$ be a Tychonoff space and let $F \subset \beta X \setminus X$ be a compact set. Then the sets $X \times F$ and $\Delta \subset X \times \beta X$ have disjoint clopen neighbourhoods $U$ and $V$ respectively, in $X \times \beta X$. One can assume that $V = (X \times \beta X) \setminus U$. Consider the function $f : X \times \beta X \to I$ equal to 0 on $V$ and to 1 on $U$. We define a pseudometric $\rho$ on $X$ with

$$\rho(x,y) = \sup_{z \in \beta X} |f(x,z) - f(y,z)|.$$

One can prove that the topology $\Omega_\rho$ induced by the pseudometric $\rho$ is weaker than the original topology $\Omega$, that is $\Omega_\rho \subset \Omega$ ([3] 5.1.38). It is not difficult to see that with this choice of $f$ we have that the cover $\{B(x, \frac{1}{2}) : x \in X\}$ of $X$ is an open disjoint cover, where by $B(x, \frac{1}{2})$ we mean the neighbourhood ball with respect to $\rho$, center at $x$ and radius $\frac{1}{2}$. Finally, for every $x \in X$ and $y \in B(x, \frac{1}{2})$ we have that $f(x,y) = |f(x,y) - f(y,y)| \leq \rho(x,y) < \frac{1}{2}$. Thus if $y \in [B(x, \frac{1}{2})]_{\beta X}$ then $f(x,y) \leq \frac{1}{2}$ which shows that $[B(x, \frac{1}{2})]_{\beta X} \cap F = \emptyset$ for every $x \in X$, since $f(x,z) = 1$ for $z \in F$. Consequently $X$ is a superparacompact space. □

## 3. Other Superparacompact type properties

**Definition 3.1.** A space $X$ is said to be *supermetacompact* if for every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a point-finite clopen refinement.

The following propositions are not difficult to prove.

**Proposition 3.1.** *For a space $X$ the following are equivalent:*

1. *The space $X$ is supermetacompact;*
2. *Every directed open cover of the space $X$ has a point-finite clopen refinement.*

**Proposition 3.2.** *For a Tychonoff space $X$ the following are equivalent:*

1. *The space $X$ is supermetacompact;*
2. *For every compact set $B \subset \beta X \setminus X$, there exists a point-finite clopen cover $\mathcal{W}$ of the space $X$ such that $B \cap (\bigcup [\mathcal{W}]_{\beta X}) = \emptyset$.*

Thus every superparacompact space is supermetacompact but the converse is not true as will be shown below.

Remember that if $\mathcal{U}$ is a collection of subsets of a space $X$ and $x \in X$ then $\text{ord}(x, \mathcal{U})$ (order of $x$ in $\mathcal{U}$) is the cardinality of $\{U \in \mathcal{U} : x \in U\}$.

**Definition 3.2.** A space $X$ is said to be *supersubmetacompact* if for every open cover $\mathcal{U}$ of the space $X$, there exists a collection of clopen sets $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ such that $\mathcal{G}_n$ refines $\mathcal{U}^F$ for every $n$ and for every $x \in X$ there exists an $n$ with $\text{ord}(x, \mathcal{G}_n) < \omega$.

The following propositions are not difficult to prove.

**Proposition 3.3.** *For a space $X$ the following are equivalent:*

1. *The space $X$ is supersubmetacompact;*
2. *For every directed open cover of the space $X$, there exists a sequence $\{\mathcal{G}_n\}$ of clopen refinements of $\mathcal{U}$ such that for every $x \in X$ there exists an $n$ with $\text{ord}(x, \mathcal{G}_n) < \omega$.*

**Proposition 3.4.** *For a Tychonoff space $X$ the following are equivalent:*

1. *The space $X$ is supersubmetacompact;*
2. *For every compact set $B \subset \beta X \setminus X$, there exists a sequence $\{\mathcal{W}_n\}$ of clopen covers of the space $X$, with $B \cap (\bigcup [\mathcal{W}_n]_{\beta X}) = \emptyset$ for every $n$, such that for every $x \in X$ there exists an $n$ with $\text{ord}(x, \mathcal{W}_n) < \omega$.*

Thus every supermetacompact space is supersubmetacompact but the converse is not true as will be shown below.

Finally we give the definition of weak superparacompactness.

**Definition 3.3.** A space $X$ is said to be *weakly superparacompact* if for every open cover $\mathcal{U}$ of the space $X$, $\mathcal{U}^F$ has a clopen refinement.

**Remark 3.1.** Weakly superparacompact Tychonoff spaces were introduced in [10] by means of condition (2) in Proposition 3.6.

The following propositions are not difficult to prove.

**Proposition 3.5.** *For a space $X$ the following are equivalent:*

1. *The space $X$ is weakly superparacompact;*
2. *Every directed open cover of the space $X$ has a clopen refinement.*

**Proposition 3.6.** *For a Tychonoff space $X$ the following are equivalent:*

1. *The space $X$ is weakly superparacompact;*
2. *For every compact set $B \subset \beta X \setminus X$, there exists a clopen cover $\mathcal{W}$ of the space $X$ such that $B \cap (\bigcup [\mathcal{W}]_{\beta X}) = \emptyset$.*

Thus every supersubmetacompact space is weakly superparacompact but the converse is not true as will be shown below. In [1] it was shown that in the hypothesis of Proposition 3.6 (2) any compactification $bX$ will do instead of $\beta X$.

We now give examples to show that none of the above classes are equivalent.

**Example 3.4.** A supermetacompact space which is not superparacompact.

The following space is given in [2]. Let $X = (\omega_2 \times \omega_2) \setminus \{(0,0)\}$. For $\alpha \in \omega_2 \setminus \{0\}$ we define

$$H_\alpha = \omega_2 \times \{\alpha\} \text{ and } V_\alpha = \{\alpha\} \times \omega_2.$$

The topology on $X$ is as follows: For $\alpha \in \omega_2 \setminus \{0\}$ neighbourhoods of $(0, \alpha)$ must contain $(0, \alpha)$ and all but finitely many points of $H_\alpha$. Neighbourhoods of $(\alpha, 0)$ must contain $(\alpha, 0)$ and all but finitely many points of $V_\alpha$. All other points of $X$ are isolated. Any open cover $\mathcal{U}$ has a clopen refinement $\mathcal{H}$, where $\operatorname{ord}(x, \mathcal{H}) \leq 2$ for every $x \in X$ and so the space $X$ is supermetacompact. In [2] it is shown that $X$ is not subparacompact and so cannot be superparacompact.

**Example 3.5.** A supersubmetacompact space which is not supermetacompact.

The following space is given in [2] and [4]. We need the following basic construction. If $D$ is an infinite set, a collection $\mathcal{C}$ of subsets of $D$ is said to be an *almost disjoint* collection if $|A \cap B| < \omega$ whenever $A, B \in \mathcal{C}$, $A \neq B$. Using Zorn's Lemma, there exists an uncountably infinite collection $\mathcal{A}$ of countably infinite subsets of $D$ such that $\mathcal{A}$ is an almost disjoint collection and maximal with respect to these properties. Let $\psi(D) = \mathcal{A} \cup D$ with the following topology: The points of $D$ are isolated. Basic neighbourhoods of a point $A \in \mathcal{A}$ are sets of the form $\{A\} \cup (A \setminus F)$ where $F$ is a finite subset of $D$. Now let $D = \boldsymbol{N}$. The open cover $\mathcal{U} = \{\{A\} \cup \boldsymbol{N} : A \in \mathcal{A}\}$ does not have a point-finite clopen refinement (and so neither does $\mathcal{U}^F$ since $\psi(\boldsymbol{N})$ is zero dimensional). In fact as noted in [2] this space is not even meta-Lindelöf.

To show that $X = \psi(\boldsymbol{N})$ is supersubmetacompact, let $\mathcal{U}$ be an open cover of

$X$. For every $A \in \mathcal{A}$ fix some clopen neighbourhood $V(A) = \{A\} \cup (A \setminus F(A))$ such that $V(A) \subset U(A)$ for some $U(A) \in \mathcal{U}$. Let $\mathcal{G}_n = \{\{k\} : k \in \boldsymbol{N}\} \cup \{V(A) \setminus \{n\} : A \in \mathcal{A}\}$. Then the clopen collection $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ satisfies the requirements of Definition 3.2.

**Example 3.6.** A weakly superparacompact space which is not supersubmetacompact.

The LOTS $X = [0, \omega_1[$ is weakly superparacompact [1] but not supersubmetacompact as the following proposition shows. □

**Proposition 3.7.** *For a GO-space X the following are equivalent:*

1. *The space X is superparacompact;*
2. *The space X is supermetacompact;*
3. *The space X is supersubmetacompact.*

**Proof.** We only need to show that (3) $\Longrightarrow$ (1). Let $X$ be a supersubmetacompact GO-space. Then $X$ is a submetacompact collectionwise normal space and so is paracompact. Being a paracompact GO-space, $X$ is strongly paracompact. But a strongly paracompact weakly superparacompact space is superparacompact. □

**Remark 3.2.** The gap between superparacompact Tychonoff spaces and weakly superparacompact Tychonoff spaces was shown in [10] with the space $X = S \times S$, where $S$ is the Sorgenfrey line. The space $X$ is zero dimensional and so is weakly superparacompact while it is not paracompact and consequently not superparacompact.

We now define CO-collectionwise normal spaces and show that for such spaces the covering properties of Proposition 3.7 are again equivalent. For this we need the following definition.

**Definition 3.7.** A collection $\mathcal{C}$ of subsets of a space $X$ is called *CO-discrete* if for every $x \in X$ there exists a clopen neighbourhood of $x$ which intersects at most one element of $\mathcal{C}$. A pair of subsets of a space $X$ are called *CO-disjoint* if they are CO-discrete in $X$.

**Definition 3.8.** A space $X$ is called *CO-collectionwise normal* if for every closed CO-discrete collection $\{F_\alpha : \alpha \in \mathcal{A}\}$ there exists a discrete clopen collection $\{O_\alpha : \alpha \in \mathcal{A}\}$ in $X$ such that $F_\alpha \subset O_\alpha$, $\alpha \in \mathcal{A}$.

In a similar way one can define CO-normality.

**Definition 3.9.** A space $X$ is called *CO-normal* if for every pair of CO-disjoint closed subsets of $X$ there exist clopen disjoint neighbourhoods of these subsets.

Evidently, every CO-collectionwise normal space is CO-normal. The metrizable space $X$ defined in [6] is collectionwise normal (and so also normal) but is not CO-normal (and so not CO-collectionwise normal), while the Niemytski plane is CO-collectionwise normal (and so also CO-normal) but is not normal (and so not collecionwise normal).

**Proposition 3.8.** *Every superparacompact space is CO-collectionwise normal.*

**Proof.** Let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be a closed CO-discrete collection in a superparacompact space $X$. Then for every $x \in X$ there exists a clopen neighbourhood $H_x$ which intersects at most one element of $\mathcal{F}$. Let $\mathcal{H} = \{H_x : x \in X\}$. Since $X$ is superparacompact, the cover $\mathcal{H}^F$ has a disjoint clopen refinement and so $\mathcal{H}$, being a clopen cover, has a disjoint clopen refinement $\mathcal{P}$. Let $O_\alpha = X \setminus \bigcup\{P \in \mathcal{P} : P \cap F_\alpha = \emptyset\}$. It is not difficult to see that the collection $\{O_\alpha : \alpha \in \mathcal{A}\}$ has the needed properties. □

Before we prove our main theorem of this section we need the following lemma.

**Lemma 3.9.** *Let $A$ be a closed set in $X$ such that for every $x \in X \setminus A$ there exists a clopen neighbourhood of $x$ disjoint from $A$. Let $\mathcal{G}$ be a clopen collection in $X$ which covers $A$ and is point finite on $A$. Then there exists a $\sigma$-discrete clopen collection in $X$ which covers $A$ and partially refines $\mathcal{G}$.*

**Proof.** For $x \in A$ let $W_x = \bigcap\{G \in \mathcal{G} : x \in G\}$ and for $n \in N$ let $F_n = \{x \in A : \operatorname{ord}(x, \mathcal{G}) \leq n\}$. The collection $\mathcal{Z}_1 = \{W_x \cap F_1 : x \in F_1\}$ is a closed collection. We now show that $\mathcal{Z}_1$ is CO-discrete. If $x \in X \setminus A$ then by the definition of the set $A$, $x$ has a clopen neighbourhood which does not intersect $A$. Now let $x \in A$, then it is either covered or not covered by $\mathcal{Z}_1$. If it is not covered then $\operatorname{ord}(x, \mathcal{G}) > 1$ and so $W_x$ is a clopen neighbourhood with $W_x \cap (\bigcup \mathcal{Z}_1) = \emptyset$. If it is covered by $\mathcal{Z}_1$ then $\operatorname{ord}(x, \mathcal{G}) = 1$ and $W_x$ is again a clopen neighbourhood intersecting only $W_x \cap F_1$, that is $(W_{x'} \cap F_1) \cap W_x = \emptyset$ whenever $W_x \neq W_{x'}$.

Thus there exists a clopen discrete collection $\mathcal{U}_1$ in $X$ covering $\bigcup \mathcal{Z}_1$. One can take each element of $\mathcal{U}_1$ to be contained in some element of $\mathcal{G}$. Continuing by induction, suppose that there exists a clopen cover $\bigcup_{k=1}^{n} \mathcal{U}_k$ of $F_n$ such that $\bigcup_{k=1}^{n} \mathcal{U}_k$ is a partial refinement of $\mathcal{G}$ and each $\mathcal{U}_k$, $1 \leq k \leq n$, is clopen and discrete in $X$. Let $V_n = \bigcup\{U \in \mathcal{U}_k : 1 \leq k \leq n\}$. Then $F_n \subset V_n$ and

$\mathcal{Z}_{n+1} = \{W_x \cap (F_{n+1} \setminus V_n) : x \in F_{n+1} \setminus F_n\}$ is a closed collection which, as above, one can show to be CO-discrete.

Therefore, using CO-collectionwise normality, there exists a clopen discrete collection $\mathcal{U}_{n+1}$ in $X$ covering $F_{n+1} \setminus V_n$ which one can assume to be a partial refinement of $\mathcal{G}$. We have thus defined a discrete clopen collection $\mathcal{U}_k$ for every $k \in \boldsymbol{N}$ such that $\bigcup_{k=1}^{\infty} \mathcal{U}_k$ covers $A$ and partially refines $\mathcal{G}$ as required. □

**Theorem 3.10.** *For a space $X$ the following are equivalent:*

1. *$X$ is superparacompact;*
2. *$X$ is CO-collectionwise normal and supermetacompact;*
3. *$X$ is CO-collectionwise normal and supersubmetacompact.*

**Proof.** We only need to show the implication (3) $\Longrightarrow$ (1).

Say $\mathcal{U}$ is an open cover of $X$. If $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n < \mathcal{U}^F$ is a clopen collection satisfying the definition of supersubmetacompactness, then for every $k \in \boldsymbol{N}$ let $A_{nk} = \{x \in X : \operatorname{ord}(x, \mathcal{G}_n) \leq k\}$. Then $A_{nk}$ is closed in $X$ and for every $x \in X \setminus A_{nk}$ there exists a clopen neighbourhood $U_x$ of $x$ with $U_x \cap A_{nk} = \emptyset$. We also have that $\mathcal{G}_n$ is point finite on $A_{nk}$.

By Lemma 3.9 there exists a $\sigma$-discrete clopen collection $\mathcal{H}_{nk}$ which covers $A_{nk}$ and partially refines $\mathcal{G}_n$. It follows that $\mathcal{H} = \bigcup_{n,k \in \boldsymbol{N}} \mathcal{H}_{nk}$ is a $\sigma$-discrete clopen refinement of $\mathcal{U}^F$. □

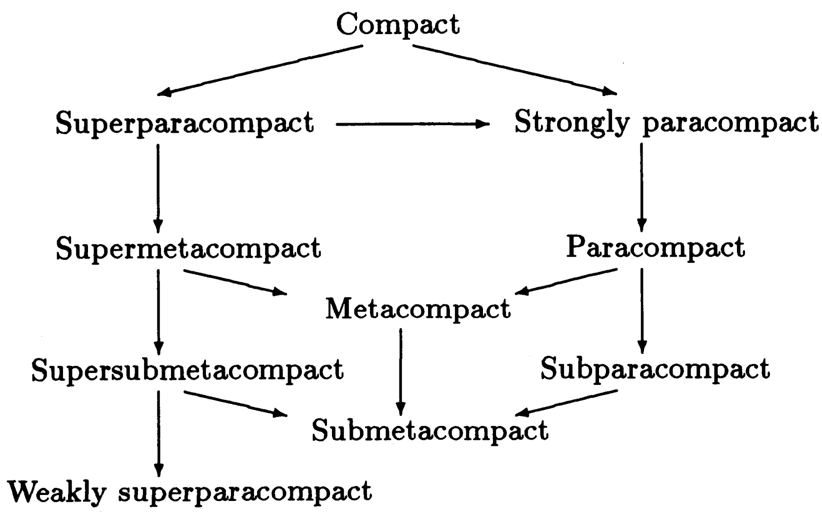


**Diagram 1**: Relations between covering properties

The relations between covering properties is shown in Diagram 1. All the arrows shown in the diagram are not reversible.

One can note that although every strongly paracompact, weakly superparacompact space is superparacompact, the space given in [6] is a completely metrizable zero dimensional space (and thus paracompact and weakly superparacompact) which is not superparacompact.

## 4. Invariance and Inverse Invariance

We now analyse the invariance of the above four covering properties under various maps and begin by showing that the above mentioned four covering properties are not invariant under open quotient maps. The following space is from [3].

**Example 4.1.** Let $X$ be the subspace of the plane $\{(0,0)\} \cup \{(k, \frac{1}{i} + \frac{1}{i \cdot j}) : k = 0, 1,\ i = 1, 2, \ldots,\ j = i, i+1, \ldots\} \cup \{(1, \frac{1}{i}) : i = 1, 2, \ldots\} \subset \boldsymbol{R}^2$ and define on $X$ the following equivalence relation $E : (x_1, y_1) E (x_2, y_2)$ if and only if $y_1 = y_2$. The natural quotient map $q : X \to X/E = Y$ is open. The space $X$ is zero dimensional and so is weakly superparacompact. Furthermore since the space $X$ is separable and metrizable we have that $\dim X = 0$. Since a $T_2$ paracompact space $Z$ with $\dim Z = 0$ is superparacompact [10], we get that our space $X$ is superparacompact. The space $Y$ is not zero dimensional since it is a $T_2$-space but is not regular (and so neither $T_{3\frac{1}{2}}$). We now show that $Y$ is not weakly superparacompact. Consider the open cover $\mathcal{U} = \{U_i : i = 1, 2, \ldots\}$ of $Y$, where $U_0 = q([(0,0), (0, \frac{1}{2})[)$, $U_1 = q(](1, \frac{1}{2}), (1, 2)])$ and $U_i = q(](1, \frac{1}{i+1}), (1, \frac{1}{i-1})[)$ for $i = 2, 3, \ldots$. By the intervals $[\cdot, \cdot], ]\cdot, \cdot[, [\cdot, \cdot[, ]\cdot, \cdot]$ we understand intervals along the verticals $x = 0$ or $x = 1$. Also, let $p = q((0,0))$. If $\mathcal{V}$ is a clopen refinement of $\mathcal{U}^F$ then there exists a $V \in \mathcal{V}$ with $p \in V$. Say $V \subset U_0 \cup U_{i_1} \cup \cdots \cup U_{i_n}$, where one can assume that $i_1 < i_2 < \cdots < i_n$. There exists an $i' < i_n$ such that $(0, \frac{1}{i'} + \frac{1}{i' \cdot j}) \in q^{-1} V$ whenever $j \geq i'$. This implies that $q((1, \frac{1}{i'})) \in [V] \setminus V$ which contradicts the fact that $V$ is a clopen set.

Since every $T_2$ paracompact space $X$ is the image (by a perfect map) of a completely zero dimensional space [11], we get that none of the four covering properties are invariants of perfect (and closed) maps. Here by a completely zero dimensional space we understand a space whose every open cover has an open disjoint (and thus clopen) refinement (such spaces are also called ultraparacompact). Therefore every completely zero dimensional space is superparacompact.

It is known that the image of a superparacompact space by an open perfect map is superparacompact [9]. We now strengthen this result. Remember that a map $f : X \to Y$ is said to be a CO-map if the image of every clopen set in $X$ is clopen in $Y$ [1].

**Proposition 4.1.** *Let $f$ be a CO-map of a superparacompact (resp. weakly superparacompact) space $X$ onto a space $Y$. Then the space $Y$ is also superparacompact (resp. weakly superparacompact).*

**Proof.** Let $\mathcal{U}$ be an open cover of the space $Y$. Then $f^{-1}\mathcal{U}$ is an open cover of $X$ and so there exists a closure preserving clopen refinement $\mathcal{V}$ of $(f^{-1}\mathcal{U})^F$. Since the map $f$ is a CO-map we have that the cover $f\mathcal{V}$ of $Y$ is a clopen refinement of $\mathcal{U}^F$. For every subcollection $\mathcal{V}' \subset \mathcal{V}$ we have $f(\bigcup \mathcal{V}') = \bigcup f\mathcal{V}'$ and so the clopen cover $f\mathcal{V}$ is closure preserving. This proves that the space $Y$ is superparacompact by Theorem 2.2 (7). The analogous claim for weak superparacompactness is trivial. □

We thus have that superparacompactness and weak superparacompactness are an invariant of clopen maps. It is an open question whether supermetacompactness and supersubmetacompactness are invariant to clopen or even open perfect maps.

Finally we show that all the four covering properties are inverse invariant to perfect maps.

**Proposition 4.2.** *Let $f : X \to Y$ be a perfect map. If the space $Y$ is superparacompact (resp. supermetacompact, supersubmetacompact, weakly superparacompact) then so is the space $X$.*

**Proof.** We give the proof for superparacompactness, the other cases are similar.

Consider an open cover $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of $X$. For every $y \in Y$ choose a finite set $\mathcal{A}(y) \subset \mathcal{A}$ such that $f^{-1}y \subset \bigcup_{\alpha \in \mathcal{A}(y)} U_\alpha = G_y \in \mathcal{U}^F$. There exists a neighbourhood $V_y$ of $y$ such that $f^{-1}y \subset f^{-1}V_y \subset G_y$. Let $\mathcal{V} = \{V_y : y \in Y\}$. Since $Y$ is superparacompact there exists an open disjoint cover $\mathcal{B}$ which refines $\mathcal{V}^F$. The collection $f^{-1}\mathcal{B}$ is an open disjoint cover of $X$ and for every $B \in \mathcal{B}$ we have $B \subset V_{y_1} \cup \cdots \cup V_{y_n}$ for some $y_1, \ldots, y_n \in Y$. This in turn shows that $f^{-1}B \subset G_{y_1} \cup \cdots \cup G_{y_n} \in \mathcal{U}^F$ and so $f^{-1}\mathcal{B}$ refines $\mathcal{U}^F$. □

**Remark 4.1.** Proposition 4.2 for the case of Tychonoff (weakly) superparacompact spaces was proved independently in [10] while for any superparacompact space in [9].

For Hausdorff spaces we have the following corollary.

**Corollary 4.3.** *The Cartesian product $X \times Y$ of a superparacompact (resp. supermetacompact, supersubmetacompact, weakly superparacompact) space $X$ and a compact space $Y$ is superparacompact (resp. supermetacompact, supersubmetacompact, weakly superparacompact).*

Table 1 Invariants and inverse invariants of maps

| | | Super paracompact | Super metacompact | Supersub metacompact | Weakly super paracompact |
|---|---|---|---|---|---|
| Invariants | Continuous | – | – | – | – |
| | Quotient | – | – | – | – |
| | Open | – | – | – | – |
| | Closed | – | – | – | – |
| | Closed-and-open | + | ? | ? | + |
| | Perfect | – | – | – | – |
| | Open perfect | + | ? | ? | + |
| Inverse invariants | Perfect | + | + | + | + |
| | Open perfect | + | + | + | + |

Table 1 shows the invariance and inverse invariance of the above mentioned covering properties. We are left with the questions concerning supermetacompactness and supersubmetacompactness.

**Problem 4.2.** Are supermetacompactness and supersubmetacompactness an invariant of clopen maps (or at least open perfect maps)?

This will be affirmative for Tychonoff spaces if we show that a zero dimensional metacompact (resp. submetacompact) space is supermetacompact (resp. supersubmetacompact), thus:

**Problem 4.3.** Is a metacompact (resp. submetacompact) space $X$ with $ind\, X = 0$ supermetacompact (resp. supersubmetacompact)?

The space given in [6] is metrizable and thus paracompact, so every open cover has a locally finite open refinement, but it has some open cover $\mathcal{U}$ such that $\mathcal{U}^F$ does not have a clopen locally finite refinement (in this case it is equivalent to saying $\mathcal{U}$ does not have a clopen locally finite refinement) as the space is not superparacompact. It is interesting to know if this space is supermetacompact.

## 5. CO-normality in $X \times \beta X$

In this final section we give a Tamano type theorem with respect to CO-normality.

**Definition 5.1.** A space $X$ is said to be *CO-superparacompact* if every clopen cover of the space $X$ has an open finite component refinement.

The following theorem is not difficult to prove.

**Theorem 5.1.** *For every space $X$ the following conditions are equivalent:*

1. *The space is $X$ is CO-superparacompact;*
2. *Every clopen cover $\mathcal{U}$ of the space $X$ has a disjoint open refinement (i.e. an open refinement of order 0);*
3. *Every clopen cover $\mathcal{U}$ of the space $X$ has a $\sigma$-discrete clopen refinement;*
4. *Every clopen cover $\mathcal{U}$ of the space $X$ has a star-finite clopen refinement;*
5. *Every clopen cover $\mathcal{U}$ of the space $X$ has a locally finite clopen refinement;*
6. *Every clopen cover $\mathcal{U}$ of the space $X$ has a $\sigma$-locally finite clopen refinement;*
7. *Every clopen cover $\mathcal{U}$ of the space $X$ has a closure preserving clopen refinement;*
8. *Every clopen cover $\mathcal{U}$ of the space $X$ has a $\sigma$-closure preserving clopen refinement.*

Evidently, every superparacompact space and every space which is the discrete union of its components (in particular, every connected and every locally connected space) is CO-superparacompact and a CO-superparacompact, weakly superparacompact space is superparacompact. Also, from the proof of Proposition 3.8 one can see that every CO-superparacompact space is CO-collectionwise normal.

**Lemma 5.2.** *CO-superparacompactness is an invariant and inverse invariant of CO-maps.*

**Proof.** That CO-superparacompactness is an invariant of CO-maps can be easily seen from Theorem 5.1 (7) and that it is inverse invariant from Theorem 5.1 (2). □

**Lemma 5.3.** *The Cartesian product $X \times Y$ of a CO-superparacompact space $X$ and a compact space $Y$ is CO-superparacompact.*

**Theorem 5.4.** *For every Tychonoff space $X$ the following conditions are equivalent:*

1. *The space $X$ is CO-superparacompact;*
2. *The Cartesian product $X \times \beta X$ is CO-normal.*

**Proof.** The implication (1) $\Longrightarrow$ (2) follows from Lemma 5.3. We now show that (2) $\Longrightarrow$ (1).

Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be a clopen cover of a Tychonoff space $X$. Consider the clopen collection $[\mathcal{U}]_{\beta X} = \{[U_\alpha]_{\beta X} : \alpha \in \mathcal{A}\}$ in $\beta X$ and let $F = \beta X \setminus \bigcup [\mathcal{U}]_{\beta X}$. The set $F$ is closed in $\beta X$ and so is compact. It is not difficult to see that the subsets $X \times F$ and $\Delta = \{(x,x) : x \in X\} \subset X \times \beta X$ are closed and CO-disjoint in $X \times \beta X$ and so by CO-normality there exist clopen disjoint neighbourhoods of these sets.

As in the proof of Theorem 2.4 one can show that there exists an open disjoint cover $\mathcal{W} = \{W_\gamma : \gamma \in \Gamma\}$ of the space $X$ such that $F \cap (\bigcup[\mathcal{W}]_{\beta X}) = \emptyset$. Thus for every $\gamma \in \Gamma$ we have that $[W_\gamma]_{\beta X}$ is covered by a finite number of elements of $[\mathcal{U}]_{\beta X}$ which shows that $\mathcal{W} < \mathcal{U}^F$. Consequently one can modify $\mathcal{W}$ to an open disjoint refinement of $\mathcal{U}$ which proves that $X$ is CO-superparacompact. □

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