The Square of a Two-Digit Number

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Abstract: We are concerned with the square of two-digit numbers. This is basically done by the "difference of two squares" formula which gives a simple way of obtaining the result, thus: (x2 means x squared, and a2 means a squared)

$x2 - a2 = (x+a)(x-a)$

Adding a2 to both sides, we get:

$x2 = (x+a)(x-a) + a2$ --------(1)

Note that equation (1) above is true for all real x and a. Here, we're considering only two-digit natural numbers. Let us show how it works for the square of 47, we just substitute x for 47 in equation (1), to yield:

$47^2 = (47+a)(47-a) + a2$

This is true for all a.

Now comes the issue of choosing an appropriate a. To make the calculation as easy as possible, we must have two criteria for the choice of a:

(i) a must be small, in the range $0 \le a \le 9$, so that we can easily find a2.
(ii) a must be such that the product (x+a)(x-a) is as simple to calculate as possible

To satisfy the above criteria, a is chosen as follows:

Let the last digit of x be n (i.e. n = x mod 10). Then a := n, if n=0,1,2,3,4, or 5

a := 10-n, if n=6,7,8, or 9.

So, in the above example, we choose a=10-7=3

So 47^2 = (47+3)(47-3) + 3^2 = 50 x 44 + 9

We note that 50x44 can be calculated easily (=2200), and adding 9 to this we get

47^2 = 2209

The important thing is that the above calculation can be made shorter by first calculating 44x5, and then 'appending' 9 to the answer!

We give another example: 76^2

76^2 = (76+4)(76-4) + 4^2 = 80 x 72 + 16 = 5760 + 16 = 5776

In this case, after calculating 72x8, we do not append '16', but we add 1 to the answer and then 'append' 6, viz:

```
 72x
  8
----
 576+
   1
----
 577|6
```

A better idea is using the following algorithn to find the square of any two-digit number n:

Step 1) Find the smallest a such that (n+a) or (n-a) is divisible by 10. Let this multiple of 10 be t, and let u be t divided by 10.

Step 2) If t>n, then write the following:

```
(n-a)|a
   u|a
  ----
```

Otherwise, write:

```
(n+a)|a
   u|a
  ----
```

Step 3) Now work out (n-a) x u (or (n+a) x u). Let result be p.

Step 4) Work out a x a to give q.

Step 5) Result is the number written as p x 10 + q.

To illustrate how this algorithm works for the various values of x mod 10, we give three further examples:

Let's calculate 82 ^ 2:

Step 1: a=2, t=80, u=8

Step 2:

```
    84|2
     8|2
    ----
```

Step 3:

```
    84|2
     8|2
    ----
      4
```

Step 4:   3

```
    84|2
     8|2
    ----
   672 4
```

Therefore, 82^2=6724.

Another example: 46^2:

Step 1: a=4, t=50, u=5

Step 2:

```
 42|4
  5|4
 ----
```

Step 3:

```
  1
 42|4
  5|4
 ----
    6
```

Step 4:

```
 11
 42|4
  5|4
 ----
211 6
```

Therefore, 46^2=2116.

Now let's present some optimisations to the method. The algoritm is particularly easy for x mod 10 = 5 as we show in the following example. First, when the last digit is 5, we said above that we choose a=5. This results in BOTH (x+5) and (x-5) being divisible by 10. For example, in 55^2

55^2 = (55+5)(55-5) + 25 = 50 x 60 + 25

Or, in the step-by-step method:

```
  2
 50|5
  6|5
 ----
302 5
```

In this case, a simpler algorithm can be used:

Step 1: Let first digit of the number we want to square be x. Find x(x+1)

Step 2: Append '25' to the answer!

The above method is very well-known.

The other optimisation to the method is for numbers $n>90$. In these cases, the choice of a simplifies the algorithm considerably.

If we consider (93^2), we have:

$$93^2 = (93+7)(93-7) + 7^2 = 100 \times 86 + 49$$

Hence, we can just calculate 93-7, and then append 7^2 to the answer! The algorithm for these values of x is as follows:

Step 1: Let a be 100-x
Step 2: Write down (x-a)
Step 3: Append a^2 to the number just written down!

Finding the square of numbers with more than two digits is possible by using recursion. The algorithm may be developed:

$$x^2 = (x+a)(x-a) + a^2$$

we can find a^2 by the same method! For example, to find the square of 237, a possible method is: first we put a=37, thus:

$$237^2 = (237+37)(237-37) + 37^2 = 274 \times 200 + 37^2 = 54800 + 37^2$$

Then we calculate 37^2 by putting a=3, thus:
$37^2 = (37+3)(37-3) + 9 = 40 \times 34 + 9 = 1360 + 9 = 1369$
Hence, $237^2 = 54800 + 1369 = 56169$
I doubt, however, if this method is any quicker than by multiplying out in the usual way. I'm mentioning it for the sake of information.