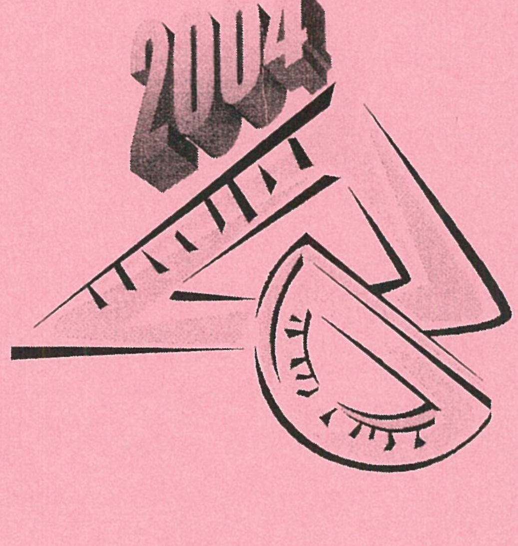
Collection IX



The Collection IX

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Proceedings of Workshop held on the 9th March 2004

Contents

Foreward

The Science and Society Directorate of the European Commission published a DVD called "Femmes de tête". The stories of established women astrophysicists, microbiologists, mathematicians, geologists and genetic engineers working on edge-cutting research in Europe are told in a witty way, highlighting the great hurdles they managed to overcome and the unfailing support of their family and colleagues. One of the protagonists Prof. Ene Ergma who is an astrophysicist and now an active MP in the Estonian Parliament acknowledged that to be a scientist is "madness" in today's world when other easier routes lead to much higher financial gain. However scientists persist in their dedicated work because "the soul sings" as they wade through the rocky paths of research. I can vouch for this exulting feeling from my experience in mathematical research. However what lifts my spirits to higher levels is when our students contract this urge for mathematical discovery.

Encounters such as we have in the "*Collection*" workshops certainly foster the aspirations of our budding mathematicians. In this session the talents of students, other than in mathematics, have also been encouraged. Romina Mamo, a local established singer and mathematics student elicited a smile from all present as she sang with great charm the lyrics of fellow student Clinton P. Cilia. The message of the song may sound negative; however it is a hard reality that the path which our students tread till graduation is adorned with thorny challenges.

Dr Irene Sciriha

The Collection IX

Faculty of Science

Department of Mathematics

Date: 9^{th} March 2004

Time: 1500 – 1700

Venue: LC 119

A seminar/workshop is being held on Tuesday 9^{th} March 2004 at 1500. Students and staff from the Department of Mathematics, Faculty of Science will present ideas from various fields of mathematics.

Keynote speakers:

Professor S. Fiorini *On Singular Trees*

Elaine Chetcuti *The Euler Phi Function for Powers of Primes*

Andrew Duncan *The Powers of Matrices and the Walk Matrix*

Angela Lombardi *The Eigenvalues of Self Complementary Graphs*

Romina Mamo *A Song for Mathematics*
Paul Clinton Cilia *Lyrics*

We shall end with a brief session for spontaneous problem posing. You are cordially invited to attend.

Abstracts of possible proofs or conjectures which you wish to share with us in this meeting, or in a future one, may be sent to Dr. I. Sciriha or Ms. A. Attard, Department of Mathematics, (marked The Collection), at any time of the year.

Dr. I. Sciriha

Organiser

Powers of the Adjacency Matrix and the Walk Matrix

Andrew Duncan

Introduction The aim of this article is to identify and prove various relations between powers of adjacency matrices of graphs and various invariant properties of graphs, in particular distance, diameter and bipartiteness. A relation between the Walk matrix of a graph and a subset of the eigenvectors of the graph will also be illustrated. A number of *Mathematica* procedures are also provided which implement the results described. Note that the procedures are only illustrative; issues of algorithmic efficiency are largely ignored.

Unless specified, all graphs are assumed to be simple and connected, that is, there is at most one edge between each pair of vertices, there are no loops, and there is at least one path between every two vertices. The adjacency matrix A or $A(G)$ of a graph G having vertex set $V = V(G) = \{1, ..., n\}$ is an $n \times n$ symmetric matrix a_{ij} such that $a_{ij} = 1$ if vertices i and j are adjacent and 0 otherwise.

Powers of the Adjacency Matrix

The following well-known result will be used frequently throughout:

Theorem 0.1 *The $(i,j)^{th}$ entry a_{ij}^k of A^k, where $A = A(G)$, the adjacency matrix of G, counts the number of walks of length k having start and end vertices i and j respectively.*

Proof For $k = 1$, $A^k = A$, and there is a walk of length 1 between i and j if and only if $a_{ij} = 1$, thus the result holds. Assume the proposition holds for $k = n$ and consider the matrix $A^{n+1} = A^n A$. By the inductive hypothesis, the $(i,j)^{th}$ entry of A^n counts the number of walks of length n between vertices i and j. Now, the number of walks of length $n+1$ between i and j equals the number of walks of length n from vertex i to each vertex v that is adjacent to j. But this is the $(i,j)^{th}$ entry of $A^n A = A^{n+1}$ the non-zero entries of the column of A corresponding to v are precisely the first neighbours of v. Thus the result follows by induction on n. ∎

The Diameter of a Graph

Definition 0.1 *The distance between two vertices i and j in a graph G, denoted d_{ij}, is the length of the shortest path between i and j. Clearly, since G is connected such a path must exist.*

Definition 0.2 *The diameter of G is defined to be* $D = \max_{(i,j)\in V}\{d_{ij}\}$.

Remark Although most of the results in this section can be found in [2], most of the proofs appearing here are different.

Lemma 0.2 *Let d_{ij} be the distance between vertices i and j in G. Then for all $k \in \mathbb{N}$, there is a walk of length $d_{ij} + 2k$ between i and j.*

Proof Let $k \in \mathbb{N}$. We shall construct a walk of length $d_{ij} + 2k$ between i and j. Let $i = v_1, v_2, \ldots, v_{p-1}, v_p = j$ be the path of length d_{ij} between i and j. After following this path, follow the cycle v_p, v_{p-1}, v_p k times. The ending vertex is j and the total length of this constructed walk is $d_{ij} + 2k$. ∎

Theorem 0.3 *[2] Let D be the diameter of the graph G. Then the matrix $A^D + A^{D-1}$ has no zero entries.*

Proof Without loss of generality, let D be even. Let $i, j \in V(G)$. If d_{ij} is even, $D - d_{ij}$ is also even, and thus by the previous lemma there is a walk of length $d_{ij} + D - d_{i,j} = D$ between i and j , and thus the (i, j) entry of A^D is greater then 0. If d_{ij} is odd, then $D - d_{ij} - 1$ is even, and thus by the previous lemma, there is a walk between i and j of length $d_{ij} + (D - d_{ij} - 1) = D - 1$. Hence the $(i, j)^{th}$ entry of A^{D-1} is greater than 0. Since the entries of A^D and A^{D-1} are non-negative, in either case, $A^D + A^{D-1}$ has no zero entries. ∎

The converse of this proposition is also true, and provides us with an algorithm for calculating the diameter of the graph.

Theorem 0.4 *[2] Let d be the smallest natural number such that, $A^d + A^{d-1}$ has no zero entries. Then $d = D$, the diameter of G.*

Proof By Theorem 0.3, d can be found such that $A^d + A^{d-1}$ has no zero entries. Hence $d \leq D$. Suppose $d < D$. Let (i, j) be a pair of vertices at a distance D apart in G. then the ijth entries of A^d and of A^{d-1} are both zero, a contradiction. Thus $d \geq D$. Also for any two vertices i, j, a_{ij}^D or a_{ij}^{D-1} is non-zero, so that $d \leq D$. Thus $d = D$ as required.

Corollary 0.5 *Let d be the smallest natural number such that $(A + I)^d$ has no zero entries. Then $d = D$ the diameter of G.*

Proof $(I + A)^d = I + dA + \ldots + A^d$, where the entries of each A^i are non-negative, and coefficients are positive. Thus the $(i, j)^{th}$ entry of $(A + I)^d$ is non-zero if and only if, $a_{ij}^k > 0$ for some $1 \leq k \leq d$. Suppose $A^d + A^{d-1}$ has a zero entry (i, j). Then there must be some $A^k, 1 \leq k \leq d - 2$ such that the $(i, j)^{th}$ entry of A^k is non-zero, i.e. There is a walk of length k from i to j. Now either $(d - k)$ is even or $(d - k - 1)$ is even. Thus by lemma 2.1, there is a walk of length $k + (d - k) = d$ or $k + (d - k - 1) = d - 1$ between i and j. This implies that the $(i, j)^{th}$ entry of $A^{d-1} + A^d$ is non-zero which is a contradiction. Thus d is the smallest natural number such that $A^d + A^{d-1}$ has no zero entries. The result follows by the previous proposition. ∎

This corollary provides us with a very simple algorithm to determine the diameter of a graph G from its adjacency matrix:

```
HasZeros[m_] := MemberQ[Flatten[m], 0]

Diam[G_] := Module[{d = 1, m2 = m = A + IdentityMatrix[Length[A]]},
        While [HasZeros[m], m = m.m2;   d++];  d]
```

Theorem 0.6 *The diameter D of a (connected) graph G is less than the number of distinct eigenvalues of the adjacency matrix.*

Proof Let the number of distinct eigenvalues of the adjacency matrix A be r. Since A is real and symmetrical, it is diagonalizable. Then the minimal polynomial $m_T(x)$ of A has degree r and $m_T(A) = \mathbf{0}$.

We will show that there exist elements in $\{I, A, \dots A^D\}$ which are not linear combinations of their predecessors, and thus show that $\{I, A, \dots A^D\}$ is linearly independent, which implies that there is no polynomial of degree D or less satisfied by A. Consider $d = d_{ij} \leq D$, for some $i, j \in V(G)$. It follows, by definition of d_{ij}, that $a^d_{ij} \neq 0$ and $a^t_{ij} = 0$ for $t < d$. Thus it is impossible that A^d is a linear combination of its predecessors. It follows that $r \geq D + 1$ as required. ∎

Tests for Bipartiteness

By observing powers of the adjacency matrix A, it is possible to determine whether G is bipartite through a simple test.

Definition 0.3 *The index of a graph G is defined to be the smallest γ such that A^γ has no zero entries.*

Note that not every graph has an index as will be seen soon. We will require the following lemma:

Lemma 0.7 *Let D be the diameter of the graph G having adjacency matrix A. If there are two columns r_i, r_j in A^D which are orthogonal, then G is necessarily bipartite.*

Proof Let r_i, r_j be two orthogonal columns in A^D. Partition V into the sets V_1 and V_2, where V_1 is the set of vertices having non-zero entries in r_i and V_2 is the set of vertices having zero entries in r_i. Then any two vertices in the same class are not adjacent. Suppose for contradiction that vertices $k, l \in V_1$ are adjacent. Then $a^D_{ki}, a^D_{li} > 0$, and thus $a^D_{kj} = a^D_{lj} = 0$. Thus the distances d_{kj} and d_{lj} are less than D, and $D - d_{kj} = D - d_{lj} = 1 \mod 2$. Thus $D - d_{kj} - 1 = D - d_{lj} - 1 = 0 \mod 2$, which implies that a^{D-1}_{kj}, $a^{D-1}_{lj} > 0$ by Lemma 0.2. But k is adjacent to l. Thus, there must be a walk of length D between vertices k, j and l, j and thus $a^D_{kj}, a^D_{lj} > 0$ which contradicts the orthogonality of r_i and r_j. A similar contradiction is obtained if one assumes that $k, l \in V_2$ are adjacent. ∎

Proposition 0.8 *Let G be a connected, non-bipartite graph, then the index γ of G exists and satisfies $D \leq \gamma \leq 2D$.*

Proof Clearly, $D \leq \gamma$, since if $\gamma < D$ then, by definition of the index γ of G, A^γ has no zero elements. Thus $A^\gamma + A^{\gamma-1}$ has no zero elements but this would contradict Theorem 0.4. Thus $D \leq \gamma$. We will now show that A^{2D} has no zero elements. Suppose, for contradiction that the $(i,j)^{th}$ entry of A^{2D} is 0. Then $a_{ij}^{2D} = A_{ij}^{2D} = A_{ij}^D A_{ij}^D = \sum_{h=1}^n a_{ih}^D a_{hj}^D = \sum_{h=1}^n a_{hi}^D a_{hj}^D = 0$. This implies that the i^{th} and j^{th} columns of A^D, a_i and a_j are orthogonal. But this implies that G is bipartite by the previous lemma, contradicting the premise that G is non-bipartite. Thus since A^{2D} has no zero entries. By the minimality of $\gamma, \gamma \leq 2D$. ∎

Proposition 0.9 *Let G be bipartite. Then the index γ of G does not exist.*

Proof Let G be bipartite, and let V_1, V_2 be the partitions. Then, for $i, j \in V_1$, all walks between i and j must contain an odd number of (not-necessarily distinct) vertices. Thus all walks between i and j must be of even length. Also, for $k \in V_1, l \in V_2$, all walks between k and l must contain an even number of (not necessarily distinct) vertices. Thus all walks between k and l must be of odd length. Thus for γ odd, the $(i,j)^{th}$ entry of A^γ is zero, and for γ even, the $(k,l)^{th}$ entry of A^γ is zero. Thus A^γ always has a zero entry. ∎

Using the above two propositions, we obtain the following corollary:

Corollary 0.10 *G is non-bipartite if and only if $D \leq \gamma \leq 2D$.* ∎

The following Mathematica function will compute the index γ of a non-bipartite graph:

```
GraphIndex[M_] :=  Module[{d := 1, M1 := M},
                    While [HasZeros[M1], d++; M1 = M1.M]; d]
```

The following algorithm will determine whether a graph G is bipartite by testing the powers of $A = A(G)$, between D and $2D$, as described in the above corollary:

```
isBipartite[G_] := Module[{d = Diam[A], result = True, m},
                    m = MatrixPower[A, d];
                    For [i = d, i <= 2d, i++,
                          If[HasZeros[m], m = m.G,
                               result = False; Break]]; result]
```

The Walk Matrix of a Graph

We will now digress and describe an interesting result that relates the number of main eigenvalues to the rank of a matrix known as the Walk Matrix of the graph. We will denote the all ones vector by $\mathbf{j} = (1, 1, \ldots, 1)^{\top}$.

Definition 0.4 *An eigenvalue is said to be non-main if it has an associated eigenvector* $\mathbf{x}$ *the sum of whose entries is not equal to 0, i.e,* $\mathbf{x}.\mathbf{j} \neq 0$

Definition 0.5 *Let G be a graph with adjacency matrix A. The walk matrix $W = W_p(A)$ of G is the $n \times p$ matrix $(j, Aj, A^2j, \ldots, A^{p-1}j)$, where p is the smallest value such that the walk matrix $W_p(A)$ attains maximum rank.*

The columns of the walk matrix define an A-cyclic subspace $U = \{a_j, \quad i \in N\}$. Since U is a subspace of R^n, it is finite dimensional, having dimension p, and it can be shown that U is generated by the basis $(j, Aj, A^2j, \ldots, A^{p-1}j)$. Thus the above definition is well defined, and p can be defined to be the smallest value such that $Rank(W_p(A)) = Rank(W_{p+1}(A)) = p$.

The graph-theoretic interpretation of a single column $A^r j$ of the walk matrix, is as follows: the i^{th} entry of $A^r j$ counts the number of walks of length r starting from vertex i. Thus using the above result, the set of these vectors is finitely generated, and every vector $A^r j$, $p \leq r$ is a linear combination of the vectors $(j, Aj, A^2j, \ldots, A^{p-1}j)$.

Lemma 0.11 *Let $A = A(G)$ be the adjacency matrix of G. Then a set of n orthonormal eigenvectors can be chosen such that for each eigenvalue, at most one corresponding eigenvector is main.*

Proof Suppose there is an eigenvalue λ with multiplicity greater than one, and let $\mathbf{x_1}, \ldots, \mathbf{x_m}$ be an orthonormal basis for the corresponding eigenspace where x_1 is main. Clearly we can assume that that $\mathbf{x_k}.\mathbf{j} \geq 0$ for all k. Now suppose $\mathbf{x_i}$ is another main vector. Then replace $\mathbf{x_i}$ with:

$$\mathbf{x_i'} = \mathbf{x_i} - \frac{\mathbf{x_i}.\mathbf{j}}{\mathbf{x_1}.\mathbf{j}}\mathbf{x_1}$$

and replace $\mathbf{x_1}$ with:

$$\mathbf{x_1'} = \frac{\mathbf{x_i}.\mathbf{j}}{\mathbf{x_1}.\mathbf{j}}\mathbf{x_i} + \mathbf{x_1}$$

and normalize accordingly. Then $\mathbf{x_1'}$ is a main eigenvector, $\mathbf{x_i'}$ is non-main, and the set $\{\mathbf{x_1'}, \mathbf{x_2}, \ldots, \mathbf{x_i'}, \ldots, \mathbf{x_n}\}$ is an orthonormal set of eigenvectors satisfying the required condition. ∎

We can now prove the following theorem relating the rank of the Walk Matrix to the number of main eigenvalues.

Theorem 0.12 *[4] The rank of the walk-matrix of a graph G is equal to the number of its main eigenvalues.*

Proof By the previous lemma, we can choose an orthonormal set of n eigenvectors such that each eigenvalue has at most one corresponding main eigenvector. Let $\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_m}$ be the set of orthonormal eigenvectors corresponding to the main eigenvalues, $\lambda_1, \ldots, \lambda_m$ of G. As before, we can assume that $a_i = \mathbf{x_i}.\mathbf{j} > 0$ for $i = 1, \ldots, m$. Then $\mathbf{j}$ can be expressed as the linear combination $\mathbf{j} = \sum_{i=1}^{m} a_i \mathbf{x_i}$. Since for any $k \geq 0, A^k\mathbf{j} = \sum_{i=1}^{m} a_i \lambda_i^k \mathbf{x_i}$, then we have that

$$U = \text{Span}(A^k\mathbf{j}, \forall k \geq 0) \subseteq \text{Span}(\mathbf{x_1}, \ldots, \mathbf{x_m})$$

Thus $\dim U =$ rank $W(G) \leq m$. We will now show that U contains a linearly independent set of m vectors, namely $\mathbf{j}, A\mathbf{j}, \ldots, A^{m-1}\mathbf{j}$.

Suppose that $\sum_{j=1}^{m} c_j A^{j-1}\mathbf{j} = 0$ for some constants $c_1, \ldots, c_m$. Then

$$0 = \sum_{j=1}^{m} c_j A^{j-1}\mathbf{j} = \sum_{j=1}^{m} c_j \sum_{i=1}^{m} a_i \lambda_i^{j-1}\mathbf{x_i} = \sum_{i=1}^{m} a_i \sum_{j=1}^{m} c_j \lambda_i^{j-1}\mathbf{x_i}$$

Then since $a_i > 0$ and $\mathbf{x_i}$, $i = 1, \ldots, m$, are orthonormal, then

$$\sum_{j=1}^{m} c_j \lambda_i^{j-1} = 0, \quad i = 1, \ldots, m$$

But the λ_i's are all distinct, and so we have a polynomial of degree $m-1$ with m distinct roots which is only possible if the polynomial is entirely 0, that is $c_j = 0$, $j = 1, \ldots, m$. Thus the set $\mathbf{j}, A\mathbf{j}, \ldots, A^{m-1}\mathbf{j}$ is linearly independent and so rank $W(G) = m$. ∎

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An Upper Bound for the Nullity of Trees and Edge-Colourings

Stanley Fiorini

Abstract

A necessary and sufficient condition for the non-singularity of the adjacency matrix of a tree is given in terms of the existence of a 1-factor in the tree. The result is used to give an upper bound for the nullity of the tree via edge-colourings of bipartite graphs.

Illustrating the basic concepts

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

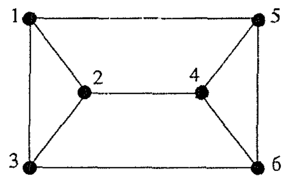

Figure 1: A *graph* G and its *adjacency matrix* $A(G)$

Edges $\{12, 46\}$ in G are *independent* because they share no vertex; they are also called a *matching*.

Independent edges $\{15, 24, 36\}$ are a *1-factor* of G because they cover all vertices; they form a *maximal* matching.

An *edge-colouring* of G is a partitioning of the edge set $E(G)$ of G into matchings, called *colour-classes*. The least number of colour-classes is the *chromatic* *index* $\chi^{/}(G)$. In the example given $\chi^{/} = 3$ and *the* partitioning (the only one possible) is {13, 24, 56} (coloured α), {12, 36, 45} (coloured β) and {15, 23, 46} (coloured γ).

If $\Delta(G)$ is the maximum valency of G, then, clearly $\Delta(G) \leqslant \chi^{/}(G)$; it has been shown by Vizing [2, pp. 30-32] that $\chi^{/}(G) \leqslant \Delta(G) + 1$.

The graph G (above) has odd circuits $\langle 1231 \rangle$, $\langle 124651 \rangle$. If all circuits are even, then G is said to be *bipartite* and the vertex set $V(G)$ of G can be partitioned into $V(G) = A \cup B$, $A \cap B = \phi$ such that $E(G) \subseteq A \times B$.

A *tree* T is a connected graph with no circuits and hence bipartite. If $|V(T)| = n$, then $|E(T)| = n - 1$ and it must have a vertex of valency 1.

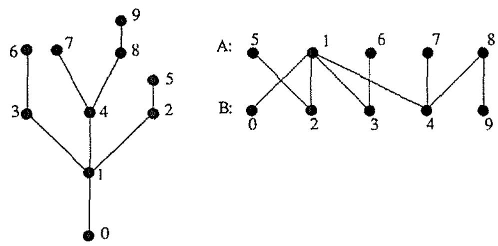

Figure 2: A *tree* T and its bipartition:

König (1916) proved that for a bipartite graph of maximum valency Δ, $\chi^{/} = \Delta$. [2, p.25]

The *spectrum* spec(G) of a graph G is the set of eigenvalues of $A(G)$; since $A(G)$ is real and symmetric, spec(G) is real. Coulson and Rushbrook (1940) proved that the spectrum of a bipartite graph is symmetric about 0. [1, p. 87]

Main Theorem: A tree T has a 1-factor if and only if $A(T)$ is non-singular.

Theorem 1: If a tree T has a 1-factor, then $A(T)$ is non-singular.

Proof: T bipartite $\Rightarrow$ spec(T) symmetric about 0

$\Rightarrow \quad A(T)$ singular if $n(T)$ is odd.

But T has a 1-factor $\Rightarrow n(T)$ even, $n(T) = 2k$.

Proceed by induction on k.

For $k = 1$, there is only one tree on 2 vertices and $\det(A(T)) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$; hence non-singular.

Assuming the assertion is true for k and considering a vertex v of valency 1 with neighbour w in a tree with $2k+2$ vertices, we label its vertices $v = v_1, w = v_2$ so that $A(T) = \begin{bmatrix} 0\ 1 & \mathbf{o} \\ 1\ 0 & \mathbf{u} \\ \mathbf{o}\ \mathbf{u}^T & A(T-v-w) \end{bmatrix}$.

By a sequence of row and column operations of the kind: $\begin{matrix} R_i \mapsto R_i - R_1 \\ C_j \mapsto C_j - C_1 \end{matrix}$

vectors $\mathbf{u}$ and $\mathbf{u}^T$ can be 'killed' without affecting the sub-matrix $A(T-v-w)$ and without changing the value of $|A(T)|$.

A final row-operation $R_1 \leftrightarrow R_2$ changes the sign of the resulting determinant and yields $|A(T)| = -|A(T-v-w)| \neq 0$, by the inductive hypothesis.

Thus $A(T)$ is non-singular. ■

Theorem 2 If a tree T has a matching M of maximum size μ (covering 2 μ vertices $v_1, \ldots, v_{2\mu}$) and if v is any other vertex, then the row R_v in $A\,(T)$ corresponding to v is linearly dependent on the rows $R_{v_1}, \ldots, R_{v_{2\mu}}$ corresponding to the vertices in the matching.

Proof: Let v have neighbours $v_{i_1}, \ldots, v_{i_s}$.

If some v_{i_t} $(1 \leqslant t \leqslant s)$ is not covered by M, then the edge $v\ v_{i_t}$ could have been added to M, contradicting maximality. Thus all of $v_{i_1}, \ldots, v_{i_t}$ are in M and deleting v from T yields a disconnected graph with s components $C_1, \ldots, C_s$ with
$v_{i_t} \in C_t\ (1 \leqslant t \leqslant s)$. Thus, $A\,(T)$ can be represented by:

$$\left(\begin{array}{cccc|c}
A(C_1) & 0 & 0 & 0 & \\
0 & A(C_2) & 0 & 0 & \\
0 & 0 & \ddots & 0 & * \\
0 & 0 & 0 & A(C_s) & \\
\hline
1 & 1 & \ldots & 1 & 0 \\
\hline
 & & * & &
\end{array}\right)$$

for an appropriate labelling of its vertices.

One notes that the top right-hand submatrix must be zero; otherwise if there exist v_j (in C_1 say) that is not covered by this matching, then there exists a path in G starting in v ending in v_j with edges alternately "not in" / "in" the matching, contradicting the maximality of M.

But by Theorem 1, the principal sub-matrix of size $2\mu \times 2\mu$ is non-singular so that by suitable elementary row-operations the first 2μ rows of $A\,(T)$ can be reduced to

$$B := \left(I_{2\mu} \,|\, *\right)$$

Thus the $(2\mu + 1)^{th}$ row corresponding to v in seen to be the sum of the rows $Rv_{i_1} + Rv_{i_2} + \ldots + Rv_{i_s}$ of B. ■

Since the vertex v was arbitrarily chosen from $V(T)\setminus V(M)$, we have the following:

Corollary: The rank $rk(T)$ of T equals 2μ. ■

The main theorem follows from this corollary and Theorem 1.

Now let $\Delta = \Delta(T)$ and $n = |V(T)|$. Since T is bipartite, T has an edge-colouring with Δ colours (by König), that is, $E(T)$ can be partitioned into Δ colour-classes $\Gamma_1, \ldots, \Gamma_\Delta$. It is clear that a colourclass consists of independent edges which form a matching. Hence the size of the largest colour class in the graph is less than that of a maximum matching.

Thus $(n-1) = |E(T)| = \sum_{i=1}^{\Delta} |\Gamma_i| \leqslant \Delta \max_{1\leqslant i \leqslant \Delta} |\Gamma_i| \leqslant \Delta\mu$,

$$\Rightarrow \quad \left\lceil \frac{n-1}{\Delta} \right\rceil \leqslant \mu$$
$$\Rightarrow \quad rk(T) \geqslant \left\lceil \frac{n-1}{2\Delta} \right\rceil$$

Thus, the nullity of T is at most $n - \left\lceil \frac{n-1}{2\Delta} \right\rceil$.

Open Problem: Investigate the nullity of bipartite graphs.

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Euler's Phi function for Powers of Primes

Elaine Chetcuti

The Phi function $\phi(n)$ is defined as the number of positive integers less than n which have no factor in common with n.

Knowing that a residue group is a set of positive integers less than n and relatively prime to n; the phi function, $\phi(n)$, can be defined as the number of elements in the residue group.

$\phi(n)$ = no. of natural numbers $< n$: (a, n) =1

Consider $\phi(4)$:

There are 2 positive integers less than 4 which have no common factor with 4 namely (1 and 3). Hence

- $\phi(4) = 2$

Consider $\phi(7)$:

There are no positive integers less than 7 which have a common factor with 7 since 7 is a prime number.

Therefore we can say that for any prime number p, $\phi(p)$ = p-1

Our attempt is to find $\phi(p^k)$

Let us consider $\varnothing(p^2)$

Consider first $\varnothing(5^2)$

Listing all positive integers less than 25, we obtain

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

1 2... p p+1...... $2p$ $2p$+1............. $3p$ $3p$+1............ $4p$

21 22 23 24 25

$4p$+1............ $5p$ (where $5p$ is p^2 in this case)

Therefore, to find $\varnothing(p^2)$, first list all positive integers less than p^2

1 2 3..... p, p+1.....$2p$,$2p$+1... $3p$,$3p$+1... .p^2

This makes us realize that $p, 2p, 3p, 4p, \ldots p^2$ are the only integers which are not coprime with p^2.

Therefore $\varnothing(p^2) = p^2 - p$

Let us now consider $\varnothing(p^3)$

The positive integers from 1 to p^3 can be divided into p sets:

1	to	p^2		($p^2 - p$ coprimes)
$p^2 + 1$	to	$2p^2$		($p^2 - p$ coprimes)
$2p^2 + 1$	to	$3p^2$		($p^2 - p$ coprimes)
......				
......				
$(p-2)p^2 + 1$	to	$(p-1)p^2$		($p^2 - p$ coprimes)
$(p-1)p^2 + 1$	to	p^3		($p^2 - p$ coprimes)

Each set has $p^2 - p$ coprimes and there are p sets.

$\Rightarrow$ total number of coprimes from 1 to $p^3 = p(p^2 - p)$

$\Rightarrow \phi(p^3) = p(p^2 - p)$

$= p^2(p - 1)$

From this we claim that $\phi(p^n) = p^{n-1}(p - 1)$

Let us prove this by the Principle of Induction

RTP: $\phi(p^n) = p^{n-1}(p - 1)$

Proof

Let $n = 1$

LHS: $\phi(p^1) = p - 1$ (as discussed earlier)

RHS: $p^{1-1}(p - 1) = p^{1-1}(p - 1) = p^0(p - 1) = (p - 1)$

$\therefore$ true for $n = 1$

Assume it is also true for $n = k$

i.e. $\phi(p^k) = p^{k-1}(p - 1)$

We need to prove it is true for $n = k + 1$

i.e. RTP $\phi(p^{k+1}) = p^k\ (p - 1)$

The positive integers from 1 to p^{k+1} can be divided into p groups as in the case of 1 to p^3 earlier on

1	to	p^k		($p^{k-1}(p-1)$ coprimes)
p^k+ 1	to	$2p^k$		($p^{k-1}(p-1)$ coprimes)
$2p^k$+ 1	to	$3p^k$		($p^{k-1}(p-1)$ coprimes)
......				
......				
(p-2)p^k+ 1	to	(p-1)p^k		($p^{k-1}(p-1)$ coprimes)
(p-1)p^k+ 1	to	p^{k+1}		($p^{k-1}(p-1)$ coprimes)

Each set has $p^{k-1}(p-1)$ coprimes and there are p sets.

$\Rightarrow$ total number of coprimes from 1 to $p^{k+1} = p(p^{k-1}(p-1))$

$\Rightarrow \phi(p^{k+1}) = p(p^{k-1}(p-1))$

$= p^k(p-1)$

As $\phi(p^n) = p^{n-1}(p-1)$ holds for $n = 1$ and whenever it is true for $n = k$, it is also true for $n = k + 1$, by the Principle of Induction, the theorem is true for all natural numbers n.

The Eigenvalues of Self Complementary Graphs

Angela Lombardi

Abstract

Self complementary graphs have many interesting properties with reference to their main and non-main eigenvalues. Eigenvalues are a special set of scalars associated with a linear system of equations (i.e., a matrix equation) that are sometimes also known as characteristic roots, proper values, or latent roots. We consider the spectra of self complementary graphs.

A *graph* has a set $\mathcal{V}$ of vertices {1,2,. . . ,n} and a set $\mathcal{E}$ of edges joining distinct pairs of vertices.

Graph Complement The complement of a graph G is the graph $\overline{G}$with the same vertex set but whose edge set consists of the edges not present in G (i.e., the complement of the edge set of G with respect to all possible edges on the vertex set of G).

Example:

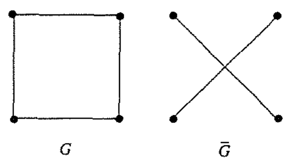


Figure 3: **Graph G and its Complement Graph**

Self Complementary Graphs : A *self-complementary* graph is a graph which is isomorphic to its graph complement.

Next are three examples of self-complementary graphs.

Example 1:

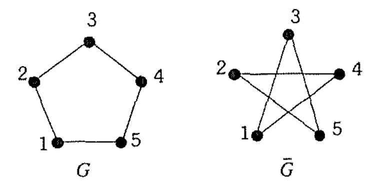


Figure 4: $G = C_5$ **and its compliment** $\overline{G}$

P: $1 \rightarrow 3$
$2 \rightarrow 5$
$3 \rightarrow 2$
$4 \rightarrow 4$
$5 \rightarrow 1$

Example 2:

Figure 5: $G = P_4$ **and its complement** $\overline{G}$

P: $1 \rightarrow 2$
$2 \rightarrow 4$
$3 \rightarrow 1$
$4 \rightarrow 3$

Example 3:

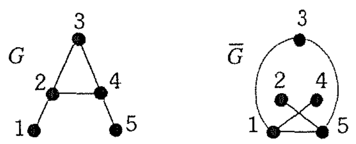


Figure 6: $G = A_G$ **and its compliment** $\overline{G}$

P: $1 \to 4$
$2 \to 1$
$3 \to 3$
$4 \to 5$
$5 \to 2$

An interesting property follows from the definitions given below of the adjacency matrix and its complement.

A is the *adjacency matrix* of a graph G, if it is the $n \times n$ symmetric matrix such that

$$a_{ij} = \begin{cases} 1 & \text{\{i,j\} is an edge of G;} \\ 0 & \text{otherwise.} \end{cases}$$

$\overline{\mathbf{A}}$ is the adjacency matrix of the *complement* $\overline{G}$ of G if it is an $n \times n$ symmetric matrix such that

$$a_{ij} = \begin{cases} 0 & \text{\{i,j\} is an edge of G;} \\ 1 & \text{otherwise.} \end{cases}$$

If **J** is the all 1 matrix and **I** is the identity matrix then

$$\overline{\mathbf{A}} + \mathbf{A} = \mathbf{J} - \mathbf{I} \tag{1}$$

Finding an Antimorphism and an Automorphism

Example1: The adjacency matrix of $\mathbf{C}_5$ is denoted by $\mathbf{A}(\mathbf{C}_5)$.

As we have shown before the mapping from $\mathbf{C}_5$ to its complement may be represented as the permutation $\mathbf{P}$ = (1 3 2 5) (4). By entering the matrices below into Mathematica and using the command Transpose[P].A.P we obtain the following matrices.

$$P = \begin{pmatrix} 0&0&0&0&1 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&1&0&0&0 \end{pmatrix}$$

$$P^T = \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \end{pmatrix}$$

$$\mathbf{A}(\mathbf{C})_5 = \begin{pmatrix} 0&1&0&0&1 \\ 1&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \end{pmatrix}$$

So $\mathbf{P}^T.\mathbf{A}.\mathbf{P}=\overline{\mathbf{A}}$

Therefore $\mathbf{P}$ is an *antimorphism* since it represents a mapping from $\mathbf{A}$ to its complement $\overline{\mathbf{A}}$

Let $\mathbf{Q} = \mathbf{P}^2$= (4) (1 3 2 5) . (4) (1 3 2 5) = (4) (1 2) (3 5)

then

$$Q = \begin{pmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \end{pmatrix}$$

So $\mathbf{Q}^{-1}.\mathbf{A}.\mathbf{Q} = \mathbf{A}$.

Therefore **Q** represents an automorphism since it is a mapping from A onto itself.

Example 2 : The adjacency matrix of $\mathbf{P}_4$ is denoted by $\mathbf{A}(\mathbf{P}_4)$.

The mapping from $\mathbf{P}_4$ to its complement maybe represented as the permutation

$\mathbf{P} = (\ 1\ 2\ 4\ 3\ )$.

So $\mathbf{P}^T.\mathbf{A}.\mathbf{P} = \overline{\mathbf{A}}$

Let $\mathbf{Q} = \mathbf{P}^2 = (\ 1\ 2\ 4\ 3\ )\ .\ (\ 1\ 2\ 4\ 3\ ) = (\ 1\ 4\ )\ (\ 2\ 3\ )$

So $\mathbf{Q}^{-1}.\mathbf{A}.\mathbf{Q} = \mathbf{A}$

Example 3 : The adjacency matrix of the graph $\mathbf{A}_G$ of Figure 6 is denoted by $\mathbf{A}(\mathbf{A}_G)$.

The mapping from $\mathbf{A}_G$ to its complement may be represented as the permutation

$\mathbf{P} = (\ 1\ 4\ 5\ 2)\ (3)$.

So $\mathbf{P}^T.\mathbf{A}.\mathbf{P} =$

Let $\mathbf{Q} = \mathbf{P}^2 = (3)(1452).(3)(1452) = (3)(15)(42)$

So $\mathbf{Q}^{-1}.\mathbf{A}.\mathbf{Q} = \mathbf{A}$

Special Eigenvalues Properties For Self Complementary Graphs:

An eigenvector is said to be *main* if it is not orthogonal to j.

Example 1: For $\mathbf{A(C)_5}$,
the eigenvalues are: $\{2, -1.61803, -1.61803, 0.618034, 0.618034\}$,
and the eigenvectors are : $\{1, 1, 1, 1, 1\}, \{-1.61803, 1.61803, -1, 0, 1\}$,
$\{-1, 1.61803, -1.61803, 1, 0\}, \{0.618034, -0.618034, -1, 0, 1\}, \{-1, -0.618034, 0.618034, 1, 0\}$

Checking if eigenvectors are main:
Since C_5 is regular the only main eigenvector $\{1, 1, 1, 1, 1\}$

$\mathbf{A(C_5)}$ has non-main eigenvalues λ_2, λ_3, λ_4 and λ_5, which can be paired off as follows:

$$\lambda_2 + \lambda_4 = \lambda_3 + \lambda_5 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = -1$$

This follows from equation 1.

Example 2: For $A(P_4)$
the eigenvalues are equal to: $\{-1.61803, 1.61803, -0.618034, 0.618034\}$,
and the corresponding eigenvectors are: $\{-1, 1.61803, -1.61803, 1\}, \{1, 1.61803, 1.61803, 1\}$,
$\{1, -0.618034, -0.618034, 1\}, \{-1, -0.618034, 0.618034, 1\}$

Checking if eigenvectors are main:
For $\lambda_2 = 1.61803$, the eigenvector $\mathbf{x_2}$ is $\{1, 1.61803, 1.61803, 1\}$ If $\mathbf{j} = \{1, 1, 1, 1, \}$ then $< \mathbf{j}, \mathbf{x_2} > \neq 0$. Hence λ_2 is main.

The non-main eigenvalues are λ_1 and λ_3, which can be paired off as follows:

$$\lambda_1 + \lambda_3 = -1$$

Example 3: For $A(A_G)$
the eigenvalues are: $\{2.30278, -1.61803, -1.30278, 0.618034, 0\}$,
and the corresponding eigenvectors are: $\{1, 2.30278, 2, 2.30278, 1\}, \{-1, 1.61803, 0, -1.61803, 1\}$,
$\{1, -1.30278, 2, -1.30278, 1\}, \{-1, -0.618034, 0, 0.618034, 1\}, \{1, 0, -1, 0, 1\}$.

The only non-main eigenvalues are λ_2 and λ_3 which can be paired off as follows:

$\lambda_2 + \lambda_3 = -1$

Justification of the results obtained:

$\overline{\mathbf{A}} + \mathbf{A} = \mathbf{J} - -\mathbf{I}$

$\Rightarrow \mathbf{A} = \mathbf{J} - -\mathbf{I} - \overline{\mathbf{A}}$

$\Rightarrow \mathbf{A}\mathbf{x}_i = \mathbf{J}\mathbf{x}_i - -\mathbf{I}\mathbf{x}_i - -\overline{\mathbf{A}}\mathbf{x}_i$

If λ_i is non-main, then $\mathbf{x}_i.\mathbf{j} = 0$

Thus λ_i $\mathbf{x}_i = \mathbf{0} - \mathbf{x}_i - -\overline{\mathbf{A}}\mathbf{x}_i$ corresponding to a non-main eigenvalue λ_i

So $\overline{\mathbf{A}}\mathbf{x}_i = (\lambda_i\ -1)\mathbf{x}_i$

Since **G** is *self complementary*, the set of eigenvalues of $\overline{\mathbf{A}}$= set of eigenvalues of **A**

For each λ_i, there exists λ_j = - $(\lambda_i + 1)$

So in *self complementary* graphs non-main eigenvalues are paired s.t. $\lambda_j + \lambda_i = - - 1$. Therefore by just looking at the eigenvalues and by pairing them off, we may find the non-main eigenvalues.

The Maths Test

Lyrics: Clinton Paul Cilia Singer: Romina Mamo

Today's the day
That I will face the test

Ring, ring goes my clock
I wake up in a total shock
Feeling down, feeling ill
Should I take myself a pill?

Wish it was over
Then I'd start to live again
But it has to be done... nothing ventured, nothing gained.

In the Maths test... I will try and do my best
Even though I'm scared to death
About Gaussian might forget
In the Maths test... I will give it my best shot
Hope I don't forget the rules
Remember the ones that scare me and you

Oh no it's half-past 8
Half an hour left to go
Will I pass? Will I fail?
What will happen I don't know?

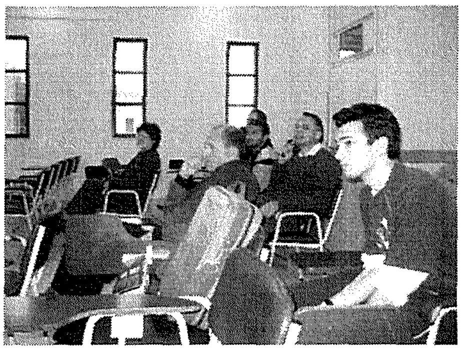
Wish it was over
Then I'd start to live again
But it has to be done... nothing ventured, nothing gained

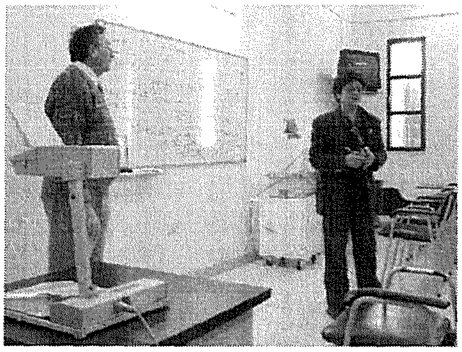
My... Oh my,
The time has come
Can't feel my knees – they're numb
Here it is... whiter than ice

In the Maths test... I will try and do my best
Even though I'm scared to death
About Gaussian might forget
In the Maths test... I will give it my best shot
Hope I don't forget the rules
Remember the ones that scare me and you

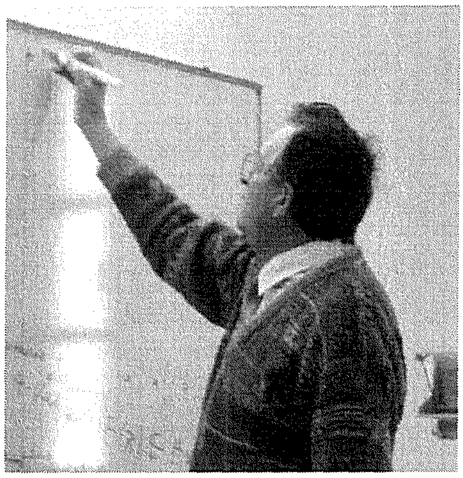
Photographs

The Audience

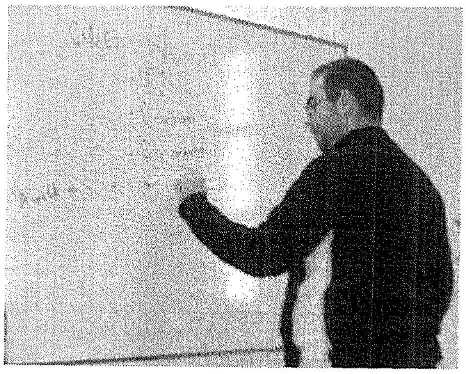




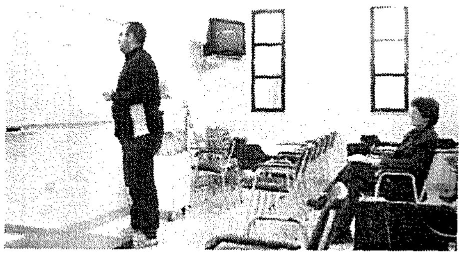
Prof. S. Fiorini and Dr. I. Sciriha



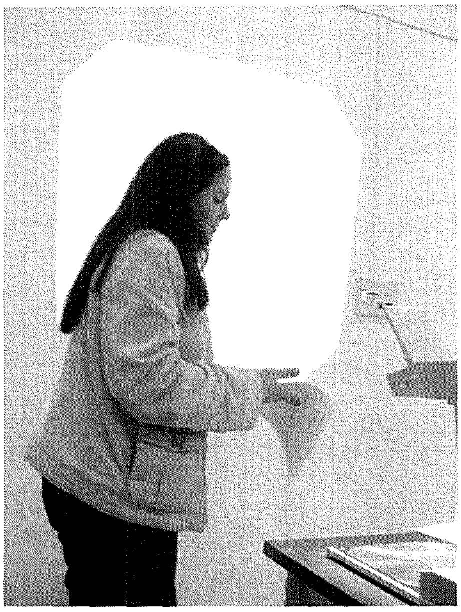
The Professor at the whiteboard



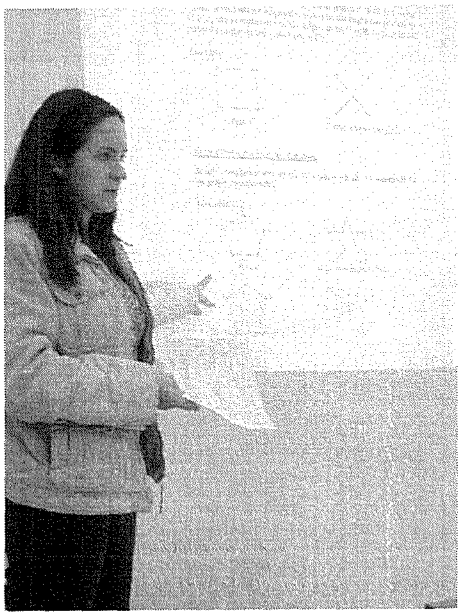
Close up of Andrew Duncan at the board



A shot of Andrew with the organiser



Angela Lombardi preparing the sheets



Explaining to the present audience