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MINIMAL BASIS FOR A VECTOR SPACE WITH AN APPLICATION TO SINGULAR GRAPHS

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Abstract

A graph is singular if its adjacency matrix is singular. In this note a parameter $\tau(G)$, termed the core-width for a singular graph G, is defined. The weight of a vector is the number of non-zero components. To determine the core-width, the bases of the nullspace of A, the adjacency matrix of G, are ordered lexicographically according to their weight; then the core-width is obtained from a minimal basis in this ordering. The core-width is unique and a minimal basis in the nullspace of the adjacency matrix of G has a unique weight sequence. We show that each term in a minimal basis is less than or equal to the corresponding term of any other basis. Corresponding to such minimal bases, certain subgraphs of G of order $\tau(G)$ are identified.

Keywords: nullity, core, core-order, weight-sequence, minimal basis, kernel eigenvector, singularity, nut graph, min-max core, core-order sequence.

1 Introduction

All the graphs we consider are **simple,** i.e. without multiple edges or loops. The **adjacency matrix** $A(G)$ or A of a graph G with ordered vertex set $\mathcal{V}(G) = \{v_1, v_2, \ldots, v_n\}$ is an $n \times n$ symmetric matrix $[a_{ij}]$ such that $a_{ij} = 1$

if v_i and v_j are adjacent and 0 otherwise. The matrix A is also represented by $(R_1^T, R_2^T, \ldots, R_n^T) = (R_1, R_2, \ldots, R_n)^T$, where R_i is the ith row vector of A corresponding to vertex v_i. Vertices which have the same neighbours are described by the same row vectors in A and are called **vertices of the same type.**

The **rank** of a graph G, denoted by $r(G)$, is the rank of its adjacency matrix. A graph is said to be **singular** if its adjacency matrix A is a singular matrix; then at least one of the eigenvalues of A is zero. There corresponds a non-zero vector v_0 such that $Av_0 = 0$. Thus the vector v_0 is an eigenvector in the nullspace of A which is denoted by $\mathcal{E}_0(A)$.

Since the adjacency matrix A is symmetric the algebraic multiplicity of any eigenvalue λ is the same as its geometric multiplicity. This common value for $\lambda = 0$ is the **nullity** of G, denoted by $\eta(G)$, and is the dimension of $\mathcal{E}_0(A)$, that is the multiplicity of the zero eigenvalue of A. It follows that the rank of G, $r(G)$ is $n(G) - \eta(G)$, where $n(G)$ is the order of the graph G.

2 The cores of a singular graph

Let V be the $n-$dimensional vector space F^n, the set of $n-$tuples $\{(\delta_1, \delta_2, \ldots, \delta_n) : \delta_i \in F, \forall i\}$, of elements of a field F.

Henceforth the adjacency matrix A of the graph G with vertices $v_1, v_2, \ldots, v_n$ will be considered to be a linear transformation acting on the vector space $V = \mathbb{R}^n$. If in V, $v = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a vector and $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_s}$ are the non-zero components in v then the subgraph of G induced by the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ is called **the subgraph of G corresponding to v.**

> **Definition 1:** *If G is a singular graph, with adjacency matrix A and v_0 is a vector in $\mathcal{E}_0(A)$, then the subgraph of G corresponding to v_0, denoted by χ_{v_0}, is said to be the* **core** *(w.r.t v_0). The number of vertices of the core is called its* **core-order**.

In [1], M. Brown *et al* defined the graph singularity $\kappa(G) = \kappa$ of a singular graph G as the least core-order in the graph. In [4], the structure of singular graphs having one core of core-order up to 5 was investigated. In this paper, the concept of a core-space and of a minimal basis, in which the core-order sequence is unique, will be discussed. The first term of the sequence is κ and its last term is defined as the core-width τ. Graphs can be classified according to τ and then more finely according to the corresponding core.

3 The Weight-Sequence of a Basis

Definition 2: *Let $v \in V$. Then the* **weight** *of v denoted by $wt(v)$ is the number of non-zero entries of v.*

Definition 3: *Let B be an ordered basis $(u_1, u_2, \ldots, u_m)$ for a subspace of dimension m. Let the weights of the vectors in B be $t_1, t_2, \ldots, t_m$ respectively. Then $\sum_{i=1}^{m} t_i$ is called the* **weight-sum** *of the basis B. If the sequence of vectors in B is such that their weights are in non-decreasing order, then the sequence of weights $t_1, t_2, \ldots, t_m$ is said to be the* **weight-sequence** *of B.*

The convention adopted will be to write **an ordered basis** such that the weights of its vectors are in **non-decreasing order.**

Definition 4: *Let W be a subspace of V of dimension $m \leq n$. If the bases of W are ordered such that their weight-sequences are in ascending lexicographic order then a basis with the minimal weight-sequence is said to be a* **minimal basis.** *All bases with weight-sequence not minimal are said to be* **non-minimal.**

The standard basis $\Big((1, 0, 0, \ldots, 0)^T, (0, 1, 0, \ldots, 0)^T, \ldots, (0, 0, 0, \ldots, 1)^T\Big)$ is a minimal basis for V. From the definition it follows that in a subspace W

(i) there exists at least one basis which is minimal,

(ii) the weight-sequence for each minimal basis for W is unique.

> **Theorem 1:** *Let W be an $m-$dimensional subspace of V. Let $B_1 = (u_1, u_2, \ldots, u_m)$ be a minimal basis and $B_2 = (w_1, w_2, \ldots, w_m)$ be another ordered basis for W with weight-sequences $t_1, t_2, \ldots, t_m$ and $s_1, s_2, \ldots, s_m$ respectively. Then $t_i \leq s_i, \forall i$.*

Proof: Since B_1 is minimal then $t_1 \leq s_1$ and $\exists k \in \{2, 3, \ldots, m+1\}$ such that $t_i \leq s_i, \forall i < k$. Suppose that $k(\leq m+1)$ is the least positive integer such that $s_k < t_k$. Since the weights in the sequences are in non-decreasing order, $s_i < t_k,\ \forall i \leq k$. Thus each $w_i,\ i \leq k$ is a non-trivial linear combination of $v_1, v_2, \ldots, v_{k-1}$; otherwise one of the w_i's can be chosen to form a basis $B = B_1 \cup \{w_i\} - v_j$, for some $j \geq k$, whose weight-sequence is lexicographically before that of B_1. But then the k linearly independent vectors $w_1, w_2, \ldots, w_k$ are spanned by the $k-1$ linearly independent vectors $v_1, v_2, \ldots, v_{k-1}$; a contradiction. Thus $t_i \leq s_i, \forall i$.

> **Corollary:** *A minimal basis for a subspace W has a minimum weight-sum.*

Given any two bases in a vector space, the vectors in one of them can always be written as a linear combination of vectors in the other. The next result then follows:

> **Lemma 1:** *Let W be a vector space of dimension m. Let $B = (w_1, w_2, \ldots, w_m)$ be a non-minimal ordered basis for W. Then by linear combinations of the w_j's a minimal basis can be obtained.*

When Lemma 1 is used to extract a minimal basis B_1 from B, then B is said to be **reduced** to B_1.

4 Applications To Graphs

> **Definition 5:** *A* **kernel eigenvector** v_0 *of a singular graph with adjacency matrix* A, *is an eigenvector in the nullspace* $\mathcal{E}_0(A)$.

Let G be a singular graph with no isolated vertices and with adjacency matrix A. It is noted that a kernel eigenvector v_0 of G, corresponds to χ_{v_0}, a core of G, the core-order of which is equal to $wt(v_0)$.

It follows that a core χ_{v_0} of G (with respect to a kernel eigenvector v_0 of G) is a vertex-induced subgraph of G which is itself singular and has a vector in its nullspace $\mathcal{E}_0\left(A(\chi)\right)$ each of whose components is non-zero. If v_0 and w_0 are two vectors in $\mathcal{E}_0(A)$ inducing cores χ_{v_0} and χ_{w_0}, then $v_0 + w_0$ is also a vector in $\mathcal{E}_0(A)$ and corresponds to another core $\chi_{v_0+w_0}$. If χ_{v_0} and χ_{w_0} have δ vertices in common and β and γ remaining vertices respectively, then $\chi_{v_0+w_0}$ has at least $\beta+\gamma$ and at most $\beta+\gamma+\delta$ vertices (the number of vertices being equal to $wt(v_0+w_0)$). Similarly if $\lambda \in \mathbb{Q}$ then λv_0 is a vector in $\mathcal{E}_0(A)$ and corresponds to a core $\chi_{(\lambda v_0)}$. Thus addition and scalar multiplication can be defined in $C_0(G)$.

> **Definition 6:** *The set of cores corresponding to the set of vectors in* $\mathcal{E}_0(A)$ *is called the* **core-space** $C_0(G)$, *which is a vector space if addition is defined by* $\chi_{v_0} + \chi_{w_0} = \chi_{(v_0+w_0)}$ *and scalar multiplication by* $\lambda\chi_{v_0} = \chi_{(\lambda v_0)}$ *such that* $\lambda\chi_{v_0} + \mu\chi_{w_0} = \chi_{(\lambda v_0+\mu w_0)}$, $\lambda, \mu \in \mathbb{Q}$ *and the zero in* $C_0(G)$ *corresponding to the zero vector in* $\mathcal{E}_0(A)$ *is the formal graph with no vertex, which we call the* **zero-graph**.

> **Definition 7:** *Let* $B = (u_1, u_2, \ldots, u_\eta)$ *be a basis for* $\mathcal{E}_0(A)$ *where* A *is the adjacency matrix of a singular graph* G. *The sequence of cores* $B' = (\chi_{u_1},\ \chi_{u_2},\ \ldots, \chi_{u_\eta})$ *is called a* **core basis** *for* G.

Definition 8: *Let G be a singular graph with adjacency matrix A. A core basis, B', for the core space $C_0(G)$ is called* **minimal** *if the corresponding basis of kernel eigenvectors $B \in \mathcal{E}_0(A)$, is minimal. Otherwise it is* **non-minimal** *and corresponds to a non-minimal basis of kernel eigenvectors.*

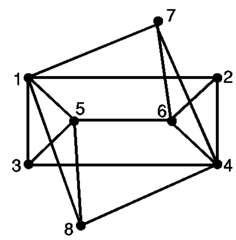


Figure 1:

The graph Y in Fig. 1 has an adjacency matrix whose nullspace $\mathcal{E}_0(Y)$ is of dimension 4. A vector $v' \in \mathcal{E}_0(Y)$ is $(1, 2, 0, 0, -1, -1, -1, 0)^T$ corresponding to the core $< v_1, v_2, v_5, v_6, v_7 >$ in $C_0(G)$ of core-order 5.

If $\eta(G) \geq 1$, then the minimum core-order is the **singularity** of G denoted by $\kappa(G)$ [1]. Therefore if among the kernel eigenvectors of G, v_0 is one with a minimal number of non-zero entries then $\kappa(G) = wt(v_0)$. For graph Y of Fig. 1, $\kappa(G) = 2$ corresponding to cores of order 2 one of which is $< v_2, v_7 >$. It had been shown in [4] that given a singular graph G with a kernel eigenvector v_0 there exist a subgraph of G which is one of a set of particular graphs of nullity one (called minimal configurations) that depend on v_0 and on its corresponding core of G. These minimal configurations have been extensively dealt with in [4].

Definition 9: *Let G be a singular graph with adjacency matrix A. The largest weight in the weight-sequence of a minimal basis for $\mathcal{E}_0(A)$, is called the* **core-width** *of G and is denoted by $\tau(G)$.*

The positive integer $\tau(G)$ is equal to the maximum order of a core in a minimal basis for the core space $C_0(G)$. That it is well defined follows from Theorem 1. In related work [3,5], it is shown that $\tau(G)$ is bounded above by $r(G)+1$. For r$(G) \geq 6$, the upperbound $r(G)+1$ could be attained. If furthermore $\kappa(G) = \tau(G) = r(G) + 1 = n$ then G is a special minimal configuration called a **nut graph**, that is a singular graph whose core is the graph itself. One such example is the graph in Fig. 2.

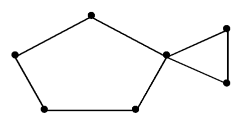


Figure 2:

Definition 10: *Let G be a graph with adjacency matrix A and core space $C_0(G)$. A core of largest order in a minimal basis for $C_0(G)$ is called* **a min-max core**.

It follows that the core-order of a min-max core is equal to the **core-width** of G, $\tau(G)$. A set Q of cores, of maximum core-order, in each basis for $C_0(G)$ contains min-max cores. These are the cores of minimum order in Q.

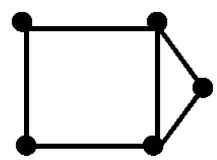
Lemma 2: *If G is a singular graph, $\tau(G) \leq r(G) + 1$.*

Proof: Let $A = (R_1, R_2, \ldots, R_r, R_{r+1}, R_{r+2}, \ldots, R_n)^T$, where $r = r(G)$ and

$R_1, R_2, \ldots, R_r$, are the linearly independent rows [2] of A. Let $B_1 = (\chi_{u_1},\ \chi_{u_2},\ \ldots, \chi_{u_\eta})$ be a minimal basis for $C_0(G)$. The eigenvector $u_\eta = (\alpha_1, \alpha_2, \ldots, \alpha_r, \beta, 0, ..)$ where $\beta \neq 0$ and not all the α_i are zero. Thus $\tau(G) = wt(u_n) \leq r + 1$.

For graph Y of Fig. 1, $\tau(Y) = 4$ corresponding to cores of order 4 one of which is $< v_1, v_2, v_3, v_4 >$. This core is therefore a min-max core of Y.

While all min-max cores have the same order not each core of order $\tau(G)$ is a min-max core. The graph Y in Fig. 1 has order 8 and nullity 4. The kernel eigenvectors $(1,1,-1,-1,0,0,0,0)^T$, $(1,1,0,0,-1,-1,0,0)^T$, $(0,1,0,0,0,0,-1,0)^T$, $(0,0,1,0,0,0,0,-1)^T$, form a minimal basis for $\mathcal{E}_0(A)$, (relative to the the labelling in the diagram). However the kernel eigenvector $(0,1,1,0,0,0,-1,-1)^T$ corresponds to a core of order $\tau(Y)$ which is not a min-max core since a basis of cores having $< v_2, v_3, v_7, v_8 >$ as a member can be reduced (in the sense of Lemma 1) by replacing it with $< v_2, v_7 >$ or $< v_3, v_8 >$.



Λ

Figure 3:

The graph Λ of Fig. 3 has nullity one and so $\kappa(\Lambda) = \tau(\Lambda) = 4$.

Definition 11: *Let* $B' = (\chi_{u_1},\ \chi_{u_2},\ \ldots, \chi_{u_\eta})$ *be a* **core basis** *for* G. *If* t_i *is the core-order of* χ_{u_i} *then the sequence* $t_1, t_2, \ldots, t_\eta$,*where* $t_i \le t_{i+1},\ 1 \le i \le \eta - 1$, *is said to be the* **core-order sequence** *of* B' *and* $\sum_1^\eta t_i$ *is said to be its* **weight-sum**.

The results in the previous sections then give:

Theorem 3: *All minimal bases of cores for the core-space of a singular graph of finite order* n, *have a unique core-order sequence and a minimum weight-sum amongst all core-sequences.*

The adjacency matrix of a singular graph G without isolated vertices, cannot have a kernel eigenvector with weight one. Thus $\kappa \geq 2$ and equality follows if the graph has vertices of the same type. From this and the definition of core-width the next result follows:

> **Lemma 3:** *Let G be a singular graph G without isolated vertices. If $s_1, s_2, \ldots, s_\eta$ is the core-order sequence of a minimal basis for $C_0(G)$, then $s_1 = \kappa \geq 2$ and $s_\eta = \tau(G) \leq r(G) + 1$.*

It is observed that the uniqueness of the weight-sequence of a minimal basis for the nullspace of $A(G)$ can be used to classify singular graphs. If a minimal basis for $\mathcal{E}_0(A)$ is known, then the corresponding minimal basis B_1 for the core-space can be determined. Furthermore, a graph with core-width $\tau(G)$ and corresponding min-max core χ_u has minimal configurations, as subgraphs, whose cores are of order $\tau(G)$ or less. Each core in B_1 corresponds to a minimal configuration from which G can be "grown" [4].

5 Reference

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