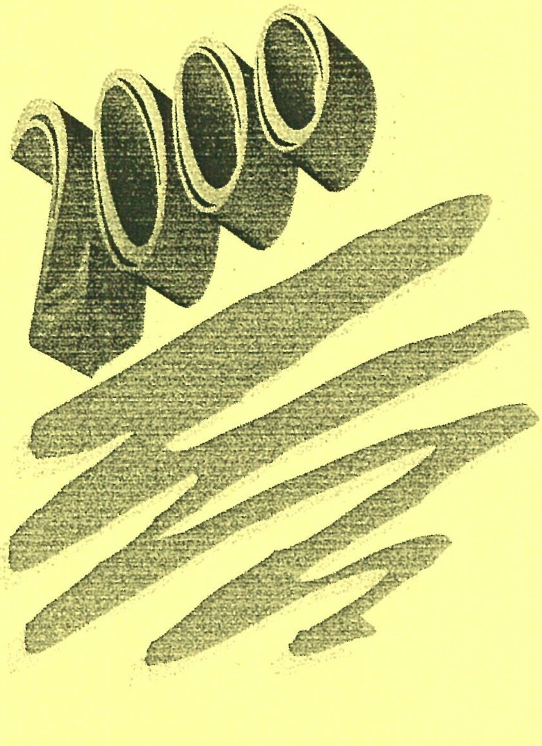
The Collection-II



The Collection-II

Editor: I. Sciriha

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Foreword

Most hobbies involve collections.
Here's one with different flavours
of what we enjoy in mathematics.

The hub of activity that followed the first two Collection workshops was very encouraging. It was interesting to note the feedback we received to questions posed during the workshops.

The conjecture posed on intersecting circles has been proved and we received three different solutions to the interesting question on equivalent sets. Reports on both are to be found in this issue. Also included in this issue is the discussion that a letter in the local press on divisibility triggered.

The contribution to mathematical ideas during the second meeting varied from balancing of dominoes to reconstructing graphs, music, divisibility and infinite sets. We look forward to our February 2001 meeting.

Dr. Irene Sciriha.
Organiser.

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The Collection-II

Editor: I. Sciriha

Department of Mathematics

Faculty of Science

The University of Malta

November 2000

Proceedings of Workshop

held on the 30th October 2000

4th October, 2000

The Collection II

A workshop is being held on Monday, 30th October 2000 from 3.00 to 4. 00 p.m. to share some interesting mathematical ideas among people who find pleasure in the elegance and preciseness of mathematics.

Venue: University of Malta
Maths and Physics Building, Department of Mathematics, Room 316.

Speakers: Ms Juanita Formosa, Mr Peter Borg, Mr Alex Farrugia, Mr James Borg, Mr. David Suda.

We shall end with a brief session for spontaneous problem posing and/or solving. You are cordially invited to attend.

Abstracts of possible proofs or conjectures which you wish to share with us in this meeting, or in a future one, may be sent to Dr. I. Sciriha or Ms. A. Attard, Department of Mathematics, (marked The Collection), at any time of the year.

Dr. I. Sciriha.
(Organisor)

p.s. European Women in Mathematics 2001[1]
10th International Meeting of **EWM**
24-30 August, 2001. Tartu, Estonia
http://www.maths.ox.ac.uk/ ewm01/
Budding, amateur and professional mathematicians who wish to become members of the EWM can contact me.

[1] **Because of unforeseen difficulties, the EWM conference will not be held in Estonia but in MALTA.**

List of Figures

Intersecting Circles and Ellipses[2]

Alastaire Farrugia

PhD candidate

Dept. of Combinatorics

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University of Waterloo

Canada

Abstract

In the February 2000 issue of The Collection, Phaedra Cassar posed the following the question: Is it possible for four or more circles to be drawn such that each new circle added has a common region with all existing regions?[3] We prove that this is not possible. Furthermore we show that for ellipses, the task is not possible for five or more ellipses.

DEFINITION 0.0.1. An r-Venn diagram is a collection of r closed curves $C_1, ..., C_r$, such that the intersection of $X_1, ..., X_{2^r}$ is non-empty and connected, where each X_i is either int(C_i) or ext(C_i).

[**Note**: we will use 'r' for the number of curves, and reserve 'n' for the number of intersection points. A region is also referred to as a face and the region outside the Venn Diagram is the infinite face.]

THEOREM 0.0.2. [4] If $C_1, ..., C_r$ are all circles, then there is no r-Venn diagram for $r > 3$; if the C_i's are all ellipses, there is no r-Venn diagram for $r > 5$.

PROOF. Given a Venn diagram we can put a vertex at each intersection point to get a planar graph. By Euler's formula for planar graphs $n - m +$

[2]Following the article "Intersecting Circles" which appeared in The Collection: 16th Feb 2000. (See p.5)

[3]See p.5.

[4]I wish to thank Frank Ruskey who gave me the necessary hint to prove this theorem.

$f = 2$ where n, m, f are the numbers of vertices (intersection points of the Venn diagram), edges and faces. Now in an r-Venn diagram $f = 2^r$, so $2^r = 2 + m - n$

We will get an upper bound on $m - n$ and thus prove the theorem.

We will consider only the case where we do NOT have three or more circles intersecting at the same point. This can be justified by geometric intuition, or by observing that if we do have three or more circles intersecting at the same point, the quantity m-n is even lower than if we don't.

The graph will be 4-regular, so $4n = 2m$, and thus $2^r = 2 + n$.

Now since every pair of circles intersect in at most two points, the number of intersection points is at most twice (r choose 2), i.e. $n \leq r(r-1)$.

Thus we must have $2^r <= 2 + r(r-1)$ which is impossible for $r = 4(2^r = 16, 2 + r(r-1) = 14)$, and thus impossible for $r \geq 4$ since 2^r increases faster than $r(r-1)$.

If we have ellipses, since each pair of ellipses can intersect at at most four points, we have $n <= 2r(r-1)$ and so $2^r <= 2+2r(r-1)$ which is impossible for $r = 6(2^r = 64, 2 + 2r(r-1) = 62)$. □

Intersecting Circles

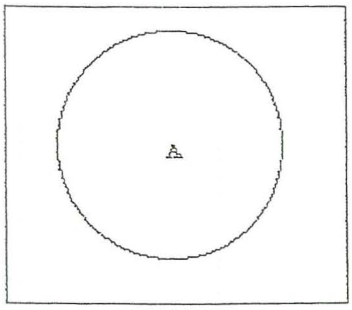
Rule: The circles must intersect in such a way that the new circle added must take in part of Each and every area of the circles and regions in the former diagram.

With two circles this is obviously possible, creating three regions.

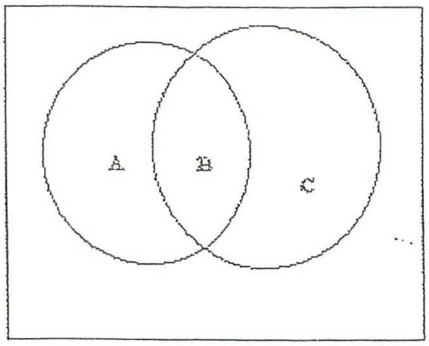
With three circles, it is also possible, with 7 regions being created.

But with four circles, one region will be enclosed entirely within the circle.

Question: Is it really impossible to draw four circles such that the rule defined above is followed? If not, why is it not possible to draw four circles to fulfill the set condition, when it is possible to do it with two or three circles? Perhaps this problem is related to the four colour theorem?



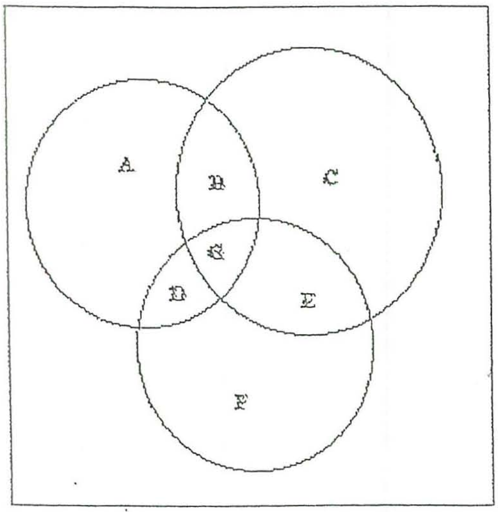
1 CIRCLE (TRIVIAL)



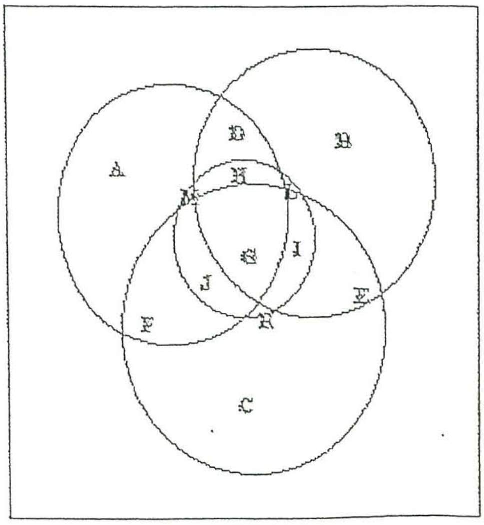
2 INT. CIRCLES

3 ENCLOSED REGIONS

Diagrams of Intersecting circles.



THREE INT. CIRCLES, CREATING 7



ONE POSSIBILITY FOR FOUR INT. CIRCLES

[5] This page appeared in The Collection of Feb 2000.

Two Simple Proofs

Alexander Farrugia B.Sc. 4th Year

Abstract

My proofs of two simple results will be presented. These are:

a) $0.99999\dot{9}... = 1$

b) The area of a circle is πr^2, where r is the radius of the circle.

a) $0.99999\dot{9}... = 1$

Remark: This simple result is usually deduced by considering the number $0.99999\dot{9}...$ as the series

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ...$$

and summing the series to infinity, whence it is equal to 1. My proof is different.

PROOF. We don't know what $0.99999\dot{9}$ ... is, so let's assign it to x.

$$x = 0.99999\dot{9} \tag{0.1}$$

Multiplying both sides by 10, we get:

$$10x = 9.99999\dot{9}$$

Subtracting 9 from both sides, we get:

$$10x - 9 = 0.99999\dot{9} \tag{0.2}$$

Comparing equations (1) and (2), we immediately note that their R.H.S is exactly the same, which implies that their L.H.S are also equal. So

$$10x - 9 = x \tag{0.3}$$

$$9x = 9 \tag{0.4}$$

$$x = 1 \tag{0.5}$$

But we started with $x = 0.99999\dot{9}$ and we ended up with $x = 1$.
So finally, $0.99999\dot{9} = 1$. □

b) The area of a circle with radius r is πr^2.

Remark: The proof that the area of a circle is π multiplied the square of its radius is usually found by integrating the area under the graph of a semicircle with radius r, remembering to double the answer in the end. My proof takes a different approach altogether.

PROOF. Consider any n-sided regular polygon. In the diagram below we have chosen a regular pentagon. (See the figure.) From the centre of the polygon we draw n lines to each of the vertices of the polygon. Call the length of each of these lines r. The polygon is thus divided into n isosceles triangles (in our case $n = 5$).

Let θ be the angle at the centre that each of these lines makes with its subsequent line. It can be easily seen that $\theta = \frac{2\pi}{n}$ (using radian measure).

Now consider finding the area of this polygon. It can be found by multiplying the area of one of the triangles by n, since obviously all of the triangles have

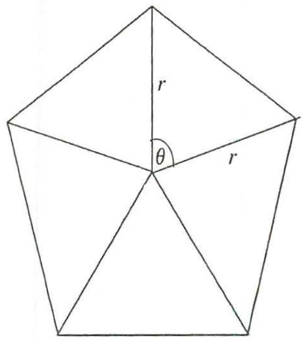


FIGURE 0.1. The Regular Pentagon.

the same area. The area of one of these triangles, knowing the length of two of its sides and the angle between them, is:

$$\frac{1}{2}r^2 sin\theta = \frac{1}{2}r^2 sin(\frac{2\pi}{n})$$

Therefore, the area of the n sided polygon is $\frac{1}{2}nr^2 sin(\frac{2\pi}{n})$.

Now comes the crucial step. If we allow n to become very large, the n sided polygon will approximate a circle. This implies that its area will be close to that of the circle. Therefore, as n approaches infinity, the area of the polygon approaches that of the circle.

Mathematically speaking,

Area of circle

$$\begin{aligned} &= \lim_{n \longrightarrow \infty} \frac{1}{2}r^2 n sin(\frac{2\pi}{n}) \\ &= \frac{1}{2}r^2 \lim_{n \longrightarrow \infty} n sin(\frac{2\pi}{n}) \end{aligned}$$

Now

$$\lim_{x \longrightarrow 0} \frac{sinx}{x} = 1.$$

Thus

$$\lim_{n \longrightarrow \infty} \frac{sin(\frac{2\pi}{n})}{\frac{2\pi}{n}} = 1$$

$$\Longrightarrow \quad lim_{n \longrightarrow \infty} n \frac{sin(\frac{2\pi}{n})}{2\pi} = 1$$

$$\Longrightarrow \quad \frac{1}{2\pi} \lim_{n \longrightarrow \infty} nsin(\frac{2\pi}{n}) = 1$$

$$\Longrightarrow \quad \lim_{n \longrightarrow \infty} nsin(2\frac{2\pi}{n}) = 2\pi$$

Therefore the area of a circle is πr^2. □

Remark: This letter appeared in the Times of the 9th Feb, 1999, in reply to a query by Mr. M. Pace of the 28th Jan, 1999.[6]

Dear Editor,

Playing with Nines

In his letter which appeared in the issue of the Times of the 28th January, 1999, Mr. M. Pace pointed out an interesting property of the number nine in the set of integers modulo ten. In general, when we work on the scale of n (where n is 3 or more) the number n-1 exhibits the same properties. So, the number 15 shows these properties in the hexadecimal scale. So does seven in the integers modulo 8.

As Mr. Pace pointed out, if a number is divisible by 9 then the sum of its digits is a multiple of 9. He discussed the product 9x7.

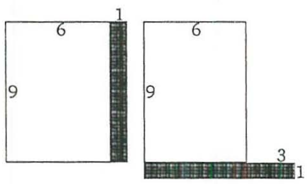


FIGURE 0.2. Areas 9x7= 10 x 6 + 3

One way to see this is to place 9 rows of 7 squares in a rectangular shape. If a column of 9 squares is removed we are left with 9 rows of 6 squares. The 9 squares are now placed as an additional row so that we end up with 10 rows of 6 squares and an extra 3 squares. The product is the area and the sum of the digits of the product is the number of squares in the additional row. It is clear that whatever the number of columns, removing a column and placing the squares removed as an additional row, gives an easy way of working out the product. Besides, the number of squares in the additional row is always nine.

[6]See p.12.

A more general way to represent the process is to write 9 as 10-1. When a non-zero number z (between 1 and 9) is multiplied by 9 the product may be written as 10z-z or as 10(z-1) + (10-z). Thus the sum of the digits is (z-1) + (10-z) = 9 whatever z is. To take Mr. Paces example namely 9x7, z is 7 and the product has z-1 as the tens digit and 10-z=3 as the units digit. The same holds true for the other example namely 33x9, writing 33 as 3 tens plus 3 units. Now if we were to work 6x7 in the octal scale, we can use the method of fingers mentioned by Mr. Pace. We hold out eight fingers and put down the sixth. The product is 42 modulo 10 which is 5 eights and 2 units or 52 modulo 8 as given by the fingers to the right and left of the sixth.

Playing with numbers can be intriguing as Mr Pace pointed out. The excitement that gripped the mathematical world two years ago when Andrew Wiles solved Fermat's Last Theorem are still vivid in our minds.

Irene Sciriha.

Any Maths Wizards?

The number nine has for decades been intriguing me as a kind of magical number but I have never really discovered the reason why. The number nine has some curious properties that are unique.

Try and multiply nine by any number and if you add up the digits of its products you always get nine. For example 9x7=63 and 6+3=9. If the number multiplied is large (say 33x9=297) the component digits of the product when added up are still multiples of nine (2+9+7=18 and 1+8=9). If you want to know if a number is divisible by nine simply add up its digits.

A well-known method of multiplying by nine which is popular with children is that of using the fingers. Thus if you want to multiply nine by seven you hold out your 10 fingers and put down your seventh finger. The number of fingers to the left of the seventh finger indicate the tens and those to the right indicate the units and you thus get an answer of 63.

I am convinced that since mathematics is logical and not metaphysical there must be some form of explanation to my query. I am therefore asking any maths wizard among the readers of this paper to put their brains in top gear and come up with an explanation.

MARIO PACE,
Attard.

[7]Mario Pace's Letter: The Times, 28th Jan 1999.

Divisibility Magic

James L. Borg[8]

Department of Mathematics

University of Malta

Abstract

To check whether a number is divisible by 9 (or 3), it is enough to check whether the sum of its digits is a multiple of 9 (or 3). More generally, if numbers are represented with base b, then the divisors of $b-1$ will have this property.

Introduction In a letter to a local newspaper [1], a reader[9] noted that the number 9 has the following "magical" property: if a number is a multiple of nine then the sum of its digits is itself a multiple of nine. He asked what the mathematical explanation of this phenomenon is.

The property mentioned, together with its converse, are well-known facts. To test an integer for divisibility by 9 we sum the digits and check whether the answer obtained is a multiple of nine. If the digit sum is still very large, the process may be repeated until a two-digit integer is obtained.

However, nine is not the only integer with this property. If we take any multiple of three, then the sum of its digits is also a multiple of three. Similarly, if the digit sum of any number is a multiple of three, then the number itself is a multiple of three.

Since the representation of a number as a string of digits depends on the base chosen, the following question immediately arises: will 3 and 9 still have the same property if numbers are represented in a different base?

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[8]Email: jlborg@maths.um.edu.mt

[9]See p.12

A quick check reveals that this is not so. For example, the number written as 45 in base 10 will be written as 140 in base 5 and 55 in base 8. In both cases, the digit sum is not a multiple of nine (or of three).

Clearly, the numbers 9 and 3 have this magical property only when integers are represented in base 10. We formulate two questions which shall be answered in the rest of this talk.

(i) Why do the numbers 3 and 9 have the above property in base 10?
(ii) Which numbers (if any) have the same property in other bases?

Main Theorem In the sequel, we shall only consider positive integers.

Let us start with a definition. Recall that the number a is represented by the string of digits $a_n a_{n-1} \dots a_0$ in base b if $a = \sum_{k=0}^{n} a_k b^k$.

Definition : The integer c has the **sum-of-digits divisibility property in base** b if for any integer a,

$$c|a \Leftrightarrow c|\sum_{k=0}^{n} a_k, \text{ where } a = \sum_{k=0}^{n} a_k b^k.$$

The theorem we shall prove is:

Theorem : The integer c has the sum-of-digits divisibility property in base b if and only if $c|(b-1)$.

Proof of Sufficiency for Base 10 Let us start with the proof for the numbers 3 and 9 i.e. in base 10.

If $a = a_n a_{n-1} \dots a_0$ then

$$\begin{aligned} a &= \sum_{k=0}^{n} a_k 10^k \\ &= \sum_{k=0}^{n} a_k (9+1)^k \\ &= \sum_{k=0}^{n} a_k (9^k + k9^{k-1} + \cdots + k9 + 1) \\ &= \sum_{k=0}^{n} a_k (9^k + k9^{k-1} + \cdots + k9) + \sum_{k=0}^{n} a_k \\ &= 9\sum_{k=0}^{n} a_k (9^{k-1} + k9^{k-2} + \cdots + k) + \sum_{k=0}^{n} a_k \end{aligned}$$

Hence 9 (or 3) divides a if and only if 9 (or 3) divides the digit sum $\sum_{k=0}^{n} a_k$. In fact, we have proved a slightly stronger result: the remainder left when a number is divided by nine is the same as the remainder left when the digit sum is divided by nine. In the language of modular arithmetic, $a \equiv \sum_{k=0}^{n} a_k \pmod 9$.

Proof of Sufficiency in an Arbitrary Base The above proof can be adapted very easily to an arbitrary base b. Then

$$\begin{aligned} a &= \sum_{k=0}^{n} a_k b^k \\ &= \sum_{k=0}^{n} a_k \left((b-1)+1\right)^k \\ &= \sum_{k=0}^{n} a_k \Big((b-1)^k + k(b-1)^{k-1} + \cdots + k(b-1) + 1\Big) \\ &= \sum_{k=0}^{n} a_k \Big((b-1)^k + k(b-1)^{k-1} + \cdots + k(b-1)\Big) + \sum_{k=0}^{n} a_k \\ &= (b-1)\sum_{k=0}^{n} a_k \Big((b-1)^{k-1} + k(b-1)^{k-2} + \cdots + k\Big) + \sum_{k=0}^{n} a_k \end{aligned}$$

Once again it is clear that $b-1$ divides a if and only if $b-1$ divides the digit sum $\sum_{k=0}^{n} a_k$. Furthermore, any number c will possess the same property if c divides $b-1$. As before, if $c|(b-1)$ then $a \equiv \sum_{k=0}^{n} a_k \pmod{c}$.

For example, in base 8, it is the number 7 which has the "magical" divisibility property, while in base 13, the numbers 2, 3, 4, 6 and 12 will have this property.

Proof of Necessity To prove necessity, it is enough to show that if c does not divide $(b-1)$, then there is *either* at least one multiple of c whose digit sum is not a multiple of c, *or* at least one non-multiple of c whose digit sum is a multiple of c.

Two cases arise:

(a) if $c < b$, then let a be a two digit number such that
   (i) the first digit is 1, and
   (ii) the second digit is $c-1$.
   Then the sum of digits is c, but the number is $b+(c-1) = c+(b-1)$ which is not a multiple of c because $b-1$ is not a multiple of c. [2]

(b) if $c \geq b$, then the digit sum of c is positive but strictly less than c, and hence cannot be a multiple of c.

Hence we prove that only those numbers which divide $b-1$ have the sum-of-digits divisibility property in base b.

Taking it Further The proof for divisibility by 9 may be adapted to obtain a proof for the divisibility test for multiples of 11. This test consists of finding the alternating sums of the digits i.e. sum the 1st, 3rd, 5th etc. digits and then sum the 2nd, 4th, 6th etc. digits. If these two sums are equal or differ by a multiple of 11, then the number is a multiple of 11. The proof of this is left as an exercise.

The more adventurous readers may even develop tests for divisibility by 7 and 13 in the same fashion.

Acknowledgments I would like to thank Irene Sciriha for inviting me to participate in this workshop. I would also like to thank Joseph Muscat for several interesting and fruitful discussions on this topic.

References 1. M. Pace, "Any Maths Wizards?" in *The Times*, 28th January 1999.

2. J. Muscat, private communication.

A Homomorphism on Musical Notes

David Suda

B.Sc. 3rd year

Abstract

The intervals between successive notes in the major and minor scales are not equal so that difficulties arose when modulating to new keys. Adjustment to the tempered scale, in which all intervals are equal, ensured portability in all keys. The tempered intervals form a group under multiplication. Moreover, the musical notes can be partitioned into equivalence classes by the octave. A homomorphism can be defined on the set of tempered intervals. The kernel is the set of exact number of octave and the range isomorphic to C_{12}.

Introduction

This article deals with applying group theory to musical notes. We start by taking note of their mathematical and physical properties.

1. The pitch of a musical note is defined by its frequency, that is the number of vibrations per second.
2. The frequencies of pure musical tones form an infinite set of real numbers. The range between 20 and 20,000Hz is within the lower and upper limits of audibility.
3. Instruments with discrete musical tones are finite subsets of this infinite set. The pianoforte, for example, has a subset with 88 elements.

4. Instruments with continuous musical tones are infinite subsets of this infinite set. Examples of these are the string instruments and some wind instruments like the trombone.

5. There is an order relation, $>$, defined on the musical notes which is antisymmetric and transitive. Given three musical notes a, b and c:

i. $a < b$ does not imply $b < a$ (antisymmetric)

ii. however, if $a < b$ and $b < c$, then $a < c$ (transitive)

6. Musical notes can also be divided into equivalence classes.

Recall that if A is a set and $\sim$ is an equivalence relation, then the equivalence class of $a \in A$ is the set: $\{x \in A : a \sim x\}$.

Also recall that three properties define an equivalence relation $\sim$.

$\forall a, b, c \in A,$

$1. a \sim a$ (reflexive)

$2. a \sim b \Longrightarrow b \sim a$ (symmetric)

$3. a \sim b$ & $b \sim c \Longrightarrow a \sim c$ (transitive).

Classes of Notes

In music, notes with the same name are part of the same equivalence class. The reason behind this is that notes with the same name are related by the following equivalence relation: $a \sim b$ if $\frac{b}{a} = 2^n, n \in \mathbb{Z}$.

1. $\sim$ is reflexive, since $a = 2^0 a$
2. $\sim$ is symmetric, since$a = 2^x b \Longrightarrow b = 2^{-x} a$
3. $\sim$ is transitive, since $a = 2^x b$ & $b = 2^y c \Longrightarrow c = 2^{x+y} a$ given that $x, y \in \mathbb{Z}$.

Thus the frequencies of successive notes in one equivalence class are in the ratio of 2:1 and are said to be an octave apart. They sound similar since they have common vibrations, the higher note doing an extra vibration in between each of two consecutive vibrations of the lower note. Notes in the same class are given the same name. Middle C on the piano is 256 Hz and the frequencies 128Hz and 512Hz are also C notes. On the tempered scale Middle C is adjusted to 261.6Hz.

Problems on Modulating

DEFINITION 0.0.3. The interval beween two notes aHz and bHz with $b > a$ is the ratio $\frac{b}{a}$.

Of the seven Greek modes, the Ionian and Aeolian modes developed into the natural major and minor scales respectively, because the tonal symmetry among the intervals contributes to the ear's ready acceptance of the scale. The intervals given by $\frac{1}{1}$, $\frac{9}{8}$,$\frac{5}{4}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{15}{8}$ and $\frac{2}{1}$ have become accepted as the ones most pleasing to the ear and correspond to unison, major 2nd, major 3rd, perfect 4th, perfect 5th, major 6th, major 7th and the octave resp.

The function $LI : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $LI : (a, b) \mapsto \log_2 \frac{b}{a}$ gives the interval in octaves. Thus if $a \sim b$ then $\frac{b}{a} = 2^n, n \in \mathbb{Z}$, the notes a, b are n octaves apart and belong to the same equivalence class. Now if we have an interval of 12 perfect fifths starting from C, we expect to obtain C again 7 octaves up, that is the interval $2^7 = 128$. But one perfect fifth is $\frac{3}{2}$ which when compounded gives $\left(\frac{3}{2}\right)^{12} = 129.7$, so that successive multiplication by $\frac{3}{2}$ is not closed under successive multiplication by 2. The discrepancy represents the interval between $B^\sharp$ and C which does not figure on the piano. Even with just intonation where each of the major triads $F-A-C$, $C-E-G$ and $G-B-D$ are in the ratio $4:5:6$, the interval between two notes, one tone, is sometimes $\frac{9}{8}$ (e.g.$C-D$) and sometimes $\frac{10}{9}$, (e.g. $D-E$), whereas one semitone (e.g. $B-C$) is $\frac{16}{15}$.

The interval between two notes, say C and D, is divided into two semitones which, in the natural scales, are not exactly equal in size since $\left(\frac{16}{15}\right)^2 \neq \frac{9}{8}$ or $\frac{10}{9}$.

The function $LI_{12} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $LI_{12} : (a, b) \mapsto \log_{2^{\frac{1}{12}}} \frac{b}{a}$ gives the interval in semitones.

In the scale of C Major the 7 semitones from C to G form the interval $\frac{3}{2}$ whereas the 7 semitones from D to A form the interval $\frac{\frac{5}{3}}{\frac{9}{8}} = \frac{40}{27} < \frac{3}{2}$. Thus modulating to the scale of D reults in a slightly flat fifth which is therefore out of tune.

The Solution: The Tempered Scale

A keyboard instrument in which the scales in all keys have equal intervals would solve this problem. The tempered scale was thus constructed in which there are 12 equal semitones in an octave. These intervals are obtained by placing eleven geometric means between 1 and 2. Since $2^{\frac{1}{12}}$ is irrational, none of the intervals (except the octave) agree with those of the natural scale. However, although the scales in all the keys are slightly out of tune, the average ear is unable to detect a discrepancy of such small dimensions.

The great advantage of the tempered scale is that the set of intervals $\{_aI_b\}$ is now a group under multiplication. Notes of the same name are octaves apart and equivalent intervals in the same class are in a ratio of $2^r : 1,\ r \in \mathbb{N}$. The interval from a to b is equivalent to that between a and $2b$. Thus $\frac{b}{a\times 2^r} \sim \frac{b}{a} \sim \frac{b\times 2^r}{a}$.

The log scale $LI(\{_aI_b\})$ gives fractions of octaves and equivalent intervals differ by an integer. The log scale $LI_{12}\left(\frac{b}{a}\right) = 12 \times LI\left(\frac{b}{a}\right)$ is the simplest and the values form a group under addition homomorphic to $C_{12} = \{0, 1, 2, 3, , 4, 5, , 6, 7, 8, 9, 10, 11\}_{x \ mod \ 12}$.

The CYCLIC GROUP C_{12}

Consider the set of consecutive notes:
$C, C^\sharp, D, D^\sharp, E, F, F^\sharp, G, G^\sharp, A, A^\sharp, B, ...$ Their set of frequencies is not a group; however the set of intervals:

FIGURE 0.3. Musical intervals on a dodecahedron.

${}_CI_C, {}_CI_{C\sharp}, {}_CI_D, {}_CI_{D\sharp}, {}_CI_E, {}_CI_F, {}_CI_{F\sharp}, {}_CI_G, {}_CI_{G\sharp}, {}_CI_A, {}_CI_{A\sharp}, {}_CI_B, \ldots.$ is a group G_C. If the intervals are taken relative to any note N, then G_N will be the same group on the tempered scale. This is the great advantage of tempered intervals: Modulation produces a set of intervals compatible with the default set.

Such a group G will always be of the form:

$\{2^0, 2^{\frac{1}{12}}, 2^{\frac{2}{12}}, 2^{\frac{3}{12}}, 2^{\frac{4}{12}}, 2^{\frac{5}{12}}, 2^{\frac{6}{12}}, 2^{\frac{7}{12}}, 2^{\frac{8}{12}}, 2^{\frac{9}{12}}, 2^{\frac{10}{12}}, 2^{\frac{11}{12}}, \ldots\}$ under multiplication and is a subgroup of $(\mathbb{Z}, \times)$.

A Homomorphism on the Set of Intervals

The homomorphism $\phi : 2^{\frac{n}{12}} \mapsto n \bmod 12$ maps the infinite set G of intervals to their equivalence classes $\{{}_CI_C, {}_CI_{C\sharp}, {}_CI_D, \ldots, {}_CI_B\}$. The kernel $\mathrm{Ker}(\phi)$ is the set consisting of a whole number of octaves. Since G is Abelian, a subgroup is normal. The isomorphism theorems imply that
i) $\phi(G)$ is isomorphic to C_{12}, the set of rotational symmetries of the regular dodecagon;
ii) every normal subgroup of C_{12} corresponds to a normal subgroup of G. Such a subgroup and its cosets represent equivalent chords.

The points on a dodecagon, starting from zero, that are $\frac{2\Pi}{12}$ $\frac{2\Pi}{6}$ and $\frac{2\Pi}{4}$ apart form subgroups of C_{12}. These are C_{12}, $\{0, 2, 4, 6, 8, 10\}$ and $\{0, 3, 6, 9\}$ respectively. The cosets of the latter are the chords of Diminished seventh.

Finally we can define a mapping $f : G \longrightarrow \mathbb{Z}$ by

$f : 2^{\frac{n}{12}} \mapsto \lfloor \frac{n}{12} \rfloor.$

The first twelve elements of the cyclic group G are mapped onto 0, the second twelve elements of the cyclic group mapped onto 1, and so on. The range $\{\lfloor\frac{n}{12}\rfloor\}$ corresponds to the octave above the default (that of middle C) to which the interval $2^{\frac{n}{12}}$ belongs. This mapping, however, is **not** a homomorphism since $f\left(2^{\frac{23}{12}}+2^{\frac{9}{12}}\right)\neq f\left(2^{\frac{23}{12}}\right)+f\left(2^{\frac{9}{12}}\right)$.

Equivalent Intervals

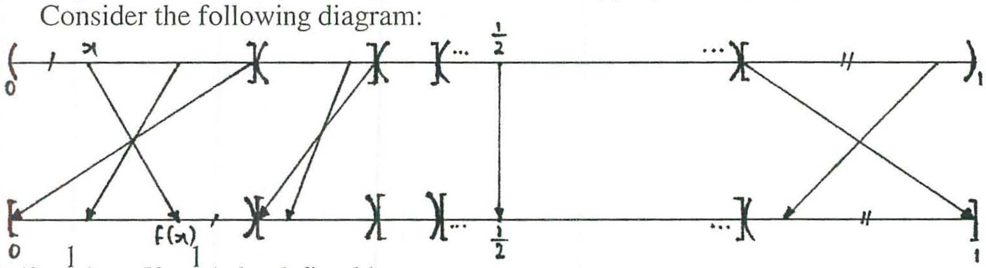
Peter Borg

To show that the open interval (0,1) and the closed interval [0,1] are equivalent.

Problem posed by Mr. James Borg.

Proof: Required to find a bijection (1 to 1 and onto mapping) from (0,1) to [0,1].

Consider the following diagram:



Let f: $(0, \frac{1}{2}) \to [0, \frac{1}{2})$ be defined by

$$f: x \mapsto \begin{cases} \frac{1}{2^2} - x & x \in (0, \frac{1}{2^2}] = I_1 \\ \frac{1}{2^2} + (\frac{1}{2^2} + \frac{1}{2^3} - x) & x \in (\frac{1}{2^2}, \frac{1}{2^2} + \frac{1}{2^3}] = I_2 \\ \frac{1}{2^2} + \frac{1}{2^3} + (\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - x) & x \in (\frac{1}{2^2} + \frac{1}{2^3}, \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}] = I_3 \\ \text{etc.} \end{cases}$$

In general, for $x \in (\frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k}, \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{k+1}}]$, $k \geq 2$, we have:

$$f(x) = 2(\frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k}) + \frac{1}{2^{k+1}} - x = 1 - \frac{1}{2^{k-1}} + \frac{1}{2^{k+1}} - x$$

and for $x \in (0, \frac{1}{2^2}]$ we have $f(x) = \frac{1}{2^2} - x$.

The first thing to note is that we have partitioned (0,1) into an infinite number of intervals I_k. This is because the sequence $(\frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k}) = (\frac{1}{2} - \frac{1}{2^k}) \to \frac{1}{2}$ as $k \to \infty$. Also, we are practically mapping an interval of type (a,b] onto an interval of type [a,b) by mapping $x \in (a,b]$ onto $a + (b - x)$; in particular b is mapped onto a. Secondly, one can easily realize that f is well-defined and injective because, on any interval I_k, a unique x value is mapped onto a unique y value. It remains to prove that f is onto. Let $y \in [0, \frac{1}{2})$. If $y \in [0, \frac{1}{2^2})$ then there exists $x \in (0, \frac{1}{2^2}]$ such that $f(x) = y$. Otherwise, there exists a $k \geq 2$ such that $(\frac{1}{2} - \frac{1}{2^k}) \leq y < (\frac{1}{2} - \frac{1}{2^{k+1}})$. So y should lie in an interval $[\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} - \frac{1}{2^{k+1}})$, but f is defined such that there exists an interval $I_k = (\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} - \frac{1}{2^{k+1}}]$ which is mapped onto $[\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} - \frac{1}{2^{k+1}})$, i.e. there exists $x \in I_k$ such that $f(x) = y$.

So we have mapped $(0, \frac{1}{2})$ onto $[0, \frac{1}{2})$. Similarly, we can map $(\frac{1}{2}, 1)$ onto $(\frac{1}{2}, 1]$. And finally, we map $\frac{1}{2} \in (0,1)$ onto $\frac{1}{2} \in [0,1]$. The proof is hence complete.

A second solution by James Borg.

If f maps the irrationals to themselves identically, the rest of the intervals $(0, 1)$ and $[0, 1]$ are equivalent since they are countable.

A third solution by Vincent Mercieca.

This solution is similar to that of Peter Borg. The intervals are divided into sub intervals with end points expressed in Ternary form.

Let f map the interval $[0, 1]$ to $(0, 1)$.

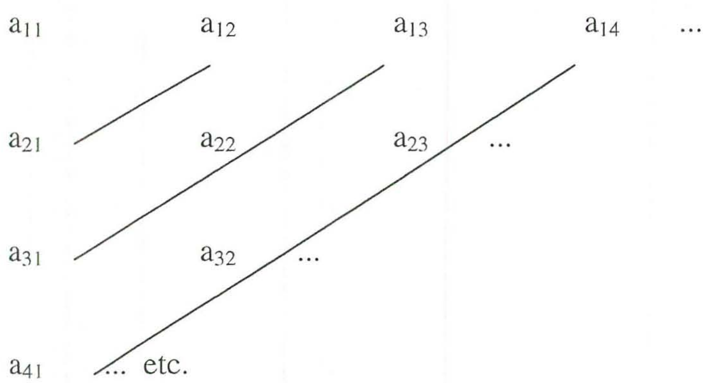
The interval $[0, 1]$ is divided into subintervals $[0, 0.1)$, $[0.1, 0.11)$, $[0.11, 0.111)$,. . . $(0.2, 1)$, $(0.12, 0.2])$, $(0.112, 0.12]$,. . . Now f(0.1)=0.11, f(.11)=0.111, . . . and f(0.2)=0.12, f(.12)=0.112, . . .

All other points are mapped identically.

Proof (1a):

Let $A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$, $A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$, ..., $A_k = \{a_{k1}, a_{k2}, a_{k3}\ldots\}$, ...

For all i, A_i is a countable set. $A = A_1 \cup A_2 \cup A_3 \cup \ldots$ is a countable union of countable sets. Hence the elements of A can be listed in the following way:



Let f: $A \to N$ be defined by $f : a_{ij} \mapsto \sum_{n=0}^{i+j-2}(n) + i = \frac{(i+j-2)(i+j-1)}{2} + i$, where $i, j > 0$.

It is required to prove that f is well-defined, one-to-one, and onto, i.e. f is bijective.

f is obviously well-defined since $f(a_{ij})$ can take only one value.

To prove f is one-to-one suppose that $f(a_{ij}) = f(a_{pq})$. Hence $\sum_{n=0}^{i+j-2}(n) + i = \sum_{n=0}^{p+q-2}(n) + p$.

Suppose $(i + j) \neq (p + q)$. Therefore either $(i + j) > (p + q)$ or $(i + j) < (p + q)$, but it is enough to just consider $(i + j) > (p + q)$. $f(a_{ij}) = f(a_{mn}) \Rightarrow p - i = \sum_{n=0}^{i+j-2}(n) - \sum_{n=0}^{p+q-2}(n)$. Hence $p - i = \sum_{n=p+q-1}^{i+j-2}(n) \geq p + q - 1$. So $1 - i \geq q$. But $i \geq 1$ $(i > 0)$, and hence $0 \geq q$. This is a contradiction since $q \geq 1$ $(q > 0)$. So $(i + j) = (p + q)$, and from $\sum_{n=0}^{i+j-2}(n) + i = \sum_{n=0}^{p+q-2}(n) + p$ it follows that $i = p$. Hence $j = q$. So $a_{pq} = a_{ij}$.

To prove f is onto let us consider any natural number k. We need to find i and j such that $f(a_{ij}) = k$, where $i, j > 0$. Let m be such that $\sum_{n=0}^{m} n = \frac{m(m+1)}{2} < k \leq \sum_{n=0}^{m+1} n = \frac{(m+1)(m+2)}{2}$.

Hence $0 < i = k - \sum_{n=0}^{m} n \leq m + 1$. Let $j = m - i + 2$ (i.e. $m = i + j - 2$). Since $m - i \geq -1$

then $j = (m - i) + 2 \geq -1 + 2 = 1$. Hence $j > 0$. So we have found i and j such that

$$f(a_{ij}) = \sum_{n=0}^{i+j-2}(n) + i, \text{ where } i, j > 0 \text{ as required.}$$

However, it was not necessary to prove onto in order to prove that A ~ N. First of all, for all i N ~ A_i. Suppose f was not onto N but onto N', an infinite proper subset of N, then we have that $A \sim N' \subset N \subset A$. In fact, by definition, an infinite set is equivalent to a proper subset of itself. So A must be equivalent to N.

Proof 1(b):

Let $a_{ij} = {}^{j}/_{i}$ in proof 1(a). The denominators in *Q* are elements of *N* and for all denominator $i \in N$ there exists a set A_i which is the set of all positive rationals with denominator i. Hence the union $A = A_1 \cup A_2 \cup A_3 \cup \ldots$ covers all the denominators and hence forms the set of all positive rationals. Hence by result 1(a) Q^+ is countable. The set Q^- of negative rationals is equivalent to the set of positive rationals, hence also countable. Again, by result 1(a), $Q = Q^+ \cup Q^-$ is countable.

2. (a) The set of real numbers, *R*, is uncountable.
   (b) The set of irrational numbers, *I*, is uncountable.

Proof 2(a):

Suppose *R* is countable. Therefore *R* is equivalent to *N*, and hence its elements can be listed. Let us just consider the real interval [0,1]. Hence let the elements in [0,1] be listed as $\{a_1, a_2, a_3, \ldots\}$. Also let each rational element in [0,1] be written in its infinite decimal expansion, e.g. 0.5 = 0.4999... Hence we have the following list:

$$a_1 = 0.a_{11}\, a_{12}\, a_{13}\, a_{14}\ldots$$

$$a_2 = 0.a_{21}\, a_{22}\, a_{23}\, a_{24}\ldots$$

$$a_3 = 0.a_{31}\, a_{32}\, a_{33}\, a_{34}\ldots$$

etc.

Let $b = 0.b_1b_2b_3\ldots$ be a real number in [0,1] such that $b_i = 1$ if $a_{ii} = 0$ and $b_i = 0$ if $a_{ii} = 1$. Hence b is not in the set $\{a_1, a_2, a_3, \ldots\}$. This is a contradiction and [0,1] is therefore uncountable. So, obviously, *R* is uncountable.

Proof 2(b):

$R = Q \cup I$. Suppose I is countable. Again, by result 1(a), this implies that R is countable since Q is also countable. So this is a contradiction and I is therefore uncountable.

3. (a) N has measure 0. Q (or any countable set) has measure 0.

   (b) $I \cap [0,1]$ has measure 1.

Some properties of Measure:

- The measure of the empty set $\varnothing$ is 0.
- For any real interval [a,b], b > a, the (Lebesgue) measure is given by (b - a).
- Let M_A denote the measure of set A. If $A = B \cup C$ then $M_A = M_B + M_C - M_{B \cap C}$.

Proof 3(a):

Let $n \in N$ be covered by a real interval of radius $\varepsilon/(2^n)$, i.e. $[n - \varepsilon/(2^n), n + \varepsilon/(2^n)]$. For any $\varepsilon > 0$, all natural numbers are covered. Taking all the covers we get that the measure of N is less than $2\sum_{n=0}^{\infty}\frac{\varepsilon}{2^n} = 2\varepsilon\left(\frac{1}{1-1/2}\right) = 4\varepsilon$. We can let ε tend to 0 without uncovering any natural number, whereby the measure tends to 0. Hence N has 0 measure.

Since (by definition) any countable set is equivalent to N, then any countable set has 0 measure. In particular, Q has 0 measure.

Proof 3(b):

The (Lebesgue) measure of the real interval [0,1] is given by 1 - 0 = 1. By definition, $I = Q^c = R \setminus Q$ (R without Q), i.e. the set of all real numbers which are not rational (irrational). $R = I \cup Q$ and $I \cap Q = \varnothing$ (empty set).

$[0,1] = [0,1] \cap R = [0,1] \cap (I \cup Q) = ([0,1] \cap I) \cup ([0,1] \cap Q)$.

Let $A = [0,1]$, $B = [0,1] \cap I$, $C = [0,1] \cap Q$. $B \cap C = \varnothing$. Hence $M_{B \cap C} = 0$. Also, since C is countable, $M_C = 0$. Therefore $M_A = M_B + M_C - M_{B \cap C} = M_B + 0 - 0 = 1$. So $M_B = 1$, i.e. the measure of the irrationals in [0,1] is 1.

Problem: <u>If M is an uncountable set in [0,1], does it necessarily have measure 1??!!</u>

Remark: This will be tackled in a future workshop.

Infinite Sets

Peter Borg

B.Sc 3rd year

Abstract

DEFINITIONS

• Two sets M and M' are **equivalent** if there exists a one-to-one correspondence between their elements, i.e. $M \sim M' \iff \exists f : M \longrightarrow M'$ such that f is bijective.

• A set M is finite if either it is empty or there exists a natural number n such that $M \sim \{1, 2, ..., n\}$; otherwise M is infinite.

• A set M is **infinite** if it is equivalent to a proper subset of itself; otherwise M is finite. M is infinite $\iff \exists M' \subset M$ s.t. $M \sim M'$.

• An infinite set M is countable if it is equivalent to the set of natural numbers, otherwise it is uncountable. That is, M is countable $\iff M \sim N$.

THEORETICAL RESULTS

1. (a) A countable union of countable sets is countable.

(b) The set of rational numbers, Q, is countable, i.e. $Q \sim N$.

A Reconstruction Game

Juanita Formosa

M.Sc. student

Abstract

We propose a game in which the number of players is 2 - the robber and the detective. The detective is not after revealing the identity of the robber but in disclosing what the robber stole, given some hints by the robber himself. The winner is the robber if the detective fails to reveal the stolen property; otherwise the detective wins. We apply this to Ulam's Reconstruction Conjecture, a problem which is still open and which states that for a graph of order three or more, it is possible to reconstruct the original graph G from the deck of one vertex-deleted subgraphs of G.

What is a Graph?

Let us begin by considering the figure below:

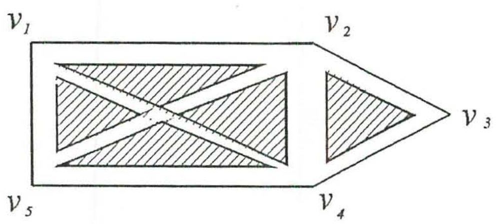


FIGURE 0.4. A road map

It is clear that it can be represented diagrammatically by means of points and lines as in Figure 0.5 below.

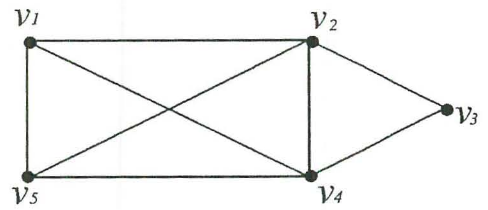


FIGURE 0.5. The corresponding graph

The points $\{v_1, v_2, \ldots, v_n\}$ are called **vertices** and the lines are called **edges**; the whole diagram is called a **graph**. Usually, we stick to the following notation:

In general a graph G has a set $V(G) = \{v_1, v_2, \ldots, v_n\}$ of n vertices and an edge set $E(G)$ of m edges such that every edge joins a pair of distinct vertices. The **degree** or **valency** of a vertex is the number of edges which have that vertex as an endpoint and corresponds in figure 1.1 to the number of roads at an intersection. thus the degree of the vertex v_2 is 4. If all the verices have tha same valency r, then G is said to be **regular**.

What is the Reconstruction Game?

Consider the following game with the following rules:

(1) Number of players is 2 - the robber and the detective. Strangely enough, in our case the detective is not after revealing the identity of the robber but in disclosing what the robber stole, given some hints by the robber himself!

(2) The winner is the robber if the detective fails to reveal the stolen property; otherwise the detective wins.

Let's say that the stolen property is the graph G given below:

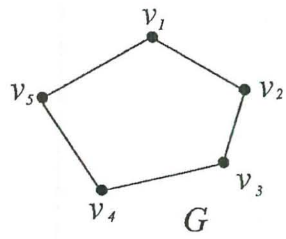


FIGURE 0.6. The stolen property

and the hints given to the detective are the five subgraphs (**cards**) below , commonly known as the **deck** $D(G)$ of G. Each card is obtained by stepwise removing v_1, v_2, ..., v_5 and any adjacent edges from G.

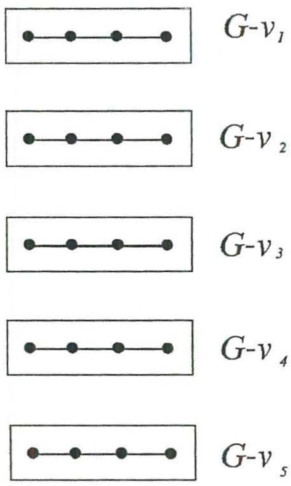


FIGURE 0.7. Deck of cards

The basic question, which is Ulam's Reconstruction Game, is very simple indeed: *is it possible to reconstruct the original graph G ?* In this particular case, the answer is yes. The method how to go about it is as follows:

From the given information deck, we deduce that $n = 5$. Also, the number of edges in each card is 4 so that the same number of edges are deleted with each vertex v_i. Thus the parent graph G is regular. Hence, it is clear that regularity of G is recognisable from $D(G)$. To recover G, it suffices to add a vertex to any one of the subgraphs in the deck and join it to those vertices having the minimum degree.

Hence, in this case the detective is the winner. The case for regular graphs is very simple but the problem has proved to be very difficult for the arbitrary graph and is still open for about half century of history. In mathematical terms, the game is called "Ulam's Reconstruction Conjecture" and it reads as follows:

Every graph with at least 3 vertices is reconstructible.

It is clear that we consider $n \geq 3$ because the problem fails for $n = 2$.

What is the Polynomial Reconstruction Game?

Now there exists a parallel reasoning using polynomials, rather than graphs. The conjecture is called "The Polynomial Reconstruction Conjecture" and it is a variant of Ulam's reconstruction conjecture originated by D. Cvetković in 1973. It states that:

Every graph with at least 3 vertices is polynomial reconstructible.

Equivalently, for $n \geq 3$, given a **p-deck** $PD(G)$ of n cards, each showing a characteristic polynomial $\phi(G - v; \lambda)$ as v runs through the n vertices of G, the characteristic polynomial $\phi(G; \lambda)$ can be recovered.

Even in this case, the problem is still open in general, although it has been solved for some classes of graghs such as regular graphs. Hence we will reconsider the previous game, this time using polynomials rather than graphs.

But let us first define what we understand by polynomial:

The **adjacency matrix** of a graph G with vertex set v_1, v_2, ..., v_n is the (0,1)symmetric $n \times n$ matrix $A(G) = (a_{ij})$ whose (ij)-entry a_{ij} is equal to the number of edges the vertex v_i to the vertex v_j. As an example, the adjacency matrices of the graph in Figure 1.2 and its vertex deleted subgraph are given below:

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A(G - v_i) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The **characteristic polynomial** of a graph G is defined to be the characteristic polynomial of its adjacency matrix $A = A(G)$ and if I denotes the identity matrix, then

$$\phi(G) = \phi(G; \lambda) = |(\lambda I - A)|$$

is a polynomial $\sum_{i=0}^{n}(a_i \lambda^{n-1})$ with integer coefficients a_i.

The characteristic polynomials of G and $G - v_i$ are $x^5 - 5x^3 + 5x - 2$ and $x^4 - 3x^2 + 1$ respectively.

A useful result which enables the recovery of most of the terms of the characteristic polynomial of the parent graph G from the $PD(G)$ is the following:

THEOREM 0.0.4.

$$\phi'(G;\lambda) = \sum_{PG} \phi(G - v_i;\lambda)$$

Thus by integrating the previous result, we obtain $\phi(G;\lambda)$, save for the constant term. This can be checked out by adding up $x^4 - 3x^2 + 1$ for all 5 subgraphs and then integrating with respect to x to obtain the characteristic polynomial of G save for the constant -2. Thus a boundary condition is required to determine $\phi(G;\lambda)$ completely.

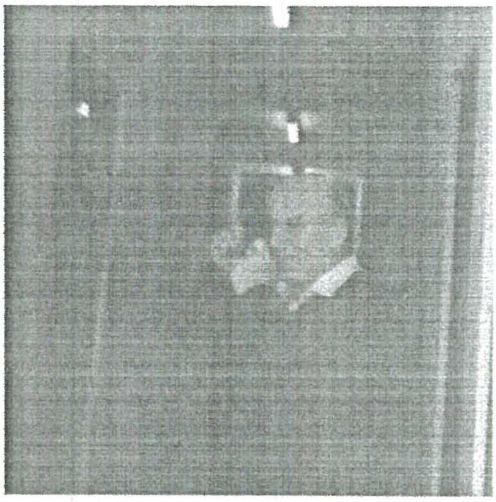
It is interesting to point out that a positive answer to the Polynomial Reconstruction Game would imply the validity of Ulam's conjecture. But this approach depends on the resolution of a major problem: which graphs are determined by their spectrum? Unfortunately, all non trivial graphs known at present to be characterised by their spectra are regular, while Ulam's conjecture is trivially true for regular graphs. Thus it would be interesting to find some non trivial classes of non regular graphs which are characterised by their spectra.

30th October

30th October, 2000.

Collection-- II

Participants



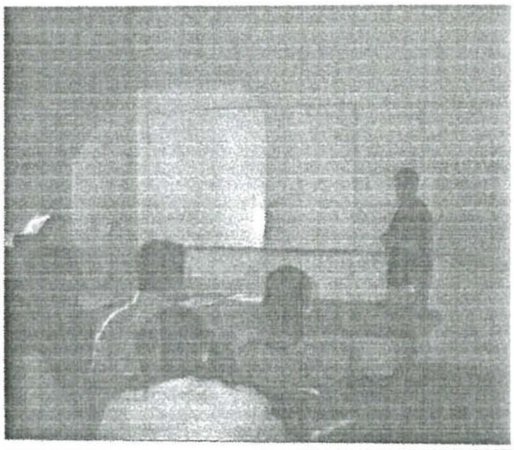
Godwin Cassar

Our Photographer

Collection -- II

The Speakers

Alex Farrugia



Juanita Formosa

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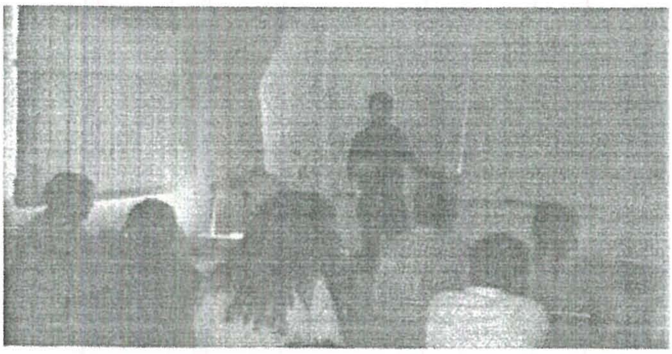

James Borg

David Suda