The Use of Choice

Dr. David Buhagiar

The aim of this note is to present two examples, one shown use of the Axiom of Choice and the other that of Zorn's Lemma in Mathematics. We begin by stating the mentioned two equivalent axioms.

Axiom of Choice 1 *Let X be a nonempty set. Then for each nonempty subset $S \subseteq X$ it is possible to* choose *some element $s \in S$. That is, there exists a function f that assigns to each nonempty set $S \subseteq X$ some representative element $f(s) \in S$. Such a function f is called a* choice function.

To state Zorn's Lemma we need the following definition.

Partially Ordered Set, Chain 1 *A set X is said to be* partially ordered *if there is a* partial ordering *defined on it, that is, a binary relation $\preceq$ that satisfies the conditions*

1. $x \preceq x$ *for every* $x \in X$,
2. *if* $x \preceq y$ *and* $y \preceq x$, *then* $x = y$,
3. *if* $x \preceq y$ *and* $y \preceq z$, *then* $x \preceq z$.

If $x, y \in X$ are such that neither $x \preceq y$ nor $y \preceq x$ holds, then they are called incomparable. *Otherwise they are called* comparable. *A* chain *or* linearly ordered set *is a partially ordered set such that every two elements of the set are comparable. An* upper bound *of a subset Y of a partially ordered set X is an element $u \in X$ such that $y \preceq u$ for every $y \in Y$. A* maximal element *of X is an element $m \in X$ such that $m \preceq x$ implies $m = x$.*

Zorn's Lemma 1 *If every chain in a partially ordered set has an upper bound, then the partially ordered set has a maximal element.*

The axiom of choice was formulated by Zermelo in 1904. Sixty years later, in 1963, Paul Cohen showed that the Axiom of Choice cannot be proved from the axioms of Zermelo-Fraenkel set theory (the standard set theoretical axioms). Thus, if we want to use the Axiom of Choice we need to include it in our set theory. It can be proved that the Axiom of Choice is equivalent to Zorn's Lemma.

Let us now see one example of the use of the Axiom of Choice and one example of the use of Zorn's Lemma.

Example 1 *Remember that a sequence* (x_n) *in a metric space* (X,d) *converges to an element* $x \in X$ *if for every positive real number* ϵ *there exists some* $n_0 \in \mathbb{N}$ *such that* $d(x_n, x) < \epsilon$ *for every natural number* $n \succeq n_0$. *Let* Y *be a subset of* X. *One usually defines closure points of* Y *in either (or both) of the following two ways:*

I $x \in X$ *is a* closure point *of* Y *if there exists a sequence* (x_n) *in* Y *which converges to* x.

II $x \in X$ *is a* closure point *of* Y *if for every positive real number* ϵ *there exists some* $y \in Y$ *with* $d(x,y) < \epsilon$.

We then go on to prove that (I) and (II) are equivalent.

Proof:

(I) $\Longrightarrow$ *(II): Given any* $\epsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $d(x_n, x) < \epsilon$ *for every* $n \succeq n_0$. *In particular,* $d(x_{n_0}, x) < \epsilon$ *and* $x_{n_0} \in Y$.

(II) $\Longrightarrow$ *(I): The usual prove proceeds as follows: Let* $Y_n = \{y \in Y : d(y,x) < \frac{1}{n}\}$. *By (II),* $Y_n \neq \emptyset$ *for all* $n \in \mathbb{N}$. *Let* (x_n) *be a sequence such that a point* $x_n \in Y_n$ *for all* $n \in \mathbb{N}$. *Then each* $x_n \in Y$ *and* (x_n) *converges to* x.

What is usually overlooked is to justify the reasons we have to assume that such a sequence (x_n) *exists. In fact, if one takes* (X,d) *to be the real line* $\mathbb{R}$ *with standard metric, it has been proved that the equivalence of (I) and (II) for all* $Y \subset \mathbb{R}$ *cannot be proved from the axioms of Zermelo-Fraenkel set theory alone. Of course, if we include the Axiom of Choice in our set theory, the fact that* $Y_n \neq \emptyset$ *for all* $n \in \mathbb{N}$ *immediately implies that such a sequence exists.*

Example 2 *One of the fundamental facts on vector spaces is that every non-zero vector space* V *has a linear basis. This result is a straightforward application of Zorn's Lemma.*

Let X *be the set of all linearly independent subsets of* V. *Since* $V \neq \{0\}$, *it has an element* $v \neq 0$ *and* $\{v\} \in X$, *so that* $X \neq \emptyset$. *We define a partial ordering on* X *by set inclusion. If* $Y \subset X$ *is a chain in* X, *the union of all the elements of* Y *gives an upper bound for* Y. *By Zorn's Lemma,* X *has a maximal element* B. *We are left to show that* B *is a linear basis for* V. *Let* $U = \mathrm{span}\ \mathrm{B}$. *Then* U *is a subspace of* V. *If* U *is a proper subspace of* V *and* $z \in V - U$, *then* $B \cup \{z\}$ *would give a linearly independent set containing* B *as a proper subset, contradicting the maximality of* B.

The above result cannot be proved in Zermelo-Fraenkel set theory alone, without using Zorn's Lemma.

Another axiom equivalent to the Axiom of Choice is the *Well-Ordering Principle*. Remember that a linear ordering $<$ on a set X is a *well-ordering* if every nonempty subset of X has a $<$-minimal element. The structure $(X, <)$ is then called a *well-ordered set*.

The Well-Ordering Principle 1 *Every set can be well-ordered.*

We end this note by asking ourselves: Is the Axiom of Choice "true"?. According to J.L.Bona in a private communication with E.Schechter in 1977,

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The Axiom of Choice is obviously true; the Well Ordering Principle
is obviously false; and who can tell about Zorn's Lemma?
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