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Constructivist Teaching: Mythical or Plausible?

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Abstract: Irrespective of the branch of constructivism they advocate, many constructivists argue that constructivism is a theory of learning, not of teaching, and therefore one cannot speak of such a thing as 'constructivist teaching' (CT). Others equate CT with a student-centred teaching methodology such as teaching for inquiry-based learning. From a radical constructivist perspective, I argue that both of these views are only partially true. The former seems to disregard the fact that teaching and learning are so interlinked that it may be virtually impossible for a teacher who strongly believes in the constructivist notion of learning not to reflect some of that belief in her/his teaching approach. The latter does not seem to acknowledge that even the most traditional and teacher-directed teaching may bring about learning, and that if learning occurs, it happens through the active construction of knowledge in the minds of the learners. Drawing on a local case study of a group of six low-performing Year 7 students (i.e., 11-year-olds) to whom I taught mathematics, I show that CT is a possibility in any classroom where the teacher is sensitive to the constructivist notion of learning. The framework I used to investigate the data was the Mathematics-Negotiation-Learner (M-N-L) framework. I devised this framework to help me to define CT and analyse the extent to which I maintain it in my lessons

Keywords: Radical Constructivism; Mathematics education; Constructivist teaching; M-N-L framework

Introduction

Radical Constructivism (RC) is built on two sets of principles about knowledge and cognition which its founder, Ernst von Glasersfeld (1990) claims to have surmised from Piaget's theory of genetic epistemology[1] (e.g., Piaget, 1985). These two sets of RC principles are that:

> **1a.** Knowledge is not passively received either through the senses or by way of communication;
> **b.** Knowledge is actively built up by the cognizing subject.
> **2a.** The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability;
> **b.** Cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality.
>
> (Glasersfeld, 1990, p. 22)

Principles 1a and 1b are shared by all branches of constructivism. It is Principles 2a and 2b that distinguish RC from other strands of constructivism. Glasersfeld claims that "those who merely speak of the construction of knowledge, but do not explicitly give up the notion that our conceptual constructions can or should in some way represent an independent, 'objective' reality, are still caught up in the traditional theory of knowledge" (Glasersfeld, 1991, p. 16). Riegler (2001) labels this latter type of constructivism *trivial*.

Like all mathematics teachers who draw their epistemological beliefs from RC theory, I need to keep in mind these two sets of principles during my teaching. Like all constructivists, I maintain that knowledge is not 'passed on' by the teacher or 'acquired' by the learner. My standpoint is that knowledge is constructed by the learner and that this development is facilitated by environments conducive to this knowledge construction, or what Steinbring (1998, p. 158) refers to as "learning offers." Being a *radical* constructivist, means that my understanding of 'knowledge' is not a mental representation

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[1] Piaget (1985) views intellectual growth as a process of adaptation to the experiential world. This happens through a process of *assimilations* and *accommodations* of perceived information to existing mental *schemas*. When humans use an established mental schema to deal with a new perception this is called assimilation. When existing schemas do not work and need to be adapted to deal with new phenomena, humans undergo a mental process called accommodation. When humans use assimilation to deal with their experiences, Piaget says that *equilibrium* has occurred. When existing mental schemas are not viable for new experiences, a mental *perturbation* occurs, creating a state of *disequilibrium* which humans feel the need to settle. The settlement of this perturbation is called *equilibration*. This occurs by modifying the existing schema to deal with the new experience through the process of accommodation, where a state of equilibrium is regained.

of an objective reality but a viable interpretation of a person's experiential reality. This implies that the mathematics I intend to teach is *my own* construction and interpretation. It also implies that whatever mathematics is developed by the students is *their own* subjective interpretation of the mathematical realities that I coordinate and facilitate in the classroom.

One of the main research questions in a case study I carried out with a group of Year 7 students was to analyse how these RC perspectives were reflected in my teaching approach. The outcome was the development of a framework which helped me analyse my constructivist teaching.

Constructivist Teaching

The argument that constructivism is a theory of knowledge construction and not of teaching has led constructivist researchers to disagree on the legitimacy of a label such as 'constructivist teaching' (CT). Usually, such a discord originates from what different people mean by the term. Engström (2014) objects to the term CT on the grounds that it is usually equated with progressive modes of teaching. Simon (1994) says that CT is a myth because constructivism is a theory of learning and, irrespective of the teaching method being used, learners will learn by constructing concepts for themselves. Simon (1995) argues that sympathising with a constructivist notion of how one learns does not translate into a set notion of how to teach. I agree with both Simon (1995) and Engström (2014) that no particular teaching method or tools can, by themselves, constitute CT.

On the other hand, I do make a case that the term CT is legitimate if it is attributed instead to a constructivist teacher's *sensitivity* towards individual students' subjective and active constructions of knowledge. Being an avid promoter of CT, Steffe repeatedly stresses the importance of teachers' *learning* about the mathematical realities of their students (for example, Steffe, 1991; Steffe & Wiegel, 1992). In the context of mathematics education, Steffe (1991) argues that RC teachers must view themselves as persons in pursuit of knowledge about bridging the mathematics *of* students (MoS), i.e., students' constructions of mathematical concepts) and the mathematics *for* students (MfS), i.e., teachers' mathematical ideas intended to be taught to a particular student or group of students.

RC teachers are concerned with building hypothetical models of students' cognitive structures (Glasersfeld & Steffe, 1991). Based on this concern, Simon (1995) presents a practical working model of how a mathematics teacher can adopt a constructivist perspective whilst teaching. Simon (1995) explains how mathematics teaching develops from what he calls a *hypothetical learning trajectory* (HLT). This is the way teachers make hypothetical predictions of the path by which learning might proceed. Simon (1995) explains that HLT consists of the teacher's:

i. learning goal which defines the direction of the lesson,
ii. plan of activities aimed to achieve the learning goal, and
iii. hypothesis of the learning process, i.e., the predictions of how students' thinking and understanding will evolve in the lesson.

These actions are 'hypothetical' because the actual learning trajectory is not knowable in advance. Glasersfeld (1994) argues that to be able to orient students' mental processes the teacher needs to have at least a hypothetical model of how the mind of a typical student operates at the outset of the lesson. I regard the use of the word 'hypothetical' (Glasersfeld & Steffe, 1991; Glasersfeld, 1994; Simon, 1995) as an acknowledgement of the fact that what learning outcomes the teacher may have in mind before the lesson starts may be changed in the course of the lesson. Such changes occur according to what the teacher learns from the students. Steffe (1991) argues that RC teachers should reflect and act upon models they build of their students' mathematical knowledge. Both Simon (1995) and Steffe (1991) suggest that constructivist mathematics teachers should help their students create connections between their mathematics and the mathematics the teacher intends to teach them. This has much in common with the *Constructivist Learning Design* proposed by Gagnon and Collay (2006).

Simon (1995) and Steffe (1991) have captured the attributes that are usually associated with fostering a mathematics teaching environment that is sensitive to constructivist notions of learning, namely to:

i. encourage students to come to an answer in diverse ways and possibly obtain multiple correct responses,
ii. appreciate and promote students' interventions in the lesson and invite them to articulate their understandings of the problem at hand,

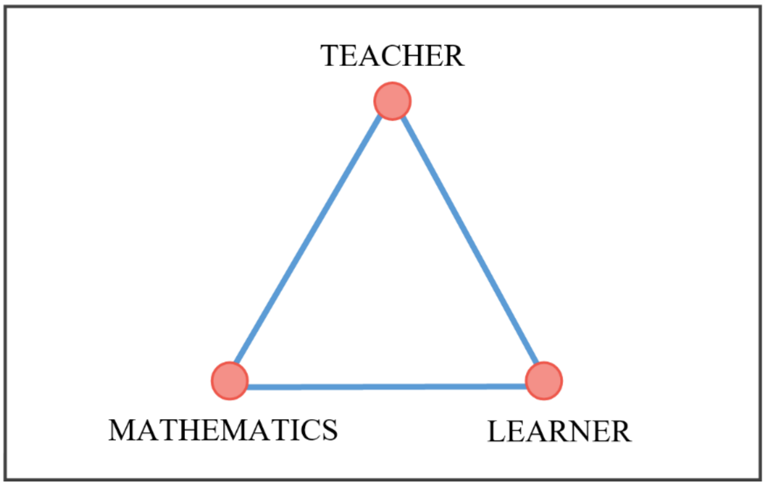
iii. allow students to describe their strategies and engage students in debates which help them refine and adjust their strategies and understandings, and
iv. learn about students' conceptual constructions and about students' own mathematical understandings through reflection on classroom experiences.

It seems, therefore, that there exists an approach, an attitude, and a standpoint in mathematics teaching which may be described as CT. This approach occurs when constructivist teachers, in their diverse preferred styles of teaching, make possible a two-way-traffic type of communication in their lessons, where both teacher and student are learners and both teacher and student are teachers (Freire, 1998). The relationship between the mathematical content, the learner, and the teacher is created by the need of learners to construct mathematical ideas and by the need of the teacher to learn about and orient students' mathematical understandings.

Mathematics, the Learner, and the Teacher

The dynamics between mathematical content, learners, and the teacher (including teaching approaches), most commonly referred to as the didactic triangle (Figure 1), has been in the limelight of French educational research since Brousseau (1997) put forward his theory of *les situations didactiques.* The latter are the didactical situations formed by this interlinked triplet within the classroom ethos.

Figure 1: Brousseau's didactic triangle



This rather simplistic diagram highlights the relationships between the three factors that establish the situation of a mathematics classroom: the teacher, the student, and the mathematics being taught and learnt. Schoenfeld (2012) identifies seven questions regarding one or more nodes of the didactic triangle and the relationships between them:

1) What is mathematics, and what version of it is the focus of classroom activities?
2) Who is the teacher, what does he or she bring to the classroom?
3) Who is the learner, what does he or she bring to the classroom?
4) What is the teacher's understanding (in a broad sense) of mathematics?
5) What is the learner's emerging understanding of mathematics?
6) What is the relationship between learner and teacher?
7) How does the teacher mediate between the learner and mathematics, shaping the learner's developing understanding of mathematics?

(Schoenfeld, 2012, p. 587)

Question 7, which is most pertinent to the subject of this paper, deals with the way the three entities relate simultaneously to each other. This question could not be tackled without considering the three triangular nodes separately (questions 1-3) and the three triangular sides, each of which connects two entities of the didactic triplet (questions 4-6). The didactic triangle even allows researchers to isolate one of the nodes of the triangle in order to elicit and expand its meaning and clarify its links with other nodes. For example, Jaworski (2012) focuses on the teacher node and identifies three interlinked activities that constructivist mathematics teachers carry out in their lessons. She calls these the *teaching triad*.

Management of Learning. This consists of the teacher's administration of the classroom activities, the students' participation in those activities, and the overall interactions fostered during the lesson. It also involves the teacher's institutional obligations and standards, assessment practices, and, most importantly, the interpretation of mathematical content.

Sensitivity to Students. This is the teacher's effort to become aware of her/his students' knowledge and thinking styles and tendencies. Such sensitivity makes students feel respected, included and cared for.

Mathematical Challenge. This is the way the teacher presents the mathematical problem to the students in a way that interests them, motivates them to learn, and promotes participation and cognitive engagement.

Jaworski's (2012) triad has much in common with ideas discussed earlier. In particular, the teacher's sensitivity to students is stressed by Steffe and Wiegel (1992) in their appeal to constructivist teachers and curriculum reformers to view mathematics knowledge as a human creation. The presentation of the 'mathematical challenge' is necessarily derived from the teacher's epistemological standpoint about the mathematical concepts she/he intends to communicate with the students. The RC teacher interprets and represents mathematical concepts as "more or less reliable ways of dealing with experiences, the only reality we know" (Glasersfeld, 1995, p. 117).

The experiences of the teacher and the students are derived from an environment which goes beyond the classroom. Chevallard (1982) introduces the notion that a didactical situation does not operate in a vacuum but is embedded within, and affected by, external social and institutional forces. The latter include government educational directives, inspecting and testing regimes and parental and community pressures. The RC teacher may well reject the idea of an *a priori* curriculum but, as Chevallard (1988) observes, the very *intention* to teach is not so much a decision of the individual teacher as it is of the society in which that teacher operates. It is society which decides what part of mathematics can be regarded as *teachable* knowledge. Chevallard (1988) argues that knowledge is inherently a tool to be put to use rather than concepts to teach and learn. He claims that it is thus an artificial enterprise to 'teach' a body of knowledge. In fact, curriculum planners need to find ways how to transform 'knowledge' from a tool to be put to use to something to be taught and learnt. He calls this the "*didactic transposition* of knowledge" (Chevallard, 1988, p. 6, original emphasis).

Once mathematical content is transformed by curriculum designers from a viable tool to a set of teachable concepts, it is the constructivist teachers' duty to "to recontextualize and repersonalize the knowledge taught to fit the student's situation" (Kang & Kilpatrick, 1992, p. 5). The RC teacher observes and reflects on the uniqueness of learners' experiential worlds and tries to find connections between the mathematical content included in the syllabus and the learners' interpretations of that content with respect to their individual experiences.

Negotiating a Link between Teachers' and Learners' Mathematics

Literature about CT, or at least about teaching from a constructivist perspective, tends to focus mostly, if not only, on the learner. In his review of research related to CT, Gash (2014) states that the emphasis is "on the child's learning rather than just focusing on what the teacher thought was important to teach" (Gash, 2014, p. 304). I agree with Gash's argument only because his inclusion of the word 'just' implies that for a constructivist teacher *both* the child and the curriculum need to be kept in mind, for both of them constitute the didactical situation (Brousseau, 1997) which puts the teacher in the classroom in the first place.

It was Dewey who was probably the first to think of the educative process as the *interaction* between these two factors. In *The Child and the Curriculum,* Dewey (1902, p. 2) points out that teaching is influenced by two forces: "an immature, undeveloped being; and certain social aims, meanings, values incarnate in the mature experiences of the adult. The educative process is the due interaction of these forces."

Although Dewey promotes the kind of education which allows learners to have control over their learning, he maintains that the teacher should focus on both the learner and the content to be taught. On the one hand, Dewey argues that it is unacceptable for a teacher to focus only on the content and forget about the needs of the learner. The teacher needs to draw attention to the *viability* of the subject content in the students' experiential worlds, something which today may be identified with RC. On the other hand, Dewey (1902) claims that if teachers focus only on the learners they will easily lose sight of what knowledge they have been entrusted to teach. Hence, the teacher needs to strike a balance between providing opportunities for learners to acquaint themselves with the topics in the curriculum and being sensitive to learners' individual interests and experiences. Dewey compares the learner and the learnt with two points and the teaching process with the interconnecting line drawn between those two points:

> The child and the curriculum are simply two limits which define a single process. Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction.
>
> (Dewey, 1902, p.16)

Figure 2 illustrates my understanding of Dewey's (1902) analogy that links the subject matter, the learner, and the teaching process. Dewey stresses that any teaching programme needs to be defined by the needs of the learner and the subject matter intended to be taught. The teacher's task is therefore to plan and proceed in assisting learners along their journey from their current situation to the state of developing knowledge about the subject matter.

Figure 2: Teaching seen as the line drawn between subject matter and learner

Dewey (1902) regards teaching as the negotiation process aimed at bringing together these two forces both of which demand the teacher's attention. In doing so, he acknowledges teachers' dual accountability to curricular and learners' requirements. Dewey's (1902) Curriculum-Teaching-Learner construct enriches constructivist frameworks such as those of Steffe (1991) and Simon (1995) because it takes into consideration the parameters within which school teachers operate, including, most importantly, the didactic contract between the teacher and the students (Brousseau, 1997). The constructivist frameworks proposed by Simon (1995), Steffe (1991), and Dewey (1902) were instrumental in my investigation of CT and the subsequent development of an analytic framework to investigate CT from a RC perspective.

Context and Methodology

The protagonists of my case study were six low-performing Year 7 students to whom I taught mathematics during the scholastic year 2014-15. Their pseudonyms were Dwayne, Dan, Jordan, Joseph, Omar, and Tony. The school had a policy of retaining mixed-ability classes for all subjects except for Mathematics, English, and Maltese. In these core subjects, students were divided according to their performance in the previous scholastic year. Those starting to attend the school at Year 7 were divided in these three subjects according to their performance in a national benchmark examination which Maltese students sit for at the end of Year 6.

The grades that my participants had obtained in the Year 6 benchmark exam, before entering the school, were between 1 and 3 standard deviations below the mean of the Year 7 cohort and hence they were in the lowest of three performance sets. The part of the Year 7 curriculum which featured in my research was that of introducing formal algebra by helping students to:

i. develop meanings for numerical and algebraic expressions,
ii. understand the use of letters as unknowns and variables, and
iii. extend their interpretation of the equals sign.

Qualitative data was collected by a number of methods, but the data concerned with CT was obtained by video-recording a series of twenty double lessons (80 minutes each) throughout the scholastic year. As Farrugia (2006) asserts, in Maltese mathematics classrooms, English is the language of written texts, while for spoken language, technical words are usually expressed in English. The main communication medium in the lessons was Maltese and we used English to read written problems or task instructions, and to say technical words like 'plus' and 'equals.' Sometimes we code-switched to English for short intervals. The transcripts were translated immediately to English and when English was used this was indicated in parenthesis.

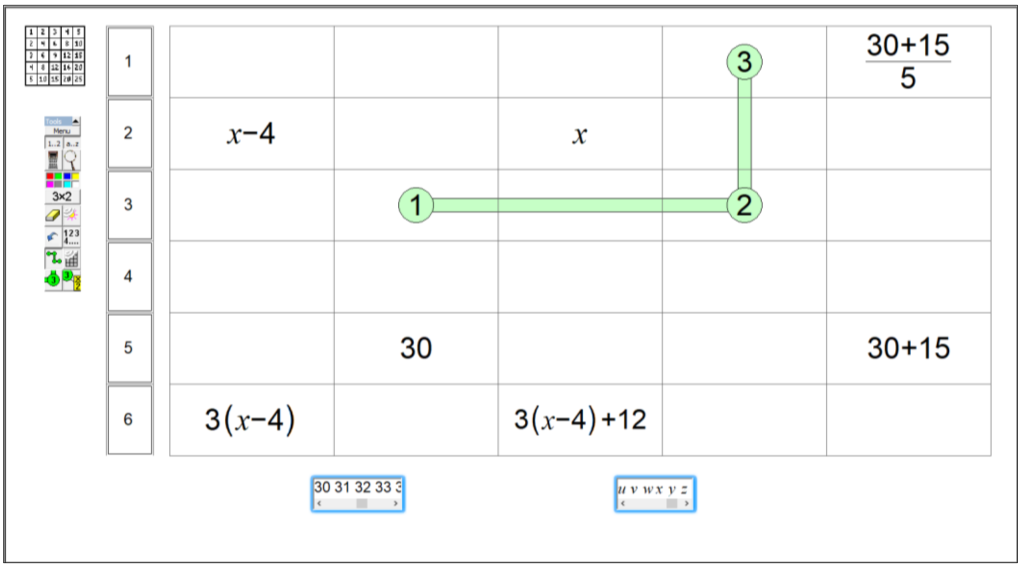
Throughout the lessons, I made use of the software package Grid Algebra[2] (GA). GA is a computer environment which is based on the multiplication grid. A typical GA interface[3] is shown in Figure 3. Only multiples of a particular number are allowed in a row. For example, in R_5C_2 (Row 5 Column 2), the number 30 is allowed because it is a multiple of 5.

The content in one cell may be dragged into another cell and GA shows the corresponding expression. For example, dragging the 30 in R_5C_2 three cells to its right to R_5C_5 is equivalent to adding 5 three times and GA shows 30+15. Right and left movements correspond respectively to adding and subtracting multiples of the row number. Movement from one row to another row corresponds to multiplication or division. For example, movement from R_2 to

[2] Developed by Dave Hewitt and distributed by Association of Teachers of Mathematics.
[3] Arrows are added to show how numerical and algebraic expressions were obtained by moving the cells.

R_6 corresponds to multiplication by 3. Similarly, movement from R_5 to R_1 corresponds to division by 5 and hence, moving the expression 30+15 from R_5C_5 to R_1C_5 results in the expression (30+15)/5 as shown in Figure 3.

Figure 3: A typical GA interface

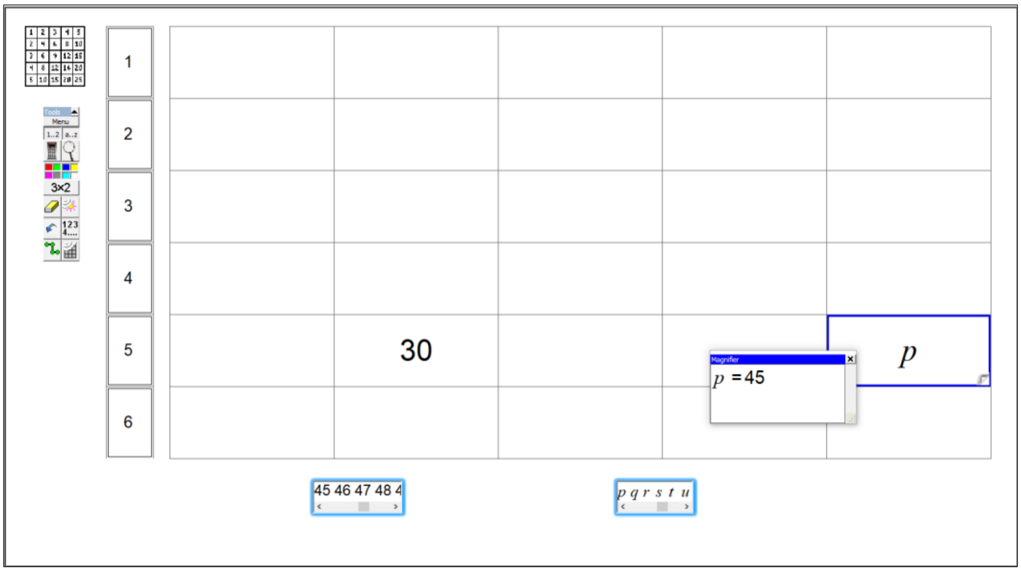


GA accepts the use of letters to represent variables or unknowns. Entering the letter x in R_2C_3 without the introduction of any other numbers in the grid, means that x represents a *variable* multiple of 2. However, if at least one number is present in the grid, that number determines the value of all the other cells in the grid. Hence, the x present in R_2C_3 in the grid shown in Figure 3, represents a *specific* multiple of 2 since there are some numbers present in the grid. Hence, it is a representation of an *unknown* (constant) rather than a variable. Evaluating neighbouring cells in Figure 3, one can see that x=14. The movements and respective creation of expressions described earlier may be similarly done with cells containing letters. Hence, moving x from R_2C_3 to R_2C_1 results in x–4, since this movement corresponds to subtracting 4. The expression x–4 may, in turn, be dragged onto R_6C_1 and, since jumping from R_2 to R_6 corresponds to a multiplication of 3, GA shows 3(x–4), and so on.

In this way, GA enables users to create and build numerical and algebraic expressions either by moving a cell and its contents from one place to another or by typing it directly with respect to its place in the multiplication grid and in relation to other expressions existing in the grid. Furthermore, it gives students the possibility to trace the movements of expressions around the grid, such as the 1-2-3 journey shown in Figure 3.

GA also allows users to input more than one expression in a single cell. Figure 4 shows a grid in which 30 is entered in R_5C_2. As previously shown, the expression in R_5C_5 should have a value of 45. In Figure 4, GA allows users to enter a letter (say, p) inside R_5C_5, along with the number 45. A feature in GA, called a magnifier, reveals the contents of this cell. As shown in Figure 4, the magnifier displays p=45 when R_5C_5 is clicked upon.

Figure 4: The magnifier feature of GA



The expression resulting in the GA magnifier was the subject of an excerpt of a lesson presented later in this paper.

The lessons were divided into two parts. The first part consisted of a class discussion about the topic at hand. The discussion was facilitated by the use of GA which was projected on the interactive whiteboard (used as a touchscreen). The second part of the lessons consisted of students working on GA tasks on their computers. While the latter was crucial in investigating students' mathematical representations and interpretations (see Borg & Hewitt, 2015), the first part was used to define and analyse CT. The framework I developed as a result of this investigation is discussed in the section that follows.

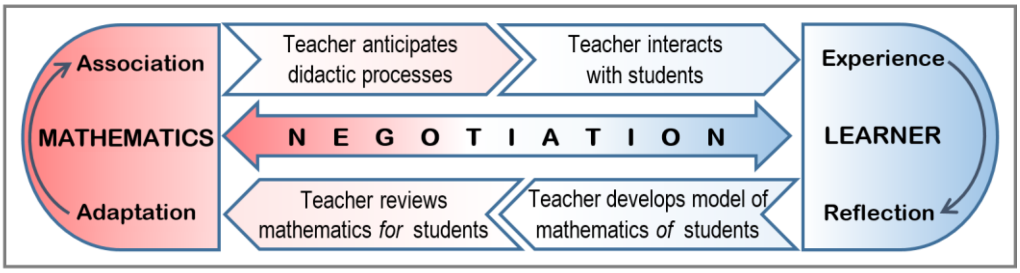
The Mathematics-Negotiation-Learner Framework

Analysing the lesson videos against the backdrop of Dewey's (1902) Curriculum-Teaching-Learner construct, I observed that I was continuously changing my purpose in the lesson due to my need to keep in mind both the mathematics I intended to teach and the mathematics being constructed by the learners. These two forces, continuously calling for my attention, necessitated negotiations from my mathematics to the learners and from the learners to my mathematics. Further analysis led to the identification of four different shifts of teaching purpose:

i. The **M-N shift**: from my mathematics to the negotiation process. This was the moment where I changed my focus from thinking about my mathematics to making hypothetical predications about the learning process (Simon, 1995). This led to interactions aimed at providing a learning offer (Steinbring, 1998) so that students could form concepts about the mathematics I intended to teach.
ii. The **N-L shift**: from the negotiation process to the learner. Here my focus shifted from interacting with the students to assisting students in their experience of mathematical phenomena. This involved helping students to make reflective abstractions (Piaget, 1985) of that mathematical experience.
iii. The **L-N shift**: from the learner to the negotiation process. This refers to the moment where I learnt something about students' mathematics (Steffe, 1991) and decided to do something about it. This negotiation was not an interaction with the students but an 'internal interaction' with myself, which led to a review of the suitability of the learning offer.
iv. The **N-M shift**: from the negotiation process to my mathematics. This was when I changed my focus from reviewing the learning offer to making associations or adaptations to my mathematics – the subset of my mental schema intended to be taught or shared with the students.

Keeping Dewey's (1902) Curriculum-Teaching-Learner construct as an overarching frame of reference, I used these shifts of focus to develop what I called the Mathematics-Negotiation-Learner (M-N-L) framework. The design and development of the M-N-L framework is discussed by Borg, Hewitt, and Jones (2016 a, b). The framework is illustrated in Figure 5.

Figure 5: The Mathematics-Negotiation-Learner framework



M-N-L builds on Dewey's (1902) Curriculum-Teaching-Learner construct by using the metaphor of two 'roads' that link (the teacher's) mathematics and the learners. These roads represent the teacher's negotiations during the lesson. The following is a description of the stages of the cycle shown in Figure 5, starting from the upper left-hand arrow that goes from mathematics to learner:

1. *The Forward-negotiation Road* 

The forward-negotiation road is formed of the teacher's actions aimed at presenting a mathematical learning offer to the students:

i. The teacher builds on models of the mathematics *of* the students (MoS) to *anticipate* possible didactic processes. The latter may help students to develop notions of the mathematics at hand, i.e., the mathematics *for* the students (MfS). Simon (1995) calls this a hypothetical learning trajectory since the teacher has no means of knowing in advance the actual didactic processes that may occur.

ii. Then, the teacher *interacts* with students by making representations of MfS intended for students' constructions of MoS. The teacher makes verbal, gestural, and written representations and coordinate goal-oriented activities and discussions. 'Interaction' includes teacher exposition and teacher-coordinated activities.

2. *Learner* 

The 'Learner' section of Figure 5 shows how this forward-negotiation road leads to students' *experience* of mathematical representations which the teacher encourages students to *reflect* upon and make abstractions. Students become learners by making abstract conceptualizations through an interplay of experience and reflection. This is reminiscent of Kolb's (1984) experiential learning construct but with an emphasis on how the teacher reacts to students representations.

3. *Backward-negotiation Road* 

i. The Learner-to-Mathematics arrow on the right shows that the teacher builds, experiential *models* of MoS. These models are experiential because they are built entirely on the experiences of the teacher and the students. Steffe emphasises that the constructivist teacher must be a keen observer in order "to construct the mathematical knowledge of his or her students." (Steffe, personal communication, October 7, 2015). Models of MoS of individual students may serve the teacher to make inferences about the possibility of similar MoS for the rest of the class.

ii. The arrow that follows on the left shows that the teacher uses these models of MoS to *review* MfS. This means that MoS serves as an assessment of whether the learning offer presented along the forward-negotiation road was appropriate for the students.

Each activity involved in the backward-negotiation road is a learning experience for the teacher.

4. *Mathematics* 

The mathematics end of the M-N-L diagram shows that the teacher revisits her/his own mathematics, to decide whether MoS can be *associated* with it either directly or by going through some kind of *adaptation or accommodation* of her/his mental schema. The settlement of this perturbation leads to a renewed MfS and a revised anticipation of the didactic processes with which the teacher starts a new forward-negotiation road.

I consider the teacher's deliberate shifts of purpose between the four elements described above to be an indication of CT. Although some exponents of CT (e.g., Steffe et al., 1983; Steffe, 1991) tend to focus almost exclusively on the teacher's learning from and about the students (backward-negotiation road), I argue that the teacher is duty-bound *to teach* and cannot learn about students' construction of knowledge without intervening to facilitate it. Nevertheless, I argue that constructivist teachers cannot just present learning offers and, like Steinbring (1998), claim that mathematics teaching is an autonomous system.

That is, CT is dependent on students' feedback and on the actions that the teacher takes based on that feedback.

The teacher's effort to balance forward- and backward-negotiations is key to sustain regular transitions from one stage to another of the M-N-L cycle, thus maintaining the two roads which bring together mathematics and learners. CT may be analysed by studying how the teacher makes transitions between successive stages of the M-N-L cycle through shifts of teaching purpose. The extent to which the teacher manages to start, maintain, and complete M-N-L cycles may be an indication of her/his success to engage in CT. When the teacher fails to complete M-N-L cycles it may indicate a failure to engage in CT. This happens when the teacher momentarily creates roadblocks in the negotiation process which hinder the shifts of teaching purpose necessary to complete M-N-L cycles. In my study, I have identified two such roadblocks; the reader is referred to Borg et al. (2016a) for a discussion of these roadblocks. In the following section, I demonstrate how I used the M-N-L framework to analyse my CT.

Analysing CT through M-N-L Cycles

In this section, I present a continuous transcript taken from the video recording of Lesson 13. This is divided into four excerpts which I use to show how I went through two successive M-N-L cycles. The main aim of the lesson was to introduce the use of letters in the GA grid. A letter in GA could represent a specific unknown or a variable quantity.

This episode occurred just 2 minutes into the lesson. As usual, the first half of the double lesson consisted of a plenary discussion. The first few minutes of class discussions consisted mainly of a teacher exposition. This was necessary since I needed to demonstrate new features of the software. Nevertheless, students' participations in such expositions were necessary since I needed students to reflect on their observations. In a typical lesson, as time went by, I usually relinquished more and more my 'control' over the discussion, where students came out to work on activities on the interactive whiteboard. This led to the second half of the double lesson where students worked in pairs on their computers. During this part of the lesson, I took on a more background, supervisory role where I assisted students only if required.

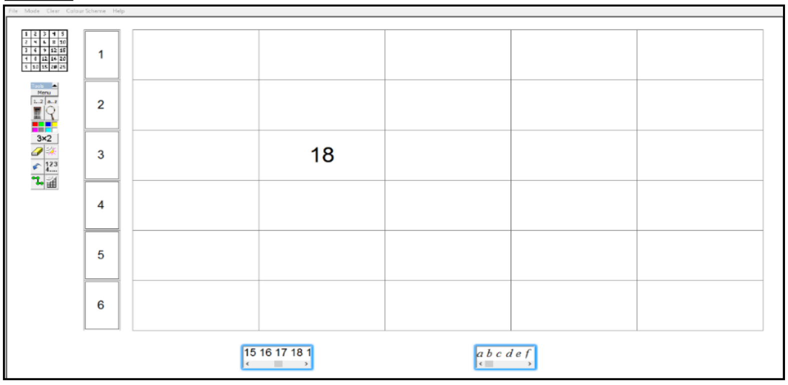
The reason for choosing this particular episode is to show that even during teacher exposition, when the teacher may be predisposed to focus more on the subject matter, CT can be achieved if the teacher is sensitive to students' knowledge constructions. This sensitivity is required for the teacher to make the necessary shifts of focus between her/his subject matter (mathematics), the negotiation process, and the learner. In this episode, a number of mathematical concepts were discussed, namely:

i. multiples of 3,
ii. letters standing for numbers and values of numerical expressions, and
iii. the meaning of the equals sign.

Excerpt 1: M-N and N-L shifts (Cycle 1)

PB: …I am going to place the number 18 here. [*Drags 18 to* R_3C_2 *- #1.*]

#1



… It [the software] will let me do it.

Joseph: Because it is in the 3-times table.

…

PB: Well done! Well done! Now, if I picked a letter at random from here [*picks the letter d and drags it to* R_3C_4] and I place it over here [*Joseph raises his hand*], that *d*, first of all, what is it symbolising? [*Pointing at Joseph…*] Come, let's see.

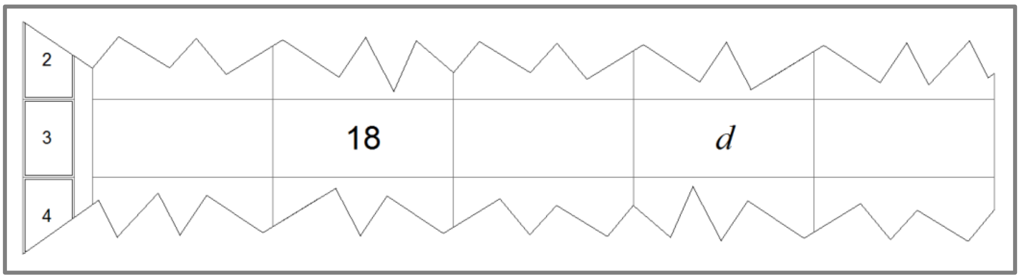
Joseph: Uh, what it is, what the answer should be. Like if you do 18 plus 3 plus 3, that is plus 6, which becomes 24, it is *d* equals 24.

This excerpt shows the beginning of an M-N-L cycle (Cycle1). At the beginning of the discussion, my initial MfS was the appreciation of the difference between variables and as unknowns. I anticipated that the students were prepared to construct notions of letters as unknowns in the GA grid by referring to neighbouring cell values. This anticipation was expressed by phrases like "I am going to…", and "…it will let me."

With this anticipation in mind, I changed my focus to start interacting with the students (M-N shift). This interaction was prompted by the fact that the number 18 could stay in cell R_3C_2. I asked questions to help students reflect on why it was allowed by GA to be there. Joseph was quick to point out that this was accepted because it was a multiple of 3. This was a cue for me that I could place a letter in the grid and I inserted *d* in a neighbouring cell (R_3C_4) and asked the students what that letter symbolised.

Here, I shifted my focus to another teaching purpose: encouraging students to reflect on mathematical phenomena (N-L shift). This reflection encouraged Joseph to suggest a meaning for *d*: "like if you do 18 plus 3 plus 3". Placing *d* in the neighbourhood of 18 (Figure 6) helped Joseph to interpret the symbol *d*, aided by the representation of its 'container', the cell R_3C_4.[4] Joseph's interpretation of the symbol *d* in association with the values of the neighbouring cells is an example of Mercer's (2000) claim that symbols (like words) gain meaning from their neighbourhood.

Figure 6: Letter gaining meaning of from its neighbourhood



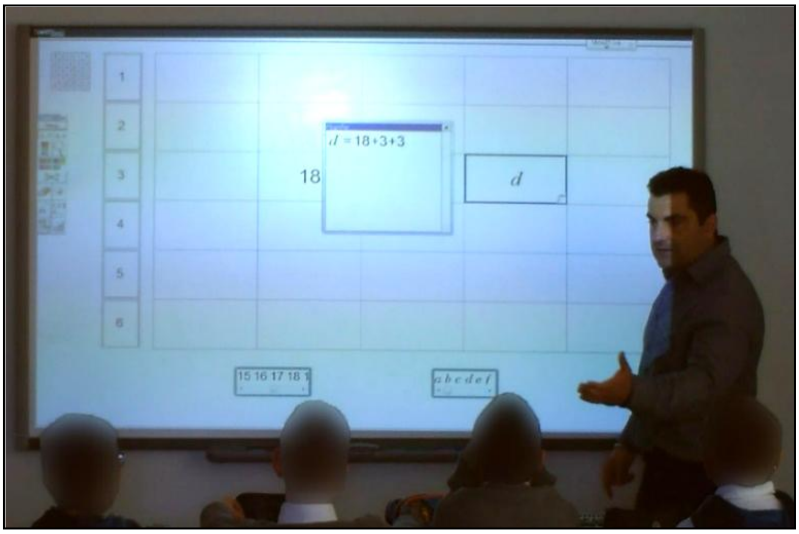
The second part of the lesson episode resumes in the following excerpt.

[4] The interplay between conceptual interpretations and pictorial, symbolical, and kinaesthetic representations are discussed by Borg and Hewitt (2015).

Excerpt 2: L-N shift (Cycle 1)

PB: [*Nodding…*] All right, so what we're saying here is that *d* is, like, the answer of when [*points to respective cells*] 18 makes plus 3 plus 3. In fact, if you do like this [*drags the 18 to* R_3C_3 *to obtain 18+3*] and like this [*moves 18+3 to* R_3C_4 *obtaining 18+3+3 on the same cell as d*] – all right? – we see *d* here and [*choosing the magnifier icon*] if we see … with the magnifier here, it is telling me exactly [*pointing to Joseph* - #2] like you told me that [*pointing to d*] *d* [*points to equals sign*] is [*points to respective numbers*] 18 plus 3 plus 3. [*Clicks on the cell to alter the expression.*] If I alter here it will tell me that [*points*] 18 plus 3 plus 3 equals *d.*

#2



In this excerpt, I changed my focus from encouraging reflection to forming a model of Joseph's interpretation of the mathematics in question, i.e., his MoS (L-N shift). At first, I confirmed aloud what Joseph seemed to be thinking: "…so what we're saying here is that…" I also made cell movements corresponding to Joseph's calculation of 18+3+3 ending on the cell containing *d*, and used GA's magnifier to help Joseph's classmates observe that what he seemed to be implying was that *d*=18+3+3 or that 18+3+3=*d*. Building a model of Joseph's and possibly other students' MoS helped me review my original MfS, that of identifying the circumstances that made *d* an unknown.

Excerpt 3: N-M shift (Cycle 1) and M-N and N-L shifts (Cycle 2)

PB: But if I want, instead of doing 18 plus 3 plus 3, I can, if I want to, erase here [*erases all expressions except 18 and d*] – OK? – I can just bring up [*pointing to the number menu*] that unique number that can be here [*the cell containing d*], a single number… What is the number?

Joseph: Twenty-four.

PB: Do we agree that it is 24?

Joseph: Yes [*the others nodding*].

When I drew students' attention to the possibility of having a single number instead of 18+3+3, Joseph proposed the number 24. At that moment, it seemed to me that Joseph, and possibly other students who were nodding to his response, were thinking of the letter *d* as being the answer of 18+3+3, i.e., 24. In the above excerpt, my focus changed again from reviewing the learning offer to associating Joseph's (and possibly other students') MoS with my mathematics (N-M shift). In order to do this, I had to make adaptations of my notion of unknown as a single fixed number to accommodate Joseph's concept of unknown as 'answer'.

This shift prompted a new M-N-L cycle, with a renewed MfS: the connection between

- a letter as a single (unknown) number due to its being the value of an expression (Joseph's MoS) and
- a letter as a single fixed (unknown) number due to its neighbourhood in the GA grid (the original MfS).

I anticipated how students could make these connections as I started off a new M-N-L cycle (Cycle 2).

My purpose shifted from anticipating these connections to interacting with students to help students develop mathematical appreciations of these connections (M-N shift). I erased all the expressions, except 18 and *d* (Figure 6). While doing so I was hoping students would observe the link between what was in cell R_3C_4 a moment earlier (18+3+3) and the single number could be inserted in that cell. Previous lessons taught me that students were very competent in assigning the right numbers in GA cells, so I figured the empty cell R_3C_4 could invoke the single number 24 in the minds of the students due to its position in relation to 18 in the 3-times table.

Excerpt 4: L-N and N-M shifts (Cycle 2)

PB: Because we're in the 3-times table and we're doing plus 3 plus 3, all right? … I bring up the 24 … I'll pick the 24 from here [*drags 24 from the number menu to R_3C_4 containing d*] … And when I go with the magnifier there it is telling me *d* equals 24. … So, *d* equals 24 and [*clicks on the cell to alter the order*] 24 equals *d*…

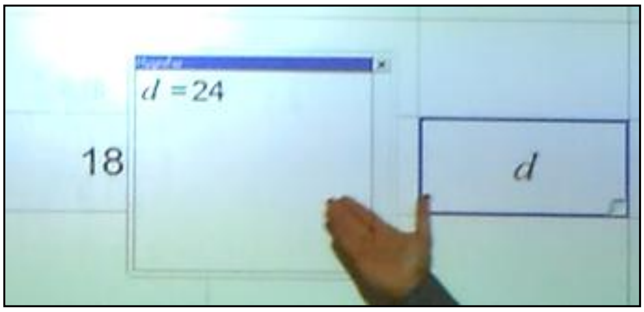
Joseph: The same.

PB: … As such, we are not seeing an answer. When you say 'answer' it's like you have done some calculation, some plus, minus…

Joseph: 18 plus 3 plus 3.

PB: We don't have any calculation, nothing, here. So now, I cannot quite say that 'equals' is 'answer.' [*Jordan shaking his head.*] So what can I say that it means there [*pointing to d*=24 - *#3*]?

#3



The equals?

Joseph: Equal *to* [*in English*].

Dwayne: They are the same in size.

With this in mind, I asked students what was the "unique number that can be" in R_3C_4. Here my purpose had changed from interacting by erasing the expression 18+3+3 to encouraging students to reflect on the single number which could be entered in that empty cell (N-L shift). It was Joseph himself who mentioned the number 24. He had already thought about it and even mentioned it earlier (see end of Excerpt 1) where it seemed he was thinking of it as the answer to 18+3+3.

In the above excerpt, I first wanted to orient students' thinking (Glasersfeld, 1991b) towards thinking of *d* as being 24 without having to think of it as the answer to a calculation. So during the experience-reflection stage, I confirmed Joseph's statement by dragging 24 into the cell containing *d* and proceeded to help students to observe and consider the mathematical statement *d*=24 which could be seen by clicking on the magnifier icon.

I knew that for some students, the equals sign was still just a symbol showing the answer of a computation. So, during the reflection exercise, I focused on the meaning of the equals sign in the expression *d*=24. When I asked what *d* 'equals' 24 meant, Joseph expressed his thinking by saying in English "equal *to*." The change from 'equals' to the more exact 'equal to' and his emphasis of the word 'to' gave the equality symbol a more a relational meaning. Dwayne immediately picked up on this and gave the response I was aiming for: "They are the same in size."

Dwayne and Joseph's feedback made me change my purpose from helping students to reflect on their mathematical observations to forming a model of these students' MoS (L-N shift). I confirmed Dwayne's response, and elaborated on his statement. I also said "Good", indicating a favourable review of Dwayne's statement. I was simultaneously making a favourable review of the outcome of my learning offer. In accepting that *d*=24 meant *d* "*is the same size as"* 24, Dwayne and possibly Joseph, seemed to have constructed an idea about the possibility of using the arbitrary letter *d* as a substitute for a constant number (unknown) irrespective of whether that number was the answer of a computation.

This led to another shift of focus: from reviewing the outcome of the learning offer to reflecting on my mathematics, i.e., my interpretation of *d*=24 (N-M shift). I knew that the neighbouring 18 meant that *d could not be anything but* 24. This concept was a subset of the original MfS. However, the original MfS included also the notion that without any other numbers in the grid, *d* would be a *variable* multiple of 3 and hence the statement *d*=24 would be viable if it were interpreted as in *d*=..., 21, 24, 27,... This prompted the onset a new M-N-L cycle in which I anticipated that students could, in this way, construct the notion of *d* as a variable.

Table 1: Summary of two complete M-N-L cycles

Mathematics	**Negotiation**		**Learner**
My concept of 'unknown' represented by a letter in a GA environment.	I **anticipate** students will develop the notion of unknown when this is contrasted with a variable.	I **interact** by placing 18 in R_3C_2 and *d* in R_3C_4. I ask students what the letter *d* may stand for.	Joseph says that 18 was allowed since it was a multiple of 3. Then he says that *d* is the answer of a computation involving 18.
The 'answer' of a calculation may also be thought of as an unknown. This holds also when the calculation is not expressed as a single number, e.g., $x=1+\sqrt{2}$	I review my original MfS and find a way how to incorporate Joseph's notion of an 'answer' within my notion of an unknown.	I create an unexpected model of Joseph's MoS concerning the letter *d*: a letter may stand for the 'answer' of a calculation.	
	I anticipate that students will link the notion of 'answer' and unknown if they can observe an example with the help of GA.	I use Joseph's explanation to show that *d* may be seen as the 'answer' of 18+3+3. Joseph says that *d* could be 24.	I help students reflect on the statement *d*=24. Joseph and Dwayne elaborate on the meaning of of the equals sign, viewing it as a relational symbol
I associate students' interpretations of *d* to my notion *d* as a variable.	I review the MfS. Dwayne and Joseph seem to interpret *d* as being *equal to* a constant.	I build a model of Joseph's and Dwayne's interpretation of the equals sign as 'same in size.'	

Table 1 above summarises how these two successive M-N-L cycles occurred by mapping each event to the respective teaching purpose. This table shows the fast toing and froing between my mathematics and my learners' knowledge constructions as I strived for CT. The arrows indicate shifts of teacher purpose. There was an average of one M-N-L cycle per 4 minutes of plenary discussion throughout the 20 lessons.

Conclusion

The M-N-L framework gives due importance to the three constituents of Brousseau's (1997) didactic situation: the learner, the teacher, and the mathematics to be taught and learnt. Based on Dewey's (1902) idea that teaching must be defined by both curriculum and learners, the M-N-L framework places the teacher as a negotiator between mathematics and the learner. The framework suggests that the main task of the constructivist teachers is to find ways how to bridge the knowledge she/he intends to teach with the knowledge being continuously constructed by the students during the lesson.

Simon's (1995) theory of teaching mathematics from a constructivist perspective was key in the formation of what I called the forward-negotiation road. The teacher's sensitivity to students' possible constructions of knowledge enables her/him to anticipate possible didactic processes and interact with students accordingly. Based on RC, M-N-L suggests that the teacher needs to make it her/his business to know whether and how the learning offer (Steinbring, 1998) makes sense to the students.

The RC teacher gives much weight to the question of viability of mathematics as experienced by the students. In this regard, Steffe's (1991) principles of (radical) CT were crucial for the formation of M-N-L's backward-negotiation road. The teacher builds models of MoS and uses them to review MfS. The teacher synthesises students' mathematics with her/his own, sometimes requiring accommodations of her/his own mathematical schema. This puts the teacher in a better position to go back to the students with a renewed MfS and a new M-N-L cycle may commence.

The formation of the M-N-L framework, inspired chiefly by the works of Dewey (1902), Steffe (1991), Simon (1995), and Jaworski (2012), and drawing on Glasersfeld's (1990) principles of RC, showed me that the idea of CT is indeed plausible. Rather than portraying it as one set notion of how to teach, the M-N-L framework presents CT as a teaching approach resulting from the teacher's sensitivity to RC notions of knowledge and learning. This sensitivity is the driving force behind the teacher's changes of purpose during the lesson necessary to keep both mathematics and learners in mind. The M-N-L framework proposes that:

i. Any learning offer presented to the students is regarded by the teacher as an attempt to facilitate students' active and subjective construction of mathematics. The teacher anticipates the possible didactic situations which may lead to students' developments of mathematical ideas. The teacher thus interacts with the students in order to orient their thinking processes. In this way, the teacher helps the students to make reflective abstractions of the mathematics in question.
ii. The RC teacher is also a learner. She/he is invested in learning about the mathematics being constructed by the students. This helps the teacher to make inferences about the success or otherwise of the current learning offer, but this exercise does not only benefit the students. When the teacher takes up the challenge of linking students' mathematics with her/his own, this enriches the teacher's own mathematical content knowledge.

The M-N-L framework is both conceptual and analytical. Besides defining CT, it also proved to be a viable tool in helping me to investigate CT in my mathematics lessons by analysing the extent to which I managed to generate and complete M-N-L cycles. It was also instrumental in identifying momentary flaws in my approach, when I created what I called 'roadblocks' (Borg et al., 2016a) that obstructed the negotiation between my mathematics and that of my students. Linking the generation and completion of M-N-L cycles with CT helped me to ascertain that these moments of failure did not render my teaching non-constructivist. Rather, such moments showed that, like anything which is not mythical, CT is not a perfect system but an endeavour of ordinary teachers who try to bring their constructivist beliefs to their daily teaching practices.

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