Non-homogeneous Markov Models for Performance Monitoring of Healthcare

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Abstract. Markov chain modelling has been previously used for hospital and community care systems, where the states in hospital care are described as phases, such as acute, rehabilitation, or long-stay and likewise social care in the community may be modelled using phases such as dependent, convalescent, or nursing home. This approach allows us to adopt a unified approach to health and community care modelling and management rather than focusing on the improvement of part of the system to the possible detriment of other components. We here extend this approach to show how the non-homogeneous Markov framework can be used to extract various metrics of interest. In particular, we use time-dependent covariates to obtain the mean and variance of the number of spells spent by a patient in hospital and in the community, and the expected total lengths of time in hospital and in the community.
Keywords: Non-homogeneous Markov Models, Healthcare Performance Monitoring.

1 Introduction

The cost of healthcare is increasing and, in addition, since there are escalating proportions of elderly people, the problem of their care is becoming increasingly important. A systems approach to healthcare planning is necessary to facilitate understanding of the process and develop a holistic method for management, monitoring and performance measurement of healthcare systems. Healthcare planning should therefore include care in the community as well as care in hospital, otherwise policies may lead to an improvement in hospital care at the expense of other components of the system.

Hospital patients may be thought of as progressing through phases such as acute care, assessment, diagnosis, rehabilitation and long-stay care. Similarly, once patients are discharged to the community they progress through phases such as dependent, convalescent, or nursing home. Such processes may be modelled using phase-type distributions [Faddy and McClean, 1999], which describe the time to absorption of a finite Markov chain in continuous time, when there is a single absorbing state and the stochastic process starts in a transient state. In addition, covariates may be incorporated into the models, thus further increasing their ability to describe complex healthcare processes.

In this paper we model stay in hospital and stay in the community as two separate phase type-distributions with transient states being phases of care in hospital and the community respectively and death being an absorbing state. Transitions can occur from all hospital phases to the first community phase, representing discharge, and from all community phases to the first hospital phase, representing admission. Transitions may also occur from all transient states to the absorbing state, death. A non-homogeneous Markov representation is used to incorporate time-dependent covariates thus improving realism of the model.

2 The Model

We have previously [Faddy and McClean, 1999,Marshall and McClean, 2003] and [McClean and Millard, 2007] modelled the movement of patients through a hospital using a Coxian phase-type model where we consider a system of n+1 states (or phases) and a Markov stochastic process $\{X(t); t \geq 0\}$ defined according to transition probabilities:

$$P\{X(t+\delta t) = i+1 | X(t) = i\} = \lambda_i \delta t + o(\delta t), \tag{1}$$

for $i = 1, 2, \ldots, n-1$ and:

$$P\{X(t+\delta t) = n+1 | X(t) = i\} = \mu_i \delta t + o(\delta t), \tag{2}$$

for $i = 1, 2, \ldots, n$. Here $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ describe sequential transitions between hospital phases $1, 2, \ldots, n$ and $\mu_1, \mu_2, \ldots, \mu_n$ describe transitions from phases $1, 2, \ldots, n$ to phase $n+1$ (Figure 1). If $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ and (at least) μ_n are all positive then phases $1, 2, \ldots, n$ are transient and phase $n+1$ (death and discharge from hospital) is absorbing. Writing the vector:

$$p = (1\ 0\ 0\ \ldots\ 0\ 0), \tag{3}$$

and matrix

$$Q = \begin{pmatrix} -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -\mu_n \end{pmatrix} \tag{4}$$

then the time spent in the transient phases, after starting in phase 1, before absorption has probability density function:

$$f(t) = p \exp(Qt) q \tag{5}$$

where:

$$q = (\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n)^T \tag{6}$$

Given data on lengths of stay, the parameters $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ and $\mu_1, \mu_2, \ldots,$ μ_{n-1} can be estimated by maximum likelyhood using the form of the density (5); starting with $n = 1$ phase, n can be increased until an adequate fit is obtained. Such an approach may also be used to model social care in the community [Xie *et al.*, 2005] and transitions between hospital and community components of the system [Taylor *et al.*, 1998], [Faddy and McClean, 1999] and [Faddy and McClean, 2005] where we define m additional community phases, with $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ describing sequential transitions between community phases 1, 2,. . . , m and $\beta_1, \beta_2, \ldots, \beta_m$ describing transitions from phases $1, 2, \ldots, m$ to $n+m+1$ (death). In addition we represent transitions between hospital phase i and the community phase 1 by $\nu_i : i = 1, \ldots, n$ and transitions between community phase i and the hospital phase 1 by $\gamma_i : i = 1, \ldots, m$ The whole system (hospital plus community) may then be represented by the matrix:

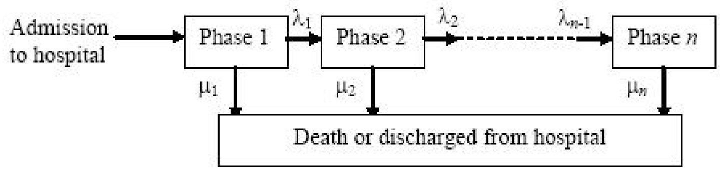

Fig. 1. The Coxian Phase-type Model

$$Q = \left(\begin{array}{ccccc|ccccc}
-(\lambda_1+\mu_1+\nu_1) & \lambda_1 & 0 & \cdot & 0 & \nu_1 & 0 & 0 & \cdot & 0 \\
0 & -(\lambda_2+\mu_2+\nu_2) & \lambda_2 & \cdot & 0 & \nu_2 & 0 & 0 & \cdot & 0 \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & \cdot & -(\mu_n+\nu_n) & \nu_n & 0 & 0 & \cdot & 0 \\
\hline
\gamma_1 & 0 & 0 & \cdot & -(\alpha_1+\beta_1+\gamma_1) & \alpha_1 & 0 & 0 & \cdot & 0 \\
\gamma_2 & 0 & 0 & \cdot & 0 & -(\alpha_2+\beta_2+\gamma_2) & \alpha_2 & 0 & \cdot & 0 \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
\gamma_m & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & -(\beta_m+\gamma_m)
\end{array}\right)$$

where, $\lambda_n = \alpha_m = 0$.

Then the time spent in the transient phases, having started in phase 1 (admission to hospital), until absorption (death) has probability density function, as before where now:

$$q = (\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n, \beta_1, \beta_2, \ldots, \beta_{m-1}, \beta_m)^T. \tag{7}$$

We now define the matrix A , corresponding to the transient states of the embedded Markov chain, representing the next transition between states of the continuous time model presented above. Here, $A = \{a_{ij}\}$ where: $a_{ij} =$ Prob (next transition is to state j | currently in state i) and,

$$A = \left(\begin{array}{cccc|cccc} 0 & \frac{\lambda_1}{(\lambda_1+\mu_1+\nu_1)} & 0 & \cdot\ 0 & \frac{\nu_1}{(\lambda_1+\mu_1+\nu_1)} & 0 & 0 & \cdot\ 0 \\ 0 & 0 & \frac{\lambda_2}{(\lambda_2+\mu_2+\nu_2)} & \cdot\ 0 & \frac{\nu_2}{(\lambda_2+\mu_2+\nu_2)} & 0 & 0 & \cdot\ 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot \\ 0 & 0 & 0 & \cdot\ 0 & \frac{\nu_n}{(\lambda_n+\mu_n+\nu_n)} & 0 & 0 & \cdot\ 0 \\ \hline \frac{\gamma_1}{(\alpha_1+\beta_1+\gamma_1)} & 0 & 0 & \cdot\ 0 & 0 & \frac{\alpha_1}{(\alpha_1+\beta_1+\gamma_1)} & 0 & \cdot\ 0 \\ \frac{\gamma_2}{(\alpha_2+\beta_2+\gamma_2)} & 0 & 0 & \cdot\ 0 & 0 & 0 & \frac{\alpha_2}{(\alpha_2+\beta_2+\gamma_2)} & \cdot\ 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ 0 \\ \frac{\gamma_m}{(\alpha_m+\beta_m+\gamma_m)} & 0 & 0 & \cdot\ 0 & 0 & 0 & 0 & \cdot\ 0 \end{array} \right).$$

Then, the expected number of entries to state j given initially in state i is ν_{ij} where, $N = \{n_{ij}\}$ is given by:

$$N = (I - A)^{-1} \tag{8}$$

Formulae for the corresponding variances and expected total times spent in hospital and the community [Iofescu, 1980] can similarly be derived and were presented in our previous paper [McClean *et al.*, 2006]. An example of such a system is presented in Example 1.

Example 1. Four hospital states and three community states: This model has been previously fitted to data [Faddy and McClean, 2005,Millard, 1991] and is illustrated in Figure 2. The data were described in [Millard, 1991].

3 The Non-homogenous Markov model

An important aspect of such an approach is the incorporation of covariates into the models so that we can take account of significant heterogeneity between patients, caused, for example, by differences between patients due to gender, age, year or socio-economic factors. Such differences may be partly addressed by stratification, where we separately model groups of patients with different covariates, e.g. we may fit different models for male and female patients. However time dependent covariates, such as age, require a different approach. In such cases we now develop a time-heterogeneous Markov model where the parameters (of the matrices Q and A) are updated every time a patient makes a transition. This can be achieved by having the transition rates $\lambda_i, \mu_i, \alpha_i$ and β_i depend log-linearly on covariates. Such dependency parameters have previously been estimated by maximum likelihood [Faddy and McClean, 1999]. The update time can then be represented

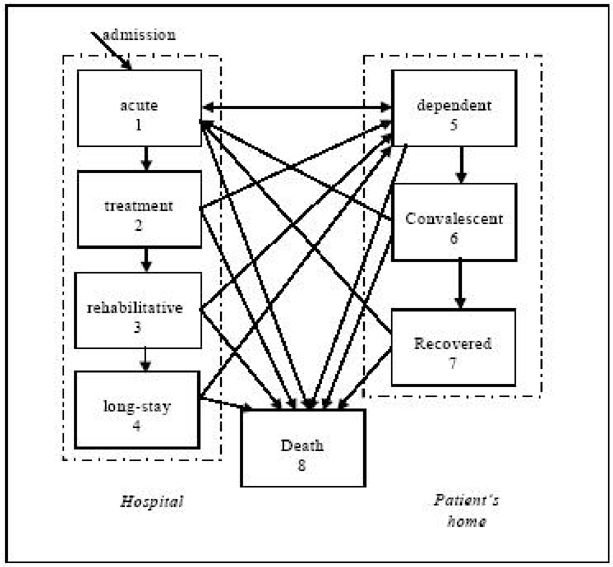

Fig. 2. The Health and Social Care Markov System

by mean time to make the corresponding transition from hospital to community or vice versa. Dependence on covariates $X = (X_1 X_2 \dots X_m)^T$ can thus be incorporated into the model by having the transition rates $\lambda_i, \mu_i, \nu_i, \alpha_i, \beta_i$ and γ_i take the form:

$$\exp(a + \mathbf{b}^T X) \tag{9}$$

with coefficient parameters a and $\mathbf{b}$ estimated for each of the transition rates [Faddy and McClean, 1999]. For the data analysed here, there are two time dependent covariates: x_1 = patients age at admission (to hospital or community care) and x_2 = year of admission. Data were also available on the different events that terminated the patient's periods of care: for hospital these were discharge or death, and for community they were re-admission to hospital or death. Given this information and a fitted phase-type distribution for the preceding period of care, probabilities θ_{ij} for event j from phase i can be obtained by conditional maximum likelihood [Faddy and McClean, 2005]. In this way, estimation of these θ_{ij} is carried out after estimation of the parameters $\lambda_i, \mu_i, \nu_i, \alpha_i, \beta_i$ and γ_i of the phase-type distribution of the time in hospital and community care, respectively. Covariate $X = (X_1 X_2 \dots X_m)^T$ dependence in the θ_{ij} probabilities can be included by putting:

$$logit(\tfrac{\theta_{ij}}{1-\theta_{i1}-\cdots-\theta_{ij-1}}) = a + \mathbf{b}^T X \tag{10}$$

and estimating parameters a and b for each θ_{ij} ($i = 1, 2, \dots,$number of phases, and $j = 1, 2, \dots,$number of events) [Faddy and McClean, 2005]. In order to implement time-dependence in the covariates we update the transition matrix every time there is a discharge or admission to hospital. The parameters $\lambda_i, \mu_i, \nu_i, \alpha_i, \beta_i$ and γ_i are then recalculated with the new updated age and updated year as covariates. Using these updated values of parameters, we recalculate the matrix A. Also, for each admission (or re-admission), we calculate the expected total time spent in hospital and for each discharge we calculate the expected total time spent in the community. This expected total time is then used to update the age and the year after each admission (or re-admission) to hospital and each discharge to the community. For each spell in hospital, we then calculate the probability of the spell ending with discharge to the community as π_d and the probability of the community spell ending with re-admission to hospital as π_r respectively, given by: $\pi_d = (1\ 0\ 0 \ldots 0)(\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{b}_1$ $\pi_r = (1\ 0\ 0 \ldots 0)(\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{b}_2$

Here $\mathbf{A}_1$ and $\mathbf{A}_2$ are sub-matrixes of $\mathbf{A}$ given by:

$$A = \left(\begin{array}{cc|cc} & \mathbf{A}_1 & \mathbf{b}_1 & 0 \\ \hline \mathbf{b}_2 & 0 & & \mathbf{A}_2 \end{array}\right) \tag{11}$$

and $\mathbf{b}_1$ and $\mathbf{b}_2$ are column vectors. The probability of a patient, initially admitted to hospital, surviving to eventual re-admitted to hospital, is then:

$$\pi_s = \pi_d \pi_r \tag{12}$$

In the case of the 7 transient state model presented in Example 1, we therefore obtain:

$$\begin{aligned} \pi_d = & \left\{\frac{\nu_1}{(\lambda_1+\mu_1+\nu_1)}\right\} + \left\{\frac{\lambda_1}{(\lambda_1+\mu_1+\nu_1)} * \frac{\nu_2}{(\lambda_2+\mu_2+\nu_2)}\right\} + \left\{\frac{\lambda_1}{(\lambda_1+\mu_1+\nu_1)} * \frac{\lambda_2}{(\lambda_2+\mu_2+\nu_2)}\right. \\ & \left. * \frac{\nu_3}{(\lambda_3+\mu_3+\nu_3)}\right\} + \left\{\frac{\lambda_1}{(\lambda_1+\mu_1+\nu_1)} * \frac{\lambda_2}{(\lambda_2+\mu_2+\nu_2)} * \frac{\lambda_3}{(\lambda_3+\mu_3+\nu_3)} * \frac{\nu_4}{(\lambda_4+\mu_4+\nu_4)}\right\} \\ & = a_{15} + a_{12} * a_{25} + a_{12} * a_{23} * a_{35} + a_{12} * a_{25} + a_{12} * a_{23} * a_{34} * a_{45} \end{aligned}$$

and

$$\begin{aligned} \pi_r = & \left\{\frac{\gamma_1}{(\alpha_1+\beta_1+\gamma_1)}\right\} + \left\{\frac{\alpha_1}{(\alpha_1+\beta_1+\gamma_1)} * \frac{\gamma_2}{(\alpha_2+\beta_2+\gamma_2)}\right\} + \left\{\frac{\alpha_1}{(\alpha_1+\beta_1+\gamma_1)} * \frac{\alpha_2}{(\alpha_2+\beta_2+\gamma_2)}\right. \\ & \left. * \frac{\gamma_3}{(\lambda_3+\mu_3+\nu_3)}\right\} \\ & = a_{51} + a_{56} * a_{61} + a_{56} * a_{67} * a_{71} \end{aligned}$$

We then calculate the expected (mean) number of admission/ re-admission to hospital

$$Ad_{mean} = 1 + \pi_s^{(1)} + (2 * \pi_s^{(1)} * \pi_s^{(2)}) + \ldots + (n * \pi_s^{(1)} * \pi_s^{(2)} * \ldots * \pi_s^{(n)}) + \ldots$$

and expected (mean) number of discharges to the community.

$$Dis_{mean} = \pi_d^{(1)} + (2 * \pi_s^{(1)} * \pi_d^{(2)}) + \ldots + (n * \pi_s^{(1)} * \pi_s^{(2)} * \ldots * \pi_d^{(n)}) + \ldots$$

where $\pi_s^{(n)}$ is the probability of surviving both of the n^{th} hospital and community spells and $\pi_d^{(n)}$ is the probability of discharge from the n^{th} hospital spell. Next we calculate the expected (mean) time in hospital:

$$Th_{mean} = (\pi_d^{(1)} * TT_1^{(1)}) + (\pi_s^{(1)} * \pi_d^{(2)} * TT_1^{(2)}) + \ldots + (n * \pi_s^{(1)} * \pi_s^{(2)} * \ldots * \pi_d^{(n)} * TT_1^{(n)}) + \ldots$$

Similarly, the expected (mean) time in the community after the first admission to hospital is:

$$Tc_{mean} = (\pi_{\tau}^{(1)} * TT_2^{(1)}) + (\pi_s^{(1)} * \pi_{\tau}^{(2)} * TT_2^{(2)}) + \ldots + (n * \pi_s^{(1)} * * \pi_s^{(2)} * \ldots * \pi_{\tau}^{(n)} * TT_2^{(n)}) + \ldots$$

where $\pi_{\tau}^{(n)}$ is the probability of re-admission to hospital after the n^{th} spell. $TT_1^{(n)}$ is the expected time in hospital at the n^{th} admission and $TT_2^{(n)}$ is the expected time in the community after the n^{th} discharge from hospital. Also, the variance of the number of admission to hospital is given by:

$$V_{hos} = \Sigma_i \{(i - Ad_{mean})^2 * pi_s^{(1)} * pi_s^{(2)} * \ldots * pi_s^{(i-1)} * pi_s^{(i)}\}$$

where i is the admission number: $i = 1, 2, 3, \ldots$. Similarly the variance of the number of discharge to the community is given by:

$$V_{dis} = \Sigma_i \{(i - Dis_{mean})^2 * pi_s^{(1)} * pi_s^{(2)} * \ldots * pi_s^{(i-1)} * pi_d^{(i)}\}$$

We here note that all the above computations are terminated when the probability of surviving the previous spells becomes very small, in this case 10^{-25} or less.

4 Results

We now obtain results for example 1, using parameters estimated from [Faddy and McClean, 2005]. The corresponding results are presented in Table 1. We here compare our previous results [McClean *et al.*, 2006] with the results from our new approach, which incorporates time-dependent covariates, namely age and year. From Table 1 we can see that incorporating time

Metrics	Performance metrics excluding time dependent covariates	Performance metrics including time dependent covariates	Change	%change
Mean number of admissions to hospital	1.73	1.7568	+0.0268	+1.55%
Variance of admissions to hospital	1.26	1.3349	+0.0749	+5.94%
Mean number of discharges to community care	1.24	1.2607	+0.0207	+1.67%
Variance of number of discharges to community care	1.51	2.3993	+0.8893	+58.89%
Mean total time in hospital (days)	60.66	43.0688	-16.5912	-27.35%
Mean total time in the community (days)	838.16	634.3818	-193.7782	-23.12%

Table 1. Results for Example 1

dependent covariates slightly increases the mean and variance of the number of admissions to hospital and number of discharges. There is a substantial decrease in the number of days spent in hospital and the community respectively, corresponding to higher death (absorbing) probabilities in all cases. This is to be expected as older patients are more likely to die during a spell in hospital or back in the community than younger patients. Our new non-homogeneous Markov model is therefore likely to be more realistic than our previous approach.

5 Conclusions and Further Work

We have described an extension to previous work that allows us to compute key performance measures for the whole patient care system, including both hospital and community components. Such an approach is particularly important for assessing the effectiveness of geriatric care systems, which typically include significant components of both hospital and community care. By including time dependent covariates, such as age and year, and utilizing a corresponding non-homogeneous Markov model, we are able to develop a more realistic model that describes key metrics.

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