Interlacing and Carbon Balls

Monique Inguanez, Sarah Buttigieg and Amanda Attard

Abstract

The Interlacing Theorem gives a relation among the eigenvalues of a $n \times n$ matrix $\mathbf{A}$ and those of a $(n-1) \times (n-1)$ principal submatrix. We deduce the Generalized Interlacing theorem which interlaces the eigenvalues of a $k \times k$ principal submatrix of $\mathbf{A}$ with those of $\mathbf{A}$. We apply this theorem to the hypothetical Carbon ball C_{40} which has two dodecahedral 6 pentagon caps.

Theorem 2.2 (The Interlacing Theorem) *Let* $\mathbf{A}$ *be an* $n \times n$ *symmetric matrix and let* $\mathbf{B}$ *be a matrix obtained from* $\mathbf{A}$ *by deleting a row and the corresponding column. Then if* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ *are the eigenvalues of* $\mathbf{A}$ $\mu_1 \geq \mu_2 \ldots \geq \mu_{n-1}$ *are those of* $\mathbf{B}$*, then* $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \ldots \geq \mu_{n-1} \geq \lambda_n$.

Theorem 2.3 (The Generalized Interlacing Theorem) *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. *Let* $\mathbf{B}$ *be a principal* $k \times k$ *submatrix of* $\mathbf{A}$ *where* $1 \leq k \leq n-1$, *having eigenvalues* $\mu_1 \geq \mu_2 \ldots \geq \mu_k$. *Then* $\lambda_i \geq \mu_i \geq \lambda_{n-k+i}$.

To deduce the Generalized Theorem from the Interlacing Theorem:
$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues for an $n \times n$ symmetric matrix $\mathbf{A}$. Let submatrices of $\mathbf{A}$ be formed such that $\mathbf{B_k}$ is a principal $k \times k$ submatrix of $\mathbf{A}$, where $k = n - j$ for $1 \leq j \leq n-1$.

So that:
$\mathbf{B_{n-1}}$ has eigenvalues $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_{n-1}$.
$\vdots$
$\mathbf{B_{n-m}}$ has eigenvalues $\pi_1 \geq \pi_2 \ldots \geq \pi_{n-m}$.
$\mathbf{B_{n-(m+1)}}$ has eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-(m+1)}$.

To prove the generalized Interlacing Theorem, induction is used. $\mathbf{B_k}$ varies from $\mathbf{B_{n-1}}$ to $\mathbf{B_1}$. So let $\mathbf{B_k}$ be represented by $\mathbf{B_{n-j}}$, where j varies from 1 to $n-1$. The induction is performed on j.

1. Verify for $j = 1$ (i.e. $k = n - 1$)

   Figure 1 visualizes the result, where, for example
   $\lambda_2 \leq \omega_1 \leq \lambda_1$:

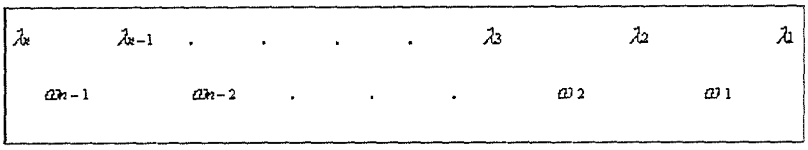


Figure 1: The above diagram visualizes the result for j = 1

$\mathbf{B_{n-1}}$ is an $(n-1) \times (n-1)$ principal submatrix of $\mathbf{A}$.
$\mathbf{B_{n-1}}$ has eigenvalues $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_{n-1}$.

Rule to prove:

$$\lambda_i \geq \omega_i \geq \lambda_{n-k+i} \iff \lambda_i \geq \omega_i \geq \lambda_{n-(n-1)+i} \iff \lambda_i \geq \omega_i \geq \lambda_{1+i}$$

This is immediately satisfied by the Interlacing Theorem which says:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \ldots \geq \mu_{n-1} \geq \lambda_n$$

2. Assume true for $j = m$ (i.e. $k = n - m$) The diagram below illustrates this assumption:

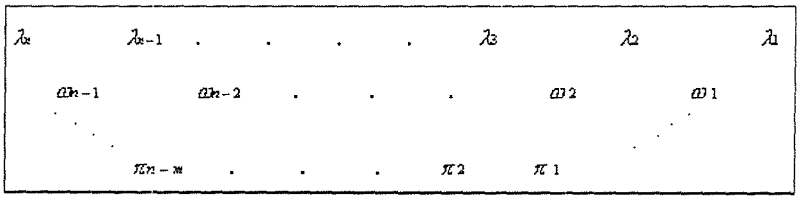


Figure 2: The above diagram visualizes the result for j = m

$\mathbf{B_{n-m}}$ is an $(n-m) \times (n-m)$ principal submatrix of $\mathbf{A}$, having eigenvalues $\pi_1 \geq \pi_2 \ldots \geq \pi_{n-m}$

Assume Generalized Theorem holds for $k = n - m$:

$$\begin{aligned} \therefore\quad & \lambda_i \geq \pi_i \geq \lambda_{n-k+i} \\ \iff\quad & \lambda_i \geq \pi_i \geq \lambda_{n-(n-m)+i} \\ \iff\quad & \lambda_i \geq \pi_i \geq \lambda_{m+i} \ldots (*) \end{aligned}$$

3. Prove true for $j = m + 1$ i.e $k = n - (m + 1)$:
The following diagram illustrates the result:

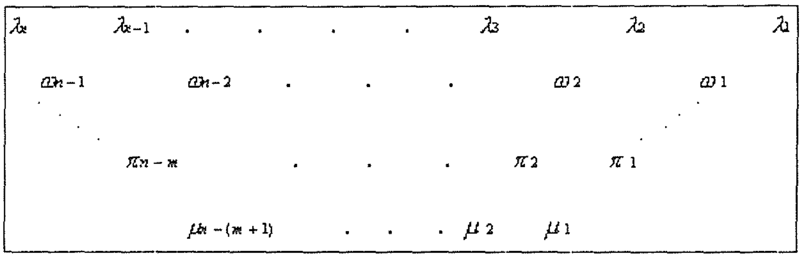


Figure 3: The above diagram visualizes the result for j = m+1

$\mathbf{B_{n-(m+1)}}$ is an $(n-(m+1)) \times (n-(m+1))$ principal submatrix of $\mathbf{A}$, having eigenvalues $\mu_1 \geq \mu_2 \geq \dots \mu_{n-(m+1)}$. It is also a submatrix of $\mathbf{B_{n-m}}$ which have eigenvalues $\pi_1 \geq \pi_2 \geq \dots \geq \pi_{n-m}$.

Rule to prove: $\lambda_i \geq \mu_i \geq \lambda_{n-k+i}$

(a) To prove $\lambda_i \geq \mu_i$:
By Interlacing Theorem,

$$\pi_1 \geq \mu_1 \geq \pi_2 \geq \mu_2 \geq \dots \mu_{n-(m+1)} \geq \pi_{n-m}$$
$$\Rightarrow \pi_i \geq \mu_i, \qquad 1 \leq i \leq n-m-1$$

But from assumption (*) we know that $\lambda_i \geq \pi_i$
Combining these two inequalities gives $\lambda_i \geq \pi_i \geq \mu_i \Rightarrow \lambda_i \geq \mu_i$ as required.

(b) To prove $\mu_i \geq \lambda_{n-k+i}$:
By Interlacing Theorem,

$$\pi_i \geq \mu_i \geq \pi_{i+1} \qquad i = 1 \dots n-(m+1)$$

By Inductive Hypotheses:

$$\pi_i \geq \lambda_{m+i} \qquad i = 1 \dots n-m$$

Thus

$$\mu_i \geq \pi_{i+1} \geq \lambda_{m+i+1}, \qquad i = 1 \dots n-(m+1)$$

But $k = n-m-1$
$\Rightarrow \mu_i \geq \pi_{i+1} \geq \lambda_{n-k-1+i+1}$
$\Rightarrow \mu_i \geq \lambda_{n-k+i}, \qquad i = 1 \dots n-(m+1)$

Since Generalized Interlacing Theorem holds for $j = 1$, and if it holds for $j = m$ then it follows for $j = m+1$, thus it holds for all integral values of $j, 1 \leq j \leq n-1$. $\blacksquare$

The dodecahedral G_{20} is a 20-vertex cubic graph embedded on sphere. It contains 12 pentagons. The dodecahedral cap Cap_{15} is an induced subgraph of G_{20} and contains 6 pentagons.

Cap_{15}

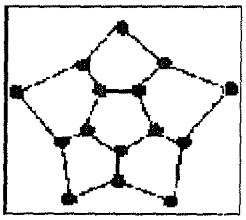


Figure 4: Cap15

Introducing a layer of hexagons between two Cap_{15} graphs gives G_{40}

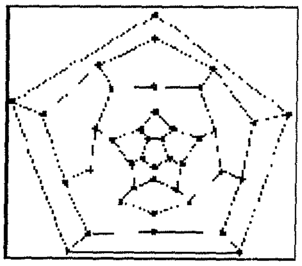


Figure 5: G40

Problem: Verify that if $\{\lambda_i, 1 \leq 1 \leq 40\}$ are the eigenvalues of G_{40} and $\{\mu_i, 1 \leq i \leq 15\}$ are those of Cap_{15}, then $\lambda_{2i} \geq \mu_i \geq \lambda_{2i+9}$; that is $\lambda_2 \geq \mu_1 \geq \lambda_{11} \geq \mu_{12} \geq \mu_6 \geq \lambda_{21} \geq \lambda_{22} \geq \mu_{11} \geq \lambda_{31} \geq \lambda_{32} \geq \mu_{16}$ and similarly for $\lambda_4 \geq \mu_2 \geq$... $\lambda_6 \geq \mu_3$...$\lambda_8 \geq \mu_4 \geq$... and $\lambda_{10} \geq \mu_5 \geq$...

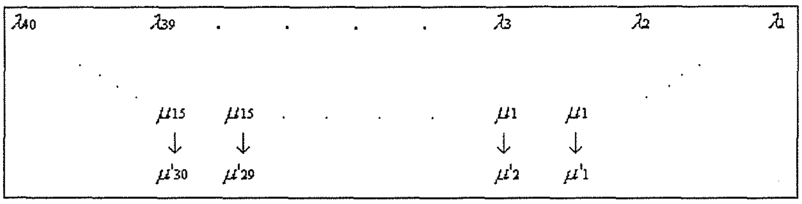
Explain how the interlacing theorem justifies this result.

G_{40} has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{40}$
Cap_{15} has eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \mu_{15}$

Included in the structure of G_{40} is Cap_{15}, which is seen twice. If from G_{40} we remove the vertices that do not belong to either Cap_{15} (i.e. we remove $40 - (15+15) = 10$ vertices), we obtain a structure with 30 vertices having eigenvalues $\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_{15}, \mu_{15}$.

$$\begin{aligned} \text{Let } \mu_{15} &= \mu'_{30} \\ \mu_{15} &= \mu'_{29} \\ &\vdots \\ \mu_1 &= \mu'_2 \\ \mu_1 &= \mu'_1 \end{aligned}$$

The following diagram illustrates this structure:



By the Generalized Interlacing Theorem:

$$\lambda_i \geq \mu'_i \geq \lambda_{n-k+i}$$

For $n = 40$ and $k = 30$

$$\lambda_i \geq \mu'_i \geq \lambda_{10+i} \ldots (**)$$

We now prove: $\lambda_{2i} \geq \mu_i \geq \lambda_{2i+9}$

- Starting from μ'_{30}:
  For $i = 30 : \lambda_{30} \geq \mu'_{30} \geq \lambda_{40} \ldots$ from (**)
  $\Rightarrow \lambda_{30} \geq \mu_{15} \geq \lambda_{40}$ (since $\mu'_{30} = \mu_{15}) \ldots (a)$

  For $i = 29 : \lambda_{29} \geq \mu'_{29} \geq \lambda_{39} \ldots$ from (**)
  $\Rightarrow \lambda_{29} \geq \mu_{15} \geq \lambda_{39}$ (since $\mu'_{30} = \mu_{15}) \ldots (b)$

  By combining a and b: $\lambda_{30} \geq \mu_{15} \geq \lambda_{39}$ which agrees with $\lambda_{2i-1} \geq \mu_i \geq \lambda_{2i+9}$
  $\Rightarrow \lambda_i \geq \mu_i \geq \lambda_{2i+9}$.

- Starting from μ'_1:
  For $i = 1 : \lambda_1 \geq \mu'_1 \geq \lambda_{11} \ldots$ from (**)
  $\Rightarrow \lambda_1 \geq \mu_1 \geq \lambda_{11}$ (since $\mu'_1 = \mu_1$) ... (c)

  For $i = 2 : \lambda_2 \geq \mu'_2 \geq \lambda_{12} \ldots$ from (**) $\Rightarrow \lambda_2 \geq \mu_1 \geq \lambda_{12}$ (since $\mu'_2 = \mu_1$) ... (d)

  By combining c and d: $\lambda_2 \geq \mu_1 \geq \lambda_{11}$ which also agrees with $\lambda 2i \geq \mu_i \geq \lambda_{2i+9}$

In general $\lambda_{2i} \geq \mu'_{2i} = \mu_i \geq \lambda_{n-k+2i}$ and $\lambda_{2i-1} \geq \mu'_{2i-1} \geq \mu_i \geq \lambda_{n-k+(2i-1)}$.
Thus for n=40 and k=30, $\lambda_{2i} \geq \mu_i \geq \lambda_{2i+9}$.
Thus we have shown that the inequality $\lambda_{2i} \geq \mu_i \geq \lambda_{2i+9}$ holds.