Two Simple Proofs

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Abstract
My proofs of two simple results will be presented. These are:
a) \(0.99999\ldots = 1\)
b) The area of a circle is \(\pi r^2\), where \(r\) is the radius of the circle.

a) \(0.99999\ldots = 1\)

Remark: This simple result is usually deduced by considering the number \(0.99999\ldots\) as the series

\[
\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots
\]

and summing the series to infinity, whence it is equal to 1. My proof is different.

Proof. We don't know what \(0.99999\ldots\) is, so let's assign it to \(x\).

(0.1) \(x = 0.99999\ldots\)

Multiplying both sides by 10, we get:

\(10x = 9.99999\ldots\)

Subtracting 9 from both sides, we get:
Comparing equations (1) and (2), we immediately note that their R.H.S is exactly the same, which implies that their L.H.S are also equal. So

\begin{align}
(0.3) & \quad 10x - 9 = x \\
(0.4) & \quad 9x = 9 \\
(0.5) & \quad x = 1 
\end{align}

But we started with \( x = 0.999999 \) and we ended up with \( x = 1 \).

So finally, \( 0.999999 \Rightarrow 1 \). \( \square \)

b) The area of a circle with radius \( r \) is \( \pi r^2 \).

**Remark:** The proof that the area of a circle is \( \pi \) multiplied the square of its radius is usually found by integrating the area under the graph of a semicircle with radius \( r \), remembering to double the answer in the end. My proof takes a different approach altogether.

**PROOF.** Consider any \( n \)-sided regular polygon. In the diagram below we have chosen a regular pentagon. (See the figure.) From the centre of the polygon we draw \( n \) lines to each of the vertices of the polygon. Call the length of each of these lines \( r \). The polygon is thus divided into \( n \) isosceles triangles (in our case \( n = 5 \)).

Let \( \theta \) be the angle at the centre that each of these lines makes with its subsequent line. It can be easily seen that \( \theta = \frac{2\pi}{n} \) (using radian measure).

Now consider finding the area of this polygon. It can be found by multiplying the area of one of the triangles by \( n \), since obviously all of the triangles have
the same area. The area of one of these triangles, knowing the length of two of its sides and the angle between them, is:

$$\frac{1}{2} r^2 \sin \theta = \frac{1}{2} r^2 \sin \left( \frac{2\pi}{n} \right)$$

Therefore, the area of the $n$ sided polygon is $\frac{1}{2}nr^2 \sin \left( \frac{2\pi}{n} \right)$.

Now comes the crucial step. If we allow $n$ to become very large, the $n$ sided polygon will approximate a circle. This implies that its area will be close to that of the circle. Therefore, as $n$ approaches infinity, the area of the polygon approaches that of the circle.

Mathematically speaking,

Area of circle

$$= \lim_{n \to \infty} \frac{1}{2} r^2 n \sin \left( \frac{2\pi}{n} \right)$$

$$= \frac{1}{2} r^2 \lim_{n \to \infty} n \sin \left( \frac{2\pi}{n} \right)$$

Now

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$
Thus

\[ \lim_{n \to \infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} = 1 \]

\[ \Rightarrow \lim_{n \to \infty} \sin\left(\frac{2\pi}{n}\right) = 1 \]

\[ \Rightarrow \frac{1}{2\pi} \lim_{n \to \infty} n \sin\left(\frac{2\pi}{n}\right) = 1 \]

\[ \Rightarrow \lim_{n \to \infty} n \sin\left(\frac{2\pi}{n}\right) = 2\pi \]

Therefore the area of a circle is \( \pi r^2 \). \( \square \)