A Problem inspired from the Cantor Set

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Required to find a subset of $[0,1] \subseteq \mathbb{R}$ which is dense, does not contain intervals of $[0,1]$, and whose measure lies between 0 and 1.

Define

$$A_1 = \left[\frac{1}{3}, \frac{2}{3}\right]$$

$$A_2 = \left[\frac{1}{3^3}, \frac{2}{3^3}\right] \cup \left[\frac{2}{3} + \frac{1}{3^3}, \frac{2}{3} + \frac{2}{3^3}\right]$$

... etc. Then define $A = \bigcup_\infty A_i \subseteq [0,1]$.

The length of the first interval: $a_0 = 1$ The length of the second interval:

$$a_1 = \frac{1 - \frac{1}{3}}{2} = \frac{1}{3}$$

The length of the third interval:

$$a_2 = \frac{a_1\left(1 - \left(\frac{1}{3}\right)^2\right)}{2} = \frac{\left(1 - \frac{1}{3}\right)\left(1 - \left(\frac{1}{3}\right)^2\right)}{2^2}$$

The length of the fourth interval:

$$a_3 = \frac{\left(1 - \frac{1}{3}\right)\left(1 - \left(\frac{1}{3}\right)^2\right)\left(1 - \left(\frac{1}{3}\right)^3\right)}{2^3}$$

Therefore $|A_1| = \frac{1}{3}a_0 = \frac{1}{3}, |A_2| = \frac{2}{3^2}a_1, \dots$

Therefore $|\bigcup_\infty A_i| = \frac{1}{3}a_0 + \frac{2}{3^2}a_1 + \frac{2^2}{3^3}a_2 + \frac{2^3}{3^4}a_3 + \cdots$

Let $S_\infty = |\bigcup_\infty A_i|$, then

$$S_\infty = \frac{1}{3} + \frac{\left(1 - \frac{1}{3}\right)}{3^2} + \frac{\left(1 - \frac{1}{3}\right)\left(1 - \left(\frac{1}{3}\right)^2\right)}{3^3} + \frac{\left(1 - \frac{1}{3}\right)\left(1 - \left(\frac{1}{3}\right)^2\right)\left(1 - \left(\frac{1}{3}\right)^3\right)}{3^4} + \cdots$$

$$S_\infty = \frac{1}{3} + \frac{(3-1)}{3^3} + \frac{(3-1)(3^2-1)}{3^6} + \frac{(3-1)(3^2-1)(3^3-1)}{3^{10}} + \cdots$$

$$\cdots + \frac{(3-1)(3^2-1)\cdots(3^{n-1}-1)}{3^{\frac{n}{2}(n+1)}} + \cdots$$

$$S_\infty < \frac{1}{3} + \frac{3}{3^3} + \frac{3 \cdot 3^2}{3^6} + \frac{3 \cdot 3^2 \cdot 3^3}{3^{10}} + \cdots$$
$$\implies S_\infty < \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots$$
$$\implies S_\infty < \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Also $S_\infty > \frac{1}{3}$, hence $\frac{1}{3} < S_\infty < \frac{1}{2}$.

Thus if we consider all the irrational numbers in these intervals A_i, then we obtain a subset of $[0, 1]$ which is dense, contains no intervals, and its measure is between $\frac{1}{3}$ and $\frac{1}{2}$.