ON POLAR CO-ORDINATES

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One of the topics mathematics A Level students find hard to master is Polar Co-Ordinates. The aim here is to help such students solve problems invovling circles, cardioids, lemniscates and their tangents.

Polar Co-Ordinates

Instead of the variables x and y of Cartesian co-ordinates (so named in honour of the French mathematician Rene Descartes, (1596 — 1650)) in Polars we introduce the variables r and T*. Instead of (x, y) a point is written (r, T) where r is a number and T an angle. As examples the points A(2, 60°), B(2 150°), C(1.5, 240°) and D(3, 330°) are plotted in *figure 1*. The horizontal line is called the polar line or initial line: it comes out from a point O called the pole or origin. The number gives the distance of the polar point concerned from O, while the angle is measured upwards from the polar line. Other examples of polar co-ordinates are points E(2, -120°), F(-1, 150°), G(-2, 210°), and H(-3, -60°) *(figure 2)*. Note that if the angle is negative then it is measured downwards from the polar line; if r is negative the point is reflected to the other side of O. Ex. 1 The student should therefore here plot for himself the above points and others of his own making. To LEARN mathematics one must DO mathematics. The student should work the exercises he meets in this discussion.

Relations Between Polar and Cartesian Co-Ordinates

If a point has co-ordinates (x, y) in cartesian form and (r, T) in its equivalent polar form the relations are given by the two equations $x = r \cos T$ and $y = r \sin T$. Useful consequences of these two are four others which the student can easily obtain on his own: $x^2 + y^2 = r^2$, $x^2 - y^2 = r^2 \cos 2T$, $2xy = r^2 \sin 2T$ and $y/x = \tan T$. It is these six relations that are utilized when problems ask to change cartesian equations into polar ones and vice-versa. The student is advised to remember the first basic two and be able to obtain the other four.

Ex. 2 Use diagrams to prove the two basic relations between cartesian and polar co-ordinates. Hint: Let the point be in each of the 4 quadrants, respectively.

Ex. 3 Starting with the cartesian forms below obtain their equivalent polar forms.

* For technical reasons, **theta** has been substituted by T throughout the article and angles are in degress instead of radians, as is more appropriate

CARTESIAN FORMS		POLAR FORMS
	lines	
$x = \pm p$		$r = \pm p \sec \theta$
$y = \pm p$		$r = \pm p \operatorname{cosec} \theta$
$y = x \tan b$		$\theta = b$ (constant angle)
	circles	
$(x \mp a)^2 + y^2 = a^2$		$r = \pm 2a \cos \theta$
$x^2 + (y \mp a)^2 = a^2$		$r = \pm 2a \sin \theta$
$x^2 + y^2 = a^2$		$r = a$
	cardioids	
$(x^2 + y^2 \mp ax)^2 = a^2(x^2 + y^2)$		$r = a(1 \pm \cos \theta)$
	lemniscates	
$a^2(x^2 - y^2) = (x^2 + y^2)^2$		$r^2 = a^2 \cos 2\theta$
$2a^2xy = (x^2 + y^2)^2$		$r^2 = a^2 \sin 2\theta$

NOTE carefully the polar lines above since they play an important part in polar problems.

Ex. 4 Starting with the polar forms otbain the cartesian forms.
Ex. 5 Sketch the polar lines and circles listed, using their familiar cartesian forms as a guide.

After that general but important introduction, we may now study certain curves in detail, starting with the cardioids.

The Cardioid $r = a(1 + \cos T)$

This curve is so called because it is heart-shaped. Points on it are ($2a$, 0°), B($3a/2$, 60°), C(a, 90°), O (0, 180°), D(a, 270), E($3a/2$, 300°) and again A (2a, 360°) *(figure 3)*. Recall that *cos* 360° = 1, *cos* 270° = 0 and *cos* 180° = -1. These results are very useful in polar curve sketching and must be memorised. NOTE that at 60° occurs the highest point and at 300° (its reflection) occurs the lowest point. Later we shall prove this.

The Cardioid $r = a(1 - \cos T)$

Points on this curve are O(0, 0°). A(a, 90°), B($3a/2$, 120°), C($2a$, 180°), D($3a/2$, 240°), E($2a$, 270°) and again O)0, 360°). Its sketch is *figure 4*.

Ex. 6 What are the co-ordinates of the highest and lowest points of $r = a(1 - \cos T)$?

Ex. 7 Sketch $r = 3(1 + \cos T)$, $r = 3a(1 - \cos T)$, showing the highest and lowest co-ordinates.

The Limacon $r = a \cos T + c$, with c greater than a

Strictly speaking $r = a(1 + \cos T)$ call it a cardioid. When c is larger than a the heart shape remains but the cusp (sharp point on a curve) dis- is also a limacon, with $c = a$. But to better distinguish between them we appears, to be replaced by a 'bend' there. As example we take the curve $r = a \cos T + 3a$. Points on it are A($4a$, 0°), B($3a$, 90°), C($2a$, 180°), D($3a$, 270°) and again A($4a$, 360°) (*figure 5*).

Ex. 8 Sketch the limacon $r = a \cos T + 2a$. Show that its cartesian equation is $(x^2 + y^2 - ax)^2 = 4a^2(x^2 + y^2)$.

Symmetry In Polar Curves

NOTE that all the above mentioned cardioids and limacons are symmetric about the x-axis (i.e., the line T = 0°). This happens whenever the curve contains, or can be written to contain, only $\cos T$ and its powers. When only $\sin T$ and its powers occur in a polar curve equation that curve is symmetric about the y-axis (i.e., the line $T = 90°$). This is important to remember since it simplifies the sketching ofpolar curves.

The Limacon $r = a \cos T + c$, with c less than a

Here the cardioid shape is retained but something is added to it. As example consider $r = 2 \cos T + 1$. Points on this curve are A(3, 0°), B(2.4, 45°), C(1, 90°), O(0, 120°), D(-10.7, 150°), E(-l, 180°), F(-0.4, 225°), G(1, 270°), H(2.4, 315°) and again A(3, 360°) (*figure 6*). NOTE that point E occurs when $\cos T = -1$, that is at $T = 180°$. The other points on the inner loop correspond to those values of T for which $2 \cos T + 1$ is negative, i.e., for $\cos T$ less than $-\frac{1}{2}$; i.e., for T between 120° and 240° (verify this).

Ex. 9 Investigate the symmetry of these polar curves, then sketch: $r = 4 \sec T$, $2r = \operatorname{cosec} T$, $r = 4 \cos T$, $r = -3 \sin T$, $r2 = 4 \cos T$ and $r2 = 4 \sin 2T$.

Ex. 10 Sketch the limacons: $r = 2 \cos T + 3$, $r = \cos T + 2$, $r = 4a \cos T + 3a$ and $r = 3a \cos T + a$.

Theorem I The highest point on the cardioid $r = a(1 + \cos T)$ occurs at T = 60°.

Proof: Refer to *figure 7*.

Let A be the highest point and (r, T) its co-ordinates. In triangle OAB, $p = r \sin T$. From calculus we argue that p will be maximum when $dp/dT = 0$. So $r \cos T + \sin T\, dr/dT = 0$. From the cardioid equation, $dr/dT = -a \sin T$. So $2 \cos^2 T + \cos T - 1 = 0$; i.e., $(2 \cos T - 1)(\cos T + 1) = 0$; i.e., $\cos T = \frac{1}{2}$ and -1. The solution we are after is, then, $T = 60°$. ////

Ex. 11 Find the co-ordinates of the highest and lowest points on $r = a + a \cos T$.

Ex. 12 Hence find the equations to the tangents parallel to the initial line for this cardioid.

Ex. 13 Show that if (r, T) represents the highest point on the limacon $r = c + a \cos T$ then T satisfies the equation $2a \cos^2 T + c \cos T - a = 0$. (The procedure is as that for the above proof.)

Theorem II The double tangent of the cardioid $r = a + a \cos T$ has the equation $4r = -a \sec T$.

Proof: *Refer to figure 8.*

Let $B(r, T)$ be one point of contact of the tangent. The equation of the double tangent AB is $r = -p \sec T$, where p is the perpendicular distance AO of the tangent from O.

First we shall find the co-ordinates of B. Imposing $dp/dT = 0$ at maximum p and using the fact that $dr/dT = -a \sin T$, we end up with the trig equation $\sin T + \sin 2T = 0$. The solutions between 0° and 360° are 0°, 120°, 180°, 240°, 360°; from which we retain $T = 120°$ at point B.

So B is the point (a/2, 120°). And from $r = -p \sec T$ we get $p = a/4$. So AB has equation $r = -p \sec T = (-a \sec T)/4$; i.e., $4r = -a \sec T$. ////

Ex. 14 Write the equations of the double tangents to the cardioids $r = 3 + 3 \cos T$, $r = 1 + \cos T$ and $r = 2a(1 + \cos T)$. Sketch your results.

Ex. 15 Prove that the equation of the double tangent to the cardioid $r = a - a \cos T$ is $4r = a \sec T$.

Ex. 16 Write the equations of the double tangents of the cardioids $r = 2(1 - \cos T)$ and $r = 4a(1 - \cos T)$. Sketch your results.

The Lemniscate $r^2 = a^2 \cos 2T$

Since r^2 is greater than or equal to zero and a^2 is positive, $\cos 2T$ must be greater than or equal to zero. So $2T$ can have values in the ranges (-90°, 90°) and (270°, 450°) inclusive. Hence, T can be in (-45°, 45°) and (135°, 225°) inclusive for the curve to exist. To this we add that since $\cos 2T = 1 - 2 \sin^2 T$, the curve is symmetric about the y-axis. Also $2T = 2 \cos^2 T - 1$ implies that the curve is also symmetric about the x-axis. Points on this curve are:

$$(\pm a, 0^\circ), \quad (0, \pm 45^\circ)$$

$$\text{and} \quad (\pm 0.7a, 30^\circ).$$

The sketch is *figure 9.*

Ex. 17 Write the equations of the tangents to these lemniscates (i) at the origin and (ii) perpendicuar to the initial line: $r^2 = a^2 \cos 2T$, $r^2 = 2 \cos 2T$. Sketch your results.

Theorem III. The highest points on the lemniscate $r^2 = a^2 \cos 2T$ occur at $T = 30°$ and $150°$.

Proof: Refer to *figure 10.*

Let A (r, T) be one of the two highest points. Let AB $= p$. In triangle OAB, $p = r \sin T$. Imposing $dp/dT = 0$ at maximum p and using $r\, dr/dT = -a^2 \sin 2T$ we get the trig equation $\cos T \cos 2T - \sin T \sin 2T = 0$, i.e., $\cos 3T = 0$. Hence $T = 30°$ is the solution we are after here. From the sketch of the lemniscate and its symmetry it is obvious that the other highest point is at $T = 150°$. ////

Ex. 18 Find the equations of the two horizontal tangents to the lemniscates: $r^2 = a^2 \cos 2T$,* $r^2 = 9 \cos 2T$ and $r^2 = 2 \cos 2T$. Sketch your results.

The lemniscate $r^2 = a^2 \sin 2T$

The sketch is that of the previous lemniscate rotated through 45° about O in an anti-clockwise direction. The shape and size remain the same. To help the student understand why this happens the following simple trig argument is given:
$r^2 = a^2 \sin 2T$ (the given curve) $= a^2 \cos (90° - 2T) = a^2 \cos 2(45° - T) = a^2 \cos 2Z$, say, which is the form of the lemniscate $r^2 = a^2 \cos 2T$. This explains why the shape and size of the given lemniscate are as those for $r^2 = a^2 \cos 2T$. Further, since the new angle is $45° - T$ the rotation is of 45° and anti-clockwise. The now-obvious sketch is left for the student.

Ex. 19 Write the equations of the two tangents at the origin for the lemniscates $r^2 = a^2 \sin 2T$, soiacT2t *sin 2T*, $r^2 = 2 \sin 2T$. Sketch your results.

Theorem IV The highest point on the lemniscate $r^2 = a^2 \sin 2T$ occurs at $T = 60°$.

The proof is left as an exercise. The procedure is as in Theorem III.

Ex. 20 Find the equations of the two tangents to $r^2 = a^2 \sin 2T$ which are parallel to the polar line.

Theorrem V The two vertical tangents to the lemniscate $r^2 = a^2 \sin 2T$ are given by the equation,

$$2r\sqrt{2} = \pm 3^{\frac{5}{4}} a \sec \theta,$$

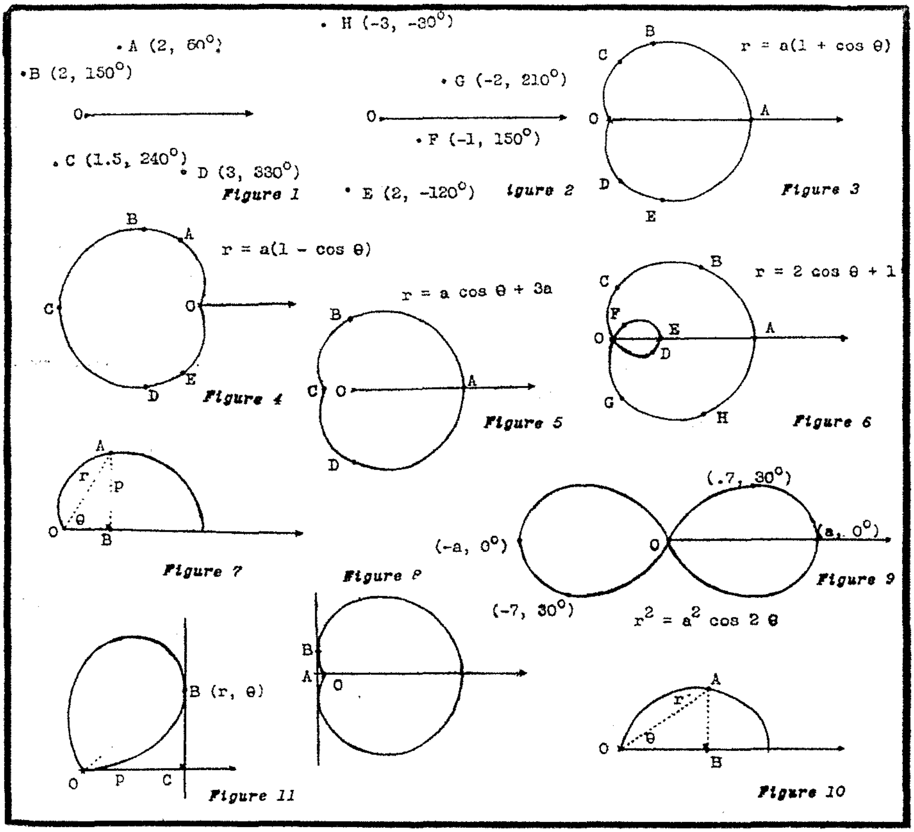
$$\text{i.e., } r = \pm 0.81 \sec \theta.$$

Proof: Refer to *figure 11.*

Let (B(r, T) be the contact point of one tangent. We first find the co-ordinates of B by imposing $dp/dT = 0$ at maximum p. Using $r\,dr/dT = a^2 \cos 2T$, one ends up with the equation $\cos 3T = 0$, solutions of which are $T = 30°, 90°, 150°$ and $270°$. For point B we retain $T = 30°$. The co-ordinates of B are, then,

$$(3^{\frac{1}{4}} a/\sqrt{2},\ 30°).$$

* Answer: $2.828r = \pm a \operatorname{cosec} \theta$



By triangle OBC

$$p = 3^{\frac{3}{4}}\, a / \, 2\sqrt{2}\ .$$

Since the tangents are given by

$$r = \pm\, p \sec \theta,$$

the required result follows. ////

In closing I would like to re-emphasise the importance of the exercises given. These exercises cannot be separted from the discussions encountered without weakening the student's ability to understand these same discussions: To LEARN mathematics one must DO mathematics.

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