The Collection

2000

The Collection

edited by

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Foreword

> A **lemma** is a result which helps in the proof of a future theorem but is not deemed to be of sufficient importance by itself to warrant the title "**Theorem.**" To distinguish between the two is often a DILEMMA.

In 1998 and 1999 my students were producing interesting "new" proofs for well known results. On their suggestion and my commendation, we decided to start sharing them. This was the start of the **Collection workshops** and the **Collection.**

The following pages tell the rest of the story so far.

Dr. Irene Sciriha.
Organisor.

UNIVERSITÀ TA' MALTA
ısida – Malta
PARTIMENT TAL-MATEMATIKA

UNIVERSITY OF MALTA
Msida – Malta
DEPARTMENT OF MATHEMATICS

27th September, 1999

Dear Mathematician,

Have you ever felt proud of yourself when you produced a neat solution to a mathematical problem or an elegant proof to a mathematical truth? Do you have a hunch that some simple mathematical truth is true but is hard to prove?

Then send an abstract (summary) of your mathematical discovery to

Ms. Annabelle Attard
The Collection
University of Malta
Department of Mathematics
Msida

A workshop will be held to share these ideas among people who find pleasure in the preciseness and elegance of mathematics.

Yours sincerely,

Dr. I. Sciriha

4^{th} January, 2000

The Collection
Department of Mathematics

Collection Workshop I

A workshop is being held on Wednesday, 16^{th} February 2000 from 3.00 to 4.00 p.m. to share some interesting mathematical ideas among people who find pleasure in the elegance and preciseness of mathematics.

Venue: University of Malta, Maths and Physics Building, Department of Mathematics, Room 316

Speakers: Ms Phoedra Cassar
Mr Peter Borg
Mr Alex Farrugia
Ms Josephine Debattista
Dr. I. Sciriha

We shall end with a brief session for spontaneous problem posing and/or solving. You are cordially invited to attend.

Abstracts of possible proofs or conjectures which you wish to share with us in this meeting, or in a future one, may be sent to Dr. I. Sciriha or Ms. A. Attard, Department of Mathematics, (marked "The Collection"), at any time of the year.

Dr. I. Sciriha

p.s. For these two years I am the convenor of the European Women in Mathematics (EWM). Budding, amateur and professional mathematicians who wish to become members of the EWM can contact me.

THE APPEAL OF OPERATIVE PROOFS
RECREATION WITH RECTANGLES

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ABSTRACT. Besides offering an aesthetic attraction, the use of proofs, which rely on simple geometrical figures suggest intuitive ideas. Two such proofs are presented here. One is a recreation with a stair-like decomposition of a rectangle. The second is a simple game that proves Pythagoras Theorem.

1. DECOMPOSE 2000

During the months leading to the first use of the number 2000 in our date, every newspaper carried articles spelling doom either from the Y2K Bug that threatened a worldwide computer meltdown, or from catastrophes foretold by pessimistic sects. To dispel my fears I thought it would be a good idea to decompose the number 2000. Small fragments do not pose any danger!

We look for playful ways to decompose the number 2000 into two arithmetic progressions each containing 40 terms.

The technique we use is often applied in Discrete Mathematics where addition of terms is performed in different ways, often by displaying them as an array and then adding either the rows or the columns or both in turn.

Date: 16th February 2000.

The tools we need are simple. We use the formula for the area of a rectangle which is known to be the product of the length and the breadth. We also use cardboard, a pair of scissors and a pencil.

A 40cm by 50cm rectangle is cut out. A stair-like picture is drawn as shown in Fig.1.

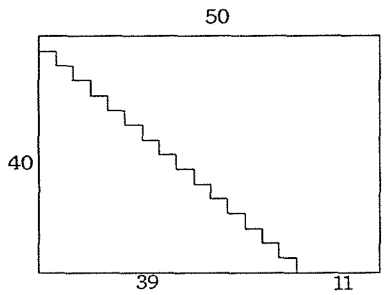


FIGURE 1. *Terms of A.P. are areas of rows.*

The terms of the A.P.s are given by the area of the rows on either side if the stairs. Thus $2000 = (0 + 1 + 2 + ... + 39) + (11 + 12 + ... + 50)$.

More trivial examples are: $2000 = (25 + 25 + ... + 25) + (25 + 25 + ... + 25)$,
$2000 = 2(0 + 1 + 2 + ... + 39) + (11 + 11 + ... + 11)$ and
$2000 = 2(1 + 2 + ... + 40) + (9 + 9 + ... + 9)$. (See Fig.2)

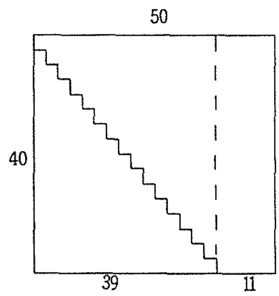


FIGURE 2. *Various divisions of a square.*

2. A Simple Proof of Pythagoras Theorem

We divide two cardboard squares shown in Figs.2 and 3, each of side $a + b$, in two different ways.

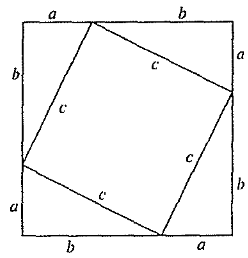


Figure 3. *Terms of A.P. are areas of rows.*

The four triangles of Fig.2 are congruent and right angled with hypotenuse length c. If they are cut out, we are left with a square of side length c.

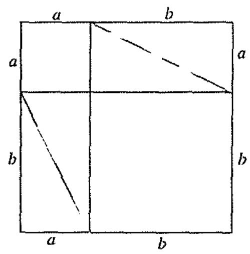


Figure 4. *Various divisions of a square.*

These four triangles fit in the rectangles $a \times b$ shown in Fig.3. The rest of the square of Fig.3 is made up of two squares of side a and b respectively.

It follows that $c^2 = a^2 + b^2$.

$\sqrt{2}$ and Eulerian Primes

Joseph Muscat
Dept of Mathematics
Faculty of Science
University of Malta.

1. A simple proof that $\sqrt{2}$ is irrational.

Suppose $\sqrt{2} = \frac{m}{n}$. Then it follows that

$$2n^2 = m^2$$

But this equates an odd product of primes to an even product of primes (because squares of integers have each prime repeated twice). This contradicts the fundamental theorem of arithmetic.

This proof can be generalised to show that $\sqrt{k}$ is irrational for k that has an odd number of primes in its decomposition. For example, $\sqrt{5}, \sqrt{8}$ etc are all irrational.

In fact it can be strengthened even further to show that $\sqrt{k}$ is irrational unless k is already a square. Simply repeat the argument considering any particular prime that divides k.

2. Write down the arithmetic sequence 41, 42, ... in a spiral form; along a diagonal we get the **Eulerian primes.**

57	56	55	54	**53**
58	45	44	**43**	52
59	46	**41**	42	51
60	**47**	48	49	50
61	62	63	...	...

One can find a formula for these numbers: $n^2 - n + 41$. These give a prime for $n = 0 \ldots 40$ (but not for all n).

There do exist polynomials of very high degree and order (approx. 20) whose range is the set of primes.

A Basic Number Theoretic Result

Peter Borg

B.Sc. 2nd Year

Abstract: We are going to give a new proof that if the greatest common divisor of any two integers a and b is an integer t, then there must exist two integers x and y such that t = xa + yb.

First of all we are going to use the following symbols:

$\exists$	-	there exist/s
$\forall$	-	for all
gcd(a,b)	-	greatest common divisor of a and b
s.t.	-	such that
Z	-	set of integers
Z^+	-	set of positive integers

Putting the above statement in a more mathematical form we have:

If gcd(a,b) = t, where a, b, t $\in$ Z, then $\exists$ x, y $\in$ Z s.t. t = xa + yb.

The following is the proof broken down into small parts:

- By definition gcd(a,b) = t means that there exist two integers λ and μ such that $\lambda t = a$ & $\mu t = b$.

  Now we claim - or rather we can show - that gcd(λ,μ) = 1.

  Suppose gcd(λ,μ) = α and $\alpha > 1$.

  $\therefore \exists \lambda 1, \mu 1 \in Z$ s.t. $\lambda 1 \alpha = \lambda$ and $\mu 1 \alpha = \mu$

  But $\lambda 1(\alpha t) = \lambda t = a$ and $\mu 1(\alpha t) = \mu t = b$

$\therefore$ gcd(a,b) = αt and $\alpha t > t$.

This is a contradiction because t is the greatest common divisor.

Hence, we have proved that gcd(λ,μ) = 1.

- We can reduce the problem to the following:

  To prove that $\exists$ x, y $\in$ Z s.t. $1 + y\mu = x\lambda$.

  This is because

  - if we are going to try to prove that $t = xa + yb$, we might as well divide throughout by t and try to prove that $\exists$ x, y $\in$ Z s.t. $x\lambda + y\mu = 1$, and
  - since y is any integer we can replace y by $-y$ and get the modified statement.

- Suppose that $\forall$ x, y $\in$ Z $x\lambda + y\mu \neq 1$. This means that $1 + y\mu = x\lambda + r$ $\forall$ x, y $\in$ Z, where r $\in$ Z.

  Now, for any value of y we can find an x and an r such that $0 < r < \lambda$ (the basic division algorithm).

  This means that there are at most (λ - 1) different remainders.

- First, we should consider the special case $\lambda = 1$.

  In this case, put x = 1 and y = 0. Therefore r = 0. Hence, in this case, we have proved the original statement.

- Consider the following sequence of equations:

  $1 + 0\mu = 0\lambda + 1$

  $1 + 1\mu = x_1\lambda + r_1$

$1 + 2\mu = x_2\lambda + r_2$

...

$1 + n\mu = x_n\lambda + r_n$

...

We have solved the case for $\lambda = 1$. Now we should consider $\lambda \neq 1$.

Since there are at most $(\lambda - 1)$ different remainders:

$\exists\, i, j \in Z^+, j - i < \lambda,$ s.t. $r_i = r_j$

$\therefore\ 1 + i\mu - x_i\lambda = 1 + j\mu - x_j\lambda$

$\therefore\ \mu(i - j) = \lambda(x_i - x_j)$

Since λ and μ have no common divisors (except 1) and $\lambda \neq 1$, the prime factorisation of λ should be contained in that of $(i - j)$. In other words, λ should divide $(j - i)$. But we know that $(j - i) < \lambda$. Hence λ does not divide $(j - i)$, and this gives a contradiction.

$\therefore$ there must exist a remainder which is equal to 0, i.e. $\exists\, r_k = 0$.

$\therefore\ \exists\, x_k, y_k \in Z$ s.t. $1 + y_k\mu = x_k\lambda$

$\therefore$ let $y = -y_k$ and $x = x_k$

$\therefore\ 1 = x\lambda + y\mu$

$\therefore$ multiplying throughout by t we get: $t = xa + yb$

$\therefore$ PROOF IS COMPLETE!

Extension: We can now go on to prove that if the greatest common divisor of two integers λ and μ is 1, then there exists an infinite number of pairs of integers x and y such that $x\lambda + y\mu = 1$.

Stated in mathematical terms, we have:

Suppose $\gcd(\lambda,\mu) = 1$, where $\lambda, \mu \in Z$, and that $(x,y) \in Z \times Z$ represents any pair of integers s.t. $x\lambda + y\mu = 1$. There exists an infinite number of pairs (x,y) s.t. $x\lambda + y\mu = 1$.

Note: The proof will consider only one infinite class of pairs:

- claim: $\gcd(x\lambda,y\mu) = 1$

  Suppose $\gcd(x\lambda,y\mu) = \alpha > 1$

  $\therefore \exists\ \alpha 1, \alpha 2 \in Z$ s.t. $\alpha 1 \alpha = x\lambda$ and $\alpha 2 \alpha = y\mu$

  $\therefore x\lambda + y\mu = \alpha 1 \alpha + \alpha 2 \alpha = \alpha(\alpha 1 + \alpha 2) = 1$

  $\therefore \alpha$ divides 1. This is a contradiction since $\alpha > 1$.

  $\therefore \gcd(x\lambda,y\mu) = 1$

- $\therefore \gcd(x^2\lambda,y^2\mu) = 1$

  $\therefore \exists\ p_1, q_1 \in Z$ s.t. $p_1(x^2\lambda) + q_1(y^2\mu) = 1$

  $\therefore$ Another pair is (p_1x^2,q_1y^2)

- $\therefore$ Similarly we can get other pairs

  $(p_nx^{n+1},q_ny^{n+1}) \quad \forall n \in Z^+$

The Square of a Two-Digit Number

Alex Farrugia
B.Sc. 3rd Year

Abstract: We are concerned with the square of two-digit numbers. This is basically done by the "difference of two squares" formula which gives a simple way of obtaining the result, thus: (x2 means x squared, and a2 means a squared)

x2 - a2 = (x+a)(x-a)

Adding a2 to both sides, we get:

x2 = (x+a)(x-a) + a2 --------(1)

Note that equation (1) above is true for all real x and a. Here, we're considering only two-digit natural numbers. Let us show how it works for the square of 47, we just substitute x for 47 in equation (1), to yield:

47^2 = (47+a)(47-a) + a2

This is true for all a.

Now comes the issue of choosing an appropriate a. To make the calculation as easy as possible, we must have two criteria for the choice of a:

(i) a must be small, in the range 0<=a<=9, so that we can easily find a2.
(ii) a must be such that the product (x+a)(x-a) is as simple to calculate as possible

To satisfy the above criteria, a is chosen as follows:

Let the last digit of x be n (i.e. n = x mod 10). Then a := n, if n=0,1,2,3,4, or 5

a := 10-n, if n=6,7,8, or 9.

So, in the above example, we choose a=10-7=3

So 47^2 = (47+3)(47-3) + 3^2 = 50 x 44 + 9

We note that 50x44 can be calculated easily (=2200), and adding 9 to this we get

47^2 = 2209

The important thing is that the above calculation can be made shorter by first calculating 44x5, and then 'appending' 9 to the answer!

We give another example: 76^2

76^2 = (76+4)(76-4) + 4^2 = 80 x 72 + 16 = 5760 + 16 = 5776

In this case, after calculating 72x8, we do not append '16', but we add 1 to the answer and then 'append' 6, viz:

```
 72x
  8
----
 576+
   1
----
 577|6
```

A better idea is using the following algorithn to find the square of any two-digit number n:

Step 1) Find the smallest a such that (n+a) or (n-a) is divisible by 10. Let this multiple of 10 be t, and let u be t divided by 10.

Step 2) If t>n, then write the following:

```
(n-a)|a
   u|a
  ----
```

Otherwise, write:

```
(n+a)|a
   u|a
  ----
```

Step 3) Now work out (n-a) x u (or (n+a) x u). Let result be p.

Step 4) Work out a x a to give q.

Step 5) Result is the number written as p x 10 + q.

To illustrate how this algorithm works for the various values of x mod 10, we give three further examples:

Let's calculate 82 ^ 2:

Step 1: a=2, t=80, u=8

Step 2:

```
84|2
 8|2
----
```

Step 3:

```
84|2
 8|2
----
  4
```

Step 4: 3

```
84|2
 8|2
----
672 4
```

Therefore, 82^2=6724.

Another example: 46^2:

Step 1: a=4, t=50, u=5

Step 2:

```
42|4
 5|4
----
```

Step 3:

```
  1
42|4
 5|4
----
   6
```

Step 4:

```
11
42|4
 5|4
----
211 6
```

Therefore, 46^2=2116.

Now let's present some optimisations to the method. The algoritm is particularly easy for x mod 10 = 5 as we show in the following example. First, when the last digit is 5, we said above that we choose a=5. This results in BOTH (x+5) and (x-5) being divisible by 10. For example, in 55^2

55^2 = (55+5)(55-5) + 25 = 50 x 60 + 25

Or, in the step-by-step method:

```
  2
50|5
 6|5
----
302 5
```

In this case, a simpler algorithm can be used:

Step 1: Let first digit of the number we want to square be x. Find x(x+1)

Step 2: Append '25' to the answer!

The above method is very well-known.

The other optimisation to the method is for numbers n>90. In these cases, the choice of a simplifies the algorithm considerably.

If we consider (93^2), we have:

93^2 = (93+7)(93-7) + 7^2 = 100 x 86 + 49

Hence, we can just calculate 93-7, and then append 7^2 to the answer! The algorithm for these values of x is as follows:

Step 1: Let a be 100-x
Step 2: Write down (x-a)
Step 3: Append a2 to the number just written down!

Finding the square of numbers with more than two digits is possible by using recursion. The algorithm may be developed:

x2 = (x+a)(x-a) + a2

we can find a2 by the same method! For example, to find the square of 237, a possible method is: first we put a=37, thus:

237^2 = (237+37)(237-37) + 37^2 = 274 x 200 + 37^2 = 54800 + 37^2

Then we calculate 37^2 by putting a=3, thus:
37^2 = (37+3)(37-3) + 9 = 40 x 34 + 9 = 1360 + 9 = 1369
Hence, 237^2 = 54800 + 1369 = 56169
I doubt, however, if this method is any quicker than by multiplying out in the usual way. I'm mentioning it for the sake of information.

PYTHAGORAS THEOREM

Josephine Debattista
St. Joseph School

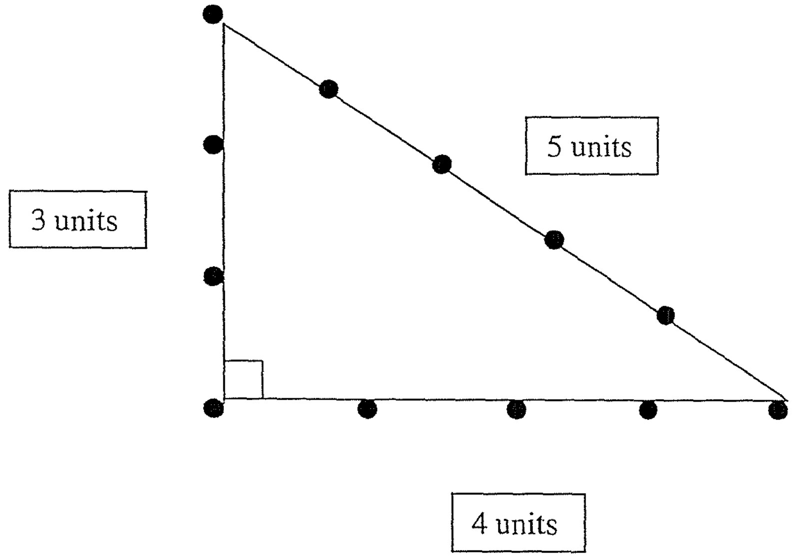
Historical background

Pythagoras 580 BC was a Greek mathematician who became famous for formulating Pythagoras Theorem but its principles were known earlier.

The ancient Egyptians wanted to lay out square (90°) corners to their fields. To solve this problem about 2000 BC they discovered the 'magic' of the 3-4-5 triangle.

Problem solving

A loop of rope with 12 equally spaced knots can be used to show the solution to the Egyptians' problem. Three sticks stretched the rope to form a triangle with 3, 4 and 5 units. The side of 5 units we call the hypotenuse and the angle opposite is equal to 90°.



Conclusion

Around 500 BC the Greeks learnt this trick from the Egyptians and explored the 3-4-5 triangle and found that: $5 \times 5 = 3 \times 3 + 4 \times 4$

Pythagoras generalised this rule to apply to all right angled triangles.

CROSSING LINES

Phaedra Cassar

Abstract: Two questions are tackled. In the first we investigate the regions into which the plane is divided by crossing lines.

1.CROSSING LINES

Rule: Lines must all cross each other at least once from a point distinct from a previous intersection point.

a.. When two lines intersect, the plane is divided into four open regions

b. When three lines intersect, the plane is divided into 7 regions, one of which is enclosed in a triangular shape. (4+3)

c. When four lines intersect, the plane is divided into 3 enclosed regions (two triangles, one quad.) and 8 open regions i.e. 11 regions in all. (7+4)

d. When 5 lines intersect, 10 open regions and 6 closed regions are formed. (16 in all, 11+5)

(i) 10 open regions and 6 closed regions (3 triangle, 3 quad)

(ii) 10 open regions and 6 closed regions (5 triangle, one 5 sided figure)

(iii) 10 open regions and 6 closed regions (4 triangle, 1 quad, one five sided)

Questions

Are these results exhaustive?

Are there more ways to draw five crossing lines?

What happens when more lines intersect?

We can predict how many regions, closed or open, are created using formulae derived from the inherent patterns pointed out. But what about the shapes of the aforesaid regions – is there any specific way we can work out a prediction? Furthermore, how many possible ways are there to draw five or more crossing lines? Are there really only two ways to cross 5 lines? What about higher numbers of crossing lines?

What is the situation if this problem had to be extended to three dimension, lines or planes?

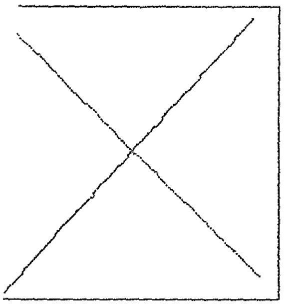
Another interesting area to explore is the number of intersection points – a pattern is immediately discernible.

TABLE OF RESULTS

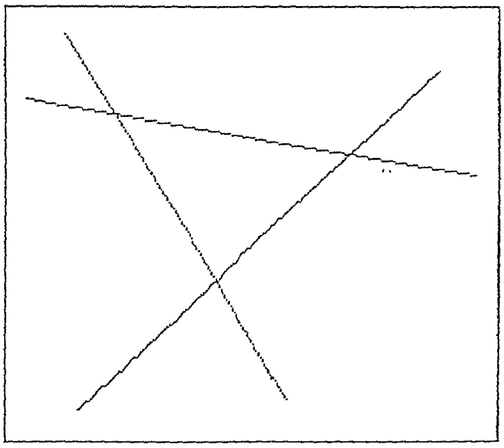
No of lines	Open Regions	Closed Regions	Total No. of Regions
1	2		2
2	4		4
3	6	1	7
4	8	3	11
5	10	6	16

Intersection Points

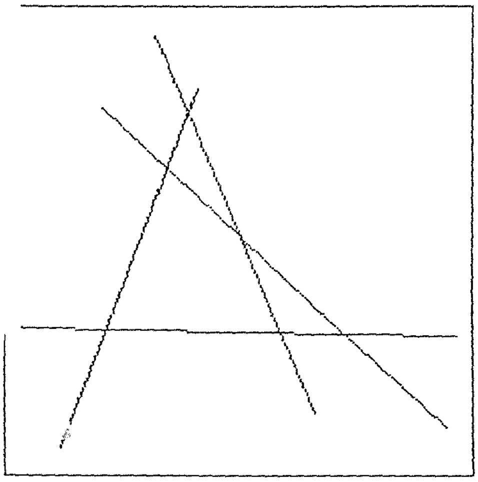
No. of lines	Intersection Points
1	0
2	1
3	3
4	6
5	10



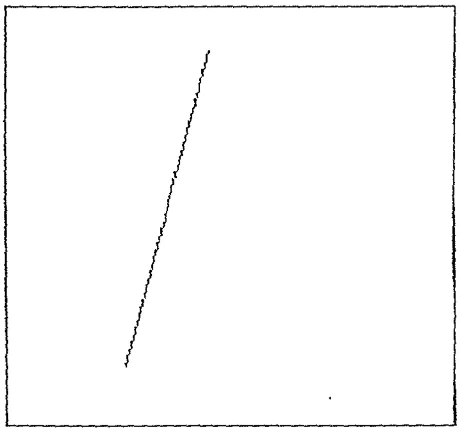
2 INTERSECTING LINES



3 INTERSECTING LINES

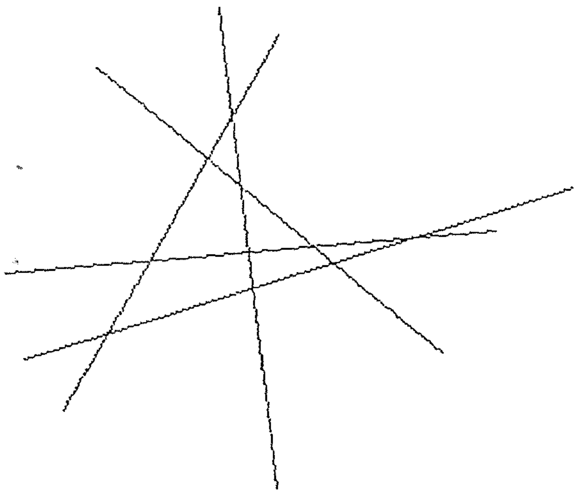


4 INTERSECTING LINES

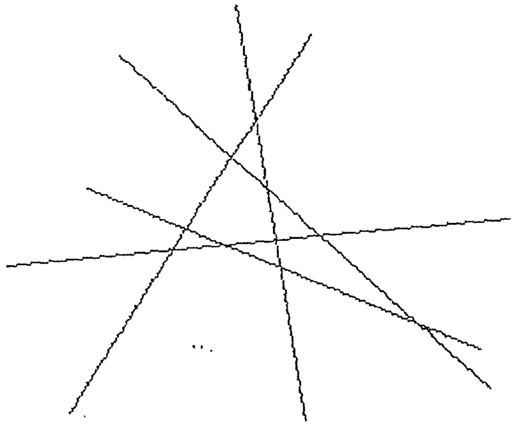


1 LINE

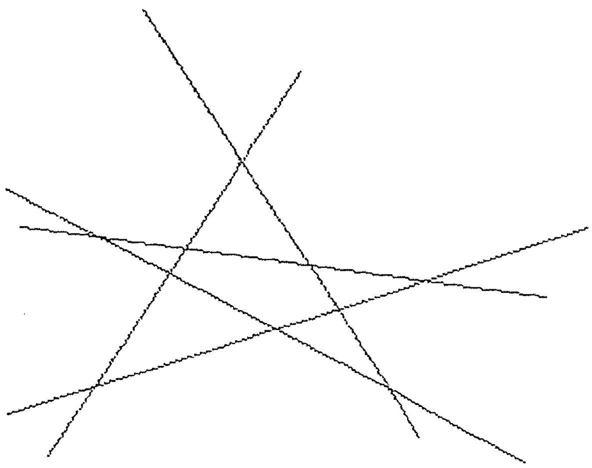
5 INTERSECTING LINES



3 TRIANGLES
3 QUADRILATERALS



4 TRIANGLES
1 QUADRILATERAL
ONE 5-SIDED FIGURE



5 TRIANGLES
ONE 5 SIDED FIGURE

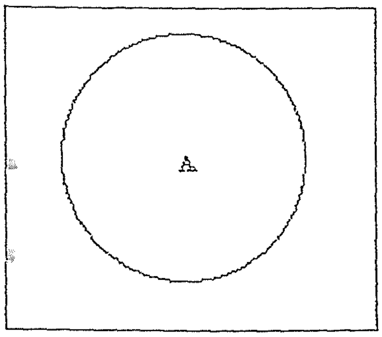
1.INTERSECTING CIRCLES

Rule: The circles must intersect in such a way that the new circle added must take in part of Each and every area of the circles and regions in the former diagram.

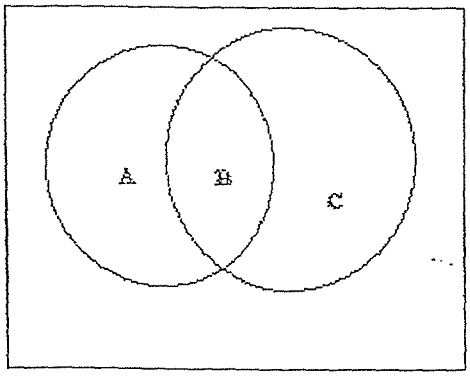
With two circles this is obviously possible, creating three regions.
With three circles, it is also possible, with 7 regions being created.
But with four circles, one region will be enclosed entirely within the circle.

Question: Is it really impossible to draw four circles such that the rule defined above is followed? If not, why is it not possible to draw four circles to fulfill the set condition, when it is possible to do it with two or three circles? Perhaps this problem is related to the four colour theorem?

Diagrams of Intersecting circles.

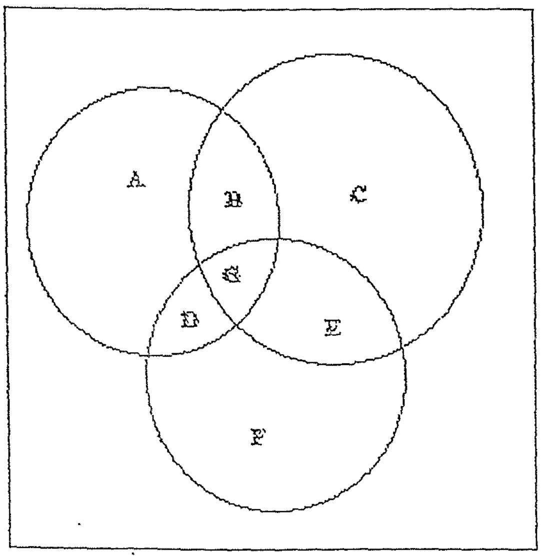


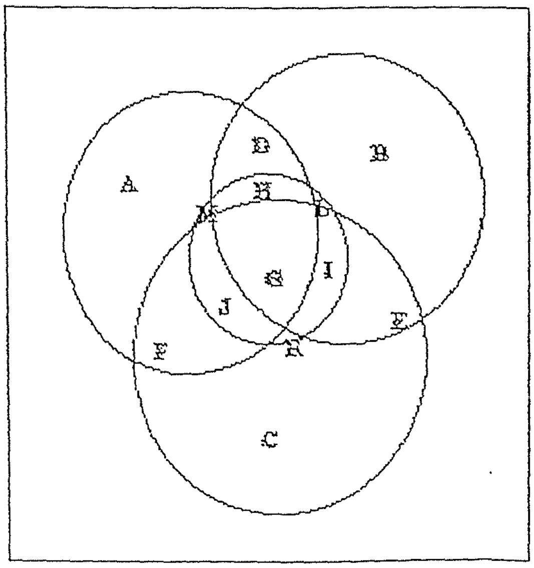
1 CIRCLE (TRIVIAL)



2 INT. CIRCLES

3 ENCLOSED REGIONS





ONE POSSIBILITY FOR FOUR INT. CIRCLES