Boolean Matrices

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In this short note, we will use Boolean matrices to help us find some interesting graph theoretical properties regarding the distance and index of a graph.

First, we will give some definitions that are necessary for the theorems which follow.

Definition: A *Boolean Matrix* is a square matrix, all of whose entries are either 0 or 1. These entries are added and multiplied together in the following fashion:

Addition	Multiplication
$x + 0 = x$	$x \cdot 0 = 0$
$x + 1 = 1$	$x \cdot 1 = x$

where $x \in \{0, 1\}$.

Definition: The *Adjacency Matrix* of a graph G is a symmetric $n \times n$ matrix, where n is the number of vertices in the graph, whose ij^{th} entry is 1 if there is an edge connecting vertex i and vertex j in the graph, and 0 otherwise, for all $1 \leq i, j \leq n$.

The following lemma is a well-known property of all adjacency matrices.

Lemma 1 *The ij^{th} entry of the r^{th} power of the adjacency matrix gives the number of walks of length r from vertex i to vertex j in G.*

In the sequel, we are not interested in the number of walks of length r between any two vertices, but rather whether there exists a walk of the same length between those two vertices or not. For this reason, we consider the adjacency matrix to be a Boolean matrix so that we can only get 0 or 1 for the entries of A^r.

Definitions: The *distance* d_{ij} between vertices i and j is the length of the shortest walk between those two vertices. The *diameter* of a graph G, denoted by D, is $max(d_{ij}) \forall i, j$.

The definition of the diameter D above can be shown to be equivalent to the following:

Lemma 2 *D is the smallest $m \in \mathbb{N}$ such that $A + A^2 + \ldots + A^m = J$, where A is the adjacency matrix of G and J is the $n \times n$ matrix whose entries are all 1.*

Proof: From Lemma 1, we know that the ij^{th} entry of A^r for all $r \in \mathbb{N}, 1 \leq i, j \leq n$ is 1 if there exists a walk of length r between vertices i and j in G, and 0 otherwise. Expanding on this knowledge, the ij^{th} entry of $A + A^2 + \ldots + A^r$ is 1 if there exists a walk of length r or less between the same two vertices, and 0 otherwise. If m is an integer such that $A + A^2 + \ldots + A^m = J$ but $A + A^2 + \ldots + A^{m-1} \neq J$, then for all pairs of vertices, there exists a walk of length less than of equal to m between them. and m is the smallest number with this property. Thus, $m = D$, as required.

A definition similar to that of the diameter will be given now.

Definition: The *walk-index* of a connected graph with an odd circuit, denoted by Γ, is the smallest $m \in \mathbb{N}$ such that $A^m = J$. That is, Γ is the smallest possible number such that there exists a walk of exactly length Γ between any pair of vertices in G.

Note that we only consider non-bipartite graphs (graphs with an odd circuit) in the above definition. This is because Γ of bipartite graphs cannot be found.

Clearly $A^{\Gamma+k} = J$ for all $k \geq 0$. It is also obvious that $\Gamma \geq D$. In fact, it can be shown that $D \leq \Gamma \leq 2D$. Thus, it might be interesting to find properties of graphs with $\Gamma = D$ and graphs with $\Gamma = 2D$. Here, we will only give a small class of graphs whose $\Gamma = 2D$. First, however, we give yet another definition.

Definition Let H and K be any two graphs. The *coalescence* of H and K, denoted by $H : K$, is the graph where r vertices of H are identified with r vertices of K, where $r \geq 1$.

Theorem: Let K be any odd circuit and let H be a connected bipartite graph. Let G be either K or $H : K$ where only one vertex v in H is identified with one vertex in K. Then $\Gamma = 2D$.

Proof: Suppose $G = K$, an odd circuit. Then $D = \frac{n-1}{2}$. If D is odd, then for all vertices v' in K, there does not exist any walk from v' to itself. If D is even, then for all pairs of adjacent vertices v_1 and v_2 in K, there does not exist any walk from v_1 to v_2. So Γ cannot be D. Thus we try for $\Gamma = D + k$ for all $k = 1, 2, \ldots, D$. In all cases except when $k = D$, the same thing as above happens. When $k = D$, i.e. when $\Gamma = 2D$, all vertices in K can be reached via a walk of length $2D = n - 1$. Thus $\Gamma = 2D$ when $G = K$.
Now suppose $G = H : K$ where only one vertex v in H is identified with a vertex in K. Consider vertex w in H with maximum distance to vertex v, and let this distance be d. Thus, the diameter of graph G is $d + \frac{n-1}{2}$ where n is the number of vertices in the odd circuit K. Now Γ cannot be odd since an odd walk from w to itself is impossible. So consider an even walk from w to a vertex adjacent to

w. Since H has no odd circuits, this walk must pass through K. Thus, this walk must be of length $d+n-1+d = 2d+n-1 = 2D$. Therefore, $\Gamma = 2D$, as required.

We end this brief note with two conjectures.

Conjecture 1: The coalescence of H and K above can have an identification of more than 1 vertex and still $\Gamma = 2D$.

Conjecture 2: The converse of the theorem above is also true.