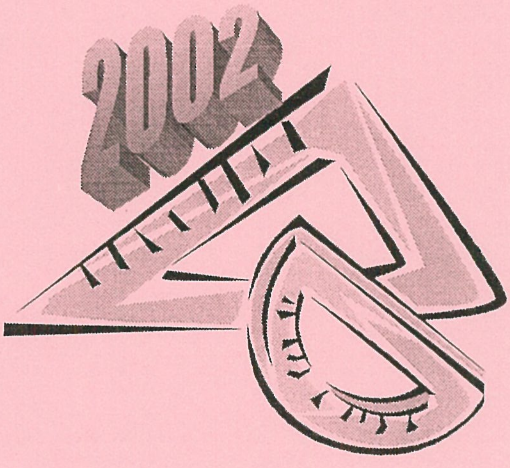
Collection V



The Collection V

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Faculty of Science

University of Malta

Proceedings of Workshop held on the 21st February 2002

Foreword

> **Discovery** and **Proof** are two different activities. They require different resources and techniques. *Archimedes*

Collection V brings together a wide scope of students' mathematical musings. Number theory is always very popular. A great deal of interest is being generated on matrices and their relation to graphs. The current research within the department on crossing lines is featured in this issue. We also survey the software used during mathematics classes in the senior schools.
Dr. Irene Sciriha.
Organizer.

The Collection V

Date:	21st February 2002
Time:	15.00 - 16.15
Venue:	Faculty of Science Department of Mathematics Room 316

A seminar/workshop is being held on Thursday 21st February 2002 at 3.00 p.m. Students and staff from the Department of Mathematics, Faculty of Science will present ideas from various fields of mathematics.

During the meeting, the software introduced in the curriculum for the MATSEC (Ordinary Level Exam) will be reviewed.

Keynote speakers:

John Baptist Gauci	On Crossing numbers
Alexander Farrugia	Boolean Matrices
Daniel Buhagiar	Introducing Software
Vincent Mercieca	A new form of primes

We shall end with a brief session for spontaneous problem posing. You are cordially invited to attend.

Abstracts of possible proofs or conjectures which you wish to share with us in this meeting, or in a future one, may be sent to Dr. I. Sciriha or Ms. A. Attard, Department of Mathematics, (marked The Collection), at any time of the year.

Dr. I. Sciriha
Organiser

Contents

NEW NECESSARY AND SUFFICIENT CONDITIONS FOR THE ZARANKIEWICZ CONJECTURE ON CROSSING NUMBERS

John Baptist Gauci

Abstract:

The purpose of this presentation is to give a historical outline about the theory of crossing numbers, and to present the current status of long standing problems on crossing numbers of complete bipartite graphs. A set of new necessary and sufficient conditions are given for the Zarankiewicz conjecture on the crossing number of complete bipartite graphs. These conditions are expressed in terms of the crossing numbers' divisibility properties and their expressibility as polynomials.

Crossing Numbers: A Historical Introduction

The study of crossing numbers originated in 1952 when Paul Turán posed what is known today as the Brick Factory Problem [1]:

> *"We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. ... All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards and unload them there. ... the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; this caused a lot of trouble and loss of time. ... The idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings?"*

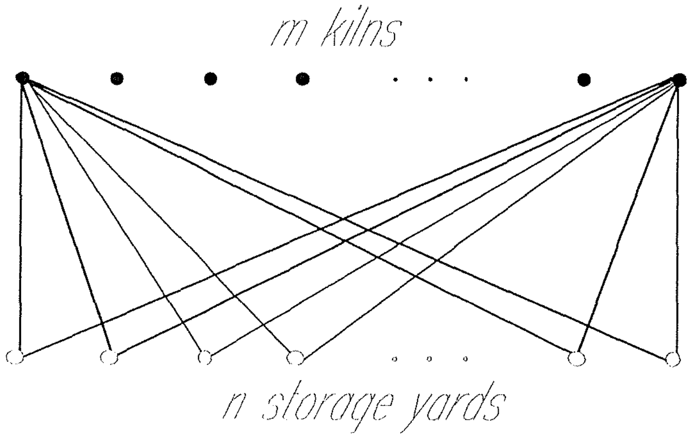
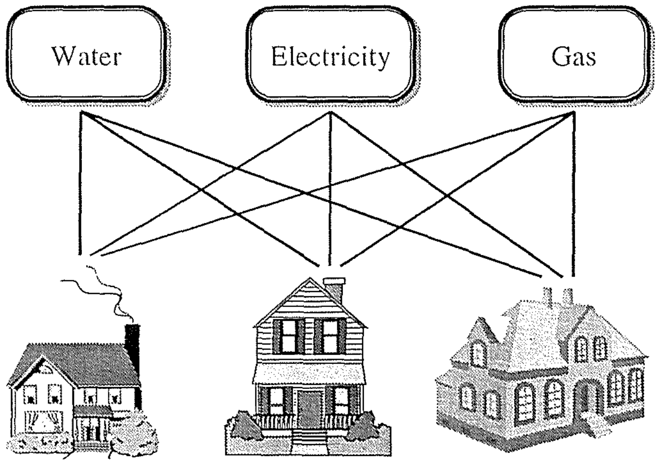


Figure 1

The graph shown in Figure 1 is the Complete Bipartite Graph $K_{m,n}$, having one set of vertices containing m vertices and the other set containing n vertices. All the vertices in the first set are connected by edges to all the vertices in the second set.

We are interested in finding the minimum number of crossings of this graph, denoted by $\nu(K_{m,n})$.

A Simple Example: The Three Houses, Three Utilities Problem



The task is to distribute the Gas, Water and Electricity to the three houses, using seperate direct pipelines from the 'utilities' to the houses. Is it possible to do so without having to cross the pipelines?

In mathematical terms, we have the complete bipartite graph $K_{3,3}$ and we want to find its crossing number $\nu(K_{3,3})$.

Note:

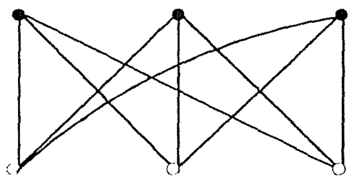
1. There are various possible drawings of $K_{3,3}$, as shown in Figure 2 below.

2. In all the four drawings shown in Figure 2, the number of crossings is ODD. In fact, Kleitman in 1971 proved the Parity Argument. This states that if the number of crossings in a particular **good** drawing is odd/even, then it is odd/even in all the good drawings.

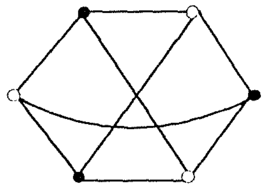
The Parity Argument (Kleitman, 1971) [2]:

Let D and D' be two **good** drawings of the complete bipartite graph $K_{m,n}$ in the plane. Then:

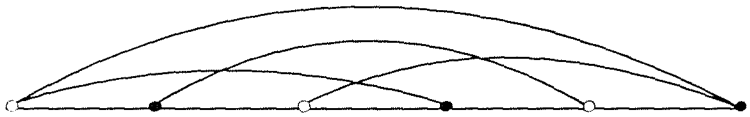
$$\nu_D(K_{m,n}) = \nu_{D'}(K_{m,n}) \pmod 2.$$



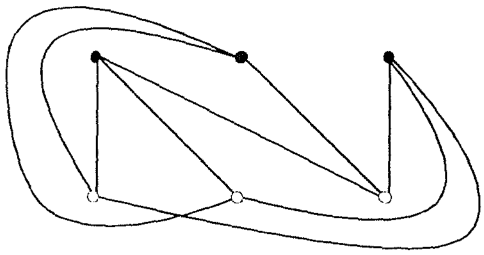
$\nu_{D_1}(K_{3,3}) = 9$



$\nu_{D_2}(K_{3,3}) = 3$



$\nu_{D_3}(K_{3,3}) = 3$



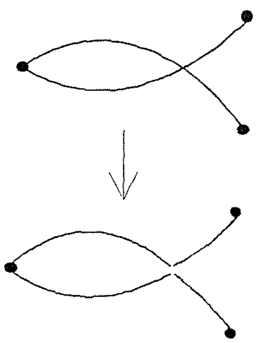
$\nu_{D_4}(K_{3,3}) = 3$

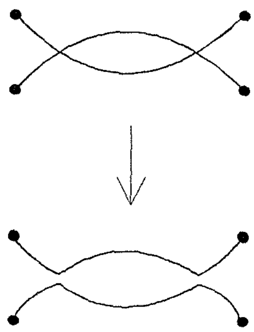
Figure 2

What is a **good** drawing? A good drawing is a drawing which does not contain any of the following bad crossings:

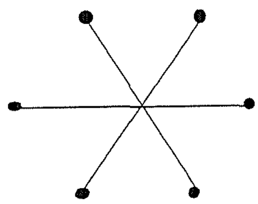
(i) an edge intersecting itself;

(ii) two edges having more than one point in common;





(iii) three edges intersecting at the same point.



To find the crossing number of $K_{3,3}$, we need to use Euler's Polyhedron Formula (stated below). Using induction on the number of vertices of a graph G, Euler showed that for any planar drawing of a graph G which has $n(G)$ vertices, $m(G)$ edges and $f(G)$ faces, then:

$$f(G) = m(G) - n(G) + 2.$$

Euler's Polyhedron Formula (Euler, 1750) [3]:

Let G be a plane drawing of a connected planar graph, and let $n(G)$, $m(G)$ and $f(G)$ denote respectively the number of vertices, edges and faces of G. Then:

$$f(G) = m(G) - n(G) + 2.$$

Lemma: $\nu(K_{3,3}) = 1$

Proof:
We consider a good drawing of $K_{3,3}$ and let α be the *least* number of edges to be deleted in order to obtain a planar drawing in which:

$$n(G) = 6,\ m(G) = 9 - \alpha$$

and, therefore, using Euler's formula we get: $f(G) = 9 - \alpha - 6 + 2 = 5 - \alpha$.
Since $K_{3,3}$ does not contain any odd circuits, then each face has at least four edges bounding it, and, thus, the girth is 4.

This implies that: $4 . f(G) \leq 2 . m(G)$

$$\Rightarrow \quad 4(5-\alpha) \leq 2(9-\alpha)$$

$$\Rightarrow \quad 20 - 4\alpha \leq 18 - 2\alpha$$

$$\Rightarrow \quad \alpha \geq 1.$$

Therefore, the least number of edges to be deleted to obtain a planar drawing of $K_{3,3}$ is one, implying that $\nu(K_{3,3}) \geq 1$.
To show that $\nu(K_{3,3}) \leq 1$, we use a clever drawing in which the crossing number is one (as the drawing D_4 shown in Figure 2).
Thus, $\nu(K_{3,3}) = 1$.

∎

Therefore, the crossing number of $K_{3,3}$ is one, and a drawing exhibiting the least number of crossings is known as an ***optimal drawing***.

This is the technique which is in general used to find the crossing number of graphs:

(i) find upperbound by using a ***clever drawing***;

(ii) get lowerbound by using *ad hoc* argument depending on the class of graphs in hand.

Zarankiewicz's Conjecture

The Polish mathematician Zarankiewicz devised a technique for drawing the complete bipartite graphs: placing the m-set of vertices on the x-axis evenly distributed about the origin, and similarly placing the n-set of vertices on the y-axis.

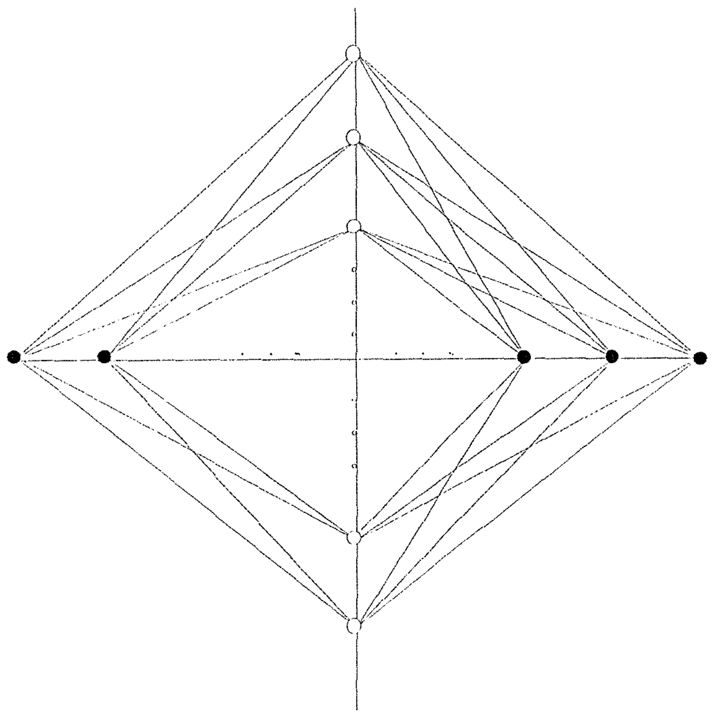


Figure 3

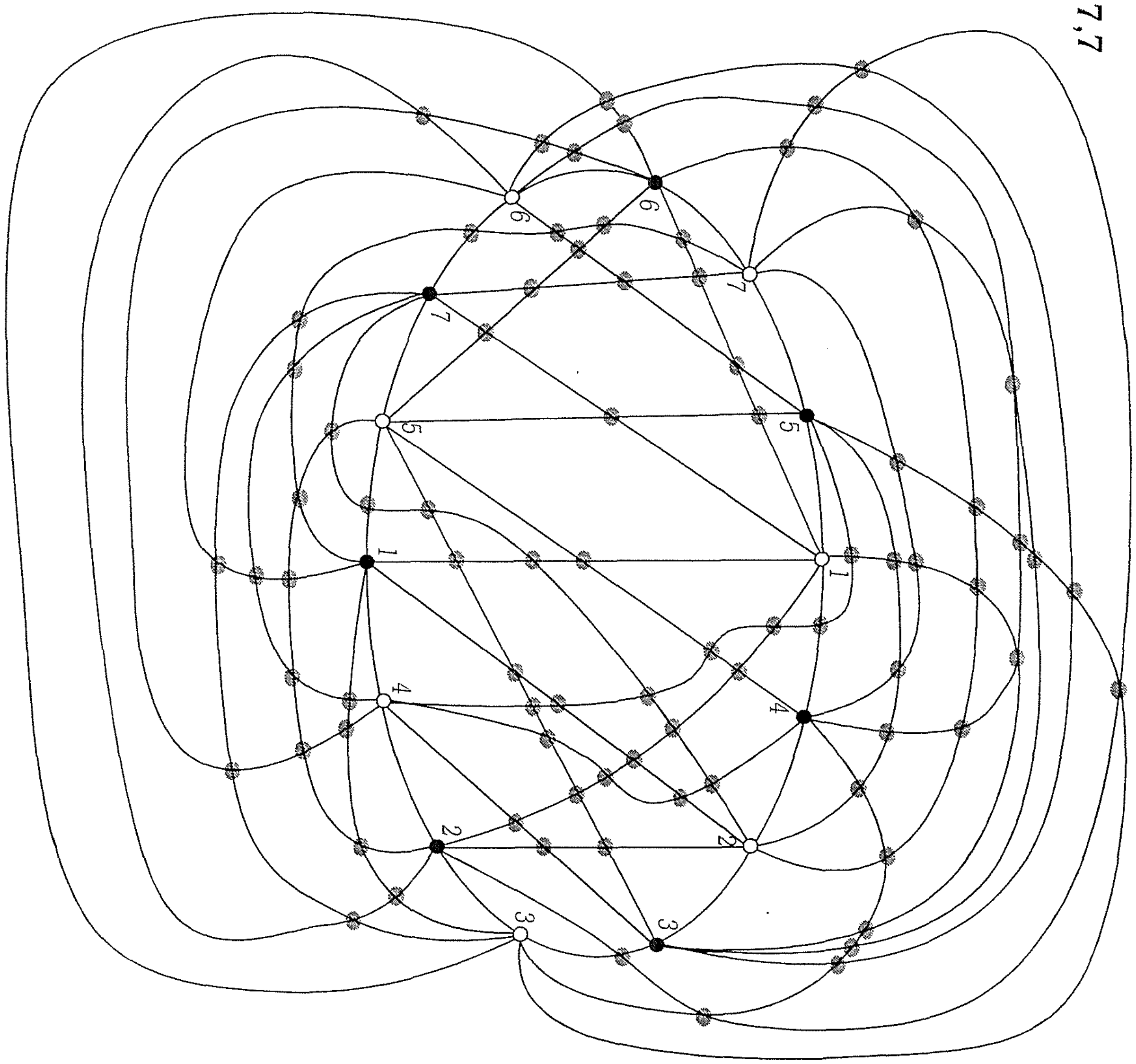
When we count the number of crossings contributed by each edge of each star, we get arithmetic progressions and using a simple counting argument, get the Zarankiewicz bound for the crossing number:

$$\zeta_{m,n} = \lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{m-1}{2} \rfloor \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor .$$

In 1953, Zarankiewicz claimed that $\zeta_{m,n}$ is in fact the crossing number of the complete bipartite graph $K_{m,n}$ [4]. In 1965 and 1966, Kainen and Ringel, respectively, spotted a flaw in the proof of Zarankiewicz (as quoted in [5]). He had assumed that in all optimal drawings of $K_{m,n}$ there are two stars that do not intersect. This is not always the case, and one possible drawing of the graph $K_{7,7}$ in which all the stars intersect and the crossing number is equal to the Zarankiewicz bound is shown in Figure 4.

Various attempts have been made throughout the years to try and solve Zarankiewicz's conjecture, but to date it has been shown that $\nu(K_{m,n}) = \zeta_{m,n}$ only when for min $\{m, n\} \leq 6$.

$K_{7,7}$



No. of crossings between stars	White	Black
1 & 2	5	2
1 & 3	1	2
1 & 4	5	2
1 & 5	7	2
1 & 6	2	3
1 & 7	3	8
2 & 3	3	5
2 & 4	5	4
2 & 5	5	2
2 & 6	2	3
2 & 7	3	4
3 & 4	3	7
3 & 5	1	6
3 & 6	8	4
3 & 7	7	1
4 & 5	5	7
4 & 6	4	7
4 & 7	1	1
5 & 6	2	4
5 & 7	3	3
6 & 7	6	4

Figure 4

New Set of Necessary and Suffiecient Conditions

The purpose of the second part of this presentation is to present a new set of necessary and sufficient conditions for the Zarankiewicz conjecture constructed by S.Fiorini and J.B.Gauci and published in the Graph Theory Notes of New York [6]. Before doing this, we will prove two basic lemmas which will be used in the main theorem.

Lemma 1:

$$\nu(K_{m,n}) \geq \frac{\binom{n}{p}}{\binom{n-2}{p-2}} \nu(K_{m,p})$$

Proof:

Let $K_{m,n}$ be represented by an optimal drawing ρ.
Select a set of p vertices from the n-set. There are $\binom{n}{p}$ such sets, each with crossing number equal to $\nu_{\rho^*}(K_{m,p})$, where ρ^* is the induced drawing.

Now, in $\binom{n}{p}\nu_{\rho^*}(K_{m,p})$, every crossing is repeated for $\binom{n-2}{p-2}$ times.

Thus, $\nu(K_{m,n}) = \nu_\rho(K_{m,n}) = \frac{\binom{n}{p}}{\binom{n-2}{p-2}} \nu_{\rho^*}(K_{m,p})$

$$\geq \frac{\binom{n}{p}}{\binom{n-2}{p-2}} \nu_{\rho[opt]}(K_{m,p}) \qquad \text{where } \rho\,[opt] \text{ is an optimal drawing for } K_{m,p}$$

$$= \frac{\binom{n}{p}}{\binom{n-2}{p-2}} \nu(K_{m,p})$$

■

Lemma 2:
If $\nu(K_{m,2s+1}) = \zeta_{m,2s+1}$, then $\nu(K_{m,2s+2}) = \zeta_{m,2s+2}$.

Proof:

$$\begin{aligned} \zeta_{m,2s+2} \;\geq\; \nu(K_{m,2s+2}) \;&\geq\; \tfrac{2s+2}{2s}\, \nu(K_{m,2s+1}) \qquad \text{with } p = 2s+1 \text{ in Lemma 1,} \\ &= \tfrac{s+1}{s}\, \zeta_{m,2s+1} \\ &= (\tfrac{s+1}{s})\, s^2 \lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{m-1}{2} \rfloor \end{aligned}$$

$$= (s+1)\, s \left\lfloor \tfrac{m}{2} \right\rfloor \left\lfloor \tfrac{m-1}{2} \right\rfloor$$
$$= \zeta_{m,2s+2}\,.$$

Thus, $\qquad \nu(K_{m,2s+2}) = \zeta_{m,2s+2}\,.$

■

We now consider the divisibility properties of $\nu(K_{m,n})$, by first assuming that $\nu(K_{m,n})$ can be expressed as four different functions $f_i(r, s) \in \mathbb{Z}[r, s]$, where:

$i = 1$	when	$m = 2r$	and	$n = 2s$;
$i = 2$	when	$m = 2r + 1$	and	$n = 2s$;
$i = 3$	when	$m = 2r$	and	$n = 2s + 1$; and
$i = 4$	when	$m = 2r + 1$	and	$n = 2s + 1$.

Clearly, if $f_i(r, s)$ is a polynomial in r and s, then $f_i(0, s) = 0 = f_i(r, 0)$, $1 \le i \le 4$, which implies that r, s and, hence, rs are factors of $f_i(r, s)$. Thus, there exists a function $g_i(r, s) \in \mathbb{Z}[r, s]$ such that $f_i(r, s)$ can be written as $rs \,.\, g_i(r, s)$. Having established this, we now proceed to give a set of necessary and sufficient conditions for the crossing number of $K_{m,n}$.

Main Theorem:

If $\nu(K_{m,n})$ can be expressed as the four different functions $f_i(r, s)$ presented above, then the following three statements are equivalent:

(i) $\quad \nu(K_{m,n}) = \zeta_{m,n}$

(ii) $\quad f_i(r, s)$ is a polynomial in r and s with integer coefficients

(iii) $\quad rs$ divides $\nu(K_{m,n})$

Proof:

(i) ⇒ *(ii)* $\quad$ Trivial.

(ii) ⇒ *(iii)* $\quad$ Discussed above.

(iii) ⇒ *(i)* $\quad$ We assume that $\nu(K_{m,n})$ can be expressed as a polynomial in r and s, and use the results of the following two theorems.

Theorem 1:

If for some n, $\nu(K_{n,n}) = \zeta_{n,n}$, then, for all $m \ge n$, $\nu(K_{m,n}) = \zeta_{m,n}$.

Proof:

We use induction on m, with the initial step being $m = n$ and assuming that the statement holds up to $m\ (> n)$. Lemma 2 proves the inductive step for $m = 2r + 1$, and, thus, leaving only the case $m = 2r$. We need to show that $\nu(K_{2r+1,n}) = \zeta_{2r+1,n}$.

Now, $\quad \nu(K_{2r+1,n}) \ \ge\ \frac{2r+1}{2r-1}\, \nu(K_{2r,n})$ $\quad$ by Lemma 1 with $q = 2r$ and $p = n$

$$= \frac{2r+1}{2r-1}\zeta_{2r,n} \quad \text{by the inductive hypothesis.}$$

If $n = 2s$, then:

$$\begin{aligned}
& \nu(K_{2r+1,2s}) = f_2(r,s) = rs\,.\,g_2(r,s) \geq \tfrac{2r+1}{2r-1}\, r(r-1)\,s(s-1) \\
\Rightarrow\quad & g_2(r,s) \geq \left\lceil \tfrac{2r+1}{2r-1}\,(r-1)(s-1) \right\rceil \qquad \text{since } g_2(r,s) \in \mathbb{Z}[r,s] \\
& \quad = \left\lceil r(s-1) - \tfrac{(s-1)}{(2r-1)} \right\rceil \\
& \quad = r(s-1) \qquad \text{since } s\text{-}1 < s = \tfrac{n}{2} \leq \tfrac{m-1}{2} = r \leq 2r-1\,.
\end{aligned}$$

Thus, $\nu(K_{2r+1,2s}) = rs\,.\,g_2(r,s) \geq r^2\,s(s-1) = \zeta_{2r+1,2s}$.

If $n = 2s + 1$, then:

$$\begin{aligned}
& \nu(K_{2r+1,2s+1}) = f_4(r,s) = rs\,.\,g_4(r,s) \geq \tfrac{2r+1}{2r-1}\, r(r-1)\,s^2 \\
\Rightarrow\quad & g_4(r,s) \geq \left\lceil \tfrac{2r+1}{2r-1}\,(r-1)\,s \right\rceil \qquad \text{since } g_4(r,s) \in \mathbb{Z}[r,s] \\
& \quad = \left\lceil rs - \tfrac{s}{(2r-1)} \right\rceil \\
& \quad = rs \qquad \text{since } s = \tfrac{n-1}{2} \leq \tfrac{m}{2} - 1 = r - \tfrac{1}{2} \leq 2r-1\,.
\end{aligned}$$

Thus, $\nu(K_{2r+1,2s+1}) = rs\,.\,g_4(r,s) \geq r^2 s^2 = \zeta_{2r+1,2s+1}$.

For the lower bound, clearly $\nu(K_{m,n}) \leq \zeta_{m,n}$, hence, giving equality. ■

Theorem 2:
For all values of n, $\nu(K_{n,n}) = \zeta_{n,n}$.

Proof:

We use induction on n, starting induction with $n = 2$ or 3 and assume statement true up to $n - 1$. That is, $\nu(K_{n-1,n-1}) = \zeta_{n-1,n-1}$.

Again, Lemma 2 takes care of the case when $n = 2r$, so that we need only prove the inductive step when $n = 2r + 1$.

By Theorem 1, $\nu(K_{2r+1,2r}) = \nu(K_{n,n-1}) = \zeta_{n,n-1} = \zeta_{2r+1,2r}$,
and by symmetry: $\nu(K_{2r,2r+1}) = \nu(K_{2r+1,2r})$.
Therefore, $\zeta_{2r+1,2r+1} \geq \nu(K_{2r+1,2r+1}) \geq \frac{2r+1}{2r-1}\,\nu(K_{2r+1,2r})$ with $p = 2r$ in Lemma 1,

$$\begin{aligned}
& = \tfrac{2r+1}{2r-1}\,\zeta_{2r+1,2r} \\
& = \tfrac{2r+1}{2r-1}\,r^3(r-1).
\end{aligned}$$

Thus, $\nu(K_{2r+1,2r+1}) = f_4(r,r) = r^2\,.\,g_4(r,r) \geq \frac{2r+1}{2r-1}\,r^3(r-1)$,
implying that:

$$\begin{aligned}
g_4(r,r) & \geq \left\lceil \tfrac{2r+1}{2r-1}\,r(r-1) \right\rceil \\
& = \left\lceil r^2 - \tfrac{r}{2r-1} \right\rceil \\
& = r^2\,, \qquad \text{since } r < 2r - 1.
\end{aligned}$$

Thus, $r^2 \cdot g_4(r, r) = f_4(r, r) = \nu(K_{2r+1,2r+1}) \geq r^4 = \zeta_{2r+1,2r+1}$.
Hence, result follows.

■

Conclusion:

The result of the Main Theorem can be used to rule out certain possible values for the crossing numbers of graphs that have not yet been determined, since if rs divides $\nu(K_{m,n})$, then this implies that $\nu(K_{m,n}) = \zeta_{m,n}$, and, thus, all multiples of rs which are less than $\zeta_{m,n}$ can be excluded.

Taking $K_{15,17}$ as an example, since $\zeta_{15,17} = 7^2 . 8^2 = 3136$, then $\nu(K_{15,17})$ cannot take the following values:

$3136 - 56,\ 3136 - 2(56),\ 3136 - 3(56),\ \ldots$

We note that these values are not excluded by the parity argument since they are all even.

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[4] Zarankiewicz K., On a problem of P.Turán concerning graphs, *Fundamenta Mathematicae*, 41 (1954) 137-145.

[5] Guy R.K., The decline and fall of Zarankiewicz's theorem, in *Proof Techniques in Graph Theory*, F.Harary (Ed.), Academic Press, New York, 63-69 (1969).

[6] Fiorini S. and Gauci J.B., Necessary and sufficient conditions for the Zarankiewicz conjecture on crossing numbers, *Graph Theory Notes of New York*, 41 (2001) 17-21.

Boolean Matrices

Alexander Farrugia

In this short note, we will use Boolean matrices to help us find some interesting graph theoretical properties regarding the distance and index of a graph.

First, we will give some definitions that are necessary for the theorems which follow.

Definition: A *Boolean Matrix* is a square matrix, all of whose entries are either 0 or 1. These entries are added and multiplied together in the following fashion:

Addition	Multiplication
$x + 0 = x$	$x \cdot 0 = 0$
$x + 1 = 1$	$x \cdot 1 = x$

where $x \in \{0, 1\}$.

Definition: The *Adjacency Matrix* of a graph G is a symmetric $n \times n$ matrix, where n is the number of vertices in the graph, whose ij^{th} entry is 1 if there is an edge connecting vertex i and vertex j in the graph, and 0 otherwise, for all $1 \leq i, j \leq n$.

The following lemma is a well-known property of all adjacency matrices.

Lemma 1 *The ij^{th} entry of the r^{th} power of the adjacency matrix gives the number of walks of length r from vertex i to vertex j in G.*

In the sequel, we are not interested in the number of walks of length r between any two vertices, but rather whether there exists a walk of the same length between those two vertices or not. For this reason, we consider the adjacency matrix to be a Boolean matrix so that we can only get 0 or 1 for the entries of A^r.

Definitions: The *distance* d_{ij} between vertices i and j is the length of the shortest walk between those two vertices. The *diameter* of a graph G, denoted by D, is $max(d_{ij}) \forall i, j$.

The definition of the diameter D above can be shown to be equivalent to the following:

Lemma 2 *D is the smallest $m \in \mathbb{N}$ such that $A + A^2 + \ldots + A^m = J$, where A is the adjacency matrix of G and J is the $n \times n$ matrix whose entries are all 1.*

Proof: From Lemma 1, we know that the ij^{th} entry of A^r for all $r \in \mathbb{N}, 1 \leq i, j \leq n$ is 1 if there exists a walk of length r between vertices i and j in G, and 0 otherwise. Expanding on this knowledge, the ij^{th} entry of $A + A^2 + \ldots + A^r$ is 1 if there exists a walk of length r or less between the same two vertices, and 0 otherwise. If m is an integer such that $A + A^2 + \ldots + A^m = J$ but $A + A^2 + \ldots + A^{m-1} \neq J$, then for all pairs of vertices, there exists a walk of length less than of equal to m between them. and m is the smallest number with this property. Thus, $m = D$, as required.

A definition similar to that of the diameter will be given now.

Definition: The *walk-index* of a connected graph with an odd circuit, denoted by Γ, is the smallest $m \in \mathbb{N}$ such that $A^m = J$. That is, Γ is the smallest possible number such that there exists a walk of exactly length Γ between any pair of vertices in G.

Note that we only consider non-bipartite graphs (graphs with an odd circuit) in the above definition. This is because Γ of bipartite graphs cannot be found.

Clearly $A^{\Gamma+k} = J$ for all $k \geq 0$. It is also obvious that $\Gamma \geq D$. In fact, it can be shown that $D \leq \Gamma \leq 2D$. Thus, it might be interesting to find properties of graphs with $\Gamma = D$ and graphs with $\Gamma = 2D$. Here, we will only give a small class of graphs whose $\Gamma = 2D$. First, however, we give yet another definition.

Definition Let H and K be any two graphs. The *coalescence* of H and K, denoted by $H : K$, is the graph where r vertices of H are identified with r vertices of K, where $r \geq 1$.

Theorem: Let K be any odd circuit and let H be a connected bipartite graph. Let G be either K or $H : K$ where only one vertex v in H is identified with one vertex in K. Then $\Gamma = 2D$.

Proof: Suppose $G = K$, an odd circuit. Then $D = \frac{n-1}{2}$. If D is odd, then for all vertices v' in K, there does not exist any walk from v' to itself. If D is even, then for all pairs of adjacent vertices v_1 and v_2 in K, there does not exist any walk from v_1 to v_2. So Γ cannot be D. Thus we try for $\Gamma = D + k$ for all $k = 1, 2, \ldots, D$. In all cases except when $k = D$, the same thing as above happens. When $k = D$, i.e. when $\Gamma = 2D$, all vertices in K can be reached via a walk of length $2D = n - 1$. Thus $\Gamma = 2D$ when $G = K$.
Now suppose $G = H : K$ where only one vertex v in H is identified with a vertex in K. Consider vertex w in H with maximum distance to vertex v, and let this distance be d. Thus, the diameter of graph G is $d + \frac{n-1}{2}$ where n is the number of vertices in the odd circuit K. Now Γ cannot be odd since an odd walk from w to itself is impossible. So consider an even walk from w to a vertex adjacent to

w. Since H has no odd circuits, this walk must pass through K. Thus, this walk must be of length $d + n - 1 + d = 2d + n - 1 = 2D$. Therefore, $\Gamma = 2D$, as required.

We end this brief note with two conjectures.

Conjecture 1: The coalescence of H and K above can have an identification of more than 1 vertex and still $\Gamma = 2D$.

Conjecture 2: The converse of the theorem above is also true.

The use of I.T in the mathematics classroom

Daniel Buhagiar

The use of information technology in mathematics is nowadays given a prominent place in the curriculum. Programming helps those children who find it difficult to learn mathematics in the traditional way. To these students the use of computers makes mathematics more appealing and creative. Thus Mathematics is given a new outlook. The Malta Mathematics Resource Centre set up a site, which supports teachers with resources to be abreast of what is happening locally, regarding the teaching of Mathematics. Syllabi, ICT activities and links to other Mathematics websites can easily be accessed at their site whose URL is *http://schoolnet.magnet.mt/maths*.

Logo

The principal credit for the development of LOGO is attributed to Seymour Papert. The concept of turtle graphics was born. This programming language helps children to learn the concepts of mathematics. Through commands that move the picture of a turtle on the screen, children are provided with concrete experiences of a number of concepts, which include distance, angle, shape and symmetry. It encourages new learning and teaching styles where importance is given to the "process" of learning rather than to the "product". Children are encouraged to understand the problem, devise a plan and finally check the solution.
The table below contains a summary of commands used in LOGO programming:

Name	Abbreviation
FORWARD	FD
BACKWARDS	BK
RIGHT	RT
LEFT	LT
PENUP	PU
PENDOWN	PD
HIDETURTLE	HT
SHOWTURTLE	ST
CLEARGRAPHICS	CG
SETCOLOR	SETC
SETBACKGROUND	SETBG
HOME	
REPEAT	

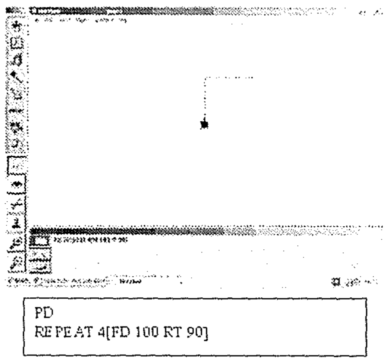


Figure 1: Various LOGO commands and a screenshot

Excel

The main aim of Excel in the mathematics classroom is to acquaint the student with the spreadsheet and its ability to store, manipulate, calculate and analyse data. For students from eleven to twelve years of age, work is done on pre-written computer stored worksheets. However at a later stage, students are encouraged to create their own work. With an electronic spreadsheet, the computer does all the calculations. It includes features such as automatic calculations, sorting and finding data, graph plotting, and built-in statistical formulae.

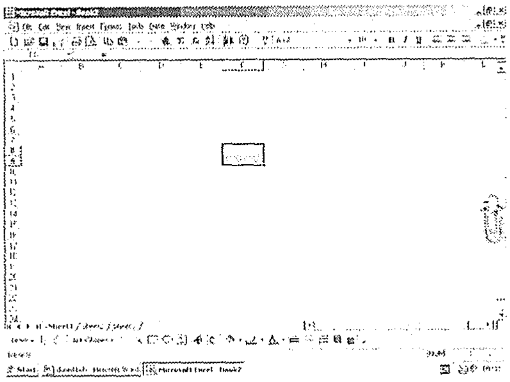


Figure 2: Microsoft Excel

Derive

Derive is a tool which eliminates the drudgery of performing long mathematical calculations. Many calculations can be worked out more efficiently and effectively than by using the traditional methods. Techniques of problem solving are emphasized. **Derive** is dedicated to algebra manipulation, equations, trigonometry, vectors, matrices and calculus although the latter three are no longer part of the present syllabus. **Derive** encourages students to be curious and provides new ways of teaching. It induces in our children a sense of motivation where they are seen as active participants rather than potential recipients of knowledge.

Cabri

Cabri provides a medium for pupils to construct elementary theorems for themselves. It is an interactive notebook for learning geometry. By moving basic points on a screen and observing changes, the students make a simple conjecture. They try to explain what they have noticed and make their own notes.
If properly used the support of computers in the mathematics classroom has the potential to make a significant contribution to pupils' learning in mathematics.

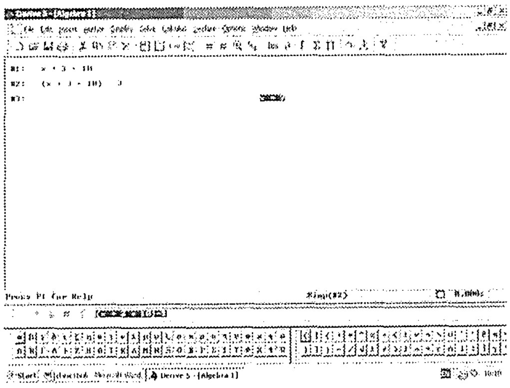


Figure 3: Derive

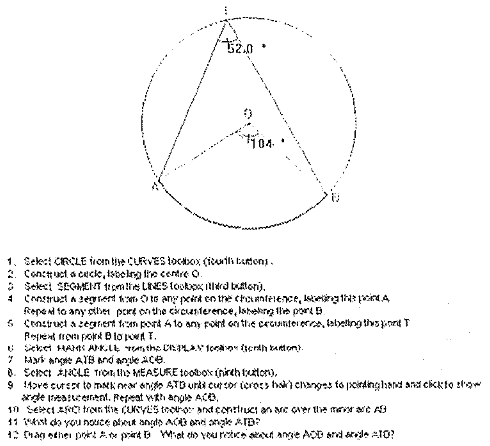


Figure 4: Cabri

Care must be taken, however, not to use computer-time at the expense of adequate exposure to mathematical theory.

A New Form Of Primes

Vincent Mercieca

Abstract: Let p_1, p_2, ...,p_k, ... denotes the ordered sequence of primes. We will consider primes of the form $p = 1 \bmod p_i$. Moreover we give a new proof to the converse of W. Iscan's Theorem, first proved by Lagrange.

Definition 1 *p is said to be prime in $\mathbb{Z}$ if $p = ab$, $a, b \in \mathbb{Z} \Rightarrow a = 1$ or $b = 1$*

Lemma 1 *If $a = b \bmod (m_1 m_2 ... m_k)$, then $a = b \bmod m_i$ $\forall i$, $1 \le i \le k$, where a, b, m_1, ..., $m_k \in \mathbb{Z}$.*

Proof:

$$\begin{aligned} a = b \bmod (m_1 m_2 ... m_k) &\Rightarrow m_1 m_2 ... m_k / (b - a) \\ &\Rightarrow m_i / (b - a) \forall i, 1 \le i \le k \\ &\Rightarrow a = b \bmod m_i \forall i, 1 \le i \le k \end{aligned} \tag{1}$$

Lemma 2 *If $gcd(a, b) = 1$, then the arithmetic progression $\langle a + bn \rangle$ contains infinitely many primes*

N.B. This theorem was first conjectured by Euler in 1785 (with $a = 1$). In 1808, Legendre claimed that he had a proof for this theorem, but later it was found to be false. Finally, in 1837, Dirichlet proved it and it was practically the birth of *Analytic Number Theory*.

Proposition: Let p_1, p_2, ..., p_k, ... be the ordered sequence of primes. Then given any prime p_k, there exists another prime p, $p > p_k$, s.t. $p = 1 \bmod p_i$ $\forall i, 1 \le i \le k$.

Proof: Consider the sequence $\langle (p_1 p_2 ... p_k) n + 1 \rangle$. This is an arithmetic progression and $gcd(p_1 p_2 ... p_k, 1) = 1$

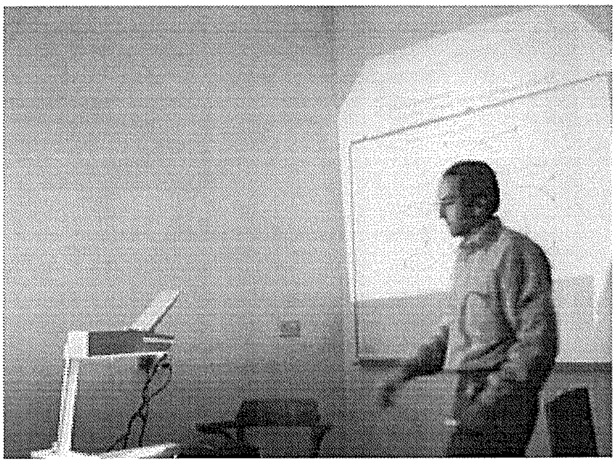
Hence by Dirichlet's Theorem $\langle (p_1 p_2 ... p_k) n + 1 \rangle$ contains infinitely many primes which are of the form $p = 1 \bmod (p_1 p_2 ... p_k) \Longrightarrow p = 1 \bmod p_i$ $\forall i, 1 \le i \le k$.

Conclusion: Twin primes are integral pairs $(p, p + 2)$ in which both members are prime. Examples are 5, 7, 11, 13, and 107, 109. It is conjectured that there are infinitely many twin primes.

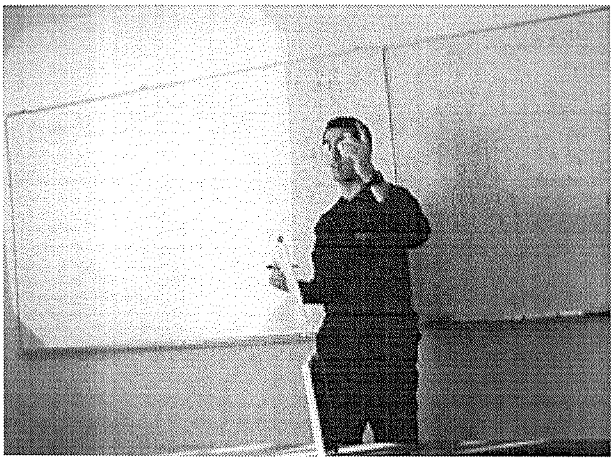
<u>Is there a relation between primes of the form $1 \bmod p_i$; $1 \le i \le k$ and twin primes?</u>

The Speakers

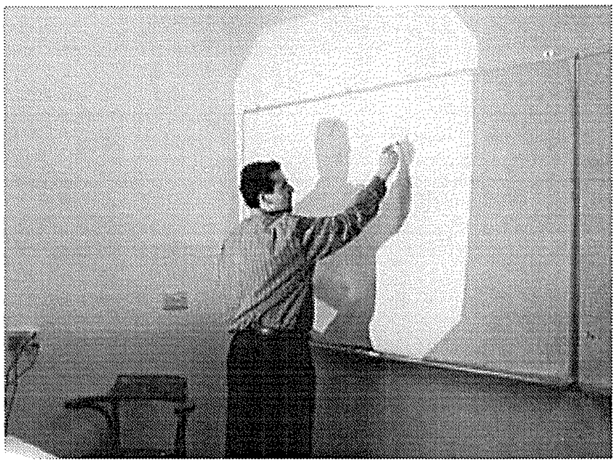
Daniel Buhagiar



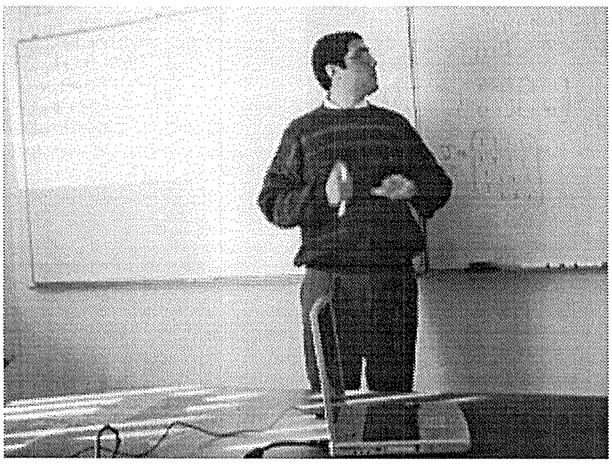
John Baptiste Gauci



Peter Borg



Vincent Mercieca



Alexander Farrugia

The Audience