Collection IV

2001

The Collection IV

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Department of Mathematics

Faculty of Science

University of Malta

Proceedings of Workshop held on the 1st November 2001

When we cannot use the compass of Mathematics or the touch of experience. . . it is certain that we cannot take a single step forward.

Voltaire

Foreword

The Collection IV, held on November 1st 2001 marked two years of pleasant activity in the realms of Mathematics within the Faculty of Science at the University of Malta. The enthusiastic crop of students, who justified the *raison d'être* of such seminars, workshops and journal, were then in their third or fourth year of their studies. Former students, then graduates in Mathematics, who were furthering their studies in the subject or its applications, lamented having missed such activities during their undergraduate years and gave their support as well.

We therefore thought it would be appropriate to show our appreciation to all those who contributed to the success of the Collection Series of seminars I to IV. Book tokens and prizes kindly offered by the Association of European Women in Math (EWM) and the Agenda Bookshop on Campus were distributed by Prof. Stanley Fiorini, head of the Mathematics department for interesting talks, articles and other contributions.

Irene Sciriha

Contents

The Collection IV

1^{st} November 2001
3.00 to 4.15pm

Venue: University of Malta
Maths and Physics Building,
Department of Mathematics,
Room 316.

A seminar/workshop is being held on Thursday 1^{st} November 2001 at 3.00 p.m. Students and staff from the Department of Mathematics, Faculty of Science will present ideas from various fields of mathematics.

During the meeting, prizes kindly presented by EWM and Agenda Bookshop, will be awarded for contributions to previous seminars.

Keynote Speakers: Ms. Maria Attard (B.Ed. Yr II Student)
Dr. David Buhagiar
Mr. Etienne Caruana B.Sc. (Hons)
Mr. David Suda (B.Sc. Yr IV Student)
Ms. Sarah Buttigieg and Ms. Monique Inguanez
Mr. Peter Borg (B.Sc. Yr IV Student)

We shall end with a brief session for spontaneous problem posing and/or solving. You are cordially invited to attend.

Abstracts of possible proofs or conjectures which you wish to share with us in this meeting, or in a future one, may be sent to Dr. I. Sciriha or Ms. A. Attard (secretary), Department of Mathematics, (marked *The Collection*), at any time of the year.

Dr. I. Sciriha
(Organiser)

Quaternion Roots

Maria Attard

We define a quaternion to be an expression $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$, and define addition and multiplication in the natural way with:

$$i^2 = j^2 = k^2 = -1 \qquad ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j \qquad (1)$$

The set of real quaternions forms a **skew field** or a **division ring** which fails to

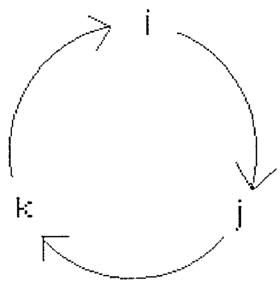


Figure 1: $ij = k$, $jk = i$, $ki = j$

be a field because commutativity under multiplication does not hold.

Theorem 1 (Fundamental Theorem of Algebra) *A polynomial equation* $c_n z^n + c_{n-1} z^{n-1} + \ldots + c_0 = 0, c_i \in \mathbb{C}\ \forall i$, *has exactly* n *roots in the field* $\mathbb{C}$.

We show that this is not the case in the skew field of real quaternions.

Lemma 2 *The quaternion* $x = bi + cj + dk$, $b, c, d \in \mathbb{R}$, *is a root of* $x^2 = -1$, *provided* $b^2 + c^2 + d^2 = 1$.

Proof:

$x = bi + cj + dk$ is a root of $x^2 = -1$ iff x satisfies the latter equation, iff $(bi + cj + dk)^2 = -1$.

Indeed, let $x = bi + cj + dk$

$$\begin{aligned} x^2 &= (bi + cj + dk)(bi + cj + dk) \\ &= b^2 i^2 + bcij + bdik + cbji + c^2 j^2 + cdjk + dbki + dckj + d^2 k^2 \end{aligned}$$

We can write $(bi)(cj) = bcij \ \forall b, c \in \mathbb{R}$ since real numbers commute.

$$\begin{aligned} \text{Hence } x^2 &= -b^2 + bcij + bdik - cbij - c^2 + cdjk - dbik - dcjk - k^2 \\ &= -(b^2 + c^2 + d^2) \text{ (using (1))} \\ &= -1 \text{ (since } b^2 + c^2 + d^2 = 1 \text{ is given)} \end{aligned}$$

Therefore $x = bi + cj + dk$ is a root of $x^2 = -1$.

Hence we can deduce the following theorem:

Theorem 3 *The Fundamental Theorem of Algebra does not hold in the set of quaternions.*

Proof:

We devise the above situation to provide a counterexample.

$x^2 = -1$ is of degree 2.

But $x^2 = -1$ has roots:

I. $x = +\sqrt{-1} = i$
II. $x = -\sqrt{-1} = -i$
III. the given quaternion $x = bi + cj + dk$ where $b^2 + c^2 + d^2 = 1$
IV. We can also show that the conjugate quaternion $-bi - cj - dk$ is also a root.

Note that we can find as many roots of the forms given in (III) and (IV) as there are sets of real numbers satisfying the condition $b^2 + c^2 + d^2 = 1$ (e.g.: $b = c = \frac{1}{2}, d = \frac{1}{\sqrt{2}}$).

Clearly, the given equation has more than two roots.

Therefore the theorem that a polynomial of degree n has n roots does not hold in the ring of real quaternions.

The Use of Choice

Dr. David Buhagiar

The aim of this note is to present two examples, one shown use of the Axiom of Choice and the other that of Zorn's Lemma in Mathematics. We begin by stating the mentioned two equivalent axioms.

Axiom of Choice 1 *Let X be a nonempty set. Then for each nonempty subset $S \subseteq X$ it is possible to* choose *some element $s \in S$. That is, there exists a function f that assigns to each nonempty set $S \subseteq X$ some representative element $f(s) \in S$. Such a function f is called a* choice function.

To state Zorn's Lemma we need the following definition.

Partially Ordered Set, Chain 1 *A set X is said to be* partially ordered *if there is a* partial ordering *defined on it, that is, a binary relation $\preceq$ that satisfies the conditions*

1. $x \preceq x$ *for every* $x \in X$,
2. *if* $x \preceq y$ *and* $y \preceq x$, *then* $x = y$,
3. *if* $x \preceq y$ *and* $y \preceq z$, *then* $x \preceq z$.

If $x, y \in X$ are such that neither $x \preceq y$ nor $y \preceq x$ holds, then they are called incomparable. *Otherwise they are called* comparable. *A* chain *or* linearly ordered set *is a partially ordered set such that every two elements of the set are comparable. An* upper bound *of a subset Y of a partially ordered set X is an element $u \in X$ such that $y \preceq u$ for every $y \in Y$. A* maximal element *of X is an element $m \in X$ such that $m \preceq x$ implies $m = x$.*

Zorn's Lemma 1 *If every chain in a partially ordered set has an upper bound, then the partially ordered set has a maximal element.*

The axiom of choice was formulated by Zermelo in 1904. Sixty years later, in 1963, Paul Cohen showed that the Axiom of Choice cannot be proved from the axioms of Zermelo-Fraenkel set theory (the standard set theoretical axioms). Thus, if we want to use the Axiom of Choice we need to include it in our set theory. It can be proved that the Axiom of Choice is equivalent to Zorn's Lemma.

Let us now see one example of the use of the Axiom of Choice and one example of the use of Zorn's Lemma.

Example 1 *Remember that a sequence* (x_n) *in a metric space* (X,d) *converges to an element* $x \in X$ *if for every positive real number* ϵ *there exists some* $n_0 \in \mathbb{N}$ *such that* $d(x_n, x) < \epsilon$ *for every natural number* $n \succeq n_0$. *Let* Y *be a subset of* X. *One usually defines closure points of* Y *in either (or both) of the following two ways:*

I $x \in X$ *is a* closure point *of* Y *if there exists a sequence* (x_n) *in* Y *which converges to* x.

II $x \in X$ *is a* closure point *of* Y *if for every positive real number* ϵ *there exists some* $y \in Y$ *with* $d(x,y) < \epsilon$.

We then go on to prove that (I) and (II) are equivalent.

Proof:

(I) $\Longrightarrow$ *(II): Given any* $\epsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $d(x_n, x) < \epsilon$ *for every* $n \succeq n_0$. *In particular,* $d(x_{n_0}, x) < \epsilon$ *and* $x_{n_0} \in Y$.

(II) $\Longrightarrow$ *(I): The usual prove proceeds as follows: Let* $Y_n = \{y \in Y : d(y,x) < \frac{1}{n}\}$. *By (II),* $Y_n \neq \emptyset$ *for all* $n \in \mathbb{N}$. *Let* (x_n) *be a sequence such that a point* $x_n \in Y_n$ *for all* $n \in \mathbb{N}$. *Then each* $x_n \in Y$ *and* (x_n) *converges to* x.

What is usually overlooked is to justify the reasons we have to assume that such a sequence (x_n) *exists. In fact, if one takes* (X,d) *to be the real line* $\mathbb{R}$ *with standard metric, it has been proved that the equivalence of (I) and (II) for all* $Y \subset \mathbb{R}$ *cannot be proved from the axioms of Zermelo-Fraenkel set theory alone. Of course, if we include the Axiom of Choice in our set theory, the fact that* $Y_n \neq \emptyset$ *for all* $n \in \mathbb{N}$ *immediately implies that such a sequence exists.*

Example 2 *One of the fundamental facts on vector spaces is that every non-zero vector space* V *has a linear basis. This result is a straightforward application of Zorn's Lemma.*

Let X *be the set of all linearly independent subsets of* V. *Since* $V \neq \{0\}$, *it has an element* $v \neq 0$ *and* $\{v\} \in X$, *so that* $X \neq \emptyset$. *We define a partial ordering on* X *by set inclusion. If* $Y \subset X$ *is a chain in* X, *the union of all the elements of* Y *gives an upper bound for* Y. *By Zorn's Lemma,* X *has a maximal element* B. *We are left to show that* B *is a linear basis for* V. *Let* $U = \text{span B}$. *Then* U *is a subspace of* V. *If* U *is a proper subspace of* V *and* $z \in V - U$, *then* $B \cup \{z\}$ *would give a linearly independent set containing* B *as a proper subset, contradicting the maximality of* B.

The above result cannot be proved in Zermelo-Fraenkel set theory alone, without using Zorn's Lemma.

Another axiom equivalent to the Axiom of Choice is the *Well-Ordering Principle*. Remember that a linear ordering $<$ on a set X is a *well-ordering* if every nonempty subset of X has a $<$-minimal element. The structure $(X, <)$ is then called a *well-ordered set*.

The Well-Ordering Principle 1 *Every set can be well-ordered.*

We end this note by asking ourselves: Is the Axiom of Choice "true"?. According to J.L.Bona in a private communication with E.Schechter in 1977,

`The Axiom of Choice is obviously true; the Well Ordering Principle is obviously false; and who can tell about Zorn's Lemma?`

Petri Nets

Etienne Caruana

Petri Nets (PN) provide a graphical approach to the modelling of communicating systems. PN have been introduced by *Carl A. Petri* in his dissertation presented in 1962. The purpose was to analyse communication systems. Further study on this subject has led to a vast applicability of PN in many sectors like in I.T. and manufacturing.

The reason for such a vast applicability of PN is due to the fact that PN hold a considerable **modelling power** as well as efficient methods for proper **performance analysis** of the system under study. In fact, PN may be shown to be effective in the modelling of concurrency, conflict and synchronisation.

Description of a PN: A PN is a bipartite, directed graph. The set N of nodes is divided into the set of **transitions** T and **places** P, i.e. $N = P \cup T$, $P \cap T = \emptyset$. Elements of $T \times P$ and $P \times T$ are said to be the **arcs** of the net. Places contain **tokens**. Tokens are usually represented by dots inside the places or by a number indicating the number of tokens that reside inside each place For

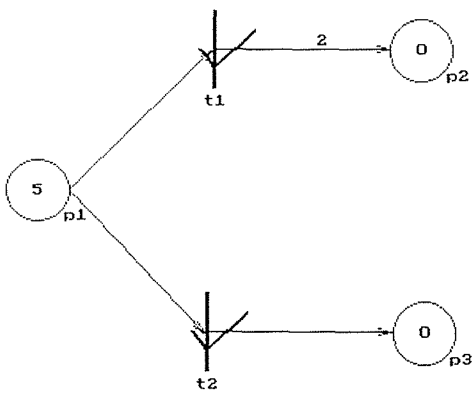


Figure 2: A simple Petri Net

example, in Figure 2:

- $P = \{p_1, p_2, p_3\}$
- $T = \{t_1, t_2\}$
- Set of input arcs $P \times T = \{(p_1, t_1), (p_1, t_2)\}$

- Set of output arcs $T \times P = \{(t_1, p_2), (t_2, p_3)\}$

Initially, this net has 5 tokens inside p_1 and 0 tokens inside p_2 and p_3.

Place p_1 is said to be an **input place** of transitions t_1 and t_2, while places p_2 and p_3 are said to be **output places** of t_1 and t_2 respectively.

Given a transition (place) t (p), then the set of output places (transitions) is denoted by $p\cdot$ ($t\cdot$), while the set of input places (transitions) is denoted by $\cdot p$ ($\cdot t$). Therefore, from Figure 1:

- $p_1 \cdot = \{t_1, t_2\}$; $t_1 \cdot = \{p_2\}$; $t_2 \cdot = \{p_3\}$
- $\cdot t_1 = \{p_1\}$; $\cdot t_2 = \{p_1\}$; $\cdot p_2 = \{t_1\}$; $\cdot p_3 = \{t_2\}$

Tokens flow from one place to another by the firing of **transitions**. The firing of each transition depends both on the number of tokens inside each place and the **multiplicity** of each arc. The multiplicity of an arc is marked by a number attached to the arc. If no number is attached to an arc, the multiplicity of the arc is taken to be one.

A transition may fire only if the number of tokens inside each of its input places is greater than or equal to the multiplicity of the corresponding arc. If the transition is in a condition to be fired, then it is said to be **enabled**. For example, in Figure 2, both t_1 and t_2 are enabled since the number of tokens inside p_1 is 5 and the multiplicity of input arcs is 1 for both.

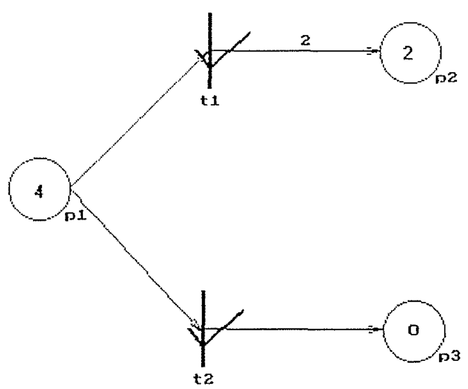


Figure 3: The Petri Net after firing t_1

When a transition fires, it removes a number of tokens from each input place equivalent to the multiplicity of its input arcs and adds a number of tokens to its output places, equivalent to the multiplicity of its output arc. Figure 3 represents the marking of the PN after firing t_1.

If we fire t_1 4 times more, the marking of p_1 will eventually decrease to 0. Then, t_1 and t_2 will not be able to fire. In that case, we say that the net is in a state of **deadlock** - a state in which no flow of tokens inside the net is possible.

Applications of PN: The following is a list of attributes of PN which make them useful for modelling:

1. Petri Nets capture the precedence relations and structural interactions of stochastic, concurrent, and asynchronous events. In addition, their graphical nature helps to visualise such complex systems.

2. Conflicts and buffer sizes can be modelled easily and efficiently.

3. Deadlocks in the system can be detected.

4. Petri net models represent a hierarchical modelling tool with a well-developed mathematical and practical foundation.

5. Various extensions of PN, such as timed PN, stochastic (timed) PN, coloured PN, and predicate/transition nets, allow for both qualitative and quantitative analyses of resource utilisation, effect of failures, and throughput rate, to name a few.

6. Petri net models give a structured framework for carrying out a systematic analysis of complex systems. Various software packages have been developed for this purpose.

In general, for modelling purposes, places represent **resources** such as machines or parts of a buffer. The meaning of a token inside a place is generally deduced according to the definition of the place. For example, a token in a place representing a state of a machine would represent the availability of the machine to do the required job while a token inside a place representing a store of a manufacturing system would represent the number of items produced by the system.

In general, a transition firing represents an activity, which begins and ends by two consecutive events (represented by places). For example, a machine representing a transition labelled 'machine repair' should have an input place representing 'machine damage' and have an output place representing 'machine working'.

The following example illustrates a Petri-Net model of a simple communication system.

Figure 4 depicts a communication system made of three robots R1, R2, R3 and two conveyors C1 and C2. Each conveyor operates on the two robots next to

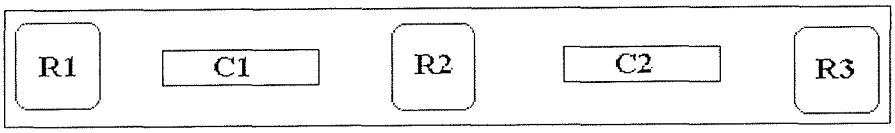


Figure 4: Three robots working with two conveyors

it. Both conveyors first use the robot on the left-hand side and then the robot on the right-hand side. No conveyor may be used by two robots at the same time, implying that R2 needs to be shared between C1 and C2. It is assumed that both conveyors remove their robots synchronously after their operations.

Tables 1 and 2 respectively give the places and the transitions which are needed to represent this system together with their description in this context.

Place	Meaning
p_1	C1 needs R1
p_2	C1 needs R2
p_3	C1 has both R1 and R2
p_4	C2 needs R2
p_5	C2 needs R3
p_6	C2 has both R2 and R3
p_7	R1 free
p_8	R2 free
p_9	R3 free

Table 1: Introducing Places

Transition	Meaning
t_1	C1 takes R1
t_2	C1 takes R2
t_3	C1 removes both R1 and R2
t_4	C2 takes R2
t_5	C2 takes R3
t_6	C2 removes both R2 and R3

Table 2: Introducing Transitions

The corresponding Petri Net would be the one shown in Figure 5. The initial marking of this net is $m_0(p_1) = m_0(p_4) = m_0(p_7) = m_0(p_8) = m_0(p_9) = 1$, with all the other places having a null marking. This marking is representing an initial situation where each robot is free and each conveyor is waiting for its left robot.

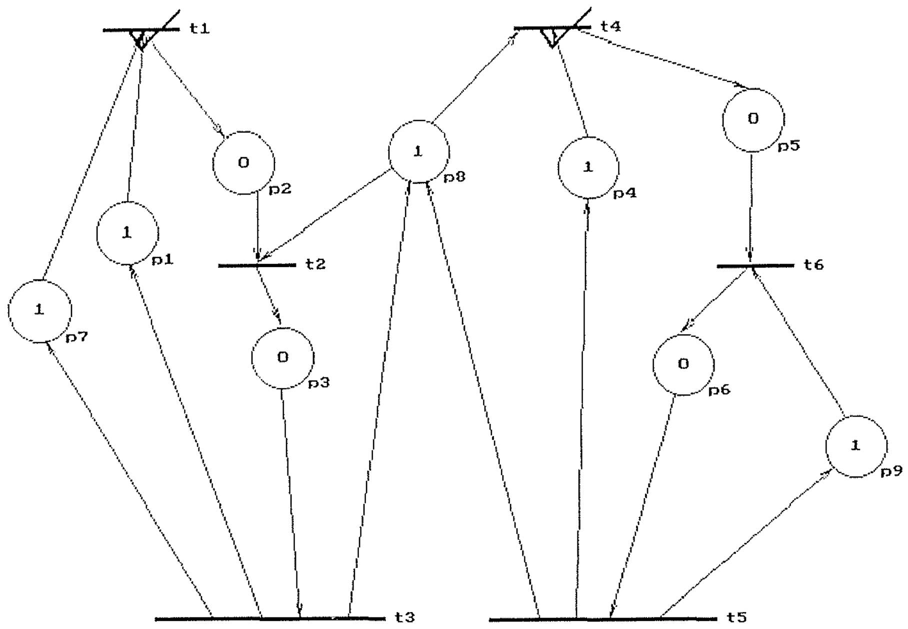


Figure 5: Petri Net of conveyor/robot problem

Note that initially the enabled transitions t_2 and t_4 are sharing the same input place, namely p_8.

Upon firing t_1, t_4 is disabled and vice versa. The sharing of place p_8 is representing a conflicting situation, where a decision has to be made of which transition to fire, analogous to deciding to either work with conveyor C1 or else with conveyor C2.

A Proof of the Cayley-Hamilton Theorem

David Suda

Theorem 1 (Cayley-Hamilton) *If $A : V \to V$ is a linear transformation, then A satisfies its characteristic polynomial $\phi(A, x)$, i.e. $\phi(A, A) = 0$.*

The standard proof to this theorem uses properties of the adjoint matrix combined with the use of the Leibnitz expansion whereby finally we get a number of simultaneous matrix equations which give us the required result. However, an alternative to this proof is the following structured proof that makes use of a lemma, a proposition and a theorem, finally giving us a neat proof of the actual theorem. In the process, several properties of linear transformations are highlighted and it is these properties which we will use to finally prove this theorem.

This theorem will be divided into four parts:

Part 1:

Lemma 2 *Suppose $f : V \to V$ is a linear transformation of an n-dimensional vector space V over a field F. If f has nullity of at least 1, then there exists a basis $v_1, v_2, \ldots v_n$ s.t. $f(v_j) \in span(v_1, v_2, \ldots v_{n-1})$, $\forall j \in \{1, 2, \ldots, n\}$. In other words, the matrix of f w.r.t. $v_1, v_2, \ldots, v_n$ is:*

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n} \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

Proof: Let r be the rank of f. Since f has a nullity of at least 1, $r < n$.

Now suppose B is a basis $v_1, v_2, \ldots, v_r$ for $Im(f)$. Extend this basis to a basis $v_1, v_2, \ldots, v_n$ for the whole of V.

Then $f(v_j) \in span(v_1, v_2, \ldots, v_r) \subseteq span(v_1, v_2, \ldots, v_{n-1})$.

Part 2:

Proposition 3 *Let $V = \mathbb{C}^n$ be the n-dimensional vector space over $\mathbb{C}$ and suppose f is a linear transformation from V to V. Then there is a basis $v_1, v_2, \ldots, v_n$ of V s.t. w.r.t this basis, the matrix of f is upper triangular.*

Proof: Assume result true for all spaces V of dimension $n-1$ over $\mathbb{C}$.

Given V of dimension n and $f : V \to V$, let λ be an eigenvalue of f. Note that $(f - \lambda I)$ has nullity of at least 1 (in the worst case, all characteristic roots are equal and the eigenspace of that characteristic root is of dimension 1). So by the previous lemma there is a basis $u_1, u_2, \ldots u_n$ s.t. $(f - \lambda I)(u_i) \in span(u_1, u_2, \ldots, u_{n-1})$, $\forall i$.

Then, the matrix of $f - \lambda I$ w.r.t this basis is of the form:

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n} \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

Hence the matrix of f w.r.t. this basis is of the form:

$$\begin{pmatrix} a_{11} + \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} + \lambda & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n} \\ 0 & 0 & \ldots & \lambda \end{pmatrix}$$

Let W be the subspace spanned by $u_1, u_2, \ldots, u_{n-1}$ and note that $f(w) \in W \; \forall w \in W$ so that the restriction of f to W is a linear transformation of W. By the induction hypothesis there is a basis $v_1, v_2, \ldots, v_{n-1}$ of W s.t. the matrix of restriction of f to W w.r.t to this basis is upper triangular, say:

$$\begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1,n-2} & b_{1,n-1} \\ 0 & b_{22} & \ldots & b_{2,n-2} & b_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & b_{n-2,n-2} & b_{n-2,n-1} \\ 0 & 0 & \ldots & 0 & b_{n-1,n-1} \end{pmatrix}$$

Let $u_n = v_n$, then $\{v_1, v_2, \ldots, v_n\}$ is a basis of V since $v_n \notin span(v_1, v_2, \ldots, v_{n-1}) = span(u_1, u_2, \ldots, u_{n-1})$. Hence $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set of size

$n = dim\ V$. Also $f(v_n) = \lambda v_n + w$ for some $w \in W$. Hence the matrix of f w.r.t. basis $v_1, v_2, \ldots, v_n$ is upper triangular too since the new matrix will now be:

$$\begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1,n-1} & b_{1n} \\ 0 & b_{22} & \ldots & b_{2,n-1} & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & \ldots & 0 & \lambda \end{pmatrix}$$

Part 3:

Theorem 4 *If A is any upper triangular $n \times n$ matrix with entries from $\mathbb{R}$ or $\mathbb{C}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the diagonal entries of A including repetitions, then the matrix $(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_n I) = 0$, the zero matrix.*

Proof: If $n = 1$, this is obvious.

By induction, we prove the theorem for all n.

$(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_n I)e_n = 0$ since $(A - \lambda_n I)e_n = w$, where $w \in W = span\{e_1, \ldots, e_{n-1}\}$ since A is upper triangular with the last row equal to $(0, 0, \ldots, \lambda_n)$.

Also, the linear transformation given by A on the subspace $W = span(e_1, e_2, \ldots, e_{n-1})$ has upper triangular matrix w.r.t. this basis, with diagonal entries $\lambda_1, \ldots, \lambda_{n-1}$. So by the induction hypothesis:

$(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_{n-1} I)e_i = 0, \forall i < n - 1$. Thus:

$(A - \lambda_n I)(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_{n-1} I)e_i = 0, \forall i < n - 1$.

Hence $(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_n I)e_i = 0, \forall i \leq n - 1$. Also $(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_n I)e_n = (A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_{n-1} I)w = 0$ since $w \in W$. Thus $(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_n I)$ is a zero transformation since its action on e_i gives $0\ \forall i \leq n$.

Part 4:

Now we are in a position to present the alternative proof of the Cayley-Hamilton Theorem, stated initially in Theorem 1.

Proof: Take $B = P^{-1}AP$ s.t. B is upper triangular, by Proposition 3.

Hence $(B-\lambda_1 I)(B-\lambda_2 I)\ldots(B-\lambda_n I)=0$, by Theorem 4.

Thus
$$\begin{aligned}(A-\lambda_1 I)(A-\lambda_2 I)\ldots(A-\lambda_n I) &= PP^{-1}(A-\lambda_1 I)PP^{-1}\ldots PP^{-1}(A-\lambda_n I)PP^{-1}\\ &= P(P^{-1}AP-\lambda_1 I)(P^{-1}AP-\lambda_2 I)\ldots(P^{-1}AP-\lambda_n I)P^{-1}\\ &= P(B-\lambda_1 I)(B-\lambda_2 I)\ldots(B-\lambda_n I)P^{-1}\\ &= 0\end{aligned}$$

Since B is upper triangular, $\phi(B,\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n)$.
Since $B=P^{-1}AP$, $\phi(A,\lambda)=\phi(B,\lambda)$. (a property of similar matrices.)
Hence $\phi(A,A)=0$. This completes the proof.

Remark: As one can see, in this case the theorem's proof all boils down to the simple use of the properties of similar matrices.

Homomorphisms and the number of Divisors

Sarah Buttigieg and Monique Inguanez

Lemma 1 *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ be defined by $f(x) = x^n$. Then f is a homomorphism.*

Proof: $f(x)f(y) = f(xy)$ $\forall x,y \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$.

Theorem 2 *Let*

$$F(p) = \sum_{m|p} m^n$$

for some $n \in \mathbb{N}$ and where p is prime and the summation runs over the divisors of p. Then F is a homomorphism under multiplication.

Proof: The divisors of p are 1 and p.

Hence

$$\begin{aligned} F(p_1)F(p_2) &= ({p_1}^n + 1^n)({p_2}^n + 1^n) \\ &= {p_1}^n {p_2}^n + 1 + {p_1}^n + {p_2}^n \end{aligned}$$

The divisors of $p_1 p_2$ are $p_1 p_2$, p_1, p_2 and 1.

Hence $F(p_1 p_2) = {p_1}^n {p_2}^n + 1 + {p_1}^n + {p_2}^n$.

Consequently, $F(p_1)F(p_2) = F(p_1 p_2)$.

This can be extended to any integer.

Corollary 3 *If f is a homomorphism under multiplication, then so is $F(n)$, defined by*

$$F(n) = \sum_{m|n} f(m)$$

where the sum is over all divisors of any integer n. Therefore, $F(x)F(y) = F(xy)$.

Application 1: Using the above corollary, we see that the number of divisors $d(n)$ of n, where

$$d(n) = \sum_{m|n} 1$$

is a homomorphism under multiplication, since $f(x) = 1$ is.

Let's consider an example: $63 = 7 \times 9$

The divisors of 63 are 63, 21, 9, 7, 3 and 1. Therefore $d(63) = 6$.

The divisors of 7 are 7 and 1. So $d(7) = 2$.

The divisors of 9 are 9, 3 and 1. So $d(9) = 3$.

By the above argument, $d(63) = d(9)d(7)$, which indeed it is, since $6 = 2 \times 3$.

Application 2: If

$$F(x) = \sum_{m|n} f(m)$$

where $f(m) = m^3$, then $F(63) = F(7)F(9)$.

The divisors of 63 are 63, 21, 9, 7, 3 and 1.
$F(63) = 63^3 + 21^3 + 9^3 + 7^3 + 3^3 + 1^3 = 260408$.

The divisors of 7 and 7 and 1, and those of 9 and 9, 3 and 1.
$F(7)F(9) = (7^3 + 1^3)(9^3 + 3^3 + 1^3) = 260408$.

This confirms our result that F is a homomorphism.

We now consider the sum of cubes of numbers.
It is "common knowledge" that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$$

Thus

$$\sum_{r=1}^{n} r^3 = \left(\frac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$$

Hence the set of numbers $\{1, 2, \ldots, n\}$ has the property that the sum of its cubes is the square of its sum. Are there any other collections of numbers with this property?

Let's consider the following argument.
Pick any number, for example 63. List the divisors of 63, and for each divisor of 63, count the number of divisors it has:

63 has **6** divisors (63, 21, 9, 7, 3, 1)
21 has **4** divisors (21, 7, 3, 1)
9 has **3** divisors (9, 3, 1)
7 has **2** divisors (7, 1)
3 has **2** divisors (3, 1)
1 has **1** divisor (1).

The resulting collection of numbers has the same property. Namely:

$$6^3 + 4^3 + 3^3 + 2^3 + 2^3 + 1^3 = 324 = (6 + 4 + 3 + 2 + 2 + 1)^2$$

$[d]^3$ is a homomorphism under multiplication, and Corollary 3 shows that

$$\sum_{m|n} d^3(m)$$

is also a homomorphism under multiplication.

Also, from Corollary 3, squaring gives that

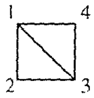
$$\left(\sum_{m|n} d(m)\right)^2$$

is a homomorphism under multiplication. Using a similar argument as before, it can finally be shown that

$$\sum_{m|n} d^3(m) = \left(\sum_{m|n} d(m)\right)^2$$

The Total Number of Non-Isomorphic Simple Graphs with n Vertices and k Edges

Definition: A graph is defined as $G := (V, E)$, $E \subseteq V \times V$, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G.

E.g. G 

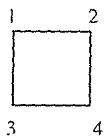
$V(G) = \{1, 2, 3, 4\}$ $E(G) = \{12, 13, 14, 23, 24\}$

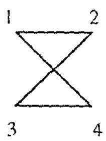
Definition: G is s.t.b. simple if it has no loops or multiple edges.

E.g.
loop (edge from 1 to 1) 

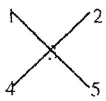
multiple edges from 2 to 3

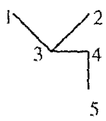
Definition: G_1 is **isomorphic** to G_2, denoted $G_1 \cong G_2$, if $\exists\ f : V(G_1) \to V(G_2)$ bijective s.t. if $v, w \in V(G_1)$ then $vw \in E(G_1) \Leftrightarrow f(v)f(w) \in E(G_2)$.

E.g. G_1 

G_2 

G_1 and G_2 are isomorphic, where
$f(1) = 1, f(2) = 2, f(3) = 4, f(4) = 3$,
so that if $vw \in E(G_1)$ then $f(v)f(w) \in E(G_2)$
e.g. $13 \in E(G_1)$ and $f(1)f(3) = 14 \in E(G_2)$

E.g. G_1 

G_2 

G_1 not isomorphic to G_2 (see Result 1)

Definition: Let $v \in V(G)$. The valency of v, denoted $\rho(v)$, is the number of edges incident to v. Also, $w \in V(G)$ is s.t.b. adjacent to v if $vw \in E(G)$.

Result 1: Let $S(G) := \{\rho(v) : v \in V(G)\}$, i.e. the set of valencies of vertices of G. Then $G_1 \cong G_2 \Rightarrow S(G_1) = S(G_2)$.

Proof: Let $v \in V(G_1)$. Thus $f(v) \in V(G_2)$.

Suppose $v_1, v_2, \ldots, v_k$ are all the vertices adjacent to v.

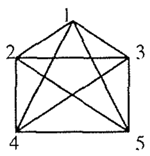
So $f(v_1), f(v_2), \ldots, f(v_k)$ are all the vertices adjacent to $f(v)$

s.t $vv_1 \in E(G_1) \Leftrightarrow f(v)f(v_1) \in E(G_2)$.

So $\forall\ v \in V(G)\ \rho(v) = \rho(f(v))$, and since f is 1 - 1 and onto we get the result.

Definition: A simple graph with n vertices is s.t.b. complete and denoted K_n if $\forall$ v, w $\in$ V(G) vw $\in$ E(G).

E.g. K_5



Result 2: Let G_k^n be the class of simple graphs with vertices labelled 1, 2, ..., n and k edges.

Then $\left|G_k^n\right| = \binom{\frac{n(n-1)}{2}}{k} = \binom{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2} - k} = \left|G^n_{n(n-1)/2-k}\right|$.

Proof: For a simple graph with n vertices we have at most $|E(K_n)| = n(n-1)/2$ edges (for all of the n vertices there are (n - 1) incident edges but every edge connects 2 vertices, hence $n(n - 1) = 2|E(K_n)|$).

So the no. of different graphs in the class G_k^n is equal to the number of ways of choosing k edges out of the n(n - 1)/2 total no. of edges.

Result 3: The isomorphism relation $\cong$ on graphs is an equivalence relation.

Proof: reflexive: $G_1 \cong G_1$ (trivial)

symmetric: Let $V(G_1) = \{v_1, v_2, ..., v_n\}$. If $G_1 \cong G_2$ then $\exists$ f : $V(G_1) \to V(G_2)$ bijective s.t. $v_iv_j \in E(G_1) \Leftrightarrow f(v_i)f(v_j) \in E(G_2)$.

f bijective $\Rightarrow V(G_2) = \{f(v_1), ..., f(v_n)\}$. Let $x_i = f(v_i)$.

Also, f bijective $\Rightarrow \exists f^{-1} : V(G_2) \to V(G_1)$ bijective s.t.

$x_ix_j = f(v_i)f(v_j) \in E(G_2) \Leftrightarrow v_iv_j = f^{-1}(f(v_i))f^{-1}(f(v_j)) \in E(G_1)$.

Hence $G_2 \cong G_1$.

transitive: Let $G_1 \cong G_2$ be defined as in the above symmetric case.

Let $G_2 \cong G_3$. Thus $\exists$ g : $V(G_2) \to V(G_3)$ bijective s.t.

$x_i x_j = f(v_i)f(v_j) \in E(G_2) \Leftrightarrow g(x_i)g(x_j) = g(f(v_i))g(f(v_j)) \in E(G_3)$.

f, g bijective $\Rightarrow$ $g_o f$ bijective where

$v_i v_j \in E(G_1) \Leftrightarrow g_o f(v_i) g_o f(v_j) \in E(G_3)$. Hence $G_1 \cong G_3$.

<u>*Definition*</u>: The complement of simple graph G with n vertices, denoted $\overline{G}$, is the graph s.t. $E(G) \cup E(\overline{G}) = E(K_n)$ and $E(G) \cap E(\overline{G}) = \phi$, i.e.

if v, w $\in$ V(G) then vw $\in$ E(G) $\Leftrightarrow$ vw $\notin E(\overline{G})$ and vw $\notin$ E(G) $\Leftrightarrow$ vw $\in E(\overline{G})$.

<u>*Result 4*</u>: $G \cong G' \Leftrightarrow \overline{G} \cong \overline{G'}$.

<u>*Proof*</u>: Let v, w $\in$ V(G) and let f be the isomorphism from G to G'.

$vw \in E(\overline{G}) \Leftrightarrow vw \notin E(G) \Leftrightarrow f(v)f(w) \notin E(G') \Leftrightarrow f(v)f(w) \in E(\overline{G'})$.

<u>*Remark*</u>: *Result 2* and *Result 4* imply that tackling the problem for the case with k edges is equivalent to tackling it for the case with [n(n-1)/2 - k] edges (i.e. number of edges of complementary graph), because from *Result 4* we easily deduce that for complementary graphs the sizes of classes of isomorphic graphs are equal (i.e. $|[G]| = |[\overline{G}]|$) and from *Result 2* we already know that the total number of different graphs are also equal (refer to *Conclusion*).

<u>*Conclusion*</u>:

Since an equivalence relation on elements of a set gives a partition of the set into equivalence classes, from the important *Result 3* we get that the set of different vertex-labelled simple graphs with n vertices and k edges is partitioned into classes of isomorphic graphs of this type. Hence if N_k^n is the number of classes of our set of graphs then there are N_k^n non-isomorphic simple graphs with n vertices and k edges. Let $[G_k^n]_i$ denote the i'th class of isomorphic graphs of this type.

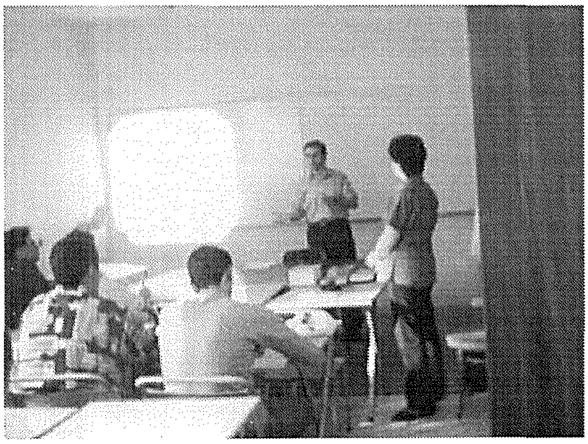
Hence $\left|G_k^n\right| = \sum_{i=1}^{N_k^n} \left|[G_k^n]_i\right| = \binom{\frac{n(n-1)}{2}}{k}$. If the class sizes were all the same then this would be an easy problem to solve, however this is not the case.

This might be an approach to finding a way of obtaining all the non-isomorphic graphs. So if

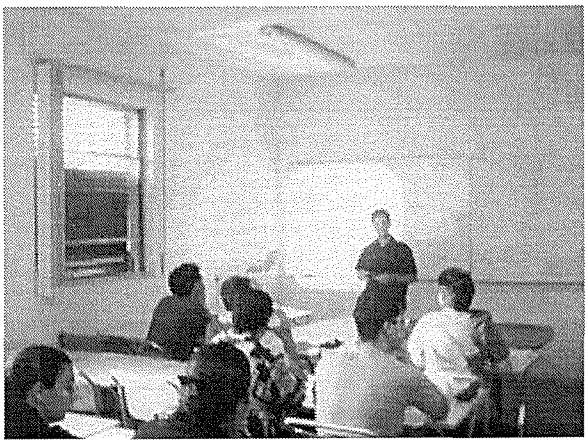
- we have an efficient straight-forward formulation which determines if 2 graphs are isomorphic or not rather than considering all the n! vertex bijections, in particular something stronger than *Result 1* which gives a helpful sufficient condition rather than a necessary one, and
- we have a formula which determines the size of the class of graphs isomorphic to any given graph,

then we have a method of finding N_k^n which works by generating non-isomorphic graphs and finding their size until the sum of sizes equals $\binom{\frac{n(n-1)}{2}}{k}$, but this would still be an algorithm rather than a formula.

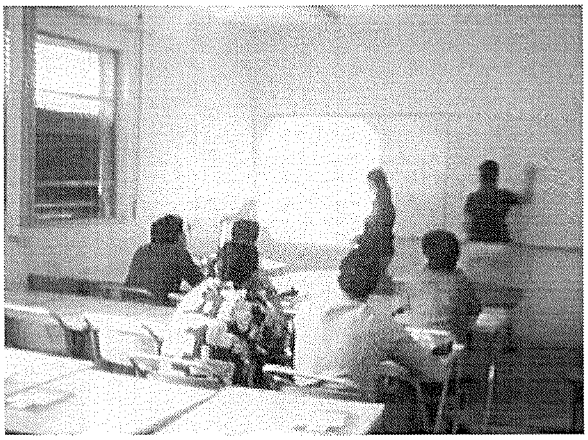
The Speakers



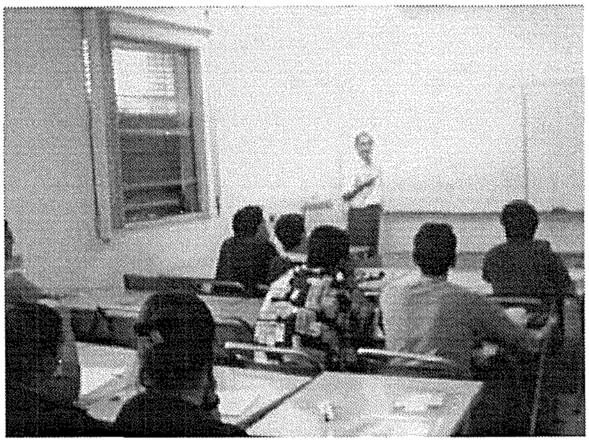
Etienne Caruana



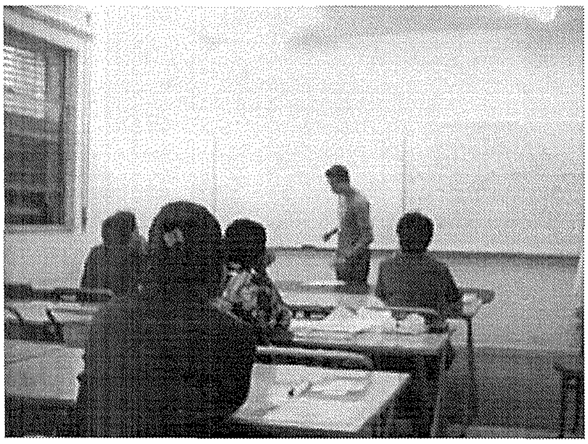
David Suda



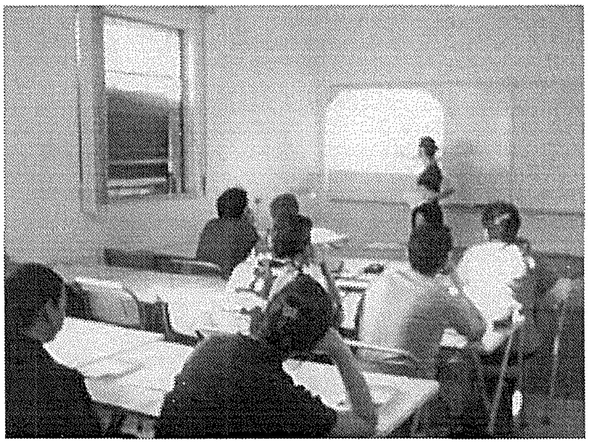
Sarah Buttigieg and Monique Inguanez



Dr. David Buhagiar



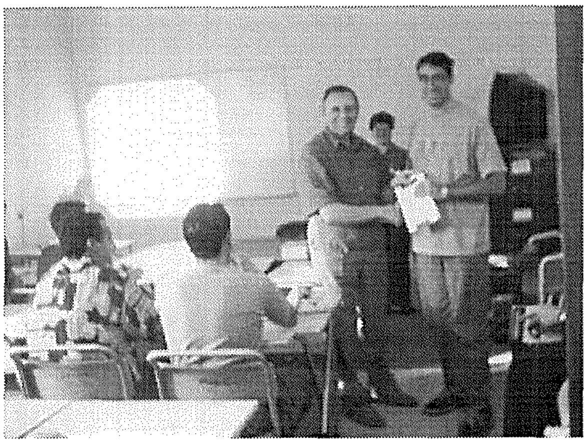
Peter Borg



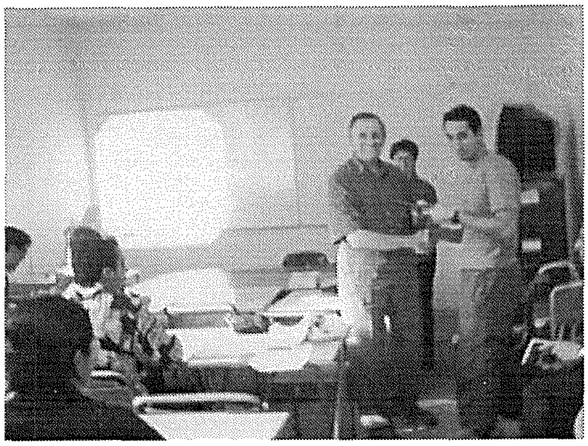
Maria Attard

The Prize Winners

Presented by Prof. S. Fiorini

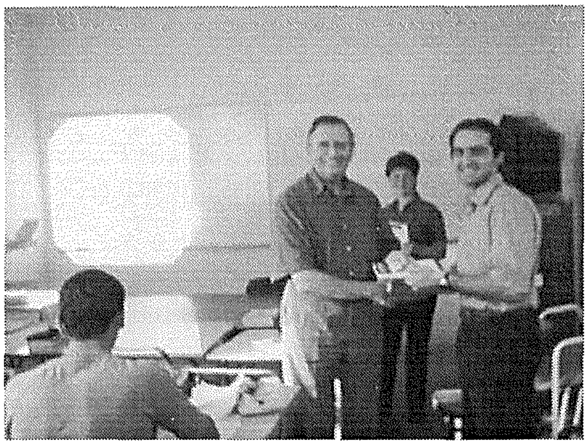


Alexander Farrugia



Peter Borg

David Suda

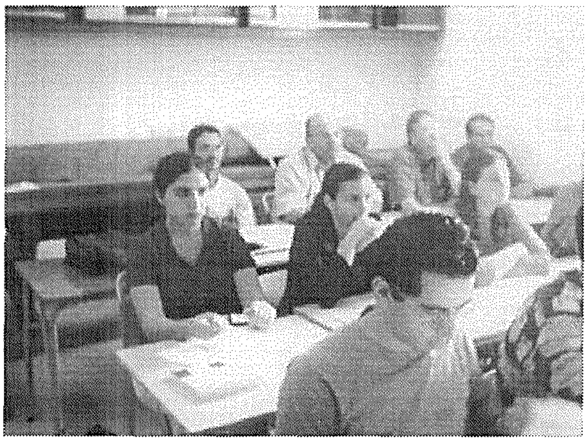


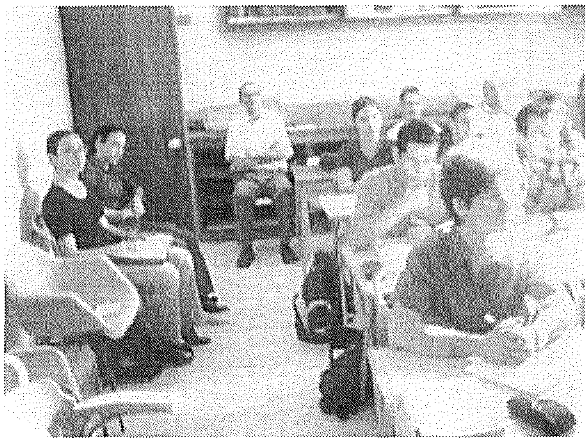
Etienne Caruana

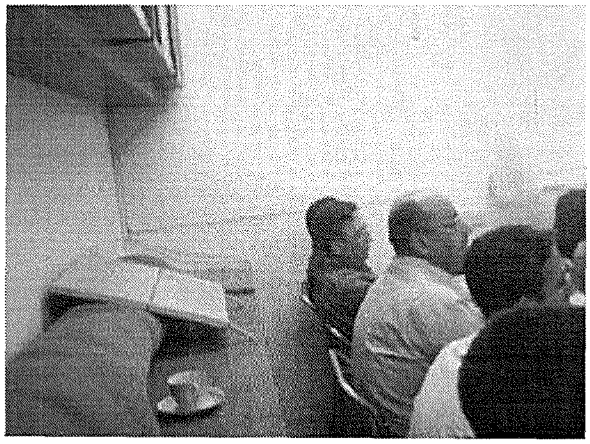
Alex Vella

Louise Casha

The Audience







Editorial Board

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