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Charged cylindrical black holes in conformal gravity

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Considering cylindrical topology, we present the static solution for a charged black hole in conformal gravity. We show that unlike the general relativistic case, there are two different solutions, both including a factor which gives rise to a linear term in the potential, which also features in the neutral case. This may have significant ramifications for particle trajectories.

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I. INTRODUCTION

Einstein's theory of general relativity has succeeded extraordinarily well in solar system observations. However, when larger length scales are investigated, an overwhelming amount of dark mass energy must be introduced in order to reproduce observations such as with galactic rotation curves and the accelerating expansion of the Universe. It may be true that most of the contributing mass energy of the Universe is nonluminous, but it may also be the case that the underlying theory contains other factors whose contribution only becomes significant on large and very large scales.

A number of proposed models have attempted to add terms and factors which only become significant on the large scales such as with modified Newtonian dynamics [1] and $f(R)$ [2] gravity. On the other hand, other proposals aim to exploit some hidden assumption or principle in general relativity. One such idea is the fourth-order conformal Weyl gravity model introduced in Refs. [3,4] which is based on the underlying principle of local conformal invariance such that the manifold remains the same under local conformal stretchings of the kind

$$g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x). \tag{1}$$

This restrictive invariance principle leads to a fourth-order theory with the unique action [4]

$$I_W = \int d^4x\sqrt{-g}L = -\alpha_c \int d^4x\sqrt{-g}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}$$
$$= -2\alpha_c \int d^4x\sqrt{-g}\left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right], \tag{2}$$

where α_c is a dimensionless coupling constant and the Weyl tensor $C_{\lambda\mu\nu\kappa}$ is given by

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu})$$
$$+ \frac{1}{6}R(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}). \tag{3}$$

This tensor also satisfies the conformal principle

$$C_{\lambda\mu\nu\kappa} \rightarrow \tilde{C}_{\lambda\mu\nu\kappa} = \Omega^2(x) C_{\lambda\mu\nu\kappa}, \tag{4}$$

due to its dependence on the metric tensor.

An immediate consequence of taking this action is that the cosmological length scale, which appears in general relativity through the cosmological constant Λ, is unnecessary here. One of the interesting consequences of conformal gravity is that a number of behaviors still emerges despite not considering a cosmological constant such as the fact that the Schwarzschild-de Sitter metric [4] is also a solution to the field equations of Weyl gravity. Besides this, it has been shown [5] that conformal gravity, despite being a fourth-order theory, still admits a Newtonian potential $1/r$ term in the field of any spherically symmetric matter distribution described by a fourth-order Poisson equation. Therefore, although the second-order Poisson equation in general relativity is sufficient to generate a Newtonian potential, it is not by any means a necessary requirement, so that Newton's law of gravity remains valid in the fourth-order Weyl gravity. The conformal action of Eq. (2) leads to the field equations [6]

$$(-g)^{-1/2} g_{\mu\alpha} g_{\nu\beta} \frac{\delta I_W}{\delta g_{\alpha\beta}} = -2\alpha_c W_{\mu\nu} = -\frac{1}{2}T_{\mu\nu}, \tag{5}$$

where $T_{\mu\nu}$ is the stress-energy tensor, and

$$W_{\mu\nu} = 2C^{\alpha}{}_{\mu\nu}{}^{\beta}{}_{;\beta\alpha} + C^{\alpha}{}_{\mu\nu}{}^{\beta}R_{\alpha\beta}, \tag{6}$$

is the Bach tensor. Thus, when the Ricci tensor $R_{\mu\nu}$ vanishes, so does $W_{\mu\nu}$, implying that any vacuum solution of general relativity also carries over to conformal gravity naturally. However, the converse does not hold in general since there are other ways by which the Bach tensor can vanish.

At present, there is a number of solutions to the conformal gravity field equations. These include the static, spherically symmetric vacuum case obtained by Mannheim and Kazanas in Ref. [4],

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7}$$

where

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$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2, \quad (8)$$

which includes the Schwarzschild ($\gamma = k = 0$) and Schwarzschild-de Sitter ($\gamma = 0$) solutions as special cases, and also its charged generalization [3,7] with

$$B(r) = (1-3\beta\gamma) - \frac{(\beta(2-3\beta\gamma) + Q^2/8\alpha_c\gamma)}{r} + \gamma r - kr^2, \quad (9)$$

where Q is the charge. Some interesting work has also been done in Refs. [8–10] where general topological solutions in Weyl gravity were investigated. The particular case of cylindrically symmetric solutions in Weyl gravity has been considered in Refs. [11,12], but due to difficulties in the complex field equations, a particular gauge is chosen that does not naturally generalize the well known cylindrically symmetric solutions in general relativity. So in Ref. [13], we derived analytically the metric for a neutral static cylindrical spacetime by adopting a gauge similar to that used by Mannheim and Kazanas for the spherically symmetric case to obtain

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2 + \alpha^2 r^2 dz^2, \quad (10)$$

where

$$A^{-1}(r) = B(r) = \frac{\beta}{r} + \sqrt{\frac{3\beta\gamma}{4}} + \frac{\gamma r}{4} + k^2 r^2. \quad (11)$$

This generalizes the static black string solution in general relativity represented by the Lemos metric [14]

$$ds^2 = -\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)dt^2 + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} + r^2 d\phi^2 + \alpha^2 r^2 dz^2, \quad (12)$$

with $k = \alpha = \sqrt{-\Lambda/3}$ and $\beta = -b/\alpha = -4GM/\alpha$, so that in our case, we have a linear γr term in the metric similar to the spherically symmetric case in Eq. (8). We now seek to consider the charged case and find that the field equations which follow by taking a nonzero stress-energy tensor in Ref. [13] have two separate solutions, each representing a conformal generalization of the charged black string solution in Einstein's second-order theory [14],

$$ds^2 = -\left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right)dt^2 + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}} + r^2 d\phi^2 + \alpha^2 dz^2, \quad (13)$$

where α and β are the same constants as in Eq. (12), and $c^2 = 4G\lambda^2$, where λ is the linear charge density along the z axis. The outline of the paper is as follows. In Sec. II, we derive the metric for the charged cylindrically symmetric spacetime in conformal gravity and compare it with the general relativity analogue in Eq. (13). In Sec. III, we discuss some of the thermodynamical properties of this new solution and then present our discussion and conclusion in Sec. IV. The signature used in this paper is $(-, +, +, +)$ and units where $G = 1 = c$ are used.

II. THE CONFORMAL CYLINDRICAL METRIC

Consider first a general line element in cylindrical coordinates (t, ρ, ϕ, z)

$$ds^2 = -a(\rho)dt^2 + b(\rho)d\rho^2 + c(\rho)d\phi^2 + d(\rho)dz^2. \quad (14)$$

The metric elements are taken to depend only on the radial coordinate since a static cylindrically symmetric background metric is not expected to be curved in the angular and axial directions.

Given the local conformal invariant symmetry, the metric in Eq. (14) can be reformulated similarly to Refs. [4,7]; that is, given an arbitrary function of a spacelike coordinate parameter r, $\rho(r)$, the metric can be written as

$$ds^2 = \rho^2(r)\left[-\frac{a(\rho)}{\rho^2(r)}dt^2 + \frac{b(\rho)}{\rho^2(r)}d\rho^2 + \frac{c(\rho)}{\rho^2(r)}d\phi^2 + \frac{d(\rho)}{\rho^2(r)}dz^2\right]. \quad (15)$$

Now a choice can be made on this dependence constrained by the aim of having an end result metric which is computationally less intensive to solve for element components. We take

$$\int \frac{d\rho}{\rho^2(r)} = -\frac{1}{\rho(r)} = \int \frac{dr}{\sqrt{a(r)b(r)}}, \quad (16)$$

which then yields the metric

$$ds^2 = \rho^2\left[-A(r)dt^2 + \frac{dr^2}{A(r)} + C(r)d\phi^2 + D(r)dz^2\right], \quad (17)$$

where $A(r) = \frac{a(r)}{\rho^2(r)}$, $C(r) = \frac{c(r)}{\rho^2(r)}$, and $D(r) = \frac{d(r)}{\rho^2(r)}$.

The metric is thus conformally related to the standard cylindrical metric for static spacetimes. Following Eq. (1), we take a transformation

$$g_{\mu\nu} \rightarrow \rho^{-2}(r) g_{\mu\nu}, \quad (18)$$

which molds the metric into

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + C(r)d\phi^2 + D(r)dz^2, \quad (19)$$

which is indeed more representational of the actual degrees of freedom enjoyed by the metric. Hence, the metric will be resolved up to an overall r-dependent conformal factor.

Through Eq. (5), it follows that $W^{\mu\nu}$ can be expressed in terms of the conformally invariant stress-energy tensor through

$$W^{\mu\nu} = \frac{1}{4\alpha_c} T^{\mu\nu}, \quad (20)$$

which implies that the mass-energy information about the system is completely contained in $W^{\mu\nu}$. Furthermore, given the stress-energy tensor of the system, it is directly proportional to $W^{\mu\nu}$.

Now, to actually calculate the $W^{\mu\nu}$ tensor, we first note that in order to achieve a conformal generalization of the charged cylindrical metric in the Lemos gauge, we take a vector potential [14]

$$A_\mu = \left(-\frac{2\lambda}{\alpha r} + \text{const}, 0, 0, 0\right), \tag{21}$$

where λ and α have the same meaning as in the Lemos metric in Eq. (13). This vector potential leads to the only nonvanishing electric field component $E_r = F_{10} = \frac{2\lambda}{\alpha r^2}$, which is covariantly conserved in general and does not depend on the conformal nature of the model being considered since Maxwell's theory is conformally invariant. The Maxwell stress-energy tensor is found to only have one nonvanishing component,

$$T^r{}_r = -\frac{2\lambda^2}{\alpha^2 r^4}, \tag{22}$$

which surprisingly is independent of metric components in Eq. (19). Besides taking cylindrical coordinates, we also take

$$C(r) = r^2, \tag{23}$$

since the background spacetime has cylindrical topology. Furthermore, since we are attempting to find a conformal generalization of the charged Lemos metric, we take the Lemos gauge

$$D(r) = \alpha^2 r^2, \tag{24}$$

in the spirit of Ref. [4].

In general relativity, the common method of finding solutions is to compare the curvature part of the field equations with the simplest representation of the stress-energy components. However, in conformal gravity, this is far too difficult for the nontrivial instances given that the curvature part is replaced by the Bach tensor in Eq. (6). The alternative method used in Ref. [4] is to consider the Euler-Lagrange equations using the general line element

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\phi^2 + \alpha^2 r^2 dz^2, \tag{25}$$

where the Lemos gauge is considered, and the coefficients of dr^2 and dt^2 are left unrelated at first.

The Euler-Lagrange equations turn out to be second order [4]

$$\sqrt{-g}W^{\mu\mu} = \frac{\delta I}{\delta g_{\mu\mu}} = \frac{\partial}{\partial g_{\mu\mu}}(\sqrt{-g}\tilde{L}) - \frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\frac{\partial \tilde{L}}{\partial (g_{\mu\mu})'}\right) + \frac{\partial^2}{\partial (x^\mu)^2}\left(\sqrt{-g}\frac{\partial \tilde{L}}{\partial (g_{\mu\mu})''}\right), \tag{26}$$

where $'$ indicates differentiation with respect to r and $\tilde{L} = R_{\mu\nu}R^{\mu\nu} - R^2/3$.

Taking the variation with respect to $A(r)$ and $B(r)$ yields

$$\begin{aligned}\sqrt{\alpha^2 r^4 AB}W^{rr} = &-\frac{\alpha^2}{48A(r)^4 B(r)^3\sqrt{r^4\alpha^2 A(r)B(r)}}[-7r^2B(r)^2A'(r)^2(rB'(r)-2B(r))^2 + 2r^2A(r)B(r)(2B(r)-rB'(r))\\ &\times(4B(r)^2A''(r) + 3rA'(r)B'(r)^2 - 2B(r)(rA''(r)B'(r) + A'(r)(2rB''(r) + B'(r)))) + A(r)^2(-7r^4B'(r)^4\\ &+4r^3B(r)B'(r)^2(3rB''(r)+5B'(r)) + 4r^2B(r)^2(r^2B''(r)^2 + B'(r)^2 - 2rB'(r)(rB^{(3)}(r) + 6B''(r))) + 16rB(r)^3\\ &\times(r(rB^{(3)}(r) + 2B''(r)) - 2B'(r)) + 16B(r)^4)],\end{aligned} \tag{27}$$

and

$$\begin{aligned}\sqrt{\alpha^2 r^4 AB}W^{tt} = &\frac{-\alpha^2}{48A(r)^4B(r)^4\sqrt{r^4\alpha^2A(r)B(r)}}[56r^3B(r)^3A'(r)^3(rB'(r)-2B(r)) + r^2A(r)B(r)^2A'(r)(57r^2A'(r)B'(r)^2\\ &-4rB(r)(13rA''(r)B'(r) + A'(r)(19rB''(r)+13B'(r))) + 4B(r)^2(26rA''(r) + 7A'(r))) + 2rA(r)^2B(r)\\ &\times(29r^3A'(r)B'(r)^3 - 6r^2B(r)B'(r)(2rA''(r)B'(r) + A'(r)(9rB''(r) + 4B'(r))) + 4rB(r)^2(A'(r)(r(6rB^{(3)}(r)\\ &+13B''(r)) - 5B'(r)) + r(4rA''(r)B''(r) + (rA^{(3)}(r) + 3A''(r))B'(r))) + 8B(r)^3(2A'(r) - r(rA^{(3)}(r) + A''(r))))\\ &+A(r)^3(-16r^3(4B^{(3)}(r) + rB^{(4)}(r))B(r)^3 + 49r^4B'(r)^4 - 4r^3B(r)B'(r)^2(29rB''(r) + 11B'(r)) + 4r^2B(r)^2\\ &\times(9r^2B''(r)^2 - 5B'(r)^2 + 2rB'(r)(6rB^{(3)}(r) + 13B''(r))) + 16B(r)^4)].\end{aligned} \tag{28}$$

The other two elements $W^{\phi\phi}$ and W^{zz} do not need to be taken into account since we have a sufficient number of constraints; these two further equations provide us with an independent check of any solution which results.

Taking the time-time and radial-radial components of the metric to be the reciprocal of each other, $A(r) = 1/B(r)$, turns Eq. (27) into

$$\begin{aligned}12r^4\frac{1}{B}W^{rr} = &-4B^2 - 4rB(-2B' + r(B'' + rB'''))\\ &+ r^2(-4B'^2 - r^2B''^2 + 2rB'(2B'' + rB''')).\end{aligned} \tag{29}$$

But by Eq. (20),

$$W^{rr} = -\frac{\lambda^2}{2\alpha_c \alpha^2 r^4} B. \tag{30}$$

This eliminates both metric tensor components and radial coordinate factors from the constraint so that

$$-\frac{6\lambda^2}{\alpha_c\alpha^2} = -4B^2 - 4rB(-2B' + r(B'' + rB''')) + r^2(-4B'^2 - r^2B''^2 + 2rB'(2B'' + rB''')). \tag{31}$$

The problem can then be solved by a number of transformations, first letting $B(r) = r^2 l(r)$ giving

$$-\frac{6\lambda^2}{\alpha_c\alpha^2} = r^6(8l'^2 - r^2 l''^2 + 2rl'(4l'' + rl''')). \tag{32}$$

Then, consider $l'(r) = y(r)$ so that

$$-\frac{6\lambda^2}{\alpha_c\alpha^2} = r^6(8y^2 - r^2y'^2 + 2ry(4y' + ry'')) \tag{33}$$

reduces the overall order of the problem and taking $y(r) = r^{-3}h(r)$ gives a second-order differential equation

$$-\frac{6\lambda^2}{\alpha_c\alpha^2} = -h^2 - r^2h'^2 + 2rh(h' + rh''). \tag{34}$$

As in Ref. [13], we consider an exponential transformation of the form $r = e^t$, which results in

$$-\frac{6\lambda^2}{\alpha_c\alpha^2} = -h^2 - \dot{h}^2 + 2h\ddot{h}, \tag{35}$$

where dots denote derivatives with respect to t.

Following $h(t) = (v(t))^2$, the equation

$$\frac{6\lambda^2}{\alpha_c\alpha^2} = v^3(v - 4\ddot{v}) \tag{36}$$

can be solved, where the first integral turns out to be

$$\frac{v^2}{2} + \frac{1}{2v^2}\frac{6\lambda^2}{\alpha_c\alpha^2} - 2(\dot{v})^2 = c_1. \tag{37}$$

This admits two separate solutions given by

$$v(t) = \sqrt{-\frac{6\lambda^2}{\alpha_c\alpha^2}e^{-(t+c_2)} + \frac{e^{t+c_2}}{4} - 2c_1 + 4c_1^2 e^{-(t+c_2)}} \tag{38}$$

and

$$v(t) = \sqrt{-\frac{6\lambda^2}{\alpha_c\alpha^2}e^{(t+c_2)} + \frac{e^{-(t+c_2)}}{4} - 2c_1 + 4c_1^2 e^{(t+c_2)}}. \tag{39}$$

This means that when all the transformations are taken in reverse and the solutions are represented with the coordinate r, the solutions for $B(r)$ turn out to be

$$B(r) = \frac{2\lambda^2}{\alpha_c\alpha^2}\frac{e^{-c_2}}{r} - \frac{e^{c_2}}{4}r + c_1 - \frac{4c_1^2 e^{-c_2}}{3r} + c_3 r^2, \tag{40}$$

and

$$B(r) = \frac{6\lambda^2}{\alpha_c\alpha^2}e^{c_2}r - \frac{e^{-c_2}}{12r} + c_1 - 4c_1^2 e^{c_2} r + c_3 r^2, \tag{41}$$

respectively, where c_1, c_2, and c_3 are constants. The solution in Eq. (40) can be represented in a form similar to Eq. (12) as

$$B(r) = \frac{u}{r} + \sqrt{\frac{3\beta\gamma}{4}} + \frac{\gamma r}{4} + k^2 r^2, \tag{42}$$

where

$$u = \beta - \frac{2\lambda^2}{\alpha_c\alpha^2\gamma}, \tag{43}$$

and $k^2 = c_3$, $\gamma = -e^{c_2}$, and $c_1 = \sqrt{\frac{3\beta\gamma}{4}}$. Similarly, Eq. (41) takes the form

$$B(r) = \frac{\beta}{r} + \sqrt{\frac{3\beta\gamma}{4}} + \frac{\gamma r}{4} - \frac{\lambda^2 r}{2\alpha_c\alpha^2\beta} + k^2r^2, \tag{44}$$

with $k^2 = c_3$, $c_1 = \sqrt{\frac{3\beta\gamma}{4}}$, and $e^{c_2} = -\frac{1}{12\beta}$.

On comparison with the general relativity case in Eq. (13), the constants in Eqs. (42) and (44) regain their regular meanings except for the γ factor, which is a measure of deviation from general relativity in conformal gravity. Also, in this representation, it is easy to see that the two solutions are a charged generalization of the neutral metric since setting λ to zero retrieves Eq. (11) in both cases.

In the first case given by Eq. (42), the coefficient u retains its dependance on β but adds charge by introducing a length scale $1/\alpha_c$ which is the coupling constant of conformal gravity as given in the action in Eq. (2). Meanwhile, the same linear term arises as in the neutral case. However, the $1/r$ dependence of the charge term in $B(r)$ is unexpected and shows a divergence between general relativity where charge produces a $1/r^2$ term and conformal gravity where charge (electromagnetism) and mass (gravitation) give rise to same contribution to the exterior geometry. This will then make the task of separating the effects on the motion of a test particle from the two contributions less straightforward than in the case of general relativity. We note that in both the second-order theory and the fourth-order theory, the only nonvanishing stress-energy tensor component has the same form, namely, $T^r{}_r \propto 1/r^4$, meaning that the different $1/r$ dependence of the charge term in $B(r)$ emerges out of the geometric part of the fourth-order theory.

The same phenomenon was observed in the spherical case [3,7], and so, since it also emerges here in the cylindrical topology, it may suggest that this could be a general feature of charged solutions in conformal gravity. This would imply, as noted in Ref. [7], that electromagnetism and gravitation may share a connection and some underlying similarity.

The second solution given by Eq. (44) also possesses the same γr term as in the neutral metric, but now the charge contribution to the external geometry features as an unexpected additional linear term in the potential. This has the unrealistic consequence that the effect from a charge distribution at the source on the motion of a point charge increases with distance. On the other hand, it may shed some light about the origin of linear terms found practically in all known solutions of conformal Weyl gravity. It is still unclear whether the γr terms in Eqs. (8) and (11), and now in Eqs. (42) and (44) have a universal origin (like the cosmological term kr^2) or are system dependent (like the mass term β/r). From Eq. (44), it seems that at least part of this is clearly system dependent.

Finally, we compare our solutions in Eqs. (42) and (44) with the earlier charged solutions in spherical geometry obtained in Ref. [3], given by

$$ds^2 = -(ar^2 + br + c + d/r)dt^2 + (ar^2 + br + c + d/r)^{-1}dr^2 + r^2 d\Omega_2^2, \quad (45)$$

where

$$3bd - c^2 + 1 + \frac{3\lambda^2}{2\alpha^2\alpha_c} = 0. \quad (46)$$

The above solution can be generalized to

$$ds^2 = -(ar^2 + br + c + d/r)dt^2 + (ar^2 + br + c + d/r)^{-1}dr^2 + r^2 d\Omega_{2,K}^2, \quad (47)$$

with

$$3bd - c^2 + K^2 + \frac{3\lambda^2}{2\alpha^2\alpha_c} = 0, \quad (48)$$

where

$$d\Omega_{2,k}^2 = \frac{d\rho^2}{1 - K\rho^2} + \rho^2 d\phi^2 \quad (49)$$

represents the metric on a unit 2 sphere ($K = 1$), a unit hyperbolic manifold ($K = -1$), or a 2 torus ($K = 0$). Comparing Eqs. (42) and (44) with Eq. (47) and putting $K = 0$, we found that both solutions satisfy Eq. (48).

III. TEMPERATURE

The most immediate path for studying the quantum nature of black holes is through a consideration of their thermodynamical properties. The horizon temperature, T_h, is the natural place to start along this line of thought. This is defined in terms of the surface gravity, κ, by the relation $T_h = \frac{\kappa}{2\pi}$, which in turn is given in terms of the Killing vector fields, χ^ν, by [15]

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu \chi^\nu)(\nabla_{[\mu}\chi_{\nu]}). \quad (50)$$

The Killing vectors will be calculated at the horizon radius. For the first solution from Eq. (38), the horizon radius is given by

$$r_{1_h} = \frac{1}{12\alpha^2\beta\alpha_c k^2}[2\lambda^2 - \alpha^2\beta\alpha_c\gamma + (\alpha^4\beta^2\alpha_c^2(\gamma^2 - 24k^2\sqrt{3\beta\gamma}) - 4\alpha^2\beta\alpha_c\gamma\lambda^2 + 4\lambda^4)/[6\alpha^4\beta^2\alpha_c^2(\gamma^2 - 12k^2\sqrt{3\beta\gamma})\lambda^2 - \alpha^6\beta^3\alpha_c^3(\gamma^3 - 36\gamma k^2\sqrt{3\beta\gamma} + 864\beta k^4) - 12\alpha^2\beta\alpha_c\gamma\lambda^4 + 4\bar{\Sigma}]^{1/3} + [-\alpha^6\beta^3\alpha_c^3(\gamma^3 - 36\gamma k^2\sqrt{3\beta\gamma} + 864\beta k^2) + 6\alpha^4\beta^2\alpha_c^2(\gamma^2 - 12k^2\sqrt{3\beta\gamma})\lambda^2 - 12\alpha^2\beta\alpha_c\gamma\lambda^4 + 44\bar{\Sigma}]^{1/3}], \quad (51)$$

where

$$\bar{\Sigma} = 2\lambda^2 + 3[3\alpha^6\beta^4\alpha_c^3k^4(\alpha^6\beta^3\alpha_c^3(\gamma^3 - 48\gamma k^2\sqrt{3\beta\gamma} + 1728\beta k^4) - 12\alpha^4\beta^2\alpha_c^2(\gamma^2 - 24k^2\sqrt{3\beta\gamma})\lambda^2 + 36\alpha^2\beta\alpha_c\gamma\lambda^4 - 32\lambda^6)]^{1/2}, \quad (52)$$

while for the second given in Eq. (39), the horizon radius is given by

$$r_{2_h} = -\frac{\gamma}{12k^2} + \frac{1}{12k^2}(\gamma^2 - 24k^2\sqrt{3\beta\gamma})\Big[-\gamma^3 + 36k^2\sqrt{3\beta\gamma^3} - 864k^4u + 12\sqrt{3k^4(-3\beta(\gamma^3 - 32k^2\sqrt{3\gamma^3\beta}) + 4u(\gamma^3 - 36k^2\sqrt{3\gamma^3\beta} + 432k^4u))}\Big]^{-1/3} + \frac{1}{12\sqrt[3]{2}k^2}\Big[-2\gamma^3 + 72k^2\sqrt{3\beta\gamma^3} - 1728k^4u + \sqrt{-4(\gamma^2 - 24k^2\sqrt{3\beta\gamma})^3 + 4(\gamma^3 - 36k^2\sqrt{3\beta\gamma^3} + 864k^4u)^2}\Big]^{-1/3}. \quad (53)$$

For either case of r_{i_h} and with the metric in Eq. (19), the following surface gravity is found:

$$\kappa = \frac{\gamma}{8} - \frac{\lambda^2}{4\alpha_c \alpha^2 \beta} - \frac{\beta}{2r_{i_h}^2} + r_{i_h} k^2. \tag{54}$$

Finally, this results in a horizon temperature

$$T_h = \frac{1}{2\pi}\left(\frac{\gamma}{8} - \frac{\lambda^2}{4\alpha_c \alpha^2 \beta} - \frac{\beta}{2r_{i_h}^2 + r_{i_h} k^2}\right), \tag{55}$$

which for $\gamma = 0$, reduces to the general relativistic result [16]. Furthermore, for a vanishing charge density, we obtain the expected result already found in Ref. [13].

IV. CONCLUSION

In this paper, we studied static charged cylindrical solutions in conformal gravity, and we found that, unlike the second-order Einstein's theory, there are two independent metric tensors which can be used to describe the external geometry.

The field equations used for conformal gravity do not feature a cosmological constant. However, both solutions contain the same cosmological term which occurs in the general relativistic solution (13).

The linear γr term arises as in the neutral case (11) and the spherically symmetric solution in Eq. (8), where it was used [4] to explain the flat rotational curves of galaxies and other large matter distributions.

Moreover, we found that, unlike general relativity where the modification from the charge to the static black string solution (12) is in the form of a term which behaves like $1/r^2$, in the fourth-order case, the modification to the neutral solution (11) is different. In one of the solutions, the modification behaves like $1/r$, i.e., like a Newtonian term, as in the spherically symmetric solution (9) derived earlier by Mannheim and Kazanas, while in the second solution, the modification is in the form of an additional linear term in the metric.

The solutions found in this paper will not be applicable to most astrophysical sources since, for the most part, they are organized with spherical symmetry, but there are a number of other sources which may be amenable to this background metric description. Another place where cylindrical symmetry may arise is on the very small scale since exotic forms of collapsing matter fields may take place here. However, this type of collapse detection would, in all likelihood, be in the form of Hawking radiation, which is one possible avenue of future development. Moreover, there may also be stringlike applications in a number of other theories such as with the use of the AdS/CFT correspondence duality.

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