Intersecting Circles and Ellipses[2]

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Abstract

In the February 2000 issue of The Collection, Phaedra Cassar posed the following the question: Is it possible for four or more circles to be drawn such that each new circle added has a common region with all existing regions?[3] We prove that this is not possible. Furthermore we show that for ellipses, the task is not possible for five or more ellipses.

DEFINITION 0.0.1. An r-Venn diagram is a collection of r closed curves $C_1, ..., C_r$, such that the intersection of $X_1, ..., X_{2^r}$ is non-empty and connected, where each X_i is either int(C_i) or ext(C_i).

[**Note**: we will use 'r' for the number of curves, and reserve 'n' for the number of intersection points. A region is also referred to as a face and the region outside the Venn Diagram is the infinite face.]

THEOREM 0.0.2. [4] If $C_1, ..., C_r$ are all circles, then there is no r-Venn diagram for $r > 3$; if the C_i's are all ellipses, there is no r-Venn diagram for $r > 5$.

PROOF. Given a Venn diagram we can put a vertex at each intersection point to get a planar graph. By Euler's formula for planar graphs $n - m +$

[2]Following the article "Intersecting Circles" which appeared in The Collection: 16th Feb 2000. (See p.5)

[3]See p.5.

[4]I wish to thank Frank Ruskey who gave me the necessary hint to prove this theorem.

$f = 2$ where n, m, f are the numbers of vertices (intersection points of the Venn diagram), edges and faces. Now in an r-Venn diagram $f = 2^r$, so $2^r = 2 + m - n$

We will get an upper bound on $m - n$ and thus prove the theorem.

We will consider only the case where we do NOT have three or more circles intersecting at the same point. This can be justified by geometric intuition, or by observing that if we do have three or more circles intersecting at the same point, the quantity m-n is even lower than if we don't.

The graph will be 4-regular, so $4n = 2m$, and thus $2^r = 2 + n$.

Now since every pair of circles intersect in at most two points, the number of intersection points is at most twice (r choose 2), i.e. $n \leq r(r-1)$.

Thus we must have $2^r <= 2 + r(r-1)$ which is impossible for $r = 4(2^r = 16, 2 + r(r-1) = 14)$, and thus impossible for $r \geq 4$ since 2^r increases faster than $r(r-1)$.

If we have ellipses, since each pair of ellipses can intersect at at most four points, we have $n <= 2r(r-1)$ and so $2^r <= 2+2r(r-1)$ which is impossible for $r = 6(2^r = 64, 2 + 2r(r-1) = 62)$. $\square$

Intersecting Circles

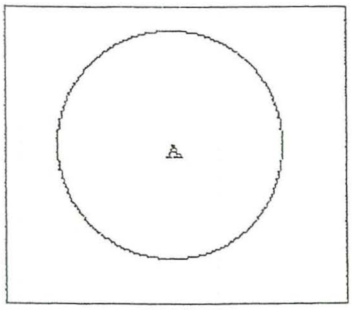
Rule: The circles must intersect in such a way that the new circle added must take in part of Each and every area of the circles and regions in the former diagram.

With two circles this is obviously possible, creating three regions.

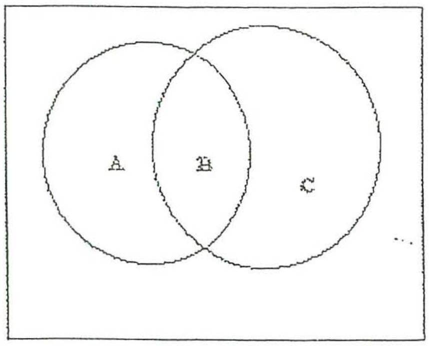
With three circles, it is also possible, with 7 regions being created.

But with four circles, one region will be enclosed entirely within the circle.

Question: Is it really impossible to draw four circles such that the rule defined above is followed? If not, why is it not possible to draw four circles to fulfill the set condition, when it is possible to do it with two or three circles? Perhaps this problem is related to the four colour theorem?



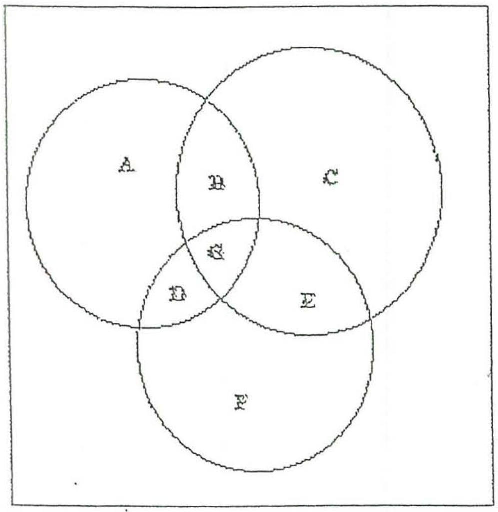
1 CIRCLE (TRIVIAL)



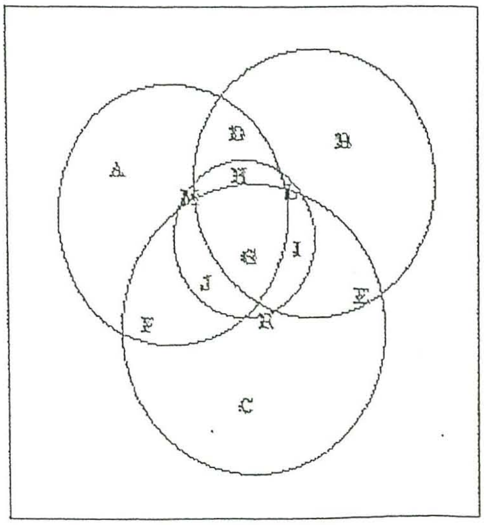
2 INT. CIRCLES

3 ENCLOSED REGIONS

Diagrams of Intersecting circles.



THREE INT. CIRCLES, CREATING 7



ONE POSSIBILITY FOR FOUR INT. CIRCLES

[5] This page appeared in The Collection of Feb 2000.