Graphs that have a weighted adjacency matrix with spectrum $\{\lambda_1^{n-2}, \lambda_1^2\}$

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Abstract

In this paper we completely characterize the graphs which have an edge weighted adjacency matrix belonging to the class of $n \times n$ involutions with spectrum equal to $\{\lambda_1^{n-2}, \lambda_2^2\}$ for some λ_1 and some λ_2. The connected graphs turn out to be the cographs constructed as the join of at least two unions of pairs of complete graphs, and possibly joined with one other complete graph.

Keywords: Minimum number of distinct eigenvalues, cographs

1. Introduction

To a graph X, we associate the collection of real $n \times n$ symmetric matrices defined by

$$S(X) = \{A \,:\, A = A^T;\text{ for } i \neq j,\ a_{ij} \neq 0 \text{ if and only if } \{i,j\} \in E(X)\}.$$

Note that there are no restrictions on the entries on the main diagonal of a matrix in $S(X)$. If $A \in S(X)$ for some graph X, then X is the *graph of A*. This family of matrices has been studied by many researchers and it is interesting to connect properties of $S(X)$ to properties of the graph. For example, there has been significant work on determining the value of the minimum rank over all matrices in $S(X)$ for a given graph X, see [3, 5, 6] and the references within.

For a real symmetric matrix A, let $q(A)$ denote the number of distinct eigenvalues of A. For a graph X, define

$$q(X) = \min\{q(A) \,:\, A \in S(X)\}.$$

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[1] Research supported by NSERC.

We say that $q(X)$ is the number of distinct eigenvalues for the graph X. There have been several recent results regarding this parameter [1, 2, 9, 12, 15]; this paper continues the work in [1].

It is easy to see that $q(X) = 1$ if and only if X is an empty graph. At the other extreme, $q(X) = |V(X)|$ if and only if X is a path [8]. There are very few known lower bounds on the value of $q(X)$ for a graph X. One of the most effective is the following simple bound; this is Theorem 3.2 from [1].

Theorem 1. *Let x and y be two vertices of a graph X at distance d. If the path of length d from x to y is unique, then $q(X) \geq d + 1$.*

The complete graph on n vertices, denoted by K_n, has $q(K_n) = 2$ (this can be achieved by the $(0, 1)$-adjacency matrix of K_n). Also the complete bipartite graph $K_{n,n}$, and the hypercube also have only two distinct eigenvalues. The next example will show that the family of graphs with only two distinct eigenvalues is very large. For graphs X and Y, the *join* of X and Y, denoted $X \triangledown Y$, is the graph with vertices $V(X) \cup V(Y)$, and edge set

$$E(X) \cup E(Y) \cup \{\{x, y\} : x \in V(X),\, y \in V(Y)\}.$$

In [1] it is shown that if X is any connected graph, then $q(X \triangledown X) = 2$. From these examples, it seems unlikely that the family of graphs X with $q(X) = 2$ can be characterized.

For any graph, the multiplicities of the eigenvalues form an integer partition of the number of vertices in the graph. In this paper, we only consider graphs in which this partition has only two parts, so is a *bipartition*. We say that $[n-i, i]$ is a *multiplicity bipartition* of X, if there exists an $A \in S(X)$ with spectrum $\{\lambda_1^{n-i}, \lambda_2^{i}\}$. The *minimal* multiplicity bipartition of a graph is $[n-i, i]$ if i is the least value such that a multiplicity bipartition $[n-i, i]$ of the graph exists. Note that if a non-empty graph X has a multiplicity bipartition, then $q(X) = 2$. The main theorem of this paper is a characterization of the graphs that have $[n-2, 2]$ as their minimal multiplicity bipartition.

Theorem 2. *Assume that X is a connected graph. The minimal multiplicity bipartition of X is $[n-2, 2]$ if and only if*

$$X = (K_{a_1} \cup K_{b_1}) \triangledown (K_{a_2} \cup K_{b_2}) \triangledown \cdots \triangledown (K_{a_k} \cup K_{b_k})$$

where $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are non-negative integers, $k > 1$, and X is not isomorphic to a complete graph, or to $(K_{a_1} \cup K_{b_1}) \triangledown K_1$.

In the next section we shall state some known results related to graphs with only two distinct eigenvalues. Section 3 presents some restrictions on graphs that have $[n-2, 2]$ as a multiplicity bipartition. In Section 4 we give constructions for matrices $A \in S(X)$ that have spectrum $\{\lambda_1^{n-2}, \lambda_2^{2}\}$ for all graphs X identified in Theorem 2. Section 5 gives the proof of Theorem 2.

2. Graphs with a multiplicity bipartition

In this section, we present a way to determine if a matrix has only two distinct eigenvalues with specific multiplicities. The first result follows from [1, Lemma 2.3].

Proposition 3. *Let X be a non-empty graph, then $q(X) = 2$ if and only if there is an $A \in S(X)$ with $A^2 = I$.*

We do not give a proof of Proposition 3. Rather we shall prove a stronger result, namely Lemma 5, which implies it. But first we need to introduce some notation. Throughout this paper, the i-th entry of a vector v will be denoted by v_i. We shall use $v(i)$ to denote different vectors (in Section 4 we shall give constructions for vectors that are based on a parameter i). The j-th entry of the vector $v(i)$ will be denoted by $v(i)_j$. We shall start with a simple theorem about the spectrum of a matrix with a specific form.

Lemma 4. *Let $\{v(1), v(2), \ldots, v(k)\}$ be a set of orthonormal vectors in $\mathbb{R}^n$ with $1 \leq k < n$, and define*

$$A = I - 2(v(1)v(1)^T + v(2)v(2)^T + \cdots + v(k)v(k)^T).$$

Then the following hold:

1. *A is real and symmetric;*
2. *the (i, j)-entry of A, where $i \neq j$, is*
$$-2\sum_{\ell=1}^{n} u(\ell)_i u(\ell)_j;$$
3. *the (i, i)-entry of A is*
$$1 - 2\sum_{\ell=1}^{n} u(\ell)_i u(\ell)_i;$$
4. *$A^2 = I$;*
5. *the spectrum of A is $\{1^{(n-k)}, -1^{(k)}\}$; so $q(A) = 2$;*
6. *if X is the graph of A, then $q(X) = 2$.*

Proof. The first four statements follow immediately from the definition of A. The fifth follows from the fact that the set $\{v(1), v(2), \ldots, v(k)\}$ forms a set of orthogonal eigenvectors of A each with eigenvalue -1. Any vector orthogonal to all of $v(1), v(2), \ldots, v(k)$ is also an eigenvector of A, but with eigenvalue 1. The final statement follows from Statement 5, and the fact that X is non-empty. $\square$

It is necessary that $k < n$ in the previous lemma, since if $k = n$, then $A = -I$ and $q(A) = 1$. The results that follow next show how the previous lemma can be used to determine if a graph has a multiplicity bipartition.

Lemma 5. *Let X be a non-empty graph. There exists $A \in S(X)$ with*

$$A = I - 2(v(1)v(1)^T + v(2)v(2)^T + \cdots + v(k)v(k)^T),$$

where $\{v(1), v(2), \ldots, v(k)\}$ is an orthonormal set of vectors (with $1 \leq k < n$), if and only if $q(X) = 2$.

Proof. By Statement 6 of Lemma 4, the condition $q(x) = 2$ is necessary.

To prove that this condition is sufficient, let $B \in S(X)$ with $q(B) = 2$. Let λ_1 and λ_2 be the eigenvalues of B. Set

$$A = \frac{2}{\lambda_1 - \lambda_2} B - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} I. \tag{1}$$

Then $A \in S(X)$ and the eigenvalues of A are -1 and 1.

By spectral decomposition, $A = 1P_1 + (-1)P_{-1}$, where P_i is the projection to the i-eigenspace. Since $P_1 + P_{-1} = I$, we have that

$$A = (I - P_{-1}) - P_{-1} = I - 2P_{-1}.$$

If $\{v(1), v(2), \ldots, v(k)\}$ is an orthonormal basis for the -1 eigenspace, then

$$P_{-1} = v(1)v(1)^T + v(2)v(2)^T + \cdots + v(k)v(k)^T.$$

Thus A has the required form. □

Statement 5 of Lemma 4 and the proof of Lemma 5 provide a way to construct an adjacency matris for any graph X with $q(X) = 2$.

Corollary 6. *A graph X has $[n-k, k]$ as a multiplicity bipartition if and only if there exist an $A \in S(X)$ such that*

$$A = I - 2(v(1)v(1)^T + v(2)v(2)^T + \cdots + v(k)v(k)^T)$$

where $\{v(1), v(2), \ldots, v(k)\}$ is an orthonormal set of vectors.

In Sections 3 and 4, we use this to determine the graphs with $[n-2, 2]$ as a multiplicity bipartition. But first we use this give information about the structure of a graph X with $q(X) = 2$. A *coclique* in a graph is a set of vertices in which no two are adjacent, a coclique is also known as an *independent set*.

Lemma 7. *Let $\{v(1), v(2), \ldots, v(k)\}$ be a set of orthonormal vectors in $\mathbb{R}^n$ and*

$$A = I - 2(v(1)v(1)^T + v(2)v(2)^T + \cdots + v(k)v(k)^T).$$

Let X be the graph of A. Then, provided that X does not contain any isolated vertices, the graph of A does not contain a coclique of size $k+1$.

Proof. Let X be the graph of A and label the vertices in X by $1, \dots, n$. Assume that vertices $1, 2, \dots, k+1$ form a coclique in X. The (i,j)-entry of A for $i \neq j$ is $-2\sum_{\ell=1}^{k} v(\ell)_i v(\ell)_j$. So for all $1 \leq i, j \leq k+1$

$$\sum_{\ell=1}^{k} v(\ell)_i v(\ell)_j = 0. \tag{2}$$

Consider the vectors $x(i) = (v(1)_i, v(2)_i, \dots, v(k)_i)$ for $i = 1, 2, \dots, k+1$. The vector $x(i)$ cannot be the zero vector, since that would imply the vertex i is an isolated vertex in X. From Equation 2, if $i \neq j$, then $x(i)$ is orthogonal to $x(j)$. But this implies that there is a set of $k+1$ non-zero, length-k orthogonal vectors, which is clearly not possible. Hence X does not have a coclique with more than k vertices. □

With this lemma, we can characterize the graphs that have $[n-1, 1]$ as a multiplicity bipartition.

Corollary 8. *The graph X has $[n-1, 1]$ as a multiplicity bipartition if and only if X is a complete graph with isolated vertices.*

Proof. If X has $[n-1, 1]$ as a multiplicity bipartition, then, by Corollary 6, there is a matrix $A \in S(X)$ with $A = I - 2uu^T$. From Lemma 7, any two vertices in X are adjacent, unless one of them is isolated.

Conversely, let X be the graph with a clique of size n and k isolated points. Let J_n be the $n \times n$ matrix and I_n the $n \times n$ identity matrix and consider the matrix

$$A = \left(\begin{array}{c|c} J_n - I_n & 0 \\ \hline 0 & -I_k \end{array}\right).$$

The spectrum of A is $\{n-1, -1^{n+k-1}\}$ and $A \in S(X)$. Since X is non-empty, it has multiplicity bipartition $[n-1, 1]$. □

In the previous corollary, we found that a graph with two distinct eigenvalues and one with multiplicity 1 must be the complete graph with isolated vertices. This completely characterizes the graphs with minimal multiplicity bipartition $[n-1, 1]$. In the sequel, we shall give a characterization of the graphs with minimal multiplicity bipartition $[n-2, 2]$. In other words, we characterize the graphs X for which there is an $A \in S(X)$ with

$$A = I - 2(uu^T + vv^T)$$

where u and v are orthonormal vectors. If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$, then the (i,j)-entry of A is $-2(u_i u_j + v_i v_j)$. So we need to determine which zero/non-zero patterns are possible in A, with the conditions that u and v are orthogonal and normalized.

The first restriction on the graphs with $[n-2, 2]$ as a multiplicity bipartition follows directly from Lemma 7.

Corollary 9. *A connected graph X with $[n-2, 2]$ as a multiplicity bipartition does not have a coclique of size* 3.

3. Cographs

In this section, we give a major restriction on the structure of the graphs that have $[n-2,2]$ as a multiplicity bipartition.

Lemma 10. *A graph X with $[n-2,2]$ as a multiplicity bipartition does not have an induced path of length three.*

Proof. Assume that the first four vertices in X (which we simply label 1, 2, 3, 4) form an induced path of length-3 in X. Assume that there is a $A \in S(X)$ with

$$A = I - 2(uu^T + vv^T).$$

Then we have the following six equations:

$$\begin{aligned} &u_1u_3 + v_1v_3 = 0, \quad u_1u_4 + v_1v_4 = 0, \qquad u_2u_4 + v_2v_4 = 0, \\ &u_1u_2 + v_1v_2 \neq 0, \quad u_2u_3 + v_2v_3 \neq 0, \qquad u_3u_4 + v_3v_4 \neq 0. \end{aligned} \tag{3}$$

First we shall show that none of u_i for $i \in \{1, \ldots, 4\}$ can be equal to zero. First suppose $u_3 = 0$. Then $v_1v_3 = 0$. If $v_3 = 0$, then the third vertex is not adjacent to the second vertex (equation $u_2u_3 + v_2v_3 \neq 0$ cannot hold). So $v_1 = 0$. But then, since $u_1u_4 + v_1v_4 = 0$, one of u_1 or u_4 is zero. If $u_1 = 0$, then the first vertex is not adjacent to the second vertex (equation $u_1u_2 + v_1v_2 \neq 0$ cannot hold). So this implies that $u_4 = 0$. Then the equation $u_2u_4 + v_2v_4 = 0$ implies that $v_2v_4 = 0$. If $v_2 = 0$, then $u_2u_3 + v_2v_3 = 0$, which is a contradiction. Similarly, if $v_4 = 0$, then $u_3u_4 + v_3v_4 = 0$, which is also a contradiction. Thus $u_3 \neq 0$.

Similarly, we can show that u_1, u_2 and u_4 are also non-zero and also that the entries in v_i are not zero for $i \in \{1, \ldots, 4\}$.

Now we can assume that u_i and v_i are not zero for $i \in \{1, \ldots, 4\}$. With this assumption, From Equation (3), we have that

$$u_1 = -\frac{v_1v_3}{u_3} = -\frac{v_1v_4}{u_4}.$$

We set $k = \frac{v_3}{u_3} = \frac{v_4}{u_4}$. Similarly,

$$u_4 = -\frac{v_1v_4}{u_1} = -\frac{v_2v_4}{u_2},$$

and in this case we set $\ell = \frac{v_1}{u_1} = \frac{v_2}{u_2}$. Thus we have that

$$v_1 = \ell u_1, \quad v_2 = \ell u_2, \quad v_3 = ku_3, \quad v_4 = ku_4.$$

Since vertices 1 and 4 are non-adjacent,

$$u_1u_4 + v_1v_4 = u_1u_4 + \ell u_1ku_4 = 0,$$

thus $1 + \ell k = 0$. But this implies that

$$u_2u_3 + v_2v_3 = u_2u_3 + \ell u_2ku_3 = (1 + \ell k)u_2u_3 = 0,$$

which is a contradiction, since vertices 2 and 3 are adjacent. □

The family of graphs that do not contain a copy of P_4 are known as the *cographs*. Cographs are a well-studied family of graphs [4, 13]. These graphs can be built recursively.

Proposition 11. *The following recursive construction defines all cographs:*

1. *a single vertex is a cograph;*
2. *the union or join of two cographs is again a cograph; and*
3. *if X is a cograph, then $\overline{X}$ is itself a cograph.*

A *cotree* is a tree that is used to represent a cograph. There is a 1-1 correspondence between cotrees and cographs; a cograph has a unique cotree, and each cotree determines a unique cograph.The leaves in the cotree correspond to vertices in the cograph. Internal nodes of a cotree are labeled with either a union or a join. The children of the nodes are connected by the operation by which the node is labelled.

Let T be the cotree of a cograph X. We can assume that below any internal node there must be at least two children, since if there is just one, then the branch can be shortened. Each child represents a subgraph of T, that corresponds to a subgraph of X that is also a cograph. If an internal node is labeled as union, then its children are either leaves, or internal nodes labeled with a join. Similarly, if an internal node is labeled with a join, then its children are either leaves or internal nodes labeled as unions.

We shall first characterize the cographs that do not contain a coclique of size three. This will give a considerable restriction on the possible graphs that have $[n-2, 2]$ as a multiplicity bipartition.

Lemma 12. *If X is a connected cograph with no coclique of size* 3*, then*

$$X = (K_{a_1} \cup K_{b_1}) \triangledown (K_{a_2} \cup K_{b_2}) \triangledown \cdots \triangledown (K_{a_k} \cup K_{b_k})$$

where $\{a_1, \dots, a_k\}$ and $\{b_1 \dots, b_k\}$ are non-negative integers.

Proof. Assume that X is a cograph with no cocliques of size three and T is the cotree for X. Since X is connected, the root of T must be a join.

If a vertex of T that is labeled with a union has three children, then any set formed by taking one vertex from the subgraph corresponding to each of the children will be a coclique in X of size three. Thus, in a cograph with no coclique of size three in X, any internal vertex labeled with a union can have at most two children, and each child must correspond to a clique in X.

If a vertex in T is labeled with a union, then it cannot have a descendent that is also labeled with a union. To see this, consider Figure 1. This is an example of a cotree in which a vertex labeled with a union has a descendant that is also labeled by a union. Any set of three vertices in which one vertex is from each of the subgraphs of X corresponding to the cotrees T_1, T_2, T_4, will form a coclique of size three in the graph X.

From these facts, the result holds. $\square$

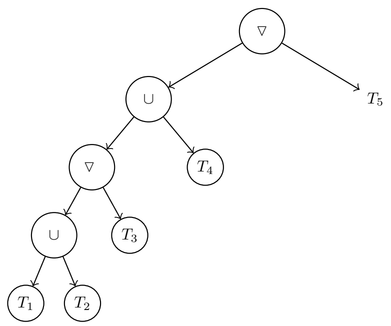

Figure 1: A cotree with an internal vertex labeled with a union with a descendent also labeled with a union.

In the following sections we shall consider cographs of the form

$$X = (K_{a_1} \cup K_{b_1}) \nabla (K_{a_2} \cup K_{b_2}) \nabla \cdots \nabla (K_{a_k} \cup K_{b_k}).$$

We shall refer to a subgraph $K_{a_i} \cup K_{b_i}$ as a *block* of X. Figure 2 shows the general form of a cotree for any such cograph.

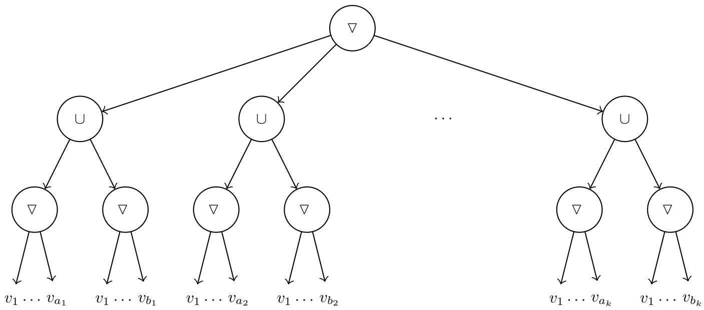

Figure 2: Cotrees for the cographs with no coclique of size three

4. Constructions

In this section, we shall give several results that are of the same style. In each result, for a given a graph X of a certain form, we construct a matrix $A \in S(X)$ with $A = I - 2(uu^T + vv^T)$ and $A^2 = I$. Note the condition from Lemma 5 that u and v both have norm 1, can be replaced with the condition that both vectors have the same norm; in this case we use the matrix $A = I - (2/\|u\|^2)(uu^T + vv^T)$. Each result in this section gives examples of two vectors u and v that they satisfy the following three conditions:

1. u and v are orthogonal;
2. u and v have the same norm;

3. $u_i u_j + v_i v_j$ is zero if and only if vertices i and j in X are non-adjacent.

Proposition 13. *Let a_i and b_i be positive integers. Suppose*

$$X = (K_{a_1} \cup K_{b_1}) \nabla (K_{a_2} \cup K_{b_2}) \nabla \cdots \nabla (K_{a_k} \cup K_{b_k}),$$

then X has $[n-2, 2]$ as a multiplicity bipartition.

Proof. For $i = 1, \dots, k$, assume that $0 < a_o \leq b_i$.

For each subgraph $K_{a_i} \cup K_{b_i}$ of X, consider the two vectors

$$u(i) = \big(\underbrace{1, 1, \dots, 1,}_{a_i \text{ times}} \underbrace{-\sqrt{\frac{a_i}{b_i}} w_i, -\sqrt{\frac{a_i}{b_i}} w_i, \dots, -\sqrt{\frac{a_i}{b_i}} w_i}_{b_i \text{ times}}\big)$$

$$v(i) = \big(\underbrace{w_i, w_i, \dots w_i,}_{a_i \text{ times}} \underbrace{\sqrt{\frac{a_i}{b_i}}, \sqrt{\frac{a_i}{b_i}}, \dots, \sqrt{\frac{a_i}{b_i}}}_{b_i \text{ times}}\big)$$

where w_i is any non-zero number. (The vertices of $K_{a_i} \cup K_{b_i}$ are sorted so that the vertices in K_{a_i} are first.) The norm of $u(i)$ equals the norm of $v(i)$, and the two vectors are orthogonal.

Let u be the vector formed by concatenating the vectors $u(1), u(2), \dots, u(k)$, and v be the vector formed from concatenating the vectors $v(1), v(2), \dots, v(k)$. Then u and v have the same norm and are orthogonal.

Finally we need to show that $A = I - (2/\|u\|^2)(uu^T + vv^T)$ is in $S(X)$.

If two vertices x, y are from the same block in this graph, say $K_{a_i} \cup K_{b_i}$, then

$$[A]_{x,y} = \begin{cases} 1 + w_i^2, & \text{if } x, y \in V(K_{a_i}); \\ 0, & \text{if } x \in V(K_{a_i}) \text{ and } y \in V(K_{b_i}); \\ \frac{a_i}{b_i}(1 + w_i^2), & \text{if } x, y \in V(K_{b_i}). \end{cases}$$

So for any x, y that are vertices in the block $K_{a_i} \cup K_{b_i}$, the (x, y)-entry of A will be zero if and only if x and y are not adjacent in X.

Next consider the two vertices x, y are from different blocks in this graph. Assume x is from $K_{a_i} \cup K_{b_i}$ and y from $K_{a_j} \cup K_{b_j}$, where $i \neq j$, then the corresponding entry in A is

$$[A]_{x,y} = \begin{cases} 1 + w_i w_j, & \text{if } x \in V(K_{a_i}), \text{ and } y \in V(K_{a_j}); \\ -\sqrt{\frac{a_j}{b_j}} w_j + \sqrt{\frac{a_j}{b_j}} w_i, & \text{if } x \in V(K_{a_i}) \text{ and } y \in V(K_{b_j}); \\ -\sqrt{\frac{a_i}{b_i}} w_i + \sqrt{\frac{a_i}{b_i}} w_j, & \text{if } x \in V(K_{b_i}) \text{ and } y \in V(K_{a_j}); \\ \sqrt{\frac{a_i a_j}{b_i b_j}} w_i w_j + \sqrt{\frac{a_i a_j}{b_i b_j}}, & \text{if } x \in V(K_{b_i}) \text{ and } y \in V(K_{b_j}). \end{cases}$$

These reduce to

$$
[A]_{x,y} = \begin{cases} 1 + w_i w_j, & \text{if } x \in V(K_{a_i}), \text{ and } y \in V(K_{a_j}); \\ \sqrt{\frac{a_j}{b_j}}(w_i - w_j), & \text{if } x \in V(K_{a_i}) \text{ and } y \in V(K_{b_j}); \\ \sqrt{\frac{a_i}{b_i}}(w_j - w_i), & \text{if } x \in V(K_{b_i}) \text{ and } y \in V(K_{a_j}); \\ \sqrt{\frac{a_i a_j}{b_i b_j}}(w_i w_j + 1), & \text{if } x \in V(K_{b_i}) \text{ and } y \in V(K_{b_j}). \end{cases} \tag{4}
$$

Since any two vertices from different blocks in X are adjacent, we need that the four values in Equation 4 are all non-zero. It suffices to choose the w_i so that $w_i w_j \neq -1$ and $w_i \neq w_j$. If we simply let $w_1, w_2, \ldots, w_k$ be any set of distinct positive real numbers, then the vectors u and v satisfy the conditions. The result follows from Corollary 6. □

In the previous result we required that all of the a_i and b_i be larger than 0. Next we shall consider the cases in which this condition is dropped. If $a_i = b_i = 0$ then $K_{a_i} \cup K_{b_i}$ has no vertices, so we do not include it in X. If $a_i = 0$, then $K_{a_i} \cup K_{b_i} = K_{b_i}$ and including it is equivalent to the join with a complete graph. In fact, if several of the $a_i = 0$, the result is equivalent to the join with one large complete graph. We shall show that we can join a complete graph on at least two vertices to the graphs in Proposition 13 without changing the multiplicity bipartition of the graph.

Proposition 14. *Let a_i and b_i be positive integers and $c \geq 2$. Suppose*

$$
X = (K_{a_1} \cup K_{b_1}) \triangledown (K_{a_2} \cup K_{b_2}) \triangledown \cdots \triangledown (K_{a_k} \cup K_{b_k}) \triangledown K_c,
$$

then X has $[n-2, 2]$ as a multiplicity bipartition.

Proof. We shall construct the vectors u and v by concatenating vectors $u(i)$ and $v(i)$ defined on the subgraphs $K_{a_i} \cup K_{b_i}$, and vectors $u(k+1)$ and $v(k+1)$ defined on the subgraph K_c. For each subgraph $K_{a_i} \cup K_{b_i}$ in X, use the vectors $u(i)$ and $v(i)$ defined in Proposition 13. Next we define the vectors $u(k+1)$ and $v(k+1)$ which are indexed by the vertices in the subgraph K_c.

If $c = 2$ simply set these two vectors to be

$$
u(k+1) = (1, 0), \quad v(k+1) = (0, 1).
$$

Since the entries of u and v are all non-zero, there will be edges between any vertex in K_2 and any vertex not in K_2.

If $c \geq 3$, define the following two vectors

$$
\begin{aligned}
u(k+1) &= \big(-1, 1, \ldots, 1, -\frac{c-2}{2} \big), \\
v(k+1) &= \big(\frac{c-2}{2}, 1, 1, \ldots, 1 \big).
\end{aligned}
$$

In Proposition 13, the values of the w_i could be any distinct positive real numbers. For the vectors u and v defined here to satisfy the condition that any vertex from K_c is adjacent to any vertex in $K_{a_i} \cup K_{b_i}$, it is necessary that the following six conditions hold:

1. $(-1)(1) + \left(\frac{c-2}{2}\right)(w_i) \neq 0$,
2. $(1)(1) + (1)(w_i) \neq 0$,
3. $(-\frac{c-2}{2})(1) + (1)(w_i) \neq 0$,
4. $(-1)\left(-\sqrt{\frac{a_i}{b_i}}w_i\right) + \left(\frac{c-2}{2}\right)\left(\sqrt{\frac{a_i}{b_i}}\right) \neq 0$,
5. $(1)\left(-\sqrt{\frac{a_i}{b_i}}w_i\right) + (1)\left(\sqrt{\frac{a_i}{b_i}}\right) \neq 0$,
6. $\left(-\frac{c-2}{2}\right)\left(-\sqrt{\frac{a_i}{b_i}}w_i\right) + (1)\left(\sqrt{\frac{a_i}{b_i}}\right) \neq 0$.

This requires simple additional restrictions on the values of w_i. Specifically, the w_i need to be positive, distinct integers that are not in the set $\{\pm 1, \pm\frac{c-2}{2}, \pm\frac{2}{c-2}\}$. Again, by Corollary 6, X has $[n-2, 2]$ as a multiplicity bipartition. □

The final case that we consider is

$$X = (K_{a_1} \cup K_{b_1})\nabla(K_{a_2} \cup K_{b_2})\nabla \cdots \nabla(K_{a_k} \cup K_{b_k})\nabla K_1.$$

This construction will be based on the construction in Proposition 14, but first we give a construction for a subgraph.

Proposition 15. *Let $X = (K_a \cup K_b)\nabla(K_c \cup K_d)\nabla K_1$, with a, b, c, d all positive integers. Then X has $[n-2, 2]$ as a multiplicity bipartition.*

Proof. Assume that $0 < a \leq b$ and $0 < c \leq d$.

For the subgraph $K_a \cup K_b$ define

$$u(1) = \big(\underbrace{1, \ldots, 1}_{a \text{ times}}, \underbrace{4, \ldots, 4}_{b \text{ times}}\big), \qquad v(1) = \big(\underbrace{2, \ldots, 2}_{a \text{ times}}, \underbrace{-2, \ldots, -2}_{b \text{ times}}\big) \tag{5}$$

and for the subgraph $K_c \cup K_d$ define

$$u(2) = \big(\underbrace{2, \ldots, 2}_{c \text{ times}}, \underbrace{9, \ldots, 9}_{d \text{ times}}\big), \qquad v(2) = \big(\underbrace{6, \ldots, 6}_{c \text{ times}}, \underbrace{-3, \ldots, -3}_{d \text{ times}}\big) \tag{6}$$

Define $u = (u(1),\, u(2),\, x)$ and $v = (v(1),\, v(2),\, y)$.

Then

$$u \cdot v = 2a - 8b + 12c - 27d + xy.$$

Since u and v must be orthogonal, it is necessary that

$$x = \frac{-2a + 8b - 12c + 27d}{y}.$$

As $0 < a \leq b$ and $0 < c \leq d$, the numerator is strictly positive; this implies that x and y have the same sign, and neither are equal to 0.

Further,

$$\|u\|^2 = 1a + 16b + 4c + 81d + x^2, \qquad \|v\|^2 = 4a + 4b + 36c + 9d + y^2.$$

The vectors u and v must have the same norm, which implies that

$$x^2 = 3a - 12b + 32c - 72d + y^2.$$

Eliminating x, this becomes

$$\left(\frac{-2a + 8b - 12c + 27d}{y}\right)^2 = 3a - 12b + 32c - 72d + y^2$$

which gives

$$0 = y^4 + (3a - 12b + 32c - 72d)y^2 - (-2a + 8b - 12c + 27d)^2. \tag{7}$$

Let $A = 3a - 12b + 32c - 72d$ and $B = -2a + 8b - 12c + 27d$. Consider the function

$$f(s) = s^2 + As - B^2$$

Then $f(2B) = 3B^2 + 2AB = B(28c - 63d)$. Since B is positive and $c \leq d$, it follows that $f(2B)$ is strictly negative. Similarly, $f(3B) = 8B^2 + 3AB = B(-7a + 28b)$, which, since $a \leq b$, is strictly positive. Thus f has at least one root r in the range $(2B, 3B)$.

Set $y = \sqrt{r}$ (since r is positive, this is possible). Then y satisfies Equation 7. We also set $x = (-2a + 8b - 12c + 27d)/y$ (this implies that both x and y are positive). At this point, we have defined the entries of u and v so that they are orthogonal and have the same norm. Finally, we need to show that $u_iu_j + v_iv_j = 0$ if and only if i and j represent a pair of non-adjacent vertices.

If one of vertices i and j is from K_a and the other from K_b, (or one from K_c and the other from K_d), then it is clear that $u_iu_j + v_iv_j = 0$. Further, for any other pair of vertices, both from $(K_a \cup K_b)\triangledown(K_c \cup K_d)$, the value of $u_iu_j + v_iv_j$ is not equal to zero.

Next consider the case where i is a vertex in K_1 and j is in $(K_a \cup K_b)\triangledown(K_c \cup K_d)$, then $u_iu_j + v_iv_j$ is one of

$$x + 2y, \quad 4x - 2y, \quad 2x + 6y, \quad 9x - 3y.$$

Since x and y are both positive, $x + 2y \neq 0$ and $2x + 6y \neq 0$. If $4x - 2y = 0$, then, since $xy = -2a + 8b - 12c + 27d$, this implies that $2(-2a + 8b - 12c + 27d) = y^2$. Similarly, if $9x - 3y = 0$, then $3(-2a + 8b - 12c + 27d) = y^2$. But, both of these are impossible, since y^2 is equal to a root of $f(s)$ strictly between $2(-2a + 8b - 12c + 27d)$ and $3(-2a + 8b - 12c + 27d)$. Thus, by Corollary 6, X has $[n - 2, 2]$ as a multiplicity bipartition. □

Proposition 16. *Let*

$$X = (K_{a_1} \cup K_{b_1})\triangledown(K_{a_2} \cup K_{b_2})\triangledown \cdots \triangledown(K_{a_k} \cup K_{b_k})\triangledown K_1,$$

where $k \geq 2$. Then X has $[n - 2, 2]$ as a multiplicity bipartition.

Proof. Use the vectors $u(1)$, $v(1)$, $u(2)$ and $v(2)$ defined in the proof of Proposition 15 for the blocks $K_{a_1} \cup K_{b_1}$ and $K_{a_2} \cup K_{b_2}$. Further, use the length-one vectors $u(k+1) = (x)$ and $v(k+1) = (y)$ that are defined in Proposition 15 for the block K_1. For all other $K_{a_i} \cup K_{b_i}$ use the vectors $u(i)$ and $v(i)$ defined in the proof of Proposition 13. Define u to be the vector formed by concatenating $u(1), \dots, u(k+1)$ and v the vectors formed by concatenating $v(1), \dots, v(k+1)$.

Provided that the w_i are distinct positive numbers not in the set

$$\{\pm 2, \pm 1/2, \pm 1/3, \pm 3, \pm x/y, \pm y/x\},$$

then $A = I - 2(uu^T + vv^T) \in S(X)$ and by Corollary 6, X has $[n-2, 2]$ as a multiplicity bipartition. $\square$

5. Proof of Main Theorem

We now have all the tools to give the exact characterization of graphs with minimal multiplicity bipartition $[n-2, 2]$.

Proof of Theorem 2. If X has $[n-2, 2]$ as a multiplicity bipartition, then by Lemma 10, it is a cograph and by Corollary 9 it has no cocliques with three vertices. Then by Lemma 12, the graph must have the form

$$X = (K_{a_1} \cup K_{b_1}) \triangledown (K_{a_2} \cup K_{b_2}) \triangledown \cdots \triangledown (K_{a_k} \cup K_{b_k})$$

(we assume that $b_i \geq a_i$ and that $(a_i, b_i) \neq (0, 0)$).

If $a_1 = 0$ and $a_2 = 0$ then

$$X = (K_0 \cup K_{b_1 + b_2}) \triangledown \cdots \triangledown (K_{a_n} \cup K_{b_n})$$

so we can assume that $a_i = 0$ for at most one i.

Assume that there is one i with $a_i = 0$. If $b_i \geq 2$, then by Proposition 14 the graph has $[n-2, 2]$ as a multiplicity bipartition. Provided that X is not the complete graph (using Corollary 8), this implies that the minimal multiplicity bipartition of X is $[n-2, 2]$.

If $a_i = 0$, $b_i = 1$ and $k \geq 3$, then by Proposition 16 the graph has $[n-2, 2]$ as a multiplicity bipartition. The fact that $k \geq 3$ ensures that the graph is not complete, so this is the minimal multiplicity bipartition.

If $a_i = 0$, $b_i = 1$ and $k = 2$, then X is equal to

$$X = (K_{a_1} \cup K_{b_1}) \triangledown K_1.$$

In this case there is a unique path with two edges from any vertex in K_{a_1} to any vertex in K_{b_1}. By Theorem 1, this implies that $q(X) \geq 3$ and that this graph has no multiplicity bipartition. Furthermore, if $a_1 = 0$, $b_1 = 1$ and $k = 1$, then X is just a single isolated vertex, in which case the spectrum contains just one eigenvalue.

Finally, if all the a_i and b_i are greater than 0, then X is not a complete graph and has minimal multiplicity bipartition $[n-2, 2]$ by Proposition 13. $\square$

We can also characterize the disconnected graphs with minimal multiplicity bipartition $[n-2,2]$.

Theorem 17. *A disconnected graph X has minimal multiplicity bipartition $[n-2,2]$ if and only if*

$$X = K_a \cup K_b \cup \overline{K_c},$$

or X is the union of a graph with minimal multiplicity bipartition $[n-2,2]$ and isolated vertices.

Proof. Assume that $X = K_a \cup K_b \cup \overline{K_c}$. By Corollary 8, there is a matrix $A \in S(K_a)$ with spectrum $\{1^1, (-1)^{a-1}\}$, and $B \in S(K_b \cup \overline{K_c})$ with spectrum $\{1^1, (-1)^{b+c-1}\}$. Then the matrix

$$C = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right).$$

is in $S(X)$ and has spectrum $\{1^2, (-1)^{a+b+c-2}\}$.

Conversely, assume that there is an $A \in S(X)$ with spectrum $\{\lambda_1^{n-2}, \lambda_2^2\}$. The spectrum for a component of X must be one of

$$\{\lambda_1^{k-2}, \lambda_2^2\}, \quad \{\lambda_1^{k-1}, \lambda_2^1\}, \quad \{\lambda_1^k\}.$$

If one component has spectrum $\{\lambda_1^{k-2}, \lambda_2^2\}$, then the spectrum of every other component must be $\{\lambda_1^{\ell_i}\}$. Thus the first component is a graph with multiplicity bipartition $[n-2,2]$ and the remaining components are isolated vertices.

If a component has spectrum $\{\lambda_1^{k-1}, \lambda_2^1\}$, then, by Corollary 8, that component is a complete graph. There can be at most two components with spectrum $\{\lambda_1^{k-1}, \lambda_2^1\}$ and the remaining components must be isolated vertices. $\square$

6. Further Work

In this paper we characterized the graphs that have minimal multiplicity bipartition $[n-1,1]$ and $[n-2,2]$. An obvious next step is to determine which graphs have $[n-k,k]$ as a multiplicity bipartition for larger values of k. This may also be a route to determining the entire family of graphs X that have $q(X) = 2$.

This problem can be generalized to any integer partition of n. Let $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ be an integer partition of n. We say that π is a *multiplicity partition* of X, if there exists an $A \in S(X)$ with spectrum $\{\xi^{\pi_1}, \xi^{\pi_2}, \dots, \xi^{\pi_k}\}$. In the case where the partition has only two parts there is a natural concept of the minimal partition. But, if π is not a bipartition, it is not clear what a minimal partition would be.

This concept relates both the minimal number of distinct eigenvalues of a graph and the maximum multiplicity for a graph. For a graph X, the value of $q(X)$ is equal to the fewest number of parts in a multiplicity partition of the graph. The maximum multiplicity is the largest size of a part in a multiplicity partition.

An interesting example is the class of even cycles. The maximum multiplicity of an eigenvalue for this graph is 2 [8] and $q(C_{2k}) = k$ [1, lemma2.7]. In [7, Thm. 3.3] it is show that for any set of numbers $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 > \ldots > \lambda_{2k-1} = \lambda_{2k}$, there is an $A \in S(C_{2k})$ with spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_{2k}\}$. This implies that $[2^k]$ is a multiplicity partition for C_{2k}. Since no multiplicity partition can have fewer parts, nor any parts of larger size, we claim that this is the minimal multiplicity partition for the even cycles.

Another interesting family of graphs to consider are the paths. Let P_n denote the path on n vertices. Every $A \in S(P_n)$ will have n distinct eigenvalues, so $[1^n]$ is a multiplicity partition of P_n. In this case, the maximum multiplicity of the path is 1 and $q(P_n) = n$, so $[1^n]$ is the only multiplicity partition for P_n. In [1, Section 7] graphs X with $q(X) = |V(X)| - 1$ are considered. These are the graph that have $[2, 1^{n-2}]$, (and possibly $[1^n]$) as a multiplicity partition, but no other multiplicity partitions.

There are also many questions related to the existence of a spectrum bipartition. For example, if a graph has λ as a multiplicity partition, does this imply that the graph also has spectrum partition μ for some other partition μ? Can these relations then be used to define an ordering on the partitions?

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