Polynomial Reconstruction Old and New Techniques *

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Abstract

The Polynomial Reconstruction Problem (PRP) asks whether for a graph G of order at least three, the characteristic polynomial can be reconstructed from the p-deck $\mathcal{PD}(G)$ of characteristic polynomials of the one-vertex-deleted subgraphs. The problem is still open in general but has been proved for certain classes of graphs. We discuss the tools and techniques most commonly used and survey the main positive results obtained so far, pointing out the classes of graphs for which we know that the PRP has a positive resolution.

Keywords: *Ulam's Reconstruction Conjecture, adjacency matrix, polynomial reconstructible, characteristic polynomial, interlacing, eigenvalues.*

1 Introduction

The **adjacency matrix** $\mathbf{A}(H)$ (or $\mathbf{A}$) of a graph H of order $n(H) = n$, having vertex set $\mathcal{V}(H) = \{v_1, v_2, \ldots, v_n\}$, is the $n \times n$ symmetric matrix $[a_{ij}]$, such

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that $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. The adjacency matrix describes H completely (up to isomorphism).
The characteristic polynomial of $\mathbf{A}(H)$ is denoted by $\phi(H, \lambda)$ $(= \phi(H))$ and

$$\phi(H, \lambda) = \mathrm{Det}(\lambda \mathbf{I} - \mathbf{A}) = \sum_{i=0}^{n} a_i \lambda^i = \prod_{i=1}^{n} (\lambda - \lambda_i). \tag{1}$$

The values $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the eigenvalues of H and form the spectrum, $Sp(H)$, of H [1, 2, 4]. If $\lambda_i = 0$ for some i, then $\mathbf{A}$ is singular and H is said to be a singular graph. Otherwise H is non-singular.

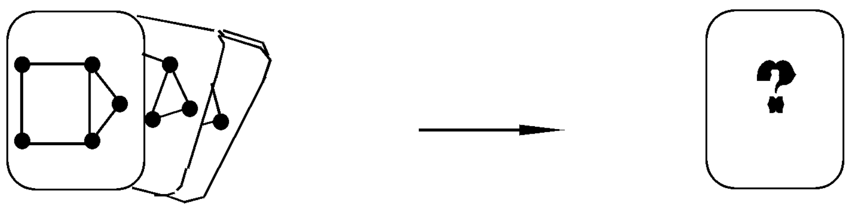

Figure 1: Ulam's RC

Ulam's Reconstruction Conjecture (RC) [9, 16] claims that a graph H, of order at least 3, can be recovered from the collection $\{H - v\}$ of the one-vertex deleted subgraphs of H (See Figure 1). A variation of the RC is the Polynomial Reconstruction Problem (PRP) which asks whether it is possible to recover the characteristic polynomial of a graph H of order at least three from the **p-deck**, $\mathcal{PD}(H)$, of H, consisting of the characteristic polynomials of the one-vertex-deleted subgraphs (with multiplicities). For each $v_i \in \mathcal{V}(H)$, there is a card in the p-deck showing $\phi(H - v_i)$ or equivalently the spectrum of $H - v_i$ (See Figure 2). It is the purpose of this article to discuss the tools and techniques usually employed for polynomial reconstruction. We survey the main positive results obtained so far, pointing out the classes of graphs for which the PRP is still open.

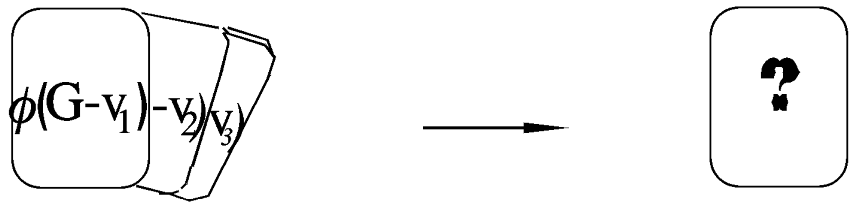

Figure 2: The PRP

In Section 2, we discuss the results proved so far relating to the PRP. A counter example pair $(H,\ G)$ to the PRP would show that the PRP has a negative result. Two such graphs H and G would have the same p-deck but a different spectrum. After recalling which information can be immediately derived from $\mathcal{PD}(H)$ in Section 3, we highlight the main tools usually utilized in polynomial reconstruction. Using these methods, the PRP has a positive resolution for regular graphs. Moreover, disconnected graphs are weakly polynomial reconstructible. The Interlacing Theorem is a very powerful tool in resolving the PRP for classes of graphs such as windmills and singular graphs with a terminal vertex, since these have repeated eigenvalues in a card of their p-deck. We proceed to derive, in Section 4, certain properties that a counter example pair $(H,\ G)$ to the PRP must have. There are classes of graphs, such as trees, which do not pair up with any graphs to give a counter example and hence must be polynomial reconstructible. In section 5, we describe new techniques that prove useful for graphs with terminal vertices. We conclude by pointing out certain classes of graphs for which the PRP is not yet resolved.

2 The PRP

The PRP, first posed by D. M. Cvetković in 1973 and later considered by I. Gutman and D. M. Cvetković in [8], asks whether it is possible to reconstruct the characteristic polynomial of a graph H, of order at least three, from the **p-deck**, $\mathcal{PD}(H)$, of H. The restriction on the order is necessary, in view of the fact that the pair of graphs on two vertices, namely, K_2 with one edge and its complement $\overline{K_2}$, form a counter example. All graphs of order at most ten and many other graphs of higher order have been shown to be polynomial reconstructible. However, the general feeling is that a counter example that answers

the PRP negatively will eventually be found.
The PRP has been shown to have a positive result for certain classes of graphs but is still open in general [5, 6, 10]. It is true for regular graphs as we show in section 3. S. Simić proved it true, in [14], for connected graphs with the smallest eigenvalue of the one-vertex-deleted subgraphs bounded below by -2. Also D. M. Cvetković and M. Lepović showed that trees are polynomial reconstructible in [7]. In [13] a disconnected graph in which one component is a tree, is shown to have no partner graph G with which it can pair up to give a counter example to the PRP. Also, in [12], graphs with more than $\lfloor \frac{n}{3} \rfloor$ pendant edges are shown to be polynomial reconstructible.
W.T.Tutte proved that the spectrum of a graph is reconstructible from the collection (deck) of its one-vertex-deleted subgraphs [15]. Thus non-isomorphic graphs, on at least 3 vertices, with the same deck have the same characteristic polynomial. This means that the PRP is still open for non-isomorphic graphs with distinct decks but the same p-decks (See Figure 3). Were the PRP to have a positive result for all graphs, then non-isomorphic graphs with non-identical decks but with the same p-decks, would be cospectral. However, a counter example pair to the PRP, would not correspond to a counter example pair to the RC since distinct characteristic polynomials must stem from different decks. Moreover, a pair of graphs, which would form a counter example to the RC, would be cospectral.
The attempt to prove the truth of the PRP for a class $\mathcal{C}$ of graphs is usually approached in two stages. Firstly, we establish whether some of the properties of a graph H, derived from its p-deck $\mathcal{PD}(H)$, are necessary and sufficient for H to lie in $\mathcal{C}$. This first stage is called the **Recognition** Stage. In the second, the **Reconstruction** Stage, we use information from the p-deck to recover the characteristic polynomial $\phi(H)$. If both stages are performed successfully, then the graphs in class $\mathcal{C}$ are said to be polynomial reconstructible and the PRP is said to be true for $\mathcal{C}$. If only the second stage is established for all graphs in $\mathcal{C}$, then the graphs in $\mathcal{C}$ are said to be weakly reconstructible.
A different technique was first used fruitfully by D. Cvetković and M. Lepović, in [7], to prove that trees are polynomial reconstructible. A graph H in a particular class is supposed to be **not** polynomial reconstructible. Then there exists a graph G such that $(H,\ G)$ is a **counter example** pair to the PRP. Thus G has the same p-deck as H but a different spectrum. This approach reveals the properties which G must have and rules out certain classes $\mathcal{C}'$ of graphs which do not allow the existence of G. The PRP would then be proved

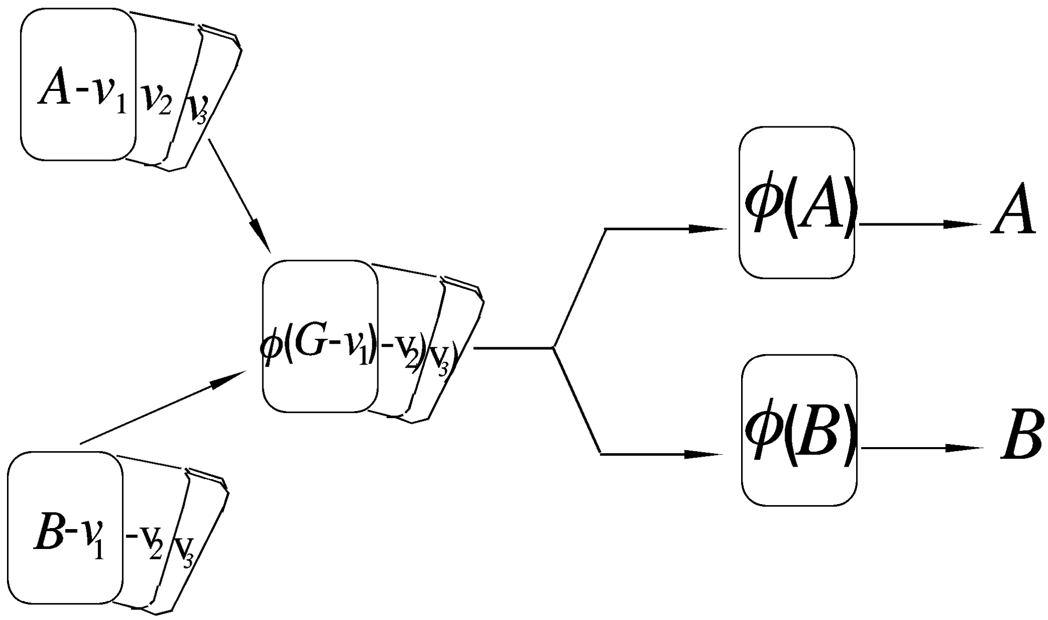

Figure 3: A counter example to the PRP

true for such classes $\mathcal{C}'$ and the existence of polynomial reconstruction would be established for $\mathcal{C}'$ without demonstrating the actual reconstruction.

3 Properties Derived from the p-deck

Remark 3.1 It is well known that if we express the characteristic polynomial $\phi(H,\lambda) = \mathrm{Det}(\lambda\mathbf{I} - \mathbf{A})$ as

$$\phi(H,\lambda) = \sum_{i=0}^{n} a_i\lambda^i = \prod_{i=1}^{n}(\lambda - \lambda_i),$$

then the number of edges of H is $-a_{n-2}$ and $a_0 = \mathrm{Det}(-\mathbf{A}) = (-1)^n \prod_{i=1}^{n} \lambda_i$. A powerful tool, that may initially give the impression that polynomial reconstruction is not as hard a problem as it is in fact proving to be, is the result $\phi'(H,\lambda) = \sum_{i=1}^{n} \phi(H-w_i,\lambda)$ [1, 3]. The following two lemmas follow immediately.

Lemma 3.2 *From the p-deck of H all the terms of the characteristic polynomial can be determined except for the constant term a_0.*

Proof: The p-deck, $\mathcal{PD}(H)$, is the collection $\{\phi(H-w_i,\lambda) : w_i \in \mathcal{V}(H)\}$. Since

$$\phi'(H,\lambda) = \sum_{i=1}^{n} \phi(H-w_i,\lambda)$$

[1, 3], then by integrating, $\phi(H)$ is determined, save for the constant term a_0 which is Det($-\mathbf{A}(H)$). ■

Remark 3.3 The degree of a vertex v is the number of edges incident to v in G. The degree sequence, dg, is the sequence of the degrees of the vertices of G for a particular labelling of G. A graph is not determined by dg.

Lemma 3.4 *The degree sequence, dg, of H is determined from the p-deck, $\mathcal{PD}(H)$.*

Proof: The degree sequence dg of H is $\{d_i\}$ where d_i is the degree of the ith vertex v_i of H. The integer d_i is the number of edges lost when v_i is deleted and works out as the difference in the coefficients of $-\lambda^{n-2}$ in $\phi(H)$ (determined as in the proof of Lemma 3.2) and of $-\lambda^{n-3}$ in $\phi(H-w_i)$ (known from the p-deck). ■

3.1 Boundary Conditions

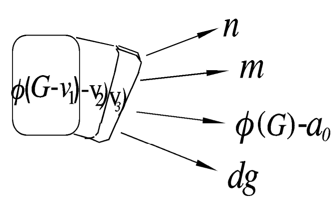

Figure 4: Data derived from the p-deck.

Remark 3.5 Though a rich source of information, the p-deck of a graph H fails to give a direct way of determining Det($\mathbf{A}(H)$) from which the constant term a_0 of the characteristic polynomial of H is derived. To recover a_0, other techniqes need to be used to yield proper boundary conditions.

Lemma 3.6 *If an eigenvalue λ_0 of G is known, then G is polynomial reconstructible.*

Proof: : Since $\phi(G, \lambda_0) = 0$, then $a_0(G)$ can be determined. ∎

Lemma 3.7 *If the graph G is regular, then the number of non-zero entries in each row of A is the degree ρ.*

Theorem 3.8 *If the graph G is regular, then G is polynomial reconstructible.*

Proof: If G is regular, then $A(1, 1, ...1)^t = \rho(1, 1, ...1)^t$. Thus ρ is an eigenvalue. Hence $a_0(G)$ is uniquely determined. ∎

Theorem 3.9 *Disconnected graphs are weakly polynomial reconstructible.*

Proof: If the graph G is known to be disconnected, then the largest eigenvalue that appears in the p-deck of G is also an eigenvalue of G. Thus $a_0(G)$ is uniquely determined. ∎

3.2 Interlacing

Remark 3.10 The following theorem has proved to be a very convenient tool, not only to determine eigenvalues of G from its p-deck but also to show that certain classes of graphs cannot be counter examples to the PRP.

The Interlacing Theorem: *If G is an n-vertex graph with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $G - v$ is a one-vertex–deleted subgraph of G with eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n-1}$, then $\lambda_i \leq \mu_i \leq \lambda_{i+1}$, $i = 1, 2, \ldots, n-1$.*

$$\begin{array}{cccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & & \lambda_{n-1} & & \lambda_n \\
\bullet & & \bullet & & \cdots & & & \bullet & & \bullet \\
 & \bullet & & \bullet & \cdots & \bullet & & & \bullet & \\
 & \mu_1 & & \mu_2 & \cdots & \mu_{n-2} & & & \mu_{n-1} &
\end{array}$$

Corollary 3.11 *Repeated eigenvalues in the p-deck $\mathcal{PD}(G)$ are enough to reconstruct G.*

Proof: If a vertex-deleted subgraph of G has two eigenvalues μ_i, μ_{i+1} equal, then, by interlacing, the eigenvalue λ_{i+1} of G is equal to μ_i. Hence $a_0(G)$ is uniquely determined. ∎

Remark 3.12 A **terminal vertex** refers to a vertex of degree one. A pendant edge $w_G v_G$ of a graph $G(\neq K_2)$ has a terminal vertex v_G and a **next-to-terminal vertex** w_G of degree at least 2. as seen in Figure 5.

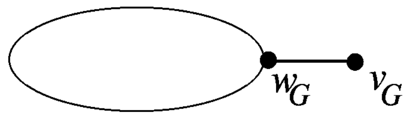

Figure 5: A graph G with a pendant edge.

Lemma 3.13 *If the graph H has a pendant edge $w_H v_H$, with terminal vertex v_H, then*

$$\phi(H) = \lambda\phi(H - v_H) - \phi(H - v_H - w_H), \quad (2)$$

Theorem 3.14 *If H is a singular graph with at least one pendant edge wv, then H is polynomial reconstructible.*

Proof: The nullity of H is the same as that of $H-w-v$. Thus the card $H-w$, which shows the union of the spectra of $H-w-v$ and of K_1, has repeated zero eigenvalues. Thus, by interlacing, we can deduce that H is polynomial reconstructible. ■

Figure 6: A Windmill: $pS.K_r$

Definition 3.15 *A* **windmill***, is the graph obtained by coalescing a complete graph K_r, $r \geq 2$, with disjoint graphs S_1, S_2, $\ldots, S_p$ at p distinct vertices of K_r, $0 \leq p \leq r$, so that these vertices are cut vertices of the graph. The subgraphs, S_1, S_2, $\ldots, S_p$, are said to be the sails and K_r, the central clique of the windmill* (See Figure 6).

Theorem 3.16 *Windmills with more than two identical sails are polynomial reconstructible.*

Proof: In [11], a windmill was shown to have repeated eigenvalues in its p-deck depending on the repeated sail. ■

4 Properties of a Counter Example

Remark 4.1 New techniques need to be developed to study certain classes of graphs for which the data acquired from the p-deck provides very little information. An approach that is yielding fruitful results is to investigate the properties of a counter example pair $(H,\ G)$ such that $\phi(G)(\neq \phi(H))$ is a reconstruction from $\mathcal{PD}(H)$. This method has been used successfully in [7], [12] and [13].

Lemma 4.2 *The characteristic polynomials of H and of G differ only in the constant term a_0.*

Proof: The result follows from Lemma 3.2. ■

Remark 4.3 It is clear that H and G are mutual partners in a counter example pair. Thus we write $a_0(H) = a_0(G) + \Delta a_0, \quad \Delta a_0 \in \mathbb{Z} - \{0\}$. *(See Figure 7.)*

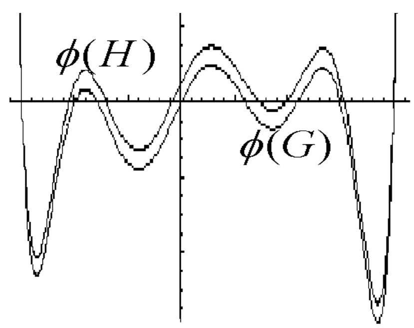

Figure 7: The polynomials $\phi(G)$ and $\phi(H)$.

Lemma 4.4 *G and H have no eigenvalues in common.*

Proof: Suppose that λ_0 is an eigenvalue found in each of the spectra of H and of G. Then $\phi(H, \lambda_0) = 0$ and $\phi(G, \lambda_0) = 0$. But by Lemma 3.2, $\phi(H, \lambda) - a_0(H) = \phi(G, \lambda) - a_0(G)$, for all values of λ. Thus $a_0(H) = a_0(G)$ and therefore $\phi(H, \lambda) = \phi(G, \lambda)$. By Lemma 4.2, this contradicts the properties of G as a counter example partner of H. ■

Lemma 4.5 *No polynomial in the p-deck $\mathcal{PD}(H)$ has repeated eigenvalues.*

Proof: If for some $w_i \in H$, $H - w_i$ has the eigenvalue λ_0 repeated, then by the interlacing theorem, it follows that each of the graphs H and G have the eigenvalue λ_0. Thus by Lemma 4.4, this contradicts the existence of G. ∎

Lemma 4.6 *The two graphs H and G are not both disconnected.*

Proof: The maximum eigenvalue of a disconnected graph is the maximum eigenvalue that appears in the deck. We recall that G and H have the same deck. Thus if both graphs are disconnected, their maximum eigenvalue is the same. This is not allowed by Lemma 4.4. ∎

Lemma 4.7 *If $n(H) = n$, each spectrum of the graphs H and G has n real eigenvalues. Also $\phi(H)$ has $\lceil \frac{n-1}{2} \rceil$ minimum values and $\lfloor \frac{n-1}{2} \rfloor$ maximum values.*

Proof: Since the adjacency matrix $\mathbf{A}$ of a graph is real and symmetric, the n eigenvalues of $\mathbf{A}$ are real. For large values of λ, $\phi(H) = O(\lambda^n)$. Thus in the range between the two larger eigenvalues, $\phi(G)$ has a minimum value. The result now follows since polynomials are continuous and by Lemma 4.5, a polynomial from the deck has only simple roots. ∎

Remark 4.8 By examining the graphs of $\phi(G)$ and $\phi(H)$ against λ, the following result follows immediately (See Figure 7).

Theorem 4.9 *Let $(G,\ H)$ be a counter example pair to the PRP and let $a_0(H) > a_0(G)$. If the eigenvalues of G are $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ and $\ell_1, \ell_2, \ldots, \ell_n$ are the eigenvalues of H, then $\Lambda_1 > \ell_1 \geq \ell_2 > \Lambda_2 \geq \Lambda_3 > \ell_3 \geq \ell_4 > \Lambda_4 \ldots \geq \Lambda_{n-1} > \ell_{n-1} \geq \ell_n > \Lambda_n$. If $\phi(G)$ has a minimum value between two successive eigenvalues of G, then H has one double eigenvalue or two simple eigenvalues in this range. There are no eigenvalues of H between every pair of successive eigenvalues of G in which range $\phi(G)$ has a maximum value.*

Theorem 4.10 *If a counter example pair $(H,\ G)$ to the PRP exists, then $|minmax\phi(G)\text{-}maxmin\phi(G)| \geq \Delta a_0 > 1$.*

Proof: The required condition is necessary for both G and H to have n real eigenvalues. ∎

Theorem 4.11 *Let ϕ denote the integral of the sum of the polynomials in the p-deck of a graph H. Let ϕ_b be the largest value of the local minima of ϕ and ϕ_t be the least value of the local maxima of ϕ. If $\lfloor \phi_t \rfloor = \lceil \phi_b \rceil$, then H is polynomial reconstructible.*

Proof: The constant term of the characteristic polynomial is an integer. The required condition ensures that $a_0(H)$ takes only one value. ■

Remark 4.12 In all examples tried so far the condition in Theorem 4.11 was found to hold. This condition is an easy criterion in the search for a counter example to the PRP.

Theorem 4.13 (Cvetković) *If $(G,\ H)$ is a counter example pair to the PRP, and H is disconnected, then H has two components only.*

Remark 4.14 The proofs of the two theorems that follow make use of the counter example technique. They show that interlacing would be violated so that counter examples cannot exist for particular classes of graphs , thus proving that these classes are polynomial reconstructible.

Theorem 4.15 (Cvetković and Lepović) *Trees are polynomial reconstructible.*

Theorem 4.16 (Sciriha and Formosa) *If a component of a disconnected graph G is a tree, then G is polynomial reconstructible.*

5 Graphs with Pendant Edges

Remark 5.1 Although the counter example method is used in [12] to study the polynomial reconstruction of a graph with at least one terminal vertex, the techniques employed have little in common with those used earlier. The relation between the geometrical structure of the cospectral one-vertex-deleted subgraphs $H - u_H$, $G - u_G$ and the particular properties of the singular cards in the p-deck are taken advantage of, to yield powerful results for this class of graphs.

Disconnected graphs with more than two components or with components of different order are polynomial reconstructible. Besides there are no counter examples to the PRP for graphs of order 10 or less [7]. Hence graphs with K_2 as a component are polynomial reconstructible.

In this section, the graph H refers to any graph with at least one terminal vertex which does not have K_2 as a component. Thus if wv is a pendant edge and v the terminal vertex, then the degree of w is at least two. We suppose that H is not polynomial reconstructible and that as a result there exists a counter example pair $(G,\ H)$ to the polynomial reconstruction problem PRP.

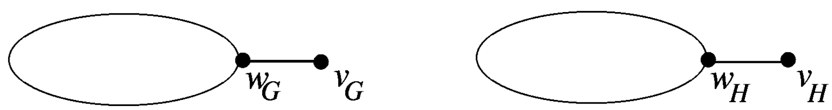

Figure 8: A Counter Example pair $(G,\ H)$.

Lemma 5.2 *For a card containing the characteristic polynomial $\phi(H-v_H)$ in the p-deck of H corresponding to a terminal vertex v_H, there exists a terminal vertex v_G in G such that $\phi(G-v_G)=\phi(H-v_H)$. (See Figure 8.)*

Proof: Since the p-decks of H and G are the same, the two graphs have the same degree sequence. Thus there is a one-to-one matching σ between the vertices in $\mathcal{V}(G)$ and those in $\mathcal{V}(H)$ such that when $\sigma(u_G)=u_H$, then $\phi(G-u_G)=\phi(H-u_H)$. From the proofs of Lemmas 3.4 and 4.2, it follows that a necessary condition is that corresponding vertices, u_G and u_H, have the same degree. This matching need not be unique. ■

Remark 5.3 We denote by w_Hv_H a pendant edge of H and by w_Gv_G the pendant edge of G such that $\phi(G-v_G)=\phi(H-v_H)$, with $v_H,\ v_G$ being terminal vertices. We refer to w_H and w_G as **next-to-terminal** (NTT) vertices.

Lemma 5.4 *If the graph H has a pendant edge w_Hv_H, with terminal vertex v_H, then $-a_0(H)$ is the coefficient of λ in $\phi(H-w_H)$.*

Proof:
This follows by comparing the constant terms in

$$\phi(H)=\lambda\phi(H-v_H)-\phi(H-v_H-w_H),$$

bearing in mind that $\phi(H-v_H-w_H)=\frac{\phi(H-w_H)}{\lambda}$[1]. ■

5.1 The Singular Cards in $\mathcal{PD}(H)$

Theorem 5.5 *If* $G,\ H$ *are non-singular, then the graphs* $G-w_G$ *and* $H-w_H$ *are singular and their characteristic polynomials differ only in the* λ *term.*

Proof: Each of the graphs $G-w_G$ and $H-w_H$ has an isolated vertex and so has nullity one. The removal of a pendant edge and its vertices from a graph leaves the nullity unchanged. Thus each of the graphs $H-v_H-w_H$ and $G-v_G-w_G$ is non-singular.
By applying equation (2) to graphs G and H in turn, and bearing in mind that $\phi(H-v_H)=\phi(G-v_G)$ as well as Remark 4.3, we deduce that

$$\phi(H)-\phi(G)=\phi(G-v_G-w_G)-\phi(H-v_H-w_H)=\Delta a_0,\ \Delta a_0\in\mathbb{Z}-\{0\}. \quad (3)$$

We now use

$$\phi(G-v_G-w_G)=\frac{\phi(G-w_G)}{\lambda} \quad (4)$$

and a similar relation for H. Thus

$$\phi(G-w_G)=\phi(H-w_H)+\lambda(\Delta a_0),\ \Delta a_0\in\mathbb{Z}-\{0\} \quad (5)$$

as required. ∎

Figure 9: The vertices x and y in $(G,\ H)$.

Theorem 5.6 *If the graph* H *has a pendant edge* w_Hv_H*, with terminal vertex* v_H*, and* $(G,\ H)$ *is a counter example pair to the PRP, then there exists a vertex* x *of* H *of the same degree as* w_H *such that* $H-x$ *and* $H-w_H$ *are singular and* $\phi(H-x)=\phi(H-w_H)+\lambda(\Delta a_0),\ \Delta a_0\in\mathbb{Z}-\{0\}$*. (See Figure 9.)*

Proof: Since G and H have the same p-deck, and $\phi(G-w_G)\neq\phi(H-w_H)$ by Theorem 5.5, there exists a vertex x of H such that $\phi(H-x)=\phi(G-w_G)$. Thus both $H-x$ and $H-w_H$ are singular. Substitution in equation (5) yields the result. ∎

Corollary 5.7 *At least two cards in the p-deck of H have nullity one.*

Proof: From the proof of Theorem 5.6, it follows that the cards for $H - w_H$ and $H - x$ have a zero eigenvalue. ■

Lemma 5.8 *Let y be a vertex of G and x a vertex of H such that $\phi(G - w_G) = \phi(H - x)$ and $\phi(H - w_H) = \phi(G - y)$. Then the four vertices $x,\ y,\ w_G, w_H$ have the same degree.*

Proof: We recall that the number of edges of a graph is the negative of the second non-zero coefficient of the characteristic polynomial. This coefficient is the same for G and H and also for $G - w_G,\ H - x,\ H - w_H$ and $G - y$. ■

Remark 5.9 Lemma 5.8 supplies an alternative proof that a graph with K_2 as a component is polynomial reconstructible. If one of the NTT vertices w_G or w_H is of degree one then both are of degree one, so that both G and H are disconnected, a contradiction by Lemma 4.6. In [12] we find sufficient conditions for graphs with at least one pendant edge to be polynomial reconstructible.

Theorem 5.10 *Vertices x and y, defined in Lemma 5.8, do not have neighbours of degree one.*

Theorem 5.11 *Let $k > 0$ and H be a graph, with k pendant edges, which is* **not** *polynomial reconstructible. Then there exist k singular cards $\{\phi(H - x_H^i) : 1 \leq i \leq k\}$ where $x_H^1, x_H^2, \ldots, x_H^k$ are vertices of degree at least two, with no neighbour of degree one. Furthermore there exist at least another k singular cards $\phi(H - w_1), \phi(H - w_2), ...$ where $w_1, w_2, ...$ are the NTT vertices (also of degree at least two).*

Proof: Since the p-decks of H and G are the same, there is a one-to-one matching σ between the vertices in $\mathcal{V}(G)$ and those in $\mathcal{V}(H)$ such that when a terminal vertex v_G^i corresponds to v_H^i under σ, then $\phi(G - v_G^i) = \phi(H - v_H^i)$ for $1 \leq i \leq k$. By Theorem 5.5, cards corresponding to the next-to-terminal vertices w_H^i and w_G^i are different and by Theorem 5.10, match with cards of vertices with no neighbour of degree one. Thus H has k terminal vertices v_H^i, k next-to-terminal vertices w_H^i corresponding to singular cards $H - w_H^i$ and k vertices x^i of degree at least two with no neighbour of degree one, also corresponding to singular cards $H - x^i$. The same holds for G. ■

Remark 5.12 The result of Lemma 5.11 leads to the polynomial reconstruction of a number of subclasses of the class of graphs with pendant edges. The following theorem establishes conditions, based on criteria that are easily recognisable from the p-deck, which are separately sufficient for polynomial reconstruction.

Theorem 5.13 *Let H be a graph with k pendant edges where $k > 0$. Each of the following conditions is separately sufficient for the characteristic polynomial of H to be recovered from the p-deck of H.*
(i) the number of singular cards in the p-deck is less than $2k$;
(ii) the number of vertices of degree at least 2 is less than $2k$.

5.2 Bipartite Graphs with Pendant Edges.

Theorem 5.14 *If H is a graph with at least one pendant edge wv, then H and G in a counter example pair to the PRP are non-singular.*

Proof: This follows from Theorem 3.14. ■

Corollary 5.15 *If H is a bipartite graph of odd order with at least one pendant edge, then H is polynomial reconstructible.*

Proof: The result follows since a bipartite graph with an odd number of vertices is singular. ■

Remark 5.16 A bipartite graph of even order n with at least one pendant edge is reconstructible if it is singular since there exists a card with repeated zero eigenvalues in its p-deck. What remains to be studied is the class of non-singular bipartite graphs of even order with at least one pendant edge. Let H be a graph in this class.
If the number of closed walks of size n can be recovered from the p-deck, then a_0 can be determined by Newton's recursive formulae and H would then be polynomial reconstructible. An alternative approach is to prove that the partner G, which together with H forms a counter example to the PRP, cannot exist. This problem is still open in general.

Remark 5.17 In searching for a counter example to the PRP, the criteria in the following two theorems should prove useful.

Theorem 5.18 *If H has $k > 0$ pendant edges, and is* **not** *polynomial reconstructible, then there exist k pairs of singular cards in $\mathcal{PD}(H)$ corresponding to $2k$ vertices with the vertices in a pair being of the same degree.*

Theorem 5.19 *Let H be a graph with $k > 0$ pendant edges. The following condition is necessary for H* **not** *to be polynomial reconstructible:*
There exists $b \in \mathbb{Z} - \{0\}$ and k pairs of singular cards in $\mathcal{PD}(H)$ with the polynomials in each pair differing by $b\lambda$.

Remark 5.20 The value of b in Theorem 5.18 is Δa_0, the difference in the constant terms of $\phi(H)$ and $\phi(G)$. Thus it is the same for all pairs
$\{(x^i, w^i), 1 \leq i \leq k\}$.

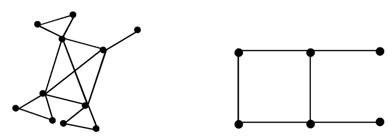

Figure 10: Polynomial Reconstructible Graphs.

Remark 5.21 We can deduce that the graphs in Figure 10, which have at least one vertex of degree one, are polynomial reconstructible by Theorem 5.18, since the first graph has only one singular one-vertex-deleted subgraph whereas in the second graph, only NTT vertices have the same degree.
We have considered various subclasses of the class of graphs with at least one terminal vertex. We have seen that a close look at the singular cards in the p-deck of a graph H with at least one terminal vertex, can give conclusive evidence that H is polynomial reconstructible.

6 Conclusion

Remark 6.1 By interlacing, we can show that polynomial reconstruction is true for a substantial number of graphs if we establish a sufficient condition for the multiplicity of an eigenvalue to increase with the deletion of at least one vertex.
There are still subclasses of the class of graphs with terminal vertices for which the PRP is not yet resolved. Among these we find the even non-singular bipartite graphs.
The singular cards in the p-deck may give conclusive evidence that a graph is polynomial reconstructible but the presence of singular cards in the p-deck is

not sufficient for the parent graph to be singular. The PRP is still open for singular graphs.
The PRP for a particular class of graphs is often hard to resolve because the p-deck of a connected graph may appear to reconstruct the characteristic polynomial of a disconnected graph. The PRP is resolved for disconnected graphs in which one component is a tree [13] but is still open for other types of components.

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