A New Form Of Primes

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Abstract: Let \( p_1, p_2, \ldots, p_k, \ldots \) denotes the ordered sequence of primes. We will consider primes of the form \( p = 1 \mod p_k \). Moreover we give a new proof to the converse of W. Lecan's Theorem, first proved by Lagrange.

Definition 1: \( p \) is said to be prime in \( \mathbb{Z} \) if \( p = ab, a, b \in \mathbb{Z} \Rightarrow a = 1 \) or \( b = 1 \).

Lemma 1: If \( a = b \mod (m_1m_2\ldots m_k) \), then \( a = b \mod m_i, \forall i, 1 \leq i \leq k \), where \( a, b, m_1, \ldots, m_k \in \mathbb{Z} \).

Proof:

\[
a = b \mod (m_1m_2\ldots m_k) \Rightarrow m_1m_2\ldots m_k/(b-a) \\
\Rightarrow m_i/(b-a) \forall i, 1 \leq i \leq k \\
\Rightarrow a = b \mod m_i, \forall i, 1 \leq i \leq k
\]

Lemma 2: If \( \gcd(a, b) = 1 \), then the arithmetic progression \( \left(a + bn\right) \) contains infinitely many primes.

N.B. This theorem was first conjectured by Euler in 1785 (with \( a = 1 \)). In 1808, Legendre claimed that he had a proof for this theorem, but later it was found to be false. Finally, in 1837, Dirichlet proved it and it was practically the birth of Analytic Number Theory.

Proposition: Let \( p_1, p_2, \ldots, p_k, \ldots \) be the ordered sequence of primes. Then given any prime \( p_k \), there exists another prime \( p, p > p_k \), s.t. \( p = 1 \mod p \), \( \forall i, 1 \leq i \leq k \).

Proof: Consider the sequence \( \{(p_1p_2\ldots p_k)n + 1\} \). This is an arithmetic progression and \( \gcd(p_1p_2\ldots p_k, 1) = 1 \).

Hence by Dirichlet's Theorem \( \{(p_1p_2\ldots p_k)n + 1\} \) contains infinitely many primes which are of the form \( p = 1 \mod (p_1p_2\ldots p_k) \) \( \Rightarrow p = 1 \mod p \), \( \forall i, 1 \leq i \leq k \).

Conclusion: Twin primes are integral pairs \( (p, p+2) \) in which both members are prime. Examples are 5, 7, 11, 13, and 107, 109. It is conjectured that there are infinitely many twin primes.

Is there a relation between primes of the form \( 1 \mod p \); \( 1 \leq i \leq k \) and twin primes?