Interlacing and Carbon Balls

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Abstract

The Interlacing Theorem gives a relation among the eigenvalues of a \( n \times n \) matrix \( A \) and those of a \( (n-1) \times (n-1) \) principal submatrix.

We deduce the Generalized Interlacing theorem which interlaces the eigenvalues of a \( k \times k \) principal submatrix of \( A \) with those of \( A \). We apply this theorem to the hypothetical Carbon ball \( C_{40} \) which has two dodecahedral 6 pentagon caps.

Theorem 2.2 (The Interlacing Theorem) Let \( A \) be an \( n \times n \) symmetric matrix and let \( B \) be a matrix obtained from \( A \) by deleting a row and the corresponding column. Then if \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) are the eigenvalues of \( A \) and \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \) are those of \( B \), then \( \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n \).

Theorem 2.3 (The Generalized Interlacing Theorem) Let \( A \) be an \( n \times n \) matrix with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \). Let \( B \) be a principal \( k \times k \) submatrix of \( A \) where \( 1 \leq k \leq n-1 \), having eigenvalues \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_k \). Then \( \lambda_1 \geq \mu_1 \geq \lambda_{n-k+1} \).

To deduce the Generalized Theorem from the Interlacing Theorem:

\( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) are the eigenvalues for an \( n \times n \) symmetric matrix \( A \). Let submatrices of \( A \) be formed such that \( B_k \) is a principal \( k \times k \) submatrix of \( A \), where \( k = n - j \) for \( 1 \leq j \leq n-1 \).

So that:

\[ B_{n-1} \text{ has eigenvalues } \omega_1 \geq \omega_2 \geq \ldots \geq \omega_{n-1}, \]
\[ \vdots \]
\[ B_{n-m} \text{ has eigenvalues } \pi_1 \geq \pi_2 \ldots \geq \pi_{n-m}, \]
\[ B_{n-(m+1)} \text{ has eigenvalues } \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-(m+1)}. \]

To prove the generalized Interlacing Theorem, induction is used. \( B_k \) varies from \( B_{n-1} \) to \( B_1 \). So let \( B_k \) be represented by \( B_{n-j} \), where \( j \) varies from 1 to \( n-1 \). The induction is performed on \( j \).

1. Verify for \( j = 1 \) (i.e. \( k = n-1 \))

Figure 1 visualizes the result, where, for example

\( \lambda_2 \leq \omega_1 \leq \lambda_1 \):
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1. Assume true for \( j = m \) (i.e. \( k = n - m \)) The diagram below illustrates this assumption:

\[
\begin{pmatrix}
\lambda & \lambda_{-1} & \ldots & \lambda_{n-1} & \lambda_{n} \\
\omega_{1} & \omega_{2} & \ldots & \omega_{n-1} & \omega_{n}
\end{pmatrix}
\]

Figure 2: The above diagram visualizes the result for \( j = m \)

\( B_{n-m} \) is an \((n - m) \times (n - m)\) principal submatrix of \( A \), having eigenvalues \( \pi_1 \geq \pi_2 \ldots \geq \pi_{n-m} \)

Assume Generalized Theorem holds for \( k = n - m \):

\[
\therefore \lambda_i \geq \pi_i \geq \lambda_{n-k+i}
\]

\( \iff \lambda_i \geq \pi_i \geq \lambda_{n-(n-m)+i} \)

\( \iff \lambda_i \geq \pi_i \geq \lambda_{m+i} \ldots (*) \)

3. Prove true for \( j = m + 1 \) i.e. \( k = n - (m + 1) \):

The following diagram illustrates the result:
Figure 3: The above diagram visualizes the result for \( j = m+1 \)

\[ \mathbf{B}_{n-(m+1)} \] is an \((n - (m + 1)) \times (n - (m + 1))\) principal submatrix of \(\mathbf{A}\), having eigenvalues \(\mu_1 \geq \mu_2 \geq \ldots \mu_{n-(m+1)}\). It is also a submatrix of \(\mathbf{B}_{n-m}\) which have eigenvalues \(\pi_1 \geq \pi_2 \geq \ldots \geq \pi_{n-m}\).

Rule to prove: \(\lambda_i \geq \mu_i \geq \lambda_{n-k+i}\)

(a) To prove \(\lambda_i \geq \mu_i\):

By Interlacing Theorem,

\[ \pi_1 \geq \mu_1 \geq \pi_2 \geq \mu_2 \geq \ldots \mu_{n-(m+1)} \geq \pi_{n-m} \]

\[ \Rightarrow \pi_i \geq \mu_i, \quad 1 \leq i \leq n-m-1 \]

But from assumption (*) we know that \(\lambda_i \geq \pi_i\)

Combining these two inequalities gives \(\lambda_i \geq \pi_i \geq \mu_i \Rightarrow \lambda_i \geq \mu_i\) as required.

(b) To prove \(\mu_i \geq \lambda_{n-k+i}\):

By Interlacing Theorem,

\[ \pi_i \geq \mu_i \geq \pi_{i+1}, \quad i = 1 \ldots n-(m+1) \]

By Inductive Hypotheses:

\[ \pi_i \geq \lambda_{m+i}, \quad i = 1 \ldots n-m \]

Thus

\[ \mu_i \geq \pi_{i+1} \geq \lambda_{m+i+1}, \quad i = 1 \ldots n-(m+1) \]

But \(k = n - m - 1\)

\[ \Rightarrow \mu_i \geq \pi_{i+1} \geq \lambda_{n-k-1+i+1} \]

\[ \Rightarrow \mu_i \geq \lambda_{n-k+i}, \quad i = 1 \ldots n-(m+1) \]
Since Generalized Interlacing Theorem holds for $j = 1$, and if it holds for $j = m$ then it follows for $j = m + 1$, thus it holds for all integral values of $j$, $1 \leq j \leq n - 1$.

The dodecahedral $G_{20}$ is a 20-vertex cubic graph embedded on sphere. It contains 12 pentagons. The dodecahedral cap $\text{Cap}_{15}$ is an induced subgraph of $G_{20}$ and contains 6 pentagons.

$\text{Cap}_{15}$

![Figure 4: Cap15](image)

Introducing a layer of hexagons between two $\text{Cap}_{15}$ graphs gives $G_{40}$

![Figure 5: G40](image)

Problem: Verify that if $\{\lambda_i, 1 \leq i \leq 40\}$ are the eigenvalues of $G_{40}$ and $\{\mu_i, 1 \leq i \leq 15\}$ are those of $\text{Cap}_{15}$, then $\lambda_{21} \geq \mu_1 \geq \lambda_{2+19}$; that is $\lambda_2 \geq \mu_1 \geq \lambda_{11} \geq \mu_{12} \geq \mu_6 \geq \lambda_{21} \geq \lambda_{22} \geq \mu_{11} \geq \lambda_{31} \geq \lambda_{32} \geq \mu_{15}$ and similarly for $\lambda_4 \geq \mu_2 \geq \ldots \lambda_6 \geq \mu_3 \ldots \lambda_9 \geq \mu_4 \geq \ldots$ and $\lambda_{10} \geq \mu_5 \geq \ldots$

Explain how the interlacing theorem justifies this result.

$G_{40}$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{40}$

$\text{Cap}_{15}$ has eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \mu_{15}$

Included in the structure of $G_{40}$ is $\text{Cap}_{15}$, which is seen twice. If from $G_{40}$ we
remove the vertices that do not belong to either \( \text{Cap}_{15} \) (i.e. we remove \( 40 - (15 + 15) = 10 \) vertices), we obtain a structure with 30 vertices having eigenvalues \( \mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_{15}, \mu_{15} \).

Let \( \mu_{15} = \mu'_{30} \)
\[
\mu_{15} = \mu'_{29} \\
\vdots \\
\mu_1 = \mu'_{2} \\
\mu_1 = \mu'_{1}
\]

The following diagram illustrates this structure:

\[
\begin{array}{cccccccc}
\lambda_0 & \lambda_0 & \cdots & \cdots & \lambda_0 & \lambda_0 & \lambda_0 \\
\downarrow & \downarrow & \cdots & \cdots & \downarrow & \downarrow \\
\mu'_{30} & \mu'_{29} & \cdots & \cdots & \mu_1 & \mu_1 & \mu_1 \\
\end{array}
\]

By the Generalized Interlacing Theorem:
\[
\lambda_i \geq \mu'_i \geq \lambda_{n-k+i}
\]

For \( n = 40 \) and \( k = 30 \)
\[
\lambda_i \geq \mu'_i \geq \lambda_{10+i} \ldots (**)
\]

We now prove: \( \lambda_{2i} \geq \mu_i \geq \lambda_{2i+9} \)

- Starting from \( \mu'_{30} \):
  - For \( i = 30 \): \( \lambda_{30} \geq \mu'_{30} \geq \lambda_{40} \ldots \) from (**)
  - \( \Rightarrow \lambda_{30} \geq \mu_{15} \geq \lambda_{40} \) (since \( \mu'_{30} = \mu_{15} \)) \ldots (a)

- For \( i = 29 \): \( \lambda_{29} \geq \mu'_{29} \geq \lambda_{39} \ldots \) from (**)
  - \( \Rightarrow \lambda_{29} \geq \mu_{15} \geq \lambda_{39} \) (since \( \mu'_{30} = \mu_{15} \)) \ldots (b)

By combining a and b: \( \lambda_{30} \geq \mu_{15} \geq \lambda_{39} \) which agrees with \( \lambda_{2i-1} \geq \mu_i \geq \lambda_{2i+9} \)
  - \( \Rightarrow \lambda_i \geq \mu_i \geq \lambda_{2i+9} \).
• Starting from $\mu'_1$:
  For $i = 1 : \lambda_1 \geq \mu'_1 \geq \lambda_{11} \ldots$ from (**)  
  $\Rightarrow \lambda_1 \geq \mu_1 \geq \lambda_{11}$ (since $\mu'_1 = \mu_1$) ... (c)

  For $i = 2 : \lambda_2 \geq \mu'_2 \geq \lambda_{12} \ldots$ from (**) $\Rightarrow \lambda_2 \geq \mu_1 \geq \lambda_{12}$ (since $\mu'_2 = \mu_1$) ...  
  (d)

By combining c and d: $\lambda_2 \geq \mu_1 \geq \lambda_{11}$ which also agrees with $\lambda_{2i} \geq \mu_i \geq \lambda_{2i+9}$

In general $\lambda_{2i} \geq \mu'_2 = \mu_i \geq \lambda_{n-k+2i}$ and $\lambda_{2i-1} \geq \mu'_{2i-1} \geq \mu_i \geq \lambda_{n-k+(2i-1)}$.
Thus for $n=40$ and $k=30$, $\lambda_{2i} \geq \mu_i \geq \lambda_{2i+9}$.
Thus we have shown that the inequality $\lambda_{2i} \geq \mu_i \geq \lambda_{2i+9}$ holds.