Homomorphisms and the number of Divisors

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Lemma 1 *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ be defined by $f(x) = x^n$. Then f is a homomorphism.*

Proof: $f(x)f(y) = f(xy)\ \forall x, y \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$.

Theorem 2 *Let*

$$F(p) = \sum_{m|p} m^n$$

for some $n \in \mathbb{N}$ and where p is prime and the summation runs over the divisors of p. Then F is a homomorphism under multiplication.

Proof: The divisors of p are 1 and p.

Hence

$$\begin{aligned} F(p_1)F(p_2) &= ({p_1}^n + 1^n)({p_2}^n + 1^n) \\ &= {p_1}^n {p_2}^n + 1 + {p_1}^n + {p_2}^n \end{aligned}$$

The divisors of p_1p_2 are p_1p_2, p_1, p_2 and 1.

Hence $F(p_1p_2) = {p_1}^n {p_2}^n + 1 + {p_1}^n + {p_2}^n$.

Consequently, $F(p_1)F(p_2) = F(p_1p_2)$.

This can be extended to any integer.

Corollary 3 *If f is a homomorphism under multiplication, then so is $F(n)$, defined by*

$$F(n) = \sum_{m|n} f(m)$$

where the sum is over all divisors of any integer n. Therefore, $F(x)F(y) = F(xy)$.

Application 1: Using the above corollary, we see that the number of divisors $d(n)$ of n, where

$$d(n) = \sum_{m|n} 1$$

is a homomorphism under multiplication, since $f(x) = 1$ is.

Let's consider an example: $63 = 7 \times 9$

The divisors of 63 are 63, 21, 9, 7, 3 and 1. Therefore $d(63) = 6$.

The divisors of 7 are 7 and 1. So $d(7) = 2$.

The divisors of 9 are 9, 3 and 1. So $d(9) = 3$.

By the above argument, $d(63) = d(9)d(7)$, which indeed it is, since $6 = 2 \times 3$.

Application 2: If

$$F(x) = \sum_{m|n} f(m)$$

where $f(m) = m^3$, then $F(63) = F(7)F(9)$.

The divisors of 63 are 63, 21, 9, 7, 3 and 1.
$F(63) = 63^3 + 21^3 + 9^3 + 7^3 + 3^3 + 1^3 = 260408$.

The divisors of 7 and 7 and 1, and those of 9 and 9, 3 and 1.
$F(7)F(9) = (7^3 + 1^3)(9^3 + 3^3 + 1^3) = 260408$.

This confirms our result that F is a homomorphism.

We now consider the sum of cubes of numbers.
It is "common knowledge" that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$$

Thus

$$\sum_{r=1}^{n} r^3 = \left(\frac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$$

Hence the set of numbers $\{1, 2, \ldots, n\}$ has the property that the sum of its cubes is the square of its sum. Are there any other collections of numbers with this property?

Let's consider the following argument.
Pick any number, for example 63. List the divisors of 63, and for each divisor of 63, count the number of divisors it has:

63 has **6** divisors (63, 21, 9, 7, 3, 1)
21 has **4** divisors (21, 7, 3, 1)
9 has **3** divisors (9, 3, 1)
7 has **2** divisors (7, 1)
3 has **2** divisors (3, 1)
1 has **1** divisor (1).

The resulting collection of numbers has the same property. Namely:

$$6^3 + 4^3 + 3^3 + 2^3 + 2^3 + 1^3 = 324 = (6 + 4 + 3 + 2 + 2 + 1)^2$$

$[d]^3$ is a homomorphism under multiplication, and Corollary 3 shows that

$$\sum_{m|n} d^3(m)$$

is also a homomorphism under multiplication.

Also, from Corollary 3, squaring gives that

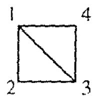
$$\left(\sum_{m|n} d(m)\right)^2$$

is a homomorphism under multiplication. Using a similar argument as before, it can finally be shown that

$$\sum_{m|n} d^3(m) = \left(\sum_{m|n} d(m)\right)^2$$

The Total Number of Non-Isomorphic Simple Graphs with n Vertices and k Edges

Definition: A graph is defined as $G := (V, E)$, $E \subseteq V \times V$, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G.

E.g. G 

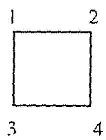
$V(G) = \{1, 2, 3, 4\}$ $E(G) = \{12, 13, 14, 23, 24\}$

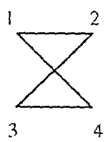
Definition: G is s.t.b. simple if it has no loops or multiple edges.

E.g.
loop (edge from 1 to 1) 

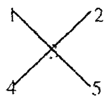
multiple edges from 2 to 3

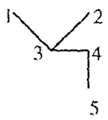
Definition: G_1 is **isomorphic** to G_2, denoted $G_1 \cong G_2$, if $\exists\, f : V(G_1) \to V(G_2)$ bijective s.t. if $v, w \in V(G_1)$ then $vw \in E(G_1) \Leftrightarrow f(v)f(w) \in E(G_2)$.

E.g. G_1 

G_2 

G_1 and G_2 are isomorphic, where
$f(1) = 1$, $f(2) = 2$, $f(3) = 4$, $f(4) = 3$,
so that if $vw \in E(G_1)$ then $f(v)f(w) \in E(G_2)$
e.g. $13 \in E(G_1)$ and $f(1)f(3) = 14 \in E(G_2)$

E.g. G_1 

G_2 

G_1 not isomorphic to G_2 (see Result 1)

Definition: Let $v \in V(G)$. The valency of v, denoted $\rho(v)$, is the number of edges incident to v. Also, $w \in V(G)$ is s.t.b. adjacent to v if $vw \in E(G)$.

Result 1: Let $S(G) := \{\rho(v) : v \in V(G)\}$, i.e. the set of valencies of vertices of G. Then $G_1 \cong G_2 \Rightarrow S(G_1) = S(G_2)$.

Proof: Let $v \in V(G_1)$. Thus $f(v) \in V(G_2)$.

Suppose $v_1, v_2, \ldots, v_k$ are all the vertices adjacent to v.

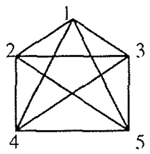
So $f(v_1), f(v_2), \ldots, f(v_k)$ are all the vertices adjacent to $f(v)$

s.t $vv_1 \in E(G_1) \Leftrightarrow f(v)f(v_1) \in E(G_2)$.

So $\forall\ v \in V(G)\ \rho(v) = \rho(f(v))$, and since f is 1 - 1 and onto we get the result.

Definition: A simple graph with n vertices is s.t.b. complete and denoted K_n if $\forall$ v, w $\in$ V(G) vw $\in$ E(G).

E.g. K_5



Result 2: Let G_k^n be the class of simple graphs with vertices labelled 1, 2, ..., n and k edges.

Then $\left|G_k^n\right| = \binom{\frac{n(n-1)}{2}}{k} = \binom{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2} - k} = \left|G^n_{n(n-1)/2-k}\right|$.

Proof: For a simple graph with n vertices we have at most $|E(K_n)| = n(n-1)/2$ edges (for all of the n vertices there are (n - 1) incident edges but every edge connects 2 vertices, hence $n(n-1) = 2|E(K_n)|$).

So the no. of different graphs in the class G_k^n is equal to the number of ways of choosing k edges out of the n(n - 1)/2 total no. of edges.

Result 3: The isomorphism relation $\cong$ on graphs is an equivalence relation.

Proof: reflexive: $G_1 \cong G_1$ (trivial)

symmetric: Let $V(G_1) = \{v_1, v_2, ..., v_n\}$. If $G_1 \cong G_2$ then $\exists\ f : V(G_1) \to V(G_2)$ bijective s.t. $v_iv_j \in E(G_1) \Leftrightarrow f(v_i)f(v_j) \in E(G_2)$.

f bijective $\Rightarrow V(G_2) = \{f(v_1), ..., f(v_n)\}$. Let $x_i = f(v_i)$.

Also, f bijective $\Rightarrow \exists\ f^{-1} : V(G_2) \to V(G_1)$ bijective s.t.

$x_ix_j = f(v_i)f(v_j) \in E(G_2) \Leftrightarrow v_iv_j = f^{-1}(f(v_i))f^{-1}(f(v_j)) \in E(G_1)$.

Hence $G_2 \cong G_1$.

transitive: Let $G_1 \cong G_2$ be defined as in the above symmetric case.

Let $G_2 \cong G_3$. Thus $\exists$ g : $V(G_2) \rightarrow V(G_3)$ bijective s.t.

$x_i x_j = f(v_i)f(v_j) \in E(G_2) \Leftrightarrow g(x_i)g(x_j) = g(f(v_i))g(f(v_j)) \in E(G_3)$.

f, g bijective $\Rightarrow g_o f$ bijective where

$v_i v_j \in E(G_1) \Leftrightarrow g_o f(v_i) g_o f(v_j) \in E(G_3)$. Hence $G_1 \cong G_3$.

Definition: The complement of simple graph G with n vertices, denoted $\overline{G}$, is the graph s.t. $E(G) \cup E(\overline{G}) = E(K_n)$ and $E(G) \cap E(\overline{G}) = \phi$, i.e.

if v, w $\in$ V(G) then $vw \in E(G) \Leftrightarrow vw \notin E(\overline{G})$ and $vw \notin E(G) \Leftrightarrow vw \in E(\overline{G})$.

Result 4: $G \cong G' \Leftrightarrow \overline{G} \cong \overline{G'}$.

Proof: Let v, w $\in$ V(G) and let f be the isomorphism from G to G'.

$vw \in E(\overline{G}) \Leftrightarrow vw \notin E(G) \Leftrightarrow f(v)f(w) \notin E(G') \Leftrightarrow f(v)f(w) \in E(\overline{G'})$.

Remark: *Result 2* and *Result 4* imply that tackling the problem for the case with k edges is equivalent to tackling it for the case with [n(n-1)/2 - k] edges (i.e. number of edges of complementary graph), because from *Result 4* we easily deduce that for complementary graphs the sizes of classes of isomorphic graphs are equal (i.e. $|[G]| = |[\overline{G}]|$) and from *Result 2* we already know that the total number of different graphs are also equal (refer to *Conclusion*).

Conclusion:

Since an equivalence relation on elements of a set gives a partition of the set into equivalence classes, from the important *Result 3* we get that the set of different vertex-labelled simple graphs with n vertices and k edges is partitioned into classes of isomorphic graphs of this type. Hence if N_k^n is the number of classes of our set of graphs then there are N_k^n non-isomorphic simple graphs with n vertices and k edges. Let $[G_k^n]_i$ denote the i'th class of isomorphic graphs of this type.

Hence $\left|G_k^n\right| = \sum_{i=1}^{N_k^n} \left|[G_k^n]_i\right| = \binom{\frac{n(n-1)}{2}}{k}$. If the class sizes were all the same then this would be an easy problem to solve, however this is not the case.

This might be an approach to finding a way of obtaining all the non-isomorphic graphs. So if

- we have an efficient straight-forward formulation which determines if 2 graphs are isomorphic or not rather than considering all the n! vertex bijections, in particular something stronger than *Result 1* which gives a helpful sufficient condition rather than a necessary one, and
- we have a formula which determines the size of the class of graphs isomorphic to any given graph,

then we have a method of finding N_k^n which works by generating non-isomorphic graphs and finding their size until the sum of sizes equals $\binom{\frac{n(n-1)}{2}}{k}$, but this would still be an algorithm rather than a formula.