Graph theory results of independence, domination, covering and Turán type

by

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Abstract

We obtain graph theory results of independence, domination, covering or Turán type, most of which are bounds on graph parameters.

We first investigate the smallest number $\lambda(G)$ of vertices and the smallest number $\lambda_{\mathrm{e}}(G)$ of edges that need to be deleted from a non-empty graph G so that the resulting graph has a smaller maximum degree. Generalising the classical Turán problem, we then investigate the smallest number $\lambda_{\mathrm{c}}(G, k)$ of edges that need to be deleted from a non-empty graph G so that the resulting graph contains no k-clique. Similarly, we address the recent problem of Caro and Hansberg of eliminating all k-cliques of G by deleting the smallest number $\iota(G, k)$ of closed neighbourhoods of vertices of G, establishing in particular a sharp bound on $\iota(G, k)$ that solves a problem they posed.

Similarly to the problem of determining $\lambda(G)$, the classical domination problem is to determine the size of a smallest set X of vertices of G such that the degree of each vertex v of the graph obtained by deleting X from G is smaller than the degree of v in G (that is, each vertex in $V(G)\backslash X$ is adjacent to some vertex in X). We add the condition that the vertices in $V(G)\backslash X$ have pairwise different numbers of neighbours in X, and we denote the size of X by $\gamma_{ir}(G)$. We also consider the further modification that $V(G)\backslash X$ is an independent set of G, and we denote the size of $V(G)\backslash X$ by $\alpha_{ir}(G)$.

We obtain several sharp bounds on the graph parameters $\lambda(G)$, $\lambda_{\mathrm{e}}(G)$, $\lambda_{\mathrm{c}}(G, k)$, $\iota(G, k)$, $\gamma_{ir}(G)$, and $\alpha_{ir}(G)$ in terms of basic graph parameters such as the order, the size, the minimum degree, and the maximum degree of G. We also characterise the extremal structures for some of the bounds.

Declaration

I hereby declare that the work in this thesis is the account of my research. The work in Chapters 2, 3 and 4 is joint work with my supervisor, Prof. Peter Borg. The work in Chapter 5 is joint work with my supervisor, Prof. Peter Borg, and with Dr. Pawaton Kaemawichanurat. The work in Chapters 6 and 7 is joint work with my supervisor, Prof. Peter Borg, and with Prof. Yair Caro.

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Embarking on this Ph.D. course had always been an academic dream of mine since much earlier on. However this journey and my academic path in general would not have been possible if it was not for a number of people, some of which I am obliged to mention.

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This thesis is dedicated to my parents,

as undoubtedly, I wouldn't be where I am today

if it was not for their love and sacrifices.

Contents

Chapter 1

Introduction

1.1 Overview

We start this thesis with a brief description of the work presented in it. The main definitions and notation are provided in the next section.

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We first investigate the smallest number $\lambda(G)$ of vertices and the smallest number $\lambda_{\mathrm{e}}(G)$ of edges that need to be deleted from a non-empty graph G so that the resulting graph has a smaller maximum degree. Generalising the classical Turán problem, we then investigate the smallest number $\lambda_{\mathrm{c}}(G,k)$ of edges that need to be deleted from a non-empty graph G so that the resulting graph contains no k-clique. Similarly, we address the recent problem of Caro and Hansberg of eliminating all k-cliques of G by deleting the smallest number $\iota(G,k)$ of closed neighbourhoods of vertices of G, establishing in particular a sharp bound on $\iota(G,k)$ that solves a problem they posed.

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We obtain several sharp bounds on the graph parameters $\lambda(G)$, $\lambda_{\mathrm{e}}(G)$, $\lambda_{\mathrm{c}}(G,k)$, $\iota(G,k)$, $\gamma_{ir}(G)$, and $\alpha_{ir}(G)$ in terms of basic graph parameters such as the order, the size, the minimum degree, and the maximum degree of G. We also characterise the extremal structures for some of the bounds.

A more detailed outline of the contents of the thesis is provided in Section 1.3.

1.2 Basic definitions and notation

In this section, we define some basic graph theory concepts and notation that will be used throughout the thesis. We shall use capital letters such as X to denote sets or graphs, and small letters such as x to denote non-negative integers or functions or elements of a set. The set $\{1, 2, \dots\}$ of positive integers is denoted by $\mathbb{N}$. For any $n \in \mathbb{N}$, the set $\{1, \dots, n\}$ is denoted by $[n]$. It is to be assumed that arbitrary sets are finite. For a set X, the set of r-element subsets of X is denoted by $\binom{X}{r}$. For any set X, the *power set of* X, denoted by $\mathcal{P}(X)$, (or 2^X), is the set of all subsets of X.

A *graph* G is a pair (X, Y), where X is a set, called the *vertex set of* G, and Y is a subset of $\binom{X}{2}$ and is called the *edge set of* G. The vertex set of G and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. It is to be assumed that arbitrary graphs have non-empty vertex sets. An element of $V(G)$ is called a *vertex of* G, and an element of $E(G)$ is called an *edge of* G. We may represent an edge $\{v, w\}$ by vw. If vw is an edge of G, then v and w are said to be *adjacent in* G, and we say that w is a *neighbour of* v *in* G (and vice-versa). An edge vw is said to be *incident to* x if $x = v$ or $x = w$.

For any $v \in V(G)$, $N_G(v)$ denotes the set of neighbours of v in G, $N_G[v]$ denotes $N_G(v) \cup \{v\}$ and is called the *closed neighbourhood of* v *in* G, $E_G(v)$ denotes the set of edges of G that are incident to v, and $d_G(v)$ denotes $|N_G(v)|$ $(= |E_G(v)|)$ and is called the *degree of* v *in* G. For $X \subseteq V(G)$, we denote $\bigcup_{v \in X} N_G(v)$, $\bigcup_{v \in X} N_G[v]$ and $\bigcup_{v \in X} E_G(v)$ by $N_G(X)$, $N_G[X]$ and $E_G(X)$ respectively. The *minimum degree of* G is $\min\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\delta(G)$. The *maximum degree of* G is $\max\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\Delta(G)$. Let $M(G)$ denote the set of vertices of G of degree $\Delta(G)$. If $G = (\emptyset, \emptyset)$, then we take both $\delta(G)$ and $\Delta(G)$ to be 0. If a vertex v of a graph G has only one neighbour in G, then v is called a *leaf of* G.

If H is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is said to be a *subgraph of* G, and we say that G *contains* H. For $X \subseteq V(G)$, $(X, E(G) \cap \binom{X}{2})$ is called the *subgraph of* G *induced by* X and is denoted by $G[X]$. For a set S, $G - S$ denotes the subgraph of G obtained by removing from G the vertices in S and all edges incident to them, that is, $G - S = G[V(G) \backslash S]$. We may abbreviate $G - \{v\}$ to $G - v$. For $L \subseteq E(G)$, $G - L$ denotes the subgraph of G obtained by removing from G the edges in L, that is, $G - L = (V(G), E(G)\backslash L)$. We may abbreviate $G - \{e\}$ to $G - e$.

If $n \geq 2$ and $v_1, v_2, \dots, v_n$ are the distinct vertices of a graph G with $E(G) = \{v_i v_{i+1} \colon i \in [n-1]\}$, then G is called a *v_1v_n-path* or simply a *path*. The path $([n], \{\{1,2\}, \dots, \{n-1, n\}\})$ is denoted by P_n, and is called the *n-path*. For a path P, the *length of P*, denoted by $l(P)$, is $|V(P)| - 1$ (the number of edges of P).

For a graph G and $u, v \in V(G)$, the *distance of v from u*, denoted by $d_G(u,v)$, is given by $d_G(u,v) = 0$ if $u = v$, $d_G(u,v) = \min\{l(P) \colon P \text{ is a } uv\text{-path}, G \text{ contains } P\}$ if G contains a uv-path, and $d_G(u,v) = \infty$ if G contains no uv-path.

Where no confusion arises, the subscript G may be omitted from any of the notation that uses it; for example, $N_G(v)$ may be abbreviated to $N(v)$.

A graph G is *connected* if for every $u, v \in V(G)$ with $u \neq v$, G contains a uv-path. A *component* of G is a maximal connected subgraph of G (that is, one that is not a subgraph of any other connected subgraph of G). It is easy to see that if H and K are distinct components of a graph G, then H and K have no common vertices (and therefore no common edges). If H is a component of G, $v \in V(G)$, and $V(H) = \{v\}$, then H is called a *singleton of G* and v is called an *isolated vertex of G*. Note that for $v \in V(G)$, v is an isolated vertex of G if and only if $d_G(v) = 0$.

If $G, G_1, \dots, G_r$ are graphs such that $V(G) = \bigcup_{i=1}^r V(G_i)$ and $E(G) = \bigcup_{i=1}^r E(G_i)$, then we say that G is the *union of $G_1, \dots, G_r$*.

If $X_1, \dots, X_s$ are sets such that no r of $X_1, \dots, X_s$ have a common element, then $X_1, \dots, X_s$ are said to be *r-wise disjoint*. Graphs $G_1, \dots, G_s$ are said to be *r-wise vertex-disjoint* if $V(G_1), \dots, V(G_s)$ are r-wise disjoint.

Graphs $G_1, \ldots, G_s$ are said to be *r-wise edge-disjoint* if $E(G_1), \ldots, E(G_s)$ are r-wise disjoint. We may use the term *pairwise* instead of *2-wise*.

It is easy to see that if $G_1, \ldots, G_r$ are the distinct components of G, then $G_1, \ldots, G_r$ are pairwise vertex-disjoint and hence pairwise edge-disjoint, and G is the union of $G_1, \ldots, G_r$.

If $n \geq 3$ and $v_1, v_2, \ldots, v_n$ are the distinct vertices of a graph G with $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$, then G is called a *cycle*. The cycle $([n], \{\{1, 2\}, \ldots, \{n-1, n\}, \{n, 1\}\})$ is denoted by C_n. A *triangle* is a copy of C_3.

A graph G is a *tree* if G is a connected graph that contains no cycles. If $|V(G)| = k + 1$ and $E(G) = \{xv \colon v \in V(G)\backslash\{x\}\}$ for some $x \in V(G)$, then G is called a *k-star*, or simply a *star*, with *centre* x. The k-star $(\{0\} \cup [k], \{\{0, i\} \colon i \in [k]\})$ is denoted by $K_{1,k}$. A copy H of $K_{1,k}$ will be called a *k-star* or simply a *star*, and, if $k \geq 2$, then the vertex of H of degree k will be called the *centre of H*. Thus a star is a tree. A *forest* is a graph whose components are trees.

If G is a graph, then the *complement of* G, denoted by $\bar{G}$, is the graph $(V(G), \binom{V(G)}{2} \setminus E(G))$. Thus, two distinct vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

If $C \subseteq V(G)$ such that every two distinct vertices in C are adjacent in G, then C is said to be a *clique of* G. If C is a clique of G and $|C| = k$, then C is said to be a *k-clique of* G. Let $\mathcal{C}_k(G)$ denote the set of distinct k-cliques of G. The size of a largest clique of G is called the *clique number of* G and is denoted by $\omega(G)$.

For $I \subseteq V(G)$, we say that I *is an independent set of* G if for every two distinct vertices $u, v \in I$, u is not adjacent to v in G. The *independence number of* G, denoted by $\alpha(G)$, is the size of a largest independent set of G.

A graph G is *complete* if every two vertices of G are adjacent (that is, $E(G) = \binom{V(G)}{2}$, or equivalently $V(G)$ is a clique of G). The complete graph $([n], \binom{[n]}{2})$ is denoted by K_n. A graph G is *empty* if no two vertices of G are adjacent (that is, $E(G) = \emptyset$, or equivalently $V(G)$ is an independent set of G). The empty graph $([n], \emptyset)$ is denoted by E_n. A graph G is a *singleton* if $|V(G)| = 1$, in which case G is complete and empty.

If G is a graph such that $V(G)$ is partitioned into two non-empty sets V_1 and V_2 such that every edge of G has one vertex in V_1 and the other in V_2 (or rather, V_1 and V_2 are independent sets of G), then we say that G is a *bipartite graph* with *partite sets* V_1 and V_2. A bipartite graph G with partite sets V_1, V_2 and with $E(G) = \{uv \colon u \in V_1, v \in V_2\}$ is said to be *complete*; the complete bipartite graph G with partite sets $[s]$ and $[s+1, s+t]$ is denoted by $K_{s,t}$.

A graph G is *regular* if the degrees of its vertices are the same. If $k \in \{0\} \cup \mathbb{N}$ and the degree of each vertex of G is k, then G is called *k-regular*.

Let H be a graph. A graph G is a *copy of* H if there exists a bijection $f : V(G) \to V(H)$ such that $E(H) = \{f(u)f(v) \colon uv \in E(G)\}$, and we write $G \simeq H$. Thus, a copy of H is a graph obtained by relabeling the vertices of H.

For $A, D \subseteq V(G)$, we say that D *dominates* A *in* G if for every $v \in A$, v is in D or v has a neighbour in G that is in D. A *dominating set of* G is a set that dominates $V(G)$ in G. The *domination number of* G, denoted by $\gamma(G)$, is the size of a smallest dominating set of G.

For $L \subseteq E(G)$ and $X \subseteq V(G)$, we say that L *is an edge cover of* X *in* G if for each $v \in X$ with $d_G(v) > 0$, v is incident to at least one edge in L. An edge cover of $V(G)$ in G is called an *edge cover of* G. The *edge covering number of* G is the size of a smallest edge cover of G and is denoted by $\beta'(G)$.

Given an injective function $f \colon X \to Y$, the set $\{\{x, y\} \colon x \in X, y = f(x)\}$ is called a *matching from* X *into* Y. A set M is called a *matching of* G if for some $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, M is a matching from X into Y. A matching M of G is *perfect* if $V(G) = \bigcup_{e \in M} e$. The *matching number of* G is the size of a largest matching of G and is denoted by $\alpha'(G)$.

1.3 Background and outline of the thesis

We shall now give an outline of the thesis together with some background from the literature. The problems considered in Chapters 2 and 3 can be generalised as follows. Given a graph G and a certain graph parameter ρ, we investigate the size of a smallest set of vertices or edges whose removal from G yields a graph with a smaller value of ρ. More formally, given a graph G and a certain graph parameter ρ, we investigate the size of a smallest set X of vertices or edges such that $\rho(G - X) < \rho(G)$. In chapters 4 and 5, we consider the stronger condition that the value of the parameter becomes smaller than a given non-negative integer k. In Chapters 2 and 3, ρ is the maximum degree, and in chapters 4 and 5, ρ is the clique number. In Chapters 6 and 7 we consider a variant of independence and domination respectively. The work in this thesis can be classified as work of independence, domination, covering or Turán type. These are classical areas of extremal graph theory and are widely studied. We will give some background on each of them and then the work in each of the subsequent chapters in more detail.

The domination number of a graph is one of the most extensively studied parameters in extremal graph theory. Many of the main results of the classical domination bound can be seen in [21, 22, 29–31]. Numerous variants have been studied; many of the earliest ones are referenced in [32], but nowadays there are several others. We consider a number of variations of the classical domination problem throughout this thesis.

The independence number of a graph is also extensively studied (see [5, 13, 24, 25, 27, 34, 37, 53]). The notions of independence and domination are closely related. We observe that for a given graph G, a maximal independent set of G is a dominating set of G. Thus, any maximal independent set of G is necessarily also a minimal dominating set. Numerous variants of independence have been studied throughout the years; see, for example, [3, 16, 19, 28].

Another widely-studied area of extremal graph theory is Turán theory, the aim of which is mainly to establish the maximum number of edges a graph G can have if it does not contain a copy of a given graph F (that is, F is forbidden from being a subgraph of G). This has its origins in the classical theorem of Turán [52], which solves the problem for the case where F is the complete graph K_k and characterises the extremal graphs. The special case $k = 3$ had been established by Mantel [42]. In [1], Aigner provides a brief insight to the problem and discusses some of the known proofs of this theorem. Several variants of this problem have been studied; see, for example, [2, 33, 44, 51]. In [36], Keevash surveys known results and methods, and discusses some open problems. Observe that a clique of G is an independent set of $\bar{G}$, and vice versa. Thus, the problem of reducing the clique number to a value less than k is equivalent to the problem of reducing the independence number to a value less than k.

Edge covering type problems are a special case of *set covering* type problems, which are the most prominent *covering* type problems. In 1959, Gallai [26] established the immediate connection between edge covering and matching by showing that $\alpha'(G) + \beta'(G) = |V(G)|$ for every graph G without isolated vertices. There is also a connection between edge covering and independence. Indeed, in 1916, Kőnig [38] proved that for any bipartite graph G without isolated vertices, $\alpha(G) = \beta'(G)$. In general, $\alpha(G) \leq \beta'(G)$ for any graph G without isolated vertices because, if X is an edge cover of G and I is an independent set of G, then each vertex of I is in some edge in X, and no edge in X contains more than one vertex in I. In [48], Paschos surveys some approximation algorithms for some of these covering type problems.

In Chapter 2, we investigate the minimum number of vertices that need to be removed from a graph so that the new graph obtained has a smaller maximum degree. Recall that $M(G)$ denotes the set of vertices of G of degree $\Delta(G)$. We call a subset R of $V(G)$ a *Δ-reducing set of G* if $\Delta(G-R) < \Delta(G)$ or $V(G) = R$ (note that $V(G)$ is the smallest Δ-reducing set of G if and only if $\Delta(G) = 0$). Note that R is a Δ-reducing set of G if and only if $M(G) \subseteq N_G[R]$. Let $\lambda(G)$ denote the size of a smallest Δ-reducing set of G. We provide several sharp bounds for $\lambda(G)$ in terms of basic graph parameters such as the order $|V(G)|$, the size $|E(G)|$, the maximum degree $\Delta(G)$, the number of vertices of maximum degree $|M(G)|$, and other graph parameters.

Note that D dominates $M(G)$ in G if and only if D is a Δ-reducing set of G. Therefore $\lambda(G) = \min\{|D|\colon D \text{ dominates } M(G) \text{ in } G\}$. Thus, the problem we consider is a variation of the classical domination problem defined above.

In Chapter 3, we investigate the minimum number of edges that need to be removed from a graph so that the new graph obtained has a smaller maximum degree. We call a subset L of $E(G)$ a *Δ-reducing edge set of G* if $\Delta(G-L) < \Delta(G)$ or $\Delta(G) = 0$. We denote the size of a smallest Δ-reducing edge set of G by $\lambda_{\mathrm{e}}(G)$. We provide several bounds and equations for $\lambda_{\mathrm{e}}(G)$. Note that L is a Δ-reducing edge set of G if and only if L is an edge cover of $M(G)$ in G. Thus, the problem we consider is an *edge covering* type problem.

In Chapter 4, we consider a generalisation of the classical problem of Turán [52]. We investigate the smallest number of edges that need to be removed from a non-empty graph G so that the resulting graph does not contain k-cliques. We call $L \subseteq E(G)$ a *k-clique reducing edge set of G* if $\omega(G-L) < k$. We denote the size of a smallest k-clique reducing edge set of G by $\lambda_{\mathrm{c}}(G,k)$. That is, $\lambda_{\mathrm{c}}(G,k) = \min\{|L|\colon L \subseteq E(G), \omega(G-L) < k\}$. We call $\lambda_{\mathrm{c}}(G,k)$ the *k-clique reducing edge number of G*. We provide a number of sharp bounds and equations for $\lambda_{\mathrm{c}}(G,k)$.

In Chapter 5, we investigate the size of a smallest set of vertices that when removed together with its closed neighbourhood from a graph, we obtain a subgraph with no k-cliques. More generally, if $\mathcal{F}$ is a set of graphs and F is a copy of a graph in $\mathcal{F}$, then we call F an *$\mathcal{F}$-graph.* If G is a graph and $D \subseteq V(G)$ such that $G - N[D]$ contains no $\mathcal{F}$-graph, then D is called an *$\mathcal{F}$-isolating set of G*. Let $\iota(G,\mathcal{F})$ denote the size of a smallest $\mathcal{F}$-isolating set of G. We call $D \subseteq V(G)$ a *k-clique isolating set of G* if $G - N_G[D]$ contains no k-clique. We define $\iota(G,k)$ to be the size of a smallest k-clique isolating set of G. That is, $\iota(G,k) = \iota(G,\{K_k\}) = \min\{|D|: D \subseteq V(G), \omega(G - N_G[D]) < k\}$. The study of isolating sets was introduced recently by Caro and Hansberg [14, 15]. It is an appealing and natural generalization of the classical domination problem. Indeed, D is a $\{K_1\}$-isolating set of G if and only if D is a dominating set of G (that is, $N[D] = V(G)$), so $\iota(G,\{K_1\})$ is the domination number of G. We obtain sharp upper bounds for $\iota(G,k)$, and consequently we solve a problem of Caro and Hansberg [14].

In Chapters 6 and 7, we consider a variant of independence and domination, respectively. We consider the notions of irregular independence and irregular domination respectively, as counterparts of the notions of regular independence and regular domination (also referred to as fair domination), which were recently introduced in [17, 18]. If D is a smallest dominating set of a graph G with the condition that the vertices in $V(G) \setminus D$ have pairwise different numbers of neighbours in D, then we call D an *irregular dominating set of* G, and we denote the size of D by $\gamma_{ir}(G)$. If we consider the further modification that $V(G) \setminus D$ is an independent set of G, and assume that G does not contain isolated vertices, then $V(G) \setminus D$ is an *irregular independent set of* G, and we denote the size of a largest irregular independent set of G by $\alpha_{ir}(G)$. The formal definitions of these parameters are as follows.

If A is an independent set of a graph G such that the vertices in A have pairwise different degrees, then we call A an *irregular independent set of* G. The size of a largest irregular independent set of G will be called the *irregular independence number of* G and will be denoted by $\alpha_{ir}(G)$. If A is an independent set of a graph G such that the vertices in A have the same degree, then A is called a *regular independent set of* G. The size of a largest regular independent set of G is called the *regular independence number of* G and is denoted by $\alpha_{reg}(G)$.

If D is a dominating set of G such that $|N(u) \cap D| \neq |N(v) \cap D|$ for every two distinct vertices u and v in $V(G)\backslash D$, then we call D an *irregular dominating set of* G. The size of a smallest irregular dominating set of G will be called the *irregular domination number of* G and will be denoted by $\gamma_{ir}(G)$. If D is a dominating set of G such that $|N(u) \cap D| = |N(v) \cap D|$ for every two vertices u and v in $V(G)\backslash D$, then D is called a *regular dominating set of* G. The size of a smallest regular dominating set of G is called the *regular domination number of* G and is denoted by $\gamma_{reg}(G)$.

Trivially, these are variants of the classical independence and domination defined above. Chapters 6 and 7 are organized as follows. In Section 6.2, we prove several sharp upper bounds on $\alpha_{ir}(G)$. In Section 6.3, we characterize the graphs G with $\alpha_{ir}(G) = 1$, we determine those that are planar, and we determine those that are outerplanar. In Section 6.4, we provide sharp Nordhaus–Gaddum-type bounds for the irregular independence number. In Section 7.2, we prove several sharp lower bounds for $\gamma_{ir}(G)$, we characterize the graphs G with $\gamma_{ir}(G) \in \{n, n-1\}$, and we also provide some upper bounds for $\gamma_{ir}(G)$. In Section 7.3, we provide sharp upper bounds relating $\alpha_{ir}(G)$ to $\gamma_{ir}(G)$ or $\gamma_{ir}(\bar{G})$. In Section 7.4, we provide sharp Nordhaus–Gaddum-type bounds for the irregular domination number.

The work in Chapter 2 is published in [9, 10]. The work in Chapter 3 is published in [11]. The work in Chapters 6 and 7 is published in [12].

Chapter 2

Reducing the maximum degree of a graph by deleting vertices

2.1 Introduction

In this chapter we investigate the minimum number of vertices that need to be removed from a graph so that the new graph obtained has a smaller maximum degree. Definitions and notation from Chapter 1 will be used. Recall that $M(G)$ denotes the set of vertices of G of degree $\Delta(G)$. We call a subset R of $V(G)$ a Δ-reducing set of G if $\Delta(G - R) < \Delta(G)$ or $V(G) = R$ (note that $V(G)$ is the smallest Δ-reducing set of G if and only if $\Delta(G) = 0$). Note that R is a Δ-reducing set of G if and only if $M(G) \subseteq N_G[R]$. Let $\lambda(G)$ denote the size of a smallest Δ-reducing set of G.

We provide several sharp bounds for $\lambda(G)$. Our main results are given in the next section. In Section 2.3, we investigate $\lambda(G)$ from a structural point of view, particularly observing how this parameter changes with the removal of vertices. Some of the structural results are then used in the proofs of the main results; these proofs are given in Section 2.4.

Recall that a subset D of $V(G)$ is called a dominating set of G if $N[D] = V(G)$. A dominating set of G is a Δ-reducing set of G. Thus, the problem of minimizing the size of a Δ-reducing set is a variant of the classical domination problem; the aim is to use as few vertices as possible to dominate the vertices of maximum degree rather than all the vertices. If G is k-regular (that is, $d(v) = k$ for each $v \in V(G)$), then our problem is the same as the classical one, that is, $\lambda(G) = \gamma(G)$.

In this chapter, we present our work from our recent papers in [9] and [10]. The parameter $\lambda(G)$ was first introduced and studied in our recent paper [9]. An application is indicated in [55].

We can now move on to the next section of this chapter, where we will present our main results on $\lambda(G)$.

2.2 Main results

Our first result is a lower bound for $\lambda(G)$.

Proposition 2.2.1. *For any graph G,*

$$\lambda(G) \geq \frac{|M(G)|}{\Delta(G) + 1}.$$

Proof. Let $k = \Delta(G)$. For any $X \subseteq V(G)$, we have $|N_G[X]| \leq \sum_{v \in X} |N_G[v]| \leq (k+1)|X|$. Let S be a Δ-reducing set of G of size $\lambda(G)$. Since $M(G) \subseteq N_G[S]$, $|M(G)| \leq |N_G[S]| \leq (k+1)|S| = (k+1)\lambda(G)$. The result follows. $\square$

The bound above is sharp; for example, it is attained by complete graphs.

We now provide a number of upper bounds for $\lambda(G)$.

Proposition 2.2.2. *For any non-empty graph G,*

$$\lambda(G) \leq \min\left\{|M(G)|, \gamma(G), \frac{|E(G)|}{\Delta(G)}\right\}.$$

Proof. Obviously, $G - M(G)$ has no vertex of degree $\Delta(G)$. Thus $\lambda(G) \leq |M(G)|$.

Let D be a dominating set of G. Since every vertex in $V(G)\backslash D$ is adjacent to some vertex in D, $d_{G-D}(v) \leq d_G(v)-1 \leq \Delta(G)-1$ for each $v \in V(G-D)$. Thus $\lambda(G) \leq |D|$. Consequently, $\lambda(G) \leq \gamma(G)$.

Since G is non-empty, $\Delta(G) > 0$. Let v_1 be a vertex of G of degree $\Delta(G)$. If $\Delta(G - v_1) = \Delta(G)$, then let $v_2, \dots, v_r$ be distinct vertices of G such that $\Delta(G-\{v_1, \dots, v_r\}) < \Delta(G)$ and $d_{G-\{v_1,\dots,v_{i-1}\}}(v_i) = \Delta(G)$ for each $i \in [r]\backslash\{1\}$. If $\Delta(G - v_1) < \Delta(G)$, then let $r = 1$. Let $R = \{v_1, \dots, v_r\}$. Then $R \subseteq M(G)$, and by the choice of $v_1, \dots, v_r$, no two vertices in R are adjacent. Thus $|E(G - R)| = |E(G)| - r\Delta(G)$, and hence $|E(G)| \geq r\Delta(G)$. Therefore, we have $\lambda(G) \leq |R| = r \leq \frac{|E(G)|}{\Delta(G)}$. $\square$

Let $\overline{d}(G)$ denote the average degree $\frac{1}{|V(G)|}\sum_{v\in V(G)} d_G(v)$ of G. Proposition 2.2.2 and the handshaking lemma ($\overline{d}(G)|V(G)| = 2|E(G)|$) give us

$$\lambda(G) \leq \frac{\overline{d}(G)|V(G)|}{2\Delta(G)}. \tag{2.1}$$

It immediately follows that $\lambda(G) \leq \frac{1}{2}|V(G)|$. In Section 2.4, we characterize the cases in which the bound $\frac{1}{2}|V(G)|$ is attained.

Theorem 2.2.3. *For any non-empty graph G,*

$$\lambda(G) \leq \frac{|V(G)|}{2},$$

and equality holds if and only if G is either a disjoint union of copies of K_2 or a disjoint union of copies of C_4.

The subsequent new theorems in this section are also proved in Section 2.4. The following sharp bound is our primary contribution.

Theorem 2.2.4. *If G is a non-empty graph, $n = |V(G)|$, $k = \Delta(G)$ and $t = |M(G)|$, then*

$$\lambda(G) \leq \frac{n + (k-1)t}{2k}.$$

In [9], we pointed out four facts regarding Theorem 2.2.4. The first is that it immediately implies (2.1). Indeed, let $S = \{v \in V(G) \colon d_G(v) = 0\}$, $G' = G - S$ and $n' = |V(G')|$; then $\lambda(G') \leq \frac{n'+(k-1)t}{2k} = \frac{kt+n'-t}{2k} \leq \frac{1}{2k}\sum_{v \in V(G')} d_G(v) = \frac{1}{2k}\sum_{v \in V(G)} d_G(v) = \frac{\overline{d}(G)n}{2k}$.

Secondly, the bound in Theorem 2.2.4 can be attained in cases where $\lambda(G) = t$ and also in cases where $\lambda(G) < t$. If G is a disjoint union of t copies of $K_{1,k}$, then $\lambda(G) = t$, $n = (k+1)t$, and hence $\lambda(G) = \frac{n+(k-1)t}{2k}$. If G is one of the extremal structures in Theorem 2.2.3, then $t = n$ and $\lambda(G) = \frac{n}{2} = \frac{n+(k-1)t}{2k}$.

Thirdly, it is immediate from the proof of Theorem 2.2.4 that the inequality in the result is strict if the closed neighbourhood of some vertex of

G contains at least 3 members of $M(G)$; see (2.8).

Fourthly, since $\lambda(G) \leq t$, Theorem 2.2.4 is not useful if $t \leq \frac{n+(k-1)t}{2k}$. This occurs if and only if $t \leq \frac{n}{k+1}$. Thus, if $t \leq \frac{n+(k-1)t}{2k}$, then $\lambda(G) \leq \frac{n}{k+1}$. We have

$$\lambda(G) \leq \max\left\{\frac{n}{k+1}, \frac{n+(k-1)t}{2k}\right\}, \tag{2.2}$$

and if $\frac{n}{k+1} < \frac{n+(k-1)t}{2k}$ and $k \geq 2$, then $n < (k+1)t$ and $\lambda(G) \leq \frac{n+(k-1)t}{2k} < t$.

In [10], we managed to come up with a new proof for the bound in Theorem 2.2.4; by induction on the number of vertices, n. The new argument enabled us to characterise the extremal graphs which attain the bound. We first define a special graph.

If $k \geq 2$, $S_1, \dots, S_t$ are vertex-disjoint k-stars, and G is a graph such that $V(G) = \bigcup_{i=1}^{t} V(S_i)$, $\bigcup_{i=1}^{t} E(S_i) \subseteq E(G)$, $\Delta(G) = k$, and $|M(G)| = t$ (or, equivalently, $M(G)$ is the set of centres of $S_1, \dots, S_t$), then we call G a *special k-star t-union* and we call $S_1, \dots, S_t$ the *constituents of G*.

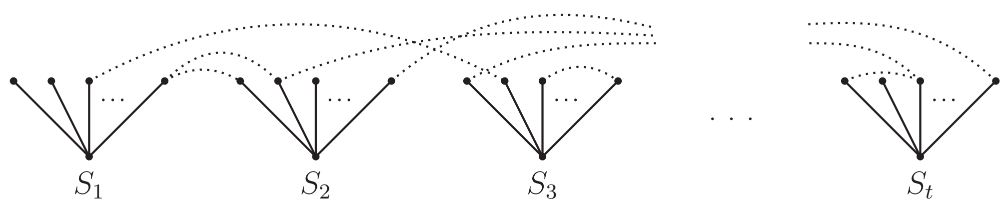

Figure 1: An illustration of a special k-star t-union.

We can now present the following Theorem.

Theorem 2.2.5. *If G is a non-empty graph, $n = |V(G)|$, $k = \Delta(G)$, and $t = |M(G)|$, then*

$$\lambda(G) \leq \frac{n+(k-1)t}{2k}.$$

Moreover, equality holds if and only if one of the following holds:

(i) $k = 1$ and each component of G is a copy of K_2,

(ii) $k = 2$ and each component of G is a copy of P_3 or C_4,

(iii) $k \geq 2$ and G is a special k-star t-union.

It turns out that if G is a tree, then, although we may have $\frac{n}{k+1} < \frac{n+(k-1)t}{2k}$ (that is, $n < (k+1)t$, as in the case of trees that are paths with at least 4 vertices), $\lambda(G) \leq \frac{n}{k+1}$ holds.

Theorem 2.2.6. *For any tree T,*

$$\lambda(T) \leq \frac{|V(T)|}{\Delta(T)+1}.$$

In [9], we pointed out that the bound is sharp. In [10], we determine the trees which attain the bound; but before stating the result, we first define a special graph.

If $S_1, \ldots, S_t$ are vertex-disjoint k-stars and T is a tree such that $V(T) = \bigcup_{i=1}^{t} V(S_i)$, $\bigcup_{i=1}^{t} E(S_i) \subseteq E(T)$, and $\Delta(T) = k$, then we call T *k-special* (it is easy to see that T has $t-1$ edges $e_1, \ldots, e_{t-1}$ such that $E(T) \setminus \bigcup_{i=1}^{t} E(S_i) = \{e_1, \ldots, e_{t-1}\}$ and, for each $i \in [t-1]$, there exist some $j, k \in [t]$ such that $j \neq k$ and $e_i = \{v_j, v_k\}$ for some leaf v_j of S_j and some leaf v_k of S_k).

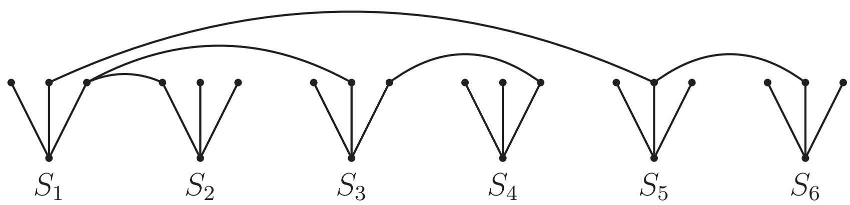

Figure 2: An illustration of a k-special tree with $k = 3$ and $t = 6$.

Theorem 2.2.7. *The bound in Theorem 2.2.6 is attained if and only if T is k-special.*

By Proposition 2.2.2, any upper bound for $\gamma(G)$ is an upper bound for $\lambda(G)$. Domination is widely studied and several bounds are known for $\gamma(G)$; see [29]. The following well-known domination bound of Reed [50] gives us $\lambda(G) \leq \frac{3}{8}|V(G)|$ when $\delta(G) \geq 3$.

Theorem 2.2.8 ([50])**.** *If G is a graph with $\delta(G) \geq 3$, then*

$$\gamma(G) \leq \frac{3}{8}|V(G)|.$$

Arnautov [6], Payan [49] and Lovász [41] independently proved that

$$\gamma(G) \leq \left(\frac{1 + \ln(\delta(G) + 1)}{\delta(G) + 1}\right) n. \tag{2.3}$$

Alon and Spencer [5] gave a probabilistic proof using Alon's well-known argument in [4]. By adapting the argument to our problem of dominating $M(G)$ rather than all of $V(G)$, we prove the following improved bound for $\lambda(G)$, replacing in particular $\delta(G)$ by $\Delta(G)$.

Theorem 2.2.9. *If G is a graph, $n = |V(G)|$, $k = \Delta(G)$ and $t = |M(G)|$, then*

$$\lambda(G) \leq \frac{n \ln(k + 1) + t}{k + 1}.$$

We now give a brief discussion on regular graphs. If G is regular, then $M(G) = V(G)$, and hence $\lambda(G) = \gamma(G)$. For a regular graph G, Theorem 2.2.9 is given by (2.3) as $\delta(G) = \Delta(G)$. Kostochka and Stodolsky [40] obtained an improvement of the bound in Theorem 2.2.8 for 3-regular graphs.

Theorem 2.2.10 ([40]). *If G is a connected 3-regular graph with $|V(G)| \geq 9$, then*

$$\gamma(G) \leq \frac{4}{11}|V(G)|.$$

Also, they showed in [39] that there exists an infinite class of connected 3-regular graphs G with $\gamma(G) > \left\lceil \frac{|V(G)|}{3} \right\rceil > \left\lceil \frac{|V(G)|}{\Delta(G)+1} \right\rceil$. This means that the lower bound in Proposition 2.2.1 is not always attained by regular graphs, and that the bound in Theorem 2.2.6 does not extend to the class of regular graphs. For regular graphs G with $\Delta(G) \leq 2$, the problem is trivial. Indeed, if such a graph G is connected, then either G has only one edge or G is a cycle. It is easy to check that $\{1+3t\colon 1+3t \in [n]\}$ is a Δ-reducing set of C_n of minimum size, and hence $\lambda(C_n) = \lceil \frac{n}{3} \rceil = \left\lceil \frac{|V(C_n)|}{\Delta(C_n)+1} \right\rceil$.

As pointed out above, a dominating set is a Δ-reducing set, so $\lambda(G) \leq \gamma(G)$. We conclude this section with a brief discussion on how the bounds above compare with well-known domination bounds. First we note that our bound $\frac{n+(k-1)t}{2k}$ on $\lambda(G)$ is at most Ore's upper bound $\frac{n}{2}$ on $\gamma(G)$ (for $\delta(G) \geq 1$) [47], and it is equal to it if and only if G is k-regular (in which case $\lambda(G) = \gamma(G)$). However, taking $\delta = \delta(G)$, we see that our bound for $k \geq 2$ is at most the classical upper bound $\frac{1+\ln(\delta+1)}{\delta+1}n$ on $\gamma(G)$ if and only if $t \leq \frac{n}{\delta+1}\left(1 + 2\ln(\delta+1) + \frac{2}{k-1}\ln(\delta+1) + \frac{k-\delta}{k-1}\right)$. Thus, the improvement offered by our bound is limited. It is interesting that, on the other hand, the upper bound $\frac{n}{k+1}$ in Theorem 2.2.6 is a basic lower bound for the domination number of any graph G with $\Delta(G) = k$ (see [29]), meaning that no domination number upper bound is better than it.

2.3 Structural results

In this section, we provide some observations on how $\lambda(G)$ is affected by the structure of G and by removing vertices or edges from G. Some of the following facts are used in the proofs of our main results.

Lemma 2.3.1. *If G is a graph, H is a subgraph of G with $\Delta(H) = \Delta(G)$, and R is a Δ-reducing set of G, then $R \cap V(H)$ is a Δ-reducing set of H.*

Proof. Let $S = R \cap V(H)$. Consider any $v \in M(H)$. Since $\Delta(H) = \Delta(G)$, $v \in M(G)$ and $N_H[v] = N_G[v]$. Since $v \in M(G)$, $u \in N_G[v]$ for some $u \in R$. Since $N_H[v] = N_G[v]$, $u \in N_H[v]$. Thus $u \in V(H)$, and hence $u \in S$. Thus $v \in N_H[S]$. The result follows. $\Box$

We point out that having $|R| = \lambda(G)$ in Lemma 2.3.1 does not guarantee that $|R \cap V(H)| = \lambda(H)$. Indeed, let $k \geq 2$, let G_1 and G_2 be copies of $K_{1,k}$ such that $V(G_1) \cap V(G_2) = \emptyset$, let G be the disjoint union of G_1 and G_2, let e be an edge of G_2, and let $H = (V(G), E(G)\backslash\{e\})$. For each $i \in [2]$, let v_i be the vertex of G_i of degree k. Let $R = \{v_1, v_2\}$. Then R is a Δ-reducing set of G of size $\lambda(G)$, $\{v_1\}$ is a Δ-reducing set of H, but $R \cap V(H) = R$.

Proposition 2.3.2. *If G is a graph and $G_1, \dots, G_r$ are the distinct components of G whose maximum degree is $\Delta(G)$, then $\lambda(G) = \sum_{i=1}^{r} \lambda(G_i)$.*

Proof. Let R be a Δ-reducing set of G of size $\lambda(G)$, and let $R_i = R \cap V(G_i)$ for each $i \in [r]$. Then $R_1, \dots, R_r$ partition R, so $|R| = \sum_{i=1}^{r} |R_i|$. By Lemma 2.3.1, $\lambda(G_i) \leq |R_i|$ for each $i \in [r]$. Suppose $\lambda(G_j) < |R_j|$ for some $j \in [r]$. Let R'_j be a Δ-reducing set of G_j of size $\lambda(G_j)$. Then $R'_j \cup \bigcup_{i \in [r]\backslash\{j\}} R_i$

is a Δ-reducing set of G that is smaller than R, a contradiction. Therefore, $\lambda(G_i) = |R_i|$ for each $i \in [r]$. Thus we have $\lambda(G) = |R| = \sum_{i=1}^{r} |R_i| = \sum_{i=1}^{r} \lambda(G_i)$. $\square$

Proposition 2.3.3. *If H is a subgraph of a graph G such that $\Delta(H) = \Delta(G)$, then $\lambda(H) \leq \lambda(G)$.*

Proof. Let R be a Δ-reducing set of G of size $\lambda(G)$. Let $S = R \cap V(H)$. By Lemma 2.3.1, $\Delta(H - S) < \Delta(G)$. Thus we have $\lambda(H) \leq |S| \leq |R| = \lambda(G)$. $\square$

Proposition 2.3.4. *If G is a graph, $v \in V(G)$ and $v \notin N_G[M(G)]$, then $\lambda(G - v) = \lambda(G)$.*

Proof. By Proposition 2.3.3, $\lambda(G - v) \leq \lambda(G)$. Let R be a Δ-reducing set of $G - v$ of size $\lambda(G - v)$. Since $v \notin N_G[M(G)]$, $M(G - v) = M(G)$. Thus R is a Δ-reducing set of G, and hence $\lambda(G) \leq \lambda(G - v)$. Hence $\lambda(G - v) = \lambda(G)$. $\square$

Proposition 2.3.5. *If v is a vertex of a graph G, then $\lambda(G) \leq 1 + \lambda(G - v)$.*

Proof. If $\Delta(G - v) < \Delta(G)$, then $\lambda(G) = 1$. Suppose $\Delta(G - v) = \Delta(G)$, so $M(G - v) \subseteq M(G)$. Let R be a Δ-reducing set of $G - v$ of size $\lambda(G - v)$. For any $x \in M(G) \backslash M(G - v)$, $x \in N_G[v]$. Thus $R \cup \{v\}$ is a Δ-reducing set of G. The result follows. $\square$

Define $M_1(G) = \{v \in M(G)\colon d_G(v, w) \leq 2$ for some $w \in M(G) \backslash \{v\}\}$ and $M_2(G) = M(G) \backslash M_1(G)$. Thus $M_2(G) = \{v \in M(G)\colon d_G(v, w) \geq 3$ for each $w \in M(G) \backslash \{v\}\}$.

Proposition 2.3.6. *For a graph G, $\lambda(G) = |M(G)|$ if and only if $M_2(G) = M(G)$.*

Proof. Suppose $\lambda(G) = |M(G)|$ and $M_2(G) \neq M(G)$. Then $M_1(G) \neq \emptyset$. Let $v \in M_1(G)$. Then $d_G(v, w) \leq 2$ for some $w \in M(G)\backslash\{v\}$. Thus $N_G[v] \cap N_G[w] \neq \emptyset$. Let $x \in N_G[v] \cap N_G[w]$. Then $(M(G)\backslash\{v, w\}) \cup \{x\}$ is a Δ-reducing set of G of size $|M(G)| - 1$, a contradiction. Therefore, if $\lambda(G) = |M(G)|$, then $M_2(G) = M(G)$.

Conversely, suppose $M_2(G) = M(G)$. Let R be a Δ-reducing set of G of size $\lambda(G)$. Then $M(G) \subseteq N_G[R]$ and $N_G[v] \cap M(G) \neq \emptyset$ for each $v \in R$. Suppose $|N_G[v] \cap M(G)| \geq 2$ for some $v \in R$. Let $x, y \in N_G[v] \cap M(G)$ with $x \neq y$. Since $x, y \in N_G[v]$, we obtain $d_G(x, y) \leq 2$, which contradicts $x, y \in M_2(G)$. Thus $|N_G[v] \cap M(G)| = 1$ for each $v \in R$. Since $M(G) \subseteq N_G[R]$, $M(G) = M(G) \cap N_G[R] = M(G) \cap \bigcup_{v \in R} N_G[v] = \bigcup_{v \in R}(N_G[v] \cap M(G))$. Thus we have $|M(G)| \leq \sum_{v \in R} |N_G[v] \cap M(G)| = \sum_{v \in R} 1 = |R|$. By Proposition 2.2.2, $|R| \leq |M(G)|$. Hence $|R| = |M(G)|$. $\square$

Proposition 2.3.7. *If G is a graph with $M_2(G) \neq M(G)$, then $\Delta(G - M_2(G)) = \Delta(G)$ and $\lambda(G) = |M_2(G)| + \lambda(G - M_2(G))$.*

Proof. We use induction on $|M_2(G)|$. The result is trivial if $|M_2(G)| = 0$. Suppose $|M_2(G)| \geq 1$. Let $x \in M_2(G)$. Since $M_2(G) \neq M(G)$, $M_1(G) \neq \emptyset$. Thus we clearly have $\Delta(G - x) = \Delta(G)$, $M_1(G - x) = M_1(G)$ and $M_2(G - x) = M_2(G)\backslash\{x\} \neq M(G - x)$. By the induction hypothesis, $\lambda(G - x) = |M_2(G - x)| + \lambda((G - x) - M_2(G - x)) = |M_2(G)| - 1 + \lambda(G - (\{x\} \cup M_2(G - x))) = |M_2(G)| - 1 + \lambda(G - M_2(G))$. By Proposition 2.3.5, $\lambda(G) \leq 1 + \lambda(G - x)$. Suppose $\lambda(G) \leq \lambda(G - x)$. Let R be a Δ-reducing set of G of size $\lambda(G)$. Then $x \in N_G[y]$ for some $y \in R$. Since $x \in M_2(G)$, $y \notin N_G[z]$ for each $z \in M(G)\backslash\{x\}$ (because otherwise we obtain $d_G(x,z) \leq 2$, a contradiction). We obtain that $R\backslash\{y\}$ is a Δ-reducing set of $G-x$ of size $\lambda(G)-1 \leq \lambda(G-x)-1$, a contradiction. Thus $\lambda(G) = 1+\lambda(G-x) = |M_2(G)| + \lambda(G-M_2(G))$. $\Box$

We conclude this section by conjecturing that for any graph G,

$$\lambda(G) \leq |M_2(G)| + \frac{\Delta(G)}{\Delta(G)+1}|M_1(G)|. \tag{2.4}$$

Equality holds if G is the following tree. Let $k \geq 3$, and let T_k be the tree with $E(T_k) = \{uv_1, \dots, uv_k, v_1x_1, \dots, v_kx_k, x_1y_{1,1}, \dots, x_1y_{1,k-1}, \dots, x_ky_{k,1}, \dots, x_k y_{k,k-1}\}$, where $u, v_1, \dots, v_k, x_1, \dots, x_k, y_{1,1}, \dots, y_{1,k-1}, \dots, y_{k,1}, \dots, y_{k,k-1}$ are the distinct vertices of T_k. We have $\Delta(T_k) = k$ and $M(T_k) = \{u, x_1, \dots, x_k\}$. Also, for each $i \in [k]$, we have $N_{T_k}[x_i] \cap N_{T_k}[u] = \{v_i\}$, and $N_{T_k}[x_i] \cap N_{T_k}[x_j] = \emptyset$ for each $j \in [k]\backslash\{i\}$. It follows that $M(T_k) = M_1(T_k)$ and that $\{v_1, \dots, v_k\}$ is a smallest Δ-reducing set of T_k. Thus $|M_1(T_k)| = k+1$, $M_2(T_k) = \emptyset$ and

$$\lambda(T_k) = k = |M_2(T_k)| + \frac{\Delta(T_k)}{\Delta(T_k)+1}|M_1(T_k)|.$$

One can easily enlarge T_k to a graph G with $\Delta(G) = k$, $M_1(G) = M_1(T_k)$, $M_2(G) \neq \emptyset$ and $\lambda(G) = |M_2(G)| + \frac{\Delta(G)}{\Delta(G)+1}|M_1(G)|$, for example, by adding copies of $K_{1,k}$ as components (and connecting components by vertex-disjoint paths of length at least 3 if G is required to be connected).

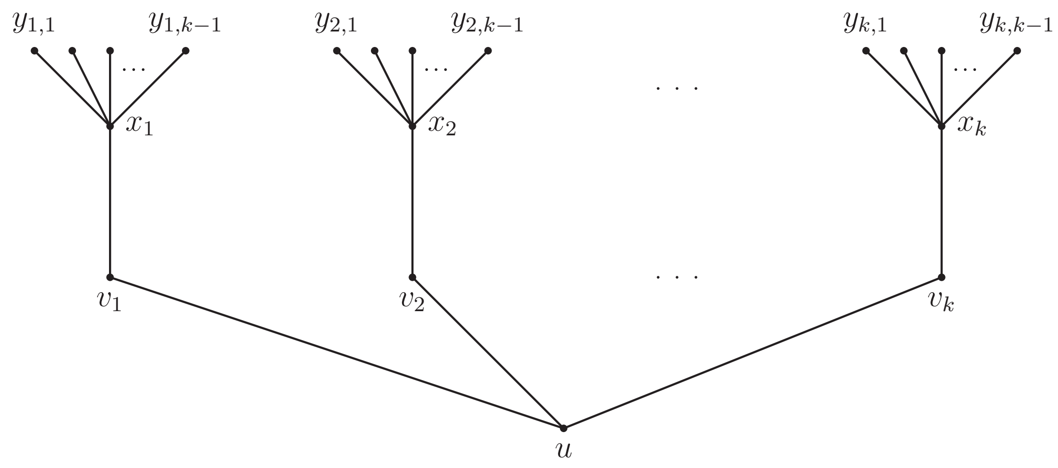

Figure 3: An illustration of the tree T_k as described above.

2.4 Proofs of the main results

We first prove Theorems 2.2.3, 2.2.4, 2.2.6 and 2.2.9.

Theorem 2.2.3. *For any non-empty graph G,*

$$\lambda(G) \leq \frac{|V(G)|}{2},$$

and equality holds if and only if G is either a disjoint union of copies of K_2 or a disjoint union of copies of C_4.

Proof of Theorem 2.2.3. Let $n = |V(G)|$ and $k = \Delta(G)$. Since G is non-empty, $k > 0$. By (2.1), $\lambda(G) \leq \frac{n}{2}$. It is straightforward that if G is either a disjoint union of copies of K_2, or a disjoint union of copies of C_4, then $\lambda(G) = \frac{n}{2}$. We now prove the converse. Thus, suppose $\lambda(G) = \frac{n}{2}$. Then, by (2.1), G is k-regular. Let $G_1, \dots, G_r$ be the distinct components of G. Consider any $i \in [r]$.

Applying the established bound to each of $G_1, \dots, G_r$, we have $\lambda(G_j) \leq \frac{|V(G_j)|}{2}$ for each $j \in [r]$. Together with Proposition 2.3.2, this gives us $\sum_{j=1}^{r} \frac{|V(G_j)|}{2} \geq \sum_{j=1}^{r} \lambda(G_j) = \lambda(G) = \frac{n}{2} = \sum_{j=1}^{r} \frac{|V(G_j)|}{2}$, and hence $\lambda(G_j) = \frac{|V(G_j)|}{2}$ for each $j \in [r]$.

Suppose $k \geq 3$. Since G is k-regular, G_i is k-regular. Thus we have $\delta(G_i) = k \geq 3$, $\lambda(G_i) = \gamma(G_i)$, and hence, by Theorem 2.2.8, $\lambda(G_i) \leq \frac{3|V(G_i)|}{8} < \frac{|V(G_i)|}{2}$, a contradiction.

Therefore, $k \leq 2$. If $k = 1$, then G_i is a copy of K_2. Suppose $k = 2$. Clearly, a 2-regular graph can only be a cycle. Thus, for some $p \geq 3$, G_i is a copy of C_p. As pointed out in Section 2.2, $\lambda(C_p) = \left\lceil \frac{p}{3} \right\rceil$. Since $\lambda(C_p) = \lambda(G_i) = \frac{|V(G_i)|}{2} = \frac{p}{2}$, it follows that $p = 4$. The result follows. $\square$

For any $m, n \in \{0\} \cup \mathbb{N}$, we denote $\{i \in \{0\} \cup \mathbb{N} \colon m \leq i \leq n\}$ by $[m, n]$. Note that $[m, n] = \emptyset$ if $m > n$.

Theorem 2.2.4. *If G is a non-empty graph, $n = |V(G)|$, $k = \Delta(G)$ and $t = |M(G)|$, then*

$$\lambda(G) \leq \frac{n + (k-1)t}{2k}.$$

Proof of Theorem 2.2.4. Since G is non-empty, $k > 0$. Let $r = \lambda(G)$ and $G_1 = G$. Let R be a Δ-reducing set of G of size r. We remove from G_1 a vertex v_1 in R whose closed neighbourhood in G_1 contains the largest number of vertices in $M(G_1)$, and we denote the resulting graph $G_1 - v_1$ by G_2. If $r \geq 2$, then we remove from G_2 a vertex v_2 in $R\backslash\{v_1\}$ whose closed neighbourhood in G_2 contains the largest number of vertices in $M(G_2)$, and we denote the resulting graph $G_2 - v_2$ by G_3. If $r \geq 3$, then we remove from G_3 a vertex v_3 in $R\backslash\{v_1, v_2\}$ whose closed neighbourhood in G_3 contains the largest number of vertices in $M(G_3)$, and we denote the resulting graph $G_3 - v_3$ by G_4. Continuing this way, we obtain $v_1, \dots, v_r$ and $G_1, \dots, G_{r+1}$ such that $R = \{v_1, \dots, v_r\}$, $G_{r+1} = G - R$, $\Delta(G_i) = k$ for each $i \in [r]$ (since $|R| = r = \lambda(G)$), $\Delta(G_{r+1}) < k$ and

$$M(G) = \bigcup_{i=1}^{r} (N_{G_i}[v_i] \cap M(G_i)). \tag{2.5}$$

For each $i \in [r]$, let $A_i = N_{G_i}[v_i] \cap M(G_i)$. The members $v_1, \dots, v_r$ of R have been labelled in such a way that

$$|A_1| \geq \cdots \geq |A_r|. \tag{2.6}$$

For every $i, j \in [r]$ with $i < j$, each member of $A_i \cap V(G_j)$ is of degree at most $k - 1$ in G_j (as its neighbour v_i in G_i is not in $V(G_j)$), and hence

$$A_i \cap A_j = \emptyset. \tag{2.7}$$

Let $I_3 = \{i \in [r] \colon |A_i| \geq 3\}$, $I_2 = \{i \in [r] \colon |A_i| = 2\}$ and $I_1 = \{i \in [r] \colon |A_i| = 1\}$. Let $r_1 = |I_1|$, $r_2 = |I_2|$ and $r_3 = |I_3|$. Then $r = r_1 + r_2 + r_3$. By (2.6), we have $I_3 = [1, r_3]$, $I_2 = [r_3 + 1, r_3 + r_2]$ and $I_1 = [r_3 + r_2 + 1, r_3 + r_2 + r_1] = [r - r_1 + 1, r]$. Let $H = G_{r-r_1+1}$.

Suppose $r_1 = 0$. Then $I_2 \cup I_3 = [r]$. By (2.5), $M(G) = \bigcup_{i \in I_2 \cup I_3} A_i$. By (2.7), it follows that $t = \sum_{i \in I_2 \cup I_3} |A_i| \geq \sum_{i \in I_2 \cup I_3} 2 = 2r$, and hence $r \leq \frac{t}{2} \leq \frac{n+(k-1)t}{2k}$.

Now suppose $r_1 \neq 0$. Then $\Delta(H) = k$. By construction, $\{v_i \colon i \in I_1\}$ is a Δ-reducing set of H, and $M(H) = \bigcup_{i \in I_1} A_i$. If we assume that H has a Δ-reducing set S of size less than $|I_1|$, then we obtain that $(R \backslash \{v_i \colon i \in I_1\}) \cup S$ is a Δ-reducing set of G of size less than $|R|$, a contradiction. Thus $\lambda(H) = |I_1|$. Together with $M(H) = \bigcup_{i \in I_1} A_i$, (2.7) gives us $|M(H)| = \sum_{i \in I_1} |A_i| = |I_1|$. By Proposition 2.3.6, $M(H) = M_2(H)$. For each $i \in I_1$, let z_i be the unique element of A_i. By (2.7), $z_i \neq z_j$ for every $i, j \in I_1$ with $i \neq j$. Since $M_2(H) = M(H) = \bigcup_{i \in I_1} A_i$, $M_2(H) = \{z_i \colon i \in I_1\}$. By definition of $M_2(H)$, it follows that for every $i, j \in I_1$ with $i \neq j$,

$$N_H[z_i] \cap N_H[z_j] = \emptyset.$$

Therefore,

$$\Big| \bigcup_{i \in I_1} N_H[z_i] \Big| = \sum_{i \in I_1} |N_H[z_i]| = (k+1)|I_1| = (k+1)r_1.$$

Let $R' = (R \backslash \{v_i \colon i \in I_1\}) \cup M(H)$. Since $|M(H)| = |I_1| = \lambda(H)$ (and $M(H)$ is a Δ-reducing set of H), R' is a Δ-reducing set of G of size $\lambda(G)$.

Let $B_1 = \bigcup_{i \in I_1} N_H[z_i]$, $B_2 = \{v_i \colon i \in I_2\}$ and $B_3 = \{v_i \colon i \in I_3\}$. Then $|B_1| = (k+1)r_1$, $|B_2| = r_2$ and $|B_3| = r_3$.

Suppose that there exists $j \in I_2$ such that $A_j \subseteq B_1 \cup B_2 \cup B_3$. Let w_1 and w_2 be the two members of A_j. Let $C = \{v_i \colon i \in I_2, i \geq j\}$. We have

$w_1, w_2 \in V(G_j) = V(G) \backslash \{v_i \colon i \in [1, j-1]\}$, so $w_1, w_2 \in B_1 \cup C$. We have $w_1, w_2 \in N_{G_j}[v_j]$ and $d_{G_j}(w_1) = d_{G_j}(w_2) = k$.

Suppose $v_j = w_1$. Since $w_1, w_2 \in B_1 \cup C$, we have $w_2 \in B_1 \cup (C \backslash \{v_j\})$. Suppose $w_2 \in B_1$. Then $w_2 \in N_H[z_i]$ for some $i \in I_1$. Since $A_j \cup \{z_i\} = \{v_j, w_2, z_i\} \subseteq N_{G_j}[w_2]$, we obtain that $(R' \backslash \{v_j, z_i\}) \cup \{w_2\}$ is a Δ-reducing set of G of size $|R'| - 1$, which contradicts $|R'| = \lambda(G)$. Thus $w_2 \in C \backslash \{v_j\}$, meaning that $w_2 = v_i$ for some $i \in I_2$ such that $i > j$. From this we obtain that $R' \backslash \{v_j\}$ is a Δ-reducing set of G of size $|R'| - 1$, a contradiction.

Therefore, $v_j \neq w_1$. Similarly, $v_j \neq w_2$. If we assume that $w_1, w_2 \in C$, then we obtain that $R' \backslash \{v_j\}$ is a Δ-reducing set of G of size $|R'| - 1$, a contradiction. Therefore, at least one of w_1 and w_2 is in B_1; we may assume that $w_1 \in B_1$. Thus $w_1 \in N_H[z_i]$ for some $i \in I_1$. If we assume that $w_2 \in C$, then we obtain that $R' \backslash \{v_j\}$ is a Δ-reducing set of G of size $|R'| - 1$, a contradiction. Thus $w_2 \in B_1$, and hence $w_2 \in N_H[z_h]$ for some $h \in I_1$. From this we obtain that $R' \backslash \{v_j\}$ is a Δ-reducing set of G of size $|R'| - 1$, a contradiction.

Therefore, $A_i \not\subseteq B_1 \cup B_2 \cup B_3$ for each $i \in I_2$. For each $i \in I_2$, let $x_i \in A_i \backslash (B_1 \cup B_2 \cup B_3)$. Let $B_4 = \{x_i \colon i \in I_2\}$. Thus $B_4 \cap (B_1 \cup B_2 \cup B_3) = \emptyset$. Since B_1, B_2 and B_3 are pairwise disjoint (by construction), it follows that $|\bigcup_{i=1}^{4} B_i| = \sum_{i=1}^{4} |B_i|$. By (2.7), $x_i \neq x_j$ for every $i, j \in I_2$ with $i \neq j$. Thus $|B_4| = r_2$.

By (2.5) and (2.7), the sets $A_1, \ldots, A_r$ partition $M(G)$. Thus $t = \sum_{i=1}^{r} |A_i|$

$\geq 3r_3 + 2r_2 + r_1 = 2r_3 + r_2 + r$, and hence $-r_3 - r_2 \geq r - t + r_3$. We have

$$
\begin{aligned}
n &\geq |\bigcup_{i=1}^{4} B_i| = \sum_{i=1}^{4} |B_i| = r_3 + 2r_2 + (k+1)r_1 \\
&= r_3 + 2r_2 + (k+1)(r - r_3 - r_2) \\
&= (k+1)r + (k-1)(-r_3 - r_2) - r_3 \geq (k+1)r + (k-1)(r - t + r_3) - r_3 \\
&= 2kr - (k-1)t + (k-2)r_3,
\end{aligned}
$$

and hence

$$r \leq \frac{n + (k-1)t - (k-2)r_3}{2k}. \tag{2.8}$$

If $k = 1$, then $r_3 = 0$. Thus $(k-2)r_3 \geq 0$, and hence $r \leq \frac{n+(k-1)t}{2k}$. $\square$

We now prove Theorem 2.2.6, making use of the following well-known fact.

Lemma 2.4.1. *Let x be a vertex of a tree T. Let $m = \max\{d_T(x,y) \colon y \in V(T)\}$, and let $D_i = \{y \in V(T) \colon d_T(x,y) = i\}$ for each $i \in \{0\} \cup [m]$. For each $i \in [m]$ and each $v \in D_i$, $N_G(v) \cap \bigcup_{j=0}^{i} D_j = \{u\}$ for some $u \in D_{i-1}$.*

Indeed, let $v \in D_i$. By definition of D_i, v can only be adjacent to vertices of distance $i-1$, i or $i+1$ from x. If v is adjacent to a vertex w of distance i, then, by considering an xv-path and an xw-path, we obtain that T contains a cycle, which is a contradiction. We obtain the same contradiction if we assume that v is adjacent to two vertices of distance $i-1$ from x.

Recall that if a vertex v of a graph G has only one neighbour in G, then v is called a *leaf of G*.

Corollary 2.4.2. *If T is a tree, $x, z \in V(T)$ and $d_T(x, z) = \max\{d_T(x, y) \colon y \in V(T)\}$, then z is a leaf of T.*

Proof. Let $D_0, D_1, \ldots, D_m$ be as in Lemma 2.4.1. Then $z \in D_m$. By Lemma 2.4.1, $N_T(z) = \{u\}$ for some $u \in D_{m-1}$. $\square$

Theorem 2.2.6. *For any tree T,*

$$\lambda(T) \leq \frac{|V(T)|}{\Delta(T) + 1}.$$

Proof of Theorem 2.2.6. Let $n = |V(T)|$ and $k = \Delta(T)$. The result is trivial for $n \leq 2$. We now proceed by induction on n. Thus consider $n \geq 3$. Since T is a connected graph, we clearly have $k \geq 2$.

Suppose that T has a leaf z whose neighbour is not in $M(T)$. Then $M(T - z) = M(T)$ and, by Proposition 2.3.4, $\lambda(T - z) = \lambda(T)$. By the induction hypothesis, $\lambda(T - z) \leq \frac{n-1}{k+1} < \frac{n}{k+1}$. Thus $\lambda(T) < \frac{n}{k+1}$.

Now suppose that each leaf of T is adjacent to a vertex in $M(T)$. Let x, m and $D_0, D_1, \ldots, D_m$ be as in Lemma 2.4.1. Let $z \in D_m$. By Corollary 2.4.2, z is a leaf of T. Let w be the neighbour of z. Then $w \in M(T)$. By Lemma 2.4.1, $w \in D_{m-1}$.

Suppose $w = x$. Then $m = 1$ and $E(T) = \{xz_1, \ldots, xz_k\}$ for some distinct vertices $z_1, \ldots, z_k$ of T. Thus $\{x\}$ is a Δ-reducing set of T, and hence $\lambda(T) = 1 = \frac{n}{k+1}$.

Now suppose $w \neq x$. Together with Lemma 2.4.1, this implies that $N_T(w) = \{v, z_1, \ldots, z_{k-1}\}$ for some $v \in D_{m-2}$ and some distinct vertices

$z_1, \ldots, z_{k-1}$ in D_m. By Corollary 2.4.2, $z_1, \ldots, z_{k-1}$ are leaves of T. Let $T' = T - v$. Then each component of T' is a tree. Let $\mathcal{K}$ be the set of components of T' whose maximum degree is k, and let $\mathcal{H}$ be the set of components of T' whose maximum degree is less than k. Let $W = \{w, z_1, \ldots, z_{k-1}\}$. Note that $(W, \{wz_1, \ldots, wz_{k-1}\})$ is in $\mathcal{H}$, and hence $W \cap \bigcup_{K \in \mathcal{K}} V(K) = \emptyset$. If $\mathcal{K} = \emptyset$, then $\{v\}$ is a Δ-reducing set of T, and hence $\lambda(T) = 1 \leq \frac{n}{k+1}$. Suppose $\mathcal{K} \neq \emptyset$. For each $K \in \mathcal{K}$, let S_K be a Δ-reducing set of K of size $\lambda(K)$. By the induction hypothesis, $|S_K| \leq \frac{|V(K)|}{k+1}$ for each $K \in \mathcal{K}$. Now $\{v\} \cup \bigcup_{K \in \mathcal{K}} S_K$ is a Δ-reducing set of T. Therefore, we have

$$\lambda(T) \leq 1 + \sum_{K \in \mathcal{K}} |S_K| \leq \frac{|W \cup \{v\}|}{k+1} + \sum_{K \in \mathcal{K}} \frac{|V(K)|}{k+1} \leq \frac{n}{k+1},$$

as required. □

In order to prove our next result, we will make use of some well-known basic results in probability theory; the following is one of these results and is referred to as the *probabilistic pigeonhole principle*. This powerful generalisation of the pigeonhole principle is also used in other probabilistic results in subsequent chapters.

Proposition 2.4.3. *If X is a random variable on a probability space (Ω, P), then there exist $\omega, \omega' \in \Omega$ such that $X(\omega) \leq \mathrm{E}[X]$ and $X(\omega') \geq \mathrm{E}[X]$.*

Theorem 2.2.9. *If G is a graph, $n = |V(G)|$, $k = \Delta(G)$ and $t = |M(G)|$, then*

$$\lambda(G) \leq \frac{n \ln (k+1) + t}{k+1}.$$

Proof of Theorem 2.2.9. We may assume that $V(G) = [n]$. Let $p = \frac{\ln(k+1)}{k+1}$. We set up n independent random experiments, and in each experiment a vertex is chosen with probability p. More formally, for each $i \in V$, let (Ω_i, P_i) be the probability space given by $\Omega_i = \{0, 1\}$, $P_i(\{1\}) = p$ and $P_i(\{0\}) = 1 - p$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$, and let $P : 2^\Omega \to [0, 1]$ such that $P(\{\omega\}) = \prod_{i=1}^{n} P_i(\{\omega_i\})$ for each $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$, and $P(A) = \sum_{\omega \in A} P(\{\omega\})$ for each $A \subseteq \Omega$. Then (Ω, P) is a probability space.

For each $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$, let S_ω be the subset of $V(G)$ such that ω is the characteristic vector of S_ω (that is, $S_\omega = \{i \in [n] \colon \omega_i = 1\}$), let T_ω be the set of vertices in $M(G)$ that are neither in S_ω nor adjacent to a vertex in S_ω (that is, $T_\omega = \{v \in M(G) \colon v \notin N_G[S_\omega]\}$), and let $D_\omega = S_\omega \cup T_\omega$. Then D_ω is a Δ-reducing set of G.

Let $X, Y : \Omega \to \mathbb{R}$ be the random variables given by $X(\omega) = |S_\omega|$ and $Y(\omega) = |T_\omega|$. For each $i \in [n]$, let $X_i : \Omega \to \mathbb{R}$ be the indicator random variable for whether vertex i is in S_ω; that is, for each $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$,

$$X_i(\omega) = \begin{cases} 1 & \text{if } i \in S_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in M(G)$, let $Y_i : \Omega \to \mathbb{R}$ be the indicator random variable for whether vertex i is in T_ω; that is, for each $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$,

$$Y_i(\omega) = \begin{cases} 1 & \text{if } i \in T_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

We have $X = \sum_{i=1}^{n} X_i$ and $Y = \sum_{i \in M(G)} Y_i$.

For each $i \in [n]$, $P(X_i = 1) = P_i(\{1\}) = p$. For each $i \in M(G)$,

$$\begin{aligned} P(Y_i = 1) &= P(\{\omega \in \Omega \colon \omega_j = 0 \text{ for each } j \in N_G[i]\}) \\ &= \prod_{j \in N_G[i]} P_j(\{0\}) = (1-p)^{|N_G[i]|} = (1-p)^{k+1}. \end{aligned}$$

For any random variable Z, let $\mathrm{E}[Z]$ denote the expected value of Z. By linearity of expectation,

$$\begin{aligned} \mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] &= \sum_{i=1}^{n} \mathrm{E}[X_i] + \sum_{i \in M(G)} \mathrm{E}[Y_i] \\ &= \sum_{i=1}^{n} P(X_i = 1) + \sum_{i \in M(G)} P(Y_i = 1) = np + t(1-p)^{k+1}. \end{aligned}$$

By Proposition 2.4.3, there exists $\omega^* \in \Omega$ such that $X(\omega^*) + Y(\omega^*) \leq np + t(1-p)^{k+1}$. Since $X(\omega^*) + Y(\omega^*) = |S_{\omega^*}| + |T_{\omega^*}| = |D_{\omega^*}|$ and $(1-p)^{k+1} \leq e^{-p(k+1)}$, $|D_{\omega^*}| \leq np + te^{-p(k+1)} = \frac{n\ln(k+1)}{k+1} + te^{-\ln(k+1)} = \frac{n\ln(k+1)}{k+1} + \frac{t}{k+1}$. $\square$

We now prove Theorems 2.2.5 and 2.2.7.

The next result implies that the bound in Theorem 2.2.5 is attained by special k-star t-unions, and that the bound in Theorem 2.2.6 is attained by k-special trees.

Lemma 2.4.4. *If $S_1, \dots, S_t$ are vertex-disjoint k-stars and G is a graph such that $V(G) = \bigcup_{i=1}^{t} V(S_i)$, $\bigcup_{i=1}^{t} E(S_i) \subseteq E(G)$, and $\Delta(G) = k$, then $|V(G)| = (k+1)t$ and $\lambda(G) = t$.*

Proof. We have $|V(G)| = \sum_{i=1}^{t} |V(S_i)| = (k+1)t$. For each $i \in [t]$, there exists a vertex x_i of S_i such that $N_{S_i}[x_i] = V(S_i)$ and $E(S_i) = E_{S_i}(x_i)$. Let $X = \{x_1, \dots, x_t\}$. Since $V(G) = \bigcup_{i=1}^{t} V(S_i) = N_G[X]$, X is a Δ-reducing set of G, so $\lambda(G) \leq |X| = t$. Now let R be a Δ-reducing set of G of size $\lambda(G)$. For each $i \in [t]$, we have $k = |V(S_i)\backslash\{x_i\}| = |N_{S_i}(x_i)| \leq |N_G(x_i)| \leq \Delta(G) = k$, so $N_G(x_i) = V(S_i)\backslash\{x_i\}$, $x_i \in M(G)$, and hence $R \cap N_G[x_i] \neq \emptyset$. We have $|R| = |R \cap V(G)| = |R \cap \bigcup_{i=1}^{t} V(S_i)| = \sum_{i=1}^{t} |R \cap V(S_i)|$ as $V(S_1), \dots, V(S_t)$ are pairwise disjoint. Thus, $|R| = \sum_{i=1}^{t} |R \cap N_G[x_i]| \geq \sum_{i=1}^{t} 1 = t$. We have $t \leq \lambda(G) \leq t$, so $\lambda(G) = t$. □

Theorem 2.2.5. *If G is a non-empty graph, $n = |V(G)|$, $k = \Delta(G)$, and $t = |M(G)|$, then*

$$\lambda(G) \leq \frac{n + (k-1)t}{2k}.$$

Moreover, equality holds if and only if one of the following holds:

(i) $k = 1$ and each component of G is a copy of K_2,

(ii) $k = 2$ and each component of G is a copy of P_3 or C_4,

(iii) $k \geq 2$ and G is a special k-star t-union.

Proof of Theorem 2.2.5. If each component of G is a copy of K_2, then $\lambda(G) = \frac{n}{2} = \frac{n+(k-1)t}{2k}$. If G has $s_1 + s_2$ components, s_1 components of G are copies of P_3, and s_2 components of G are copies of C_4, then $k = 2$, $n = 3s_1 + 4s_2$, $t = s_1 + 4s_2$, and clearly $\lambda(G) = s_1 + 2s_2 = \frac{n+(k-1)t}{2k}$. If G is a special k-star t-union, then $n = (k+1)t$ and $\lambda(G) = t = \frac{n+(k-1)t}{2k}$ by Lemma 2.4.4.

We now prove the bound in the theorem and show that it is attained only in the cases above. Since G is non-empty, $n \geq 2$. If $n = 2$, then G is a copy of K_2, so $\lambda(G) = 1 = \frac{n+(k-1)t}{2k}$. We proceed by induction on n. Thus, consider $n \geq 3$. If $k = 1$, then G is the union of vertex-disjoint copies of K_2, so $\lambda(G) = \frac{n}{2} = \frac{n+(k-1)t}{2k}$. Consider $k \geq 2$. Let $v^* \in M(G)$. We have $n \geq |N[v^*]| = k + 1$.

Suppose that $M_2(G)$ has a member u. If $\Delta(G - u) < \Delta(G)$, then $\lambda(G) = 1 \leq \frac{n+(k-1)t}{2k}$ (as $n \geq k + 1$). If $\lambda(G) = 1 = \frac{n+(k-1)t}{2k}$, then $V(G) = N[u]$, so G is a special k-star 1-union. Now suppose $\Delta(G - u) = \Delta(G)$. Then, since $u \in M_2(G)$, $M(G - u) = M(G)\backslash\{u\}$ and $v \notin N_{G-u}[M(G - u)]$ for each $v \in N(u)$. Thus, $M(G - N[u]) = M(G - u)$, $\Delta(G - N[u]) = \Delta(G - u) = k$, and $\lambda(G-N[u]) = \lambda(G-u)$ by repeated application of Proposition 2.3.4. Let $G' = G - N[u]$, $n' = |V(G')| = n - k - 1$, and $t' = |M(G')| = |M(G - u)| = t - 1$. By Proposition 2.3.5 and the induction hypothesis,

$$\lambda(G) \leq 1 + \lambda(G - u) = 1 + \lambda(G') \leq 1 + \frac{n' + (k-1)t'}{2k} = \frac{n + (k-1)t}{2k}.$$

Suppose $\lambda(G) = \frac{n+(k-1)t}{2k}$. Then $\lambda(G') = \frac{n'+(k-1)t'}{2k}$. By the induction hypothesis, G' is a special k-star $(t-1)$-union or each component of G' is a copy of P_3 or C_4. Suppose that each component of G' is a copy of P_3 or C_4. Then $k = 2$. Let u_1 and u_2 be the two members of $N(u)$. Since $u \in M_2(G)$, we have $d(u_1) = d(u_2) = 1$, so $N(u_1) = N(u_2) = \{u\}$. Thus, $G[N[u]]$ is a copy of P_3 and a component of G. Therefore, each component of G is a copy of P_3 or C_4. Now suppose that G' is a special k-star $(t-1)$-union with constituents $S_1, \dots, S_{t-1}$. Let S_t be the k-star $(N[u], E(u))$. Then $S_1, \dots, S_t$ are vertex-disjoint, $V(G) = V(G') \cup N[u] = \bigcup_{i=1}^{t} V(S_i)$, and $\bigcup_{i=1}^{t} E(S_i) \subseteq E(G)$. Thus,

G is a special k-star t-union.

Now suppose $M_2(G) = \emptyset$. Then $M(G) = M_1(G)$.

Suppose that G has a vertex u such that $N[u]$ contains at least 3 vertices in $M(G)$. If $\Delta(G-u) < \Delta(G)$, then $\lambda(G) = 1 < \frac{n+(k-1)t}{2k}$ as $n \geq k+1$, $t \geq 3$, and $k \geq 2$. Now suppose $\Delta(G-u) = \Delta(G)$. Let $n' = |V(G-u)| = n-1$ and $t' = |M(G-u)| \leq t-3$. By Proposition 2.3.5 and the induction hypothesis,

$$\begin{aligned}\lambda(G) &\leq 1 + \lambda(G-u) \leq 1 + \frac{n' + (k-1)t'}{2k} \\ &\leq 1 + \frac{(n-1) + (k-1)(t-3)}{2k} = \frac{n + (k-1)t - (k-2)}{2k} \\ &\leq \frac{n+(k-1)t}{2k}. \end{aligned} \tag{2.9}$$

Suppose $\lambda(G) = \frac{n+(k-1)t}{2k}$. Then, in (2.9), equality holds throughout. Thus, $k = 2$ (as $n + (k-1)t - (k-2) = n + (k-1)t$), $t' = t - 3$ (as $n' + (k-1)t' = (n-1)+(k-1)(t-3)$), and $\lambda(G-u) = \frac{n'+(k-1)t'}{2k}$. By the induction hypothesis, $G-u$ is a special 2-star t'-union or each component of $G-u$ is a copy of P_3 or C_4. If $G-u$ is a special 2-star t'-union, then, by definition, the constituents of $G-u$ are the components of $G-u$ (because, since $k=2$ and $|M(G-u)| = t'$, $d_{G-u}(z) = 1$ for each leaf z of any constituent), and they are copies of P_3. Therefore, in any case, each component of $G-u$ is a copy of P_3 or C_4. Let s_1 be the number of components of $G-u$ that are copies of P_3, and let s_2 be the number of components of $G-u$ that are copies of C_4. Let u_1 and u_2 be two distinct members of $N(u)$. Since $k=2$ and $|N[u] \cap M(G)| \geq 3$, $N[u] = \{u, u_1, u_2\} = N[u] \cap M(G)$. Thus, $d(u) = d(u_1) = d(u_2) = \Delta(G) = 2$. For each $i \in [2]$, $d_{G-u}(u_i) = d_G(u_i) - 1 = 1$, so u_i is a leaf of a component H_i

of $G-u$ that is a copy $(\{u_i, u_i', u_i''\}, \{u_iu_i', u_i'u_i''\})$ of P_3. Since $N(u) = \{u_1, u_2\}$ and $M_2(G) = \emptyset$, H_1 and H_2 are the only components of $G-u$ that are copies of P_3. Suppose $H_1 \neq H_2$. Then G has s_2+1 components, s_2 components of G are copies of C_4, and 1 component of G is a copy of P_7. Thus, $n = 4s_2+7$, $t = 4s_2+5$, and clearly $\lambda(G) = 2s_2+2$. We have $\lambda(G) < 2s_2+3 = \frac{n+(k-1)t}{2k}$, a contradiction. Thus, $H_1 = H_2$, and hence each component of G is a copy of C_4.

Now suppose that

$$|N[v] \cap M(G)| \leq 2 \text{ for each } v \in V(G). \tag{2.10}$$

Suppose that, for each $v \in M(G)$, $N(v)$ contains no member of $M(G)$. Let $x \in M(G)$. Since $M(G) = M_1(G)$, there exists some $w \in N(x)\backslash M(G)$ such that $y \in N(w)$ for some $y \in M(G)\backslash N[x]$. Since $x, y \in N(w)$, $N(w) \cap M(G) = \{x, y\}$ by (2.10). If $\Delta(G-w) < \Delta(G)$, then $\lambda(G) = 1 < \frac{n+(k-1)t}{2k}$ as $n \geq 3$ and $t \geq 2$. Suppose $\Delta(G-w) = \Delta(G)$. Then $M(G-w) = M(G)\backslash\{x, y\}$. Let $G' = G - \{w, x, y\}$. Since $N(x) \cap M(G) = \emptyset$, $N(y) \cap M(G) = \emptyset$, and $N(w) \cap M(G) = \{x, y\}$, we have $M(G') = M(G)\backslash\{x, y\} = M(G-w)$, $\Delta(G') = k$, and $\lambda(G') = \lambda(G - \{w, x\}) = \lambda(G-w)$ by Proposition 2.3.4 (as $y \notin N_{G-\{w,x\}}[M(G-\{w,x\})]$ and $x \notin N_{G-w}[M(G-w)]$). Let $n' = |V(G')| = n-3$ and $t' = |M(G')| = t-2$. By Proposition 2.3.5 and the induction hypothesis,

$$\lambda(G) \leq 1 + \lambda(G-w) = 1 + \lambda(G') \leq 1 + \frac{n'+(k-1)t'}{2k} < \frac{n+(k-1)t}{2k}.$$

Finally, suppose that G has a vertex u in $M(G)$ such that $N(u)$ contains a member w of $M(G)$. By (2.10), $N[u] \cap M(G) = \{u, w\} = N[w] \cap M(G)$. If $\Delta(G-u) < \Delta(G)$, then $\lambda(G) = 1 < \frac{n+(k-1)t}{2k}$ as $n \geq 3$ and $t \geq 2$. Suppose $\Delta(G-u) = \Delta(G)$. Then $M(G-u) = M(G)\backslash\{u, w\}$. Let $G' = G - \{u, w\}$. Since $N[u] \cap M(G) = \{u, w\} = N[w] \cap M(G)$, we have $M(G') = M(G)\backslash\{u, w\} = M(G-u)$, $\Delta(G') = k$, and $\lambda(G') = \lambda(G-u)$ by Proposition 2.3.4 (as $w \notin N_{G-u}[M(G-u)]$). Let $n' = |V(G')| = n-2$ and $t' = |M(G')| = t-2$. By Proposition 2.3.5 and the induction hypothesis,

$$\lambda(G) \leq 1 + \lambda(G-u) = 1 + \lambda(G') \leq 1 + \frac{n' + (k-1)t'}{2k} = \frac{n + (k-1)t}{2k}.$$

Suppose $\lambda(G) = \frac{n+(k-1)t}{2k}$. Then $\lambda(G') = \frac{n'+(k-1)t'}{2k}$. By the induction hypothesis, G' is a special k-star $(t-2)$-union or each component of G' is a copy of P_3 or C_4. Thus, $\delta(G') \geq 1$.

Suppose first that each component of G' is a copy of P_3 or C_4. Then $\Delta(G') = 2$. Since $\Delta(G) = \Delta(G')$, $d(u) = d(w) = 2$. Thus, $N(u) = \{u', w\}$ for some $u' \in V(G)\backslash\{u, w\} = V(G')$. Since $N[u] \cap M(G) = \{u, w\}$ and $k = 2$, we have $d(u') < 2$, so $N(u') = \{u\}$. We obtain $d_{G'}(u') = 0$, which contradicts $\delta(G') \geq 1$.

Now suppose that G' is a special k-star $(t-2)$-union. Let $S_1, \dots, S_{t-2}$ be the constituents of G'. Let $X = N(u)\backslash\{w\}$ and $Y = N(w)\backslash\{u\}$. Then $|X| = |Y| = k-1$ and $d_{G'}(v) < k$ for each $v \in X \cup Y$. For each $i \in [t-2]$, S_i has a vertex v_i such that $d_{S_i}(v_i) = k$. Since $\Delta(G) = k$, $d(v_i) = d_{S_i}(v_i) = k$

for each $i \in [t-2]$. Note that

$$X \cup Y \subseteq V(G')\backslash\{v_1, \ldots, v_{t'}\} = V(G')\backslash M(G') = \bigcup_{i=1}^{t-2} N(v_i). \tag{2.11}$$

Suppose $X \cap Y \neq \emptyset$. Let $x \in X \cap Y$. We have $x \in N(v_p)$ for some $p \in [t-2]$. Thus, we have $u, w, v_p \in N[x] \cap M(G)$, contradicting (2.10). Therefore, $X \cap Y = \emptyset$. Recall that we are considering $k \geq 2$. Since $|X| = |Y| = k-1$, $X \neq \emptyset \neq Y$. Let $x^* \in X$. By (2.11), $x^* \in N(v_p)$ for some $p \in [t-2]$. Consider any $y \in Y$. By (2.11), $y \in N(v_q)$ for some $q \in [t-2]$. Suppose $q \neq p$. Then $(\{v_1, \ldots, v_{t-2}\}\backslash\{v_p, v_q\}) \cup \{x^*, y\}$ is a Δ-reducing set of G of size $t-2$. We have

$$\begin{aligned} t-2 \geq \lambda(G) = \frac{n+(k-1)t}{2k} &= \frac{|\{u,w\} \cup \bigcup_{i=1}^{t-2} V(S_i)| + (k-1)t}{2k} \\ &= \frac{(2+(k+1)(t-2)) + (k-1)t}{2k} = t-1, \end{aligned}$$

a contradiction. Thus, $Y \subseteq N(v_p)$. Let $y^* \in Y$. Then $y^* \in N(v_p)$. By an argument similar to that for x^*, $X \subseteq N(v_p)$. Since $X \cap Y = \emptyset$, we have $2(k-1) = |X \cup Y| \leq |N(v_p)| = k$, so $k \leq 2$. Since $k \geq 2$, $k = 2$. Thus, since $N[u] \cap M(G) = \{u, w\}$, $N(u) = \{w, u'\}$ for some $u' \in V(G)\backslash M(G)$. Since $d(u') < k = 2$, $N(u') = \{u\}$. We obtain $d_{G'}(u') = 0$, which contradicts $\delta(G') \geq 1$. □

We now prove Theorem 2.2.7.

Theorem 2.2.7. *The bound in Theorem 2.2.6 is attained if and only if T is k-special.*

Proof of Theorem 2.2.7. By Lemma 2.4.4, $\lambda(T) = \frac{n}{k+1}$ if T is k-special. We now prove the converse. This is trivial if $n \leq 2$. We proceed by induction on n. Suppose $n \geq 3$ and $\lambda(T) = \frac{n}{k+1}$. Since T is a connected graph, we clearly have $k \geq 2$.

Suppose that T has a leaf z whose neighbour is not in $M(T)$. Then $M(T - z) = M(T)$ and, by Proposition 2.3.4, $\lambda(T - z) = \lambda(T)$. By Theorem 2.2.6, $\lambda(T - z) \leq \frac{n-1}{k+1} < \frac{n}{k+1}$. Thus, we have $\lambda(T) < \frac{n}{k+1}$, a contradiction.

Therefore, each leaf of T is adjacent to a vertex in $M(T)$. Let x, m, and $D_0, D_1, \ldots, D_m$ be as in Lemma 2.4.1. Let $z \in V(T)$ such that $d(x, z) = m$. By Corollary 2.4.2, z is a leaf of T. Let w be the neighbour of z. Then $w \in M(T)$. By Lemma 2.4.1, $w \in D_{m-1}$.

Suppose $w = x$. Then $m = 1$ and $E(T) = \{xz_1, \ldots, xz_k\}$ for some distinct vertices $z_1, \ldots, z_k$ of T. Thus, T is a k-star and hence k-special.

Now suppose $w \neq x$. Together with Lemma 2.4.1, this implies that $N(w) = \{v, z_1, \ldots, z_{k-1}\}$ for some $v \in D_{m-2}$ and some distinct vertices $z_1, \ldots, z_{k-1}$ in D_m. By Corollary 2.4.2, $z_1, \ldots, z_{k-1}$ are leaves of T. Let $T' = T - v$. Then each component of T' is a tree. Let $\mathcal{K}$ be the set of components of T' whose maximum degree is k, and let $\mathcal{H}$ be the set of components of T' whose maximum degree is less than k. Let $W = \{w, z_1, \ldots, z_{k-1}\}$. Note that $(W, \{wz_1, \ldots, wz_{k-1}\}) \in \mathcal{H}$, and hence $W \cap \bigcup_{C \in \mathcal{K}} V(C) = \emptyset$. Let S_0 be the k-star $(W \cup \{v\}, \{wv, wz_1, \ldots, wz_{k-1}\})$.

Suppose $\mathcal{K} = \emptyset$. Then $\{v\}$ is a Δ-reducing set of T, and hence $\lambda(T) = 1$. Since $\lambda(T) = \frac{n}{k+1}$, we have $n = k + 1$, so $T = S_0$. Thus, T is k-special.

Now suppose $\mathcal{K} \neq \emptyset$. Let $T_1, \ldots, T_r$ be the distinct members of $\mathcal{K}$. For each $i \in [r]$, let R_i be a Δ-reducing set of T_i of size $\lambda(T_i)$. By Theorem 2.2.6, $|R_i| \leq \frac{|V(T_i)|}{k+1}$ for each $i \in [r]$. Now $\{v\} \cup \bigcup_{i=1}^{r} R_i$ is a Δ-reducing set of T. Thus, we have

$$\lambda(T) \leq 1 + \sum_{i=1}^{r} |R_i| \leq \frac{|V(S_0)|}{k+1} + \sum_{i=1}^{r} \frac{|V(T_i)|}{k+1} \leq \frac{n}{k+1}.$$

Since $\lambda(T) = \frac{n}{k+1}$, it follows that $V(T) = V(S_0) \cup \bigcup_{i=1}^{r} V(T_i)$ and $\lambda(T_i) = \frac{|V(T_i)|}{k+1}$ for each $i \in [r]$. By the induction hypothesis, for each $i \in [r]$, T_i is k-special, so there exist vertex-disjoint k-stars $S_{i,1}, \ldots, S_{i,t_i}$ such that $V(T_i) = \bigcup_{j=1}^{t_i} V(S_{i,j})$ and $\bigcup_{j=1}^{t_i} E(S_{i,j}) \subseteq E(T_i)$. Therefore, we have $V(T) = V(S_0) \cup \bigcup_{i=1}^{r} \bigcup_{j=1}^{t_i} V(S_{i,j})$ and $E(S_0) \cup \bigcup_{i=1}^{r} \bigcup_{j=1}^{t_i} E(S_{i,j}) \subseteq E(T)$. Since $S_0, T_1, \ldots, T_r$ are vertex-disjoint, $S_0, S_{1,1}, \ldots, S_{1,t_1}, \ldots, S_{r,1}, \ldots, S_{r,t_r}$ are vertex-disjoint. Since $\Delta(T) = k$, T is k-special. $\square$

Chapter 3

Reducing the maximum degree of a graph by deleting edges

3.1 Introduction

In the previous chapter, we investigated the minimum number of vertices that need to be removed from a graph so that the new graph obtained has a smaller maximum degree. In this chapter, we investigate the minimum number of edges that need to be removed from a graph for the same purpose. The first problem is of *domination* type (see [9]), whereas the second problem is of *edge covering* type (see below). In this chapter, we present our work from our recent paper in [11].

Recall that we call a subset L of $E(G)$ a Δ-*reducing edge set of* G if $\Delta(G-L) < \Delta(G)$ or $\Delta(G) = 0$. We denote the size of a smallest Δ-reducing edge set of G by $\lambda_{\mathrm{e}}(G)$. More formally, $\lambda_{\mathrm{e}}(G) = \min\{|L|\colon L \subseteq E(G), \Delta(G-L) < \Delta(G)\}$. Note that L is a Δ-reducing edge set of G if and only if $M(G) \subseteq \bigcup_{e \in L \cap E_G(M(G))} e$ or $\Delta(G) = 0$.

We provide several bounds and equations for $\lambda_{\mathrm{e}}(G)$. Our main results are given in the next section. Before stating our results, we need to recall some definitions and notation, and make a few observations.

Let G_{e} denote the subgraph of G given by $(\bigcup_{v \in M(G)} E_G(v),\ E_G(M(G)))$ $(= (N_G[M(G)], E_G(M(G))))$. Recall that for $L \subseteq E(G)$ and $X \subseteq V(G)$, we say that L *is an edge cover of* X *in* G if for each $v \in X$ with $d_G(v) > 0$, v is incident to at least one edge in L. Note that L is a Δ-reducing edge set of G if and only if L is an edge cover of $M(G)$ in G. Thus,

$$\lambda_{\mathrm{e}}(G) = \min\{|L| \colon L \text{ is an edge cover of } M(G) \text{ in } G\}.$$

Consequently, we immediately obtain

$$\lambda_{\mathrm{e}}(G) = \lambda_{\mathrm{e}}(G_{\mathrm{e}}). \tag{3.1}$$

Definitions and notation from Chapter 1 will be used.

We are now ready to state our main results, given in the next section. In Section 3.3, we investigate $\lambda_{\mathrm{e}}(G)$ from a structural point of view; we obtain equations for $\lambda_{\mathrm{e}}(G)$ in terms of certain parameters of certain subgraphs of G, and observe how $\lambda_{\mathrm{e}}(G)$ changes with the deletion of edges. Some of the structural results are then used in the proofs of the main upper bounds presented in the next section; these proofs are given in Section 3.4.

3.2 Main results

In this section, we present our main results, most of which are bounds for $\lambda_e(G)$ in terms of basic paramaters of G. We start with a lower bound.

Proposition 3.2.1. *If G is a graph, $n = |V(G)|$, $m = |E(G)|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\lambda_e(G) \geq \max\left\{\left\lceil \frac{2m-(k-1)n}{2} \right\rceil, \left\lceil \frac{t}{2} \right\rceil\right\}.$$

Moreover, equality holds if G is complete.

Proof. Let L be a Δ-reducing edge set of G of size $\lambda_e(G)$. Since $\Delta(G-L) \leq k-1$, the handshaking lemma (applied to $G-L$) gives us $|E(G-L)| \leq \frac{(k-1)n}{2}$. Since $m = |E(G-L)| + |L| \leq \frac{(k-1)n}{2} + \lambda_e(G)$, $\lambda_e(G) \geq \left\lceil \frac{2m-(k-1)n}{2} \right\rceil$.

Since L is a Δ-reducing edge set of G, each vertex in $M(G)$ is contained in some edge in L. Thus, $M(G) \subseteq \bigcup_{e \in L} e$. Therefore, $t \leq \sum_{e \in L} |e| = 2|L|$, and hence $\lambda_e(G) \geq \left\lceil \frac{t}{2} \right\rceil$.

Suppose that G is a complete graph. Then $t = n$, $k = n-1$, and $m = \frac{n(n-1)}{2}$. Let $v_1, \ldots, v_n$ be the vertices of G. Let $X = \{v_{2i-1}v_{2i} \colon i \in \mathbb{N}, i \leq \frac{n}{2}\}$. If n is even, then X is a Δ-reducing edge set of G of size $\frac{n}{2} = \left\lceil \frac{t}{2} \right\rceil = \left\lceil \frac{2m-(k-1)n}{2} \right\rceil$. If n is odd, then $X \cup \{v_n v_1\}$ is a Δ-reducing edge set of G of size $\frac{n+1}{2} = \left\lceil \frac{t}{2} \right\rceil = \left\lceil \frac{2m-(k-1)n}{2} \right\rceil$. $\square$

In the rest of this section, we present upper bounds for $\lambda_e(G)$, the proofs of which are given in Section 3.4. For this purpose, we shall first introduce a class of graphs that attain each of these upper bounds.

For $k \geq 1$, we will call a graph G a *special k-star union* if $\Delta(G) = k$ and each non-singleton component of G is a union of k-stars that are pairwise edge-disjoint and k-wise vertex-disjoint.

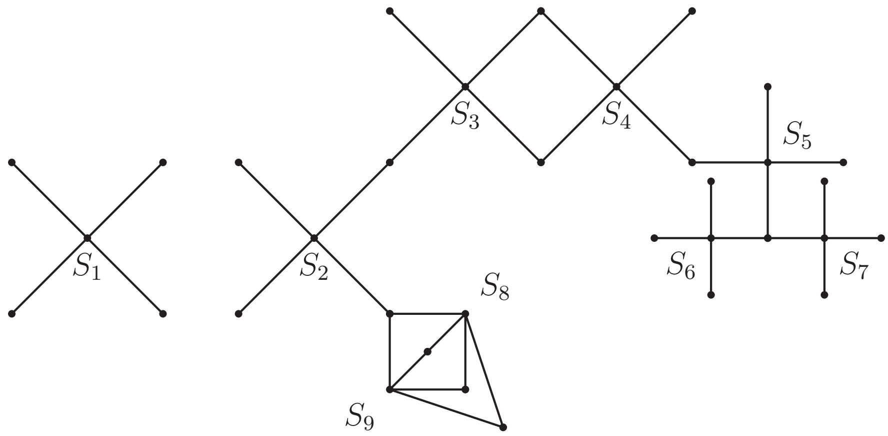

Figure 4: An illustration of special k-star union.

In Section 3.4, we prove the following.

Lemma 3.2.2. *If G is a special k-star union, $m = |E(G)|$, and $t = |M(G)|$, then $m = kt$ and $\lambda_{\mathrm{e}}(G) = t$.*

Theorem 3.2.3. *If G is a graph, $m = |E(G)|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\lambda_{\mathrm{e}}(G) \leq \frac{m + (k-1)t}{2k-1}.$$

Moreover, equality holds if and only if G is a special k-star union or each non-singleton component of G is a 2-star or a triangle.

Remark 3.2.4. By (3.1), we may take $m = |E(G_{\mathrm{e}})|$ in each of the results above, and $n = |V(G_{\mathrm{e}})|$ in Proposition 3.2.1. Note that $\Delta(G) = \Delta(G_{\mathrm{e}})$ and $M(G) = M(G_{\mathrm{e}})$. Thus, we actually have the following immediate consequence.

Corollary 3.2.5. *If G is a graph, $n = |V(G_{\mathrm{e}})|$, $m = |E(G_{\mathrm{e}})|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\max\left\{\left\lceil \frac{2m-(k-1)n}{2} \right\rceil, \left\lceil \frac{t}{2} \right\rceil\right\} \leq \lambda_{\mathrm{e}}(G) \leq \frac{m+(k-1)t}{2k-1}.$$

Moreover, the bounds are sharp.

Consider the numbers m, k, and t in Corollary 3.2.5. By the definition of G_{e}, $m \leq kt$. Let $H = G_{\mathrm{e}}$. By the handshaking lemma, $2m = \sum_{v \in V(H)} d_H(v) \geq \sum_{v \in M(G)} d_H(v) = kt$ (and equality holds if and only if G_{e} is regular). Thus,

$$\frac{kt}{2} \leq m \leq kt. \tag{3.2}$$

Using a probabilistic argument similar to that used by Alon in [4], we prove the following bound.

Theorem 3.2.6. *If G is a graph, $m = |E(G_{\mathrm{e}})|$, $k = \Delta(G) \geq 2$, and $t = |M(G)|$, then*

$$\lambda_{\mathrm{e}}(G) \leq m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right).$$

Moreover, equality holds if G_{e} is a special k-star union.

As we also show in Section 3.4, a slight adjustment of the proof of Theorem 3.2.6 yields the following weaker but simpler (and still sharp) result.

Theorem 3.2.7. *If G is a graph, $m = |E(G_{\mathrm{e}})|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\lambda_{\mathrm{e}}(G) \leq \frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right).$$

Moreover, equality holds if G_{e} is a special k-star union.

A set of pairwise disjoint edges of G is called a *matching of G*. The *matching number of G* is the size of a largest matching of G and is denoted by $\alpha'(G)$. In the next section, we prove the following result.

Theorem 3.2.8. *For every non-empty graph G,*

$$\lambda_{\mathrm{e}}(G) = |M(G)| - \alpha'(G[M(G)]).$$

If G is a regular non-empty graph, then $M(G) = V(G)$, and hence, by Theorem 3.2.8, $\lambda_{\mathrm{e}}(G) = |V(G)| - \alpha'(G)$. Thus, for a regular graph G, a lower bound for $\alpha'(G)$ yields an upper bound for $\lambda_{\mathrm{e}}(G)$, and vice-versa. For $k \geq 3$, Henning and Yeo [35] established a lower bound for $\alpha'(G)$ for all k-regular graphs G, and showed that the bound is attained for infinitely many k-regular graphs. Biedl et al. [8] had proved the bound for $k = 3$ and several other interesting lower bounds for $\alpha'(G)$. Another important lower bound for k-regular graphs with $k \geq 4$ is given by O and West [46]. The 2-regular graphs are the cycles. It is easy to see that $\{n, 1\} \cup \{\{2i, 2i+1\} \colon 1 \leq i \leq \lceil n/2 \rceil - 1\}$ is a smallest Δ-reducing edge set of C_n, so

$$\lambda_{\mathrm{e}}(C_n) = \left\lceil \frac{n}{2} \right\rceil. \tag{3.3}$$

For $k \geq 1$, we will call a tree T an *edge-disjoint k-star union* if T is a union of pairwise edge-disjoint k-stars.

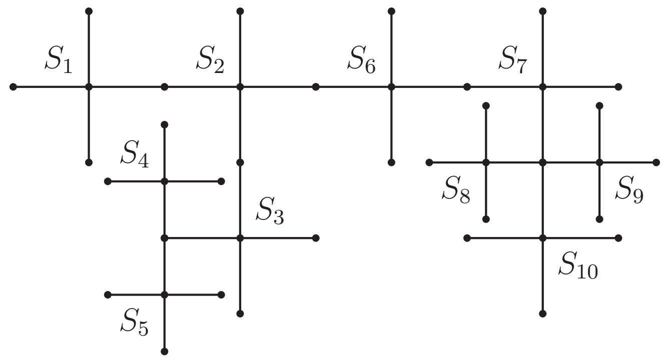

Figure 5: An illustration of an edge-disjoint k-star union.

In Section 3.4, we prove the following sharp bound for trees.

Theorem 3.2.9. *If T is a tree, $n = |V(T)|$, $m = |E(T)|$, and $k = \Delta(T) \geq 1$, then*

$$\lambda_{\mathrm{e}}(T) \leq \frac{n-1}{k} = \frac{m}{k}.$$

Moreover, equality holds if and only if T is an edge-disjoint k-star union.

The trees of maximum degree at most 2 are the paths. It is easy to see that $\{\{2i, 2i+1\} \colon 1 \leq i \leq \lceil (n-2)/2 \rceil\}$ is a smallest Δ-reducing edge set of P_n, so

$$\lambda_{\mathrm{e}}(P_n) = \left\lceil \frac{n-2}{2} \right\rceil. \tag{3.4}$$

Theorem 3.2.9 yields the following generalization.

Theorem 3.2.10. *If F is a forest, $m = |E(F)|$, and $k = \Delta(F) \geq 1$, then*

$$\lambda_{\mathrm{e}}(F) \leq \frac{m}{k}.$$

Moreover, equality holds if and only if each non-singleton component of F is an edge-disjoint k-star union.

Proof. Let $\mathcal{C}$ be the set of components of F. Let $\mathcal{D} = \{C \in \mathcal{C} \colon \Delta(C) = k\}$. Since $\Delta(F) = k$, $\mathcal{D} \neq \emptyset$. For each $D \in \mathcal{D}$, D is a tree, so $\lambda_{\rm e}(D) \leq \frac{|E(D)|}{k}$ by Theorem 3.2.9. By Proposition 3.3.7 (given in the next section), $\lambda_{\rm e}(F) = \sum_{D \in \mathcal{D}} \lambda_{\rm e}(D) \leq \sum_{D \in \mathcal{D}} \frac{|E(D)|}{k} \leq \frac{m}{k}$. If each non-singleton component of F is an edge-disjoint k-star union, then, by Theorem 3.2.9, $\lambda_{\rm e}(F) = \sum_{D \in \mathcal{D}} \frac{|E(D)|}{k} = \frac{m}{k}$. Now suppose $\lambda_{\rm e}(F) = \frac{m}{k}$. Then, by the above, $m = \sum_{D \in \mathcal{D}} |E(D)|$ and $\lambda_{\rm e}(D) = \frac{|E(D)|}{k}$ for each $D \in \mathcal{D}$. Thus, each non-singleton component of F is a member of $\mathcal{D}$, and, by Theorem 3.2.9, it is an edge-disjoint k-star union. $\Box$

By the observations in Remark 3.2.4, we may take $m = |E(G_{\rm e})|$ in Theorem 3.2.10. Thus, for the case where G is a forest, Theorem 3.2.10 improves each of the upper bounds in Corollary 3.2.5, Theorem 3.2.6, and Theorem 3.2.7. Indeed, since $m \leq kt$ (by (3.2)), we have $\frac{m+(k-1)t}{2k-1} \geq \frac{m+(k-1)(m/k)}{2k-1} = \frac{m}{k}$, $m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right) \geq m\left(1 - \frac{k-1}{k}\right) = \frac{m}{k}$, and $\frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right) \geq \frac{m}{k}$.

3.3 Structural results

In this section, we take a close look at how $\lambda_{\rm e}(G)$ is determined by the structure of G and at how it is affected by removing edges from G. Some of the following observations are used in the proofs given in the next section.

Let $M_1(G)$ denote $\{v \in M(G) \colon vw \in E(G)$ for some $w \in M(G)\backslash\{v\}\}$. Let $M_2(G)$ denote $M(G)\backslash M_1(G)$. Thus, $M_2(G) = \{v \in M(G) \colon d_G(v, w) \geq 2$ for each $w \in M(G)\backslash\{v\}\}$.

Recall the definition of an edge cover, given in Section 3.1. An edge cover of $V(G)$ in G is called an *edge cover of G*. The *edge covering number of G*

is the size of a smallest edge cover of G and is denoted by $\beta'(G)$. Clearly, $\lambda_{\mathrm{e}}(G) = \beta'(G)$ if G is regular. In general, we have the following.

Theorem 3.3.1. *For every non-empty graph G,*

$$\lambda_{\mathrm{e}}(G) = |M_2(G)| + \beta'(G[M_1(G)]).$$

Proof. We start with a few observations. Let $k = \Delta(G)$. Since G is non-empty, $k \geq 1$. For each $v \in M(G)$, G has exactly k edges incident to v. By definition of $M_2(G)$,

$$\text{for any } v \in M_2(G) \text{ and any } e \in E_G(v),\ e \notin E_G(w) \text{ for each } w \in M(G)\backslash\{v\}. \tag{3.5}$$

For any $v \in M_1(G)$, $vw \in E(G)$ for some $w \in M(G)\backslash\{v\}$, and therefore $w \in M_1(G)$ and $vw \in G[M_1(G)]$. In other words,

$$\text{for any } v \in M_1(G),\ G[M_1(G)] \text{ has at least one edge incident to } v. \tag{3.6}$$

Thus, $G[M_1(G)]$ has an edge cover.

Let K be an edge cover of $G[M_1(G)]$ of size $\beta'(G[M_1(G)])$. For each $v \in M_2(G)$, let $e_v \in E_G(v)$. Let $K' = \{e_v \colon v \in M_2(G)\} \cup K$. Then K' is a Δ-reducing edge set of G. By (3.5), $|K'| = |M_2(G)| + |K|$. Thus, $\lambda_{\mathrm{e}}(G) \leq |M_2(G)| + \beta'(G[M_1(G)])$.

Now let L be a Δ-reducing edge set of G of size $\lambda_{\mathrm{e}}(G)$. For each $v \in M(G)$, there exists some $e_v \in E_G(v)$ such that $e_v \in L$. Let $L_1 = \{e_v \colon v \in M_1(G)\}$ and $L_2 = \{e_v \colon v \in M_2(G)\}$. Then $L_1 \cup L_2$ is a Δ-reducing edge set of

G. Thus, since $L_1 \cup L_2 \subseteq L$ and $|L| = \lambda_{\mathrm{e}}(G)$, $L = L_1 \cup L_2$. By (3.5), $|L_1 \cup L_2| = |L_1| + |M_2(G)|$. Let $X = \{v \in M_1(G)\colon e_v \notin E(G[M_1(G)])\}$. By (3.6), for each $v \in M_1(G)$, there exists some $e'_v \in E_G(v)$ such that $e'_v \in E(G[M_1(G)])$. Let $L'_1 = (L_1 \backslash \{e_v\colon v \in X\}) \cup \{e'_v\colon v \in X\}$. For each $v \in X$, $e_v \cap M_1(G) = \{v\}$. Thus, L'_1 is an edge cover of $G[M_1(G)]$, and $|L'_1| \leq |L_1|$. We have $\lambda_{\mathrm{e}}(G) = |L| = |M_2(G)| + |L_1| \geq |M_2(G)| + |L'_1| \geq |M_2(G)| + \beta'(G[M_1(G)])$. Since $\lambda_{\mathrm{e}}(G) \leq |M_2(G)| + \beta'(G[M_1(G)])$, the result follows. □

We now prove Theorem 3.2.8. Using a well-known result of Gallai [26], we then show that Theorems 3.2.8 and 3.3.1 are equivalent, meaning that they imply each other.

Theorem 3.2.8. *For every non-empty graph G,*

$$\lambda_{\mathrm{e}}(G) = |M(G)| - \alpha'(G[M(G)]).$$

Proof of Theorem 3.2.8. Let $H = G[M(G)]$. Let K be a matching of H of size $\alpha'(H)$. Let X be the union of the vertices which are incident to the edges in K. That is, $X = \bigcup_{e \in K} e$. Then $X \subseteq M(G)$ and $|X| = 2|K|$. For each $v \in M(G)\backslash X$, let $e_v \in E_G(v)$. Let $K' = \{e_v\colon v \in M(G)\backslash X\}$. Then $K \cup K'$ is a Δ-reducing edge set of G. Thus, $\lambda_{\mathrm{e}}(G) \leq |K| + |K'| \leq |K| + |M(G)\backslash X| = |K| + |M(G)| - |X| = |M(G)| - |K| = |M(G)| - \alpha'(H)$.

Now let L be a Δ-reducing edge set of G of size $\lambda_{\mathrm{e}}(G)$. Then, for each

$v \in M(G)$, there exists some $e'_v \in E_G(v)$ such that $e'_v \in L$. Let J be a largest subset of L that is a matching of H. Let $Y = \bigcup_{e \in J} e$. Then $Y \subseteq M(G)$ and $|Y| = 2|J|$. Let $Y' = M(G) \backslash Y$. Let $J' = \{e'_v \colon v \in Y'\}$. If we assume that $e'_u = e'_v$ for some $u, v \in Y'$ with $u \neq v$, then we obtain that $e'_u = e'_v = uv$ and that $J \cup \{uv\}$ is a matching of H of size $|J| + 1$, which contradicts the choice of J. Thus, $|J'| = |Y'|$. Now $J \cup J' \subseteq L$ and $J \cap J' = \emptyset$. We have $\lambda_{\mathrm{e}}(G) = |L| \geq |J \cup J'| = |J| + |J'| = |J| + |Y'| = |J| + |M(G)| - |Y| = |M(G)| - |J| \geq |M(G)| - \alpha'(H)$. Since $\lambda_{\mathrm{e}}(G) \leq |M(G)| - \alpha'(H)$, the result follows. □

Proposition 3.3.2. *Theorems 3.2.8 and 3.3.1 are equivalent.*

Proof. By (3.6), $\delta(G[M_1(G)]) \geq 1$. A result of Gallai [26] tells us that $\alpha'(H) + \beta'(H) = |V(H)|$ for every graph H with $\delta(H) \geq 1$. Therefore, $\alpha'(G[M_1(G)]) + \beta'(G[M_1(G)]) = |V(G[M_1(G)])| = |M_1(G)|$. If $v, w \in M(G)$ such that $vw \in E(G)$, then $vw \in M_1(G)$. Thus, $E(G[M(G)]) = E(G[M_1(G)])$, and hence $\alpha'(G[M_1(G)]) = \alpha'(G[M(G)])$. Thus, since $|M(G)| = |M_1(G)| + |M_2(G)|$, Theorem 3.2.8 implies Theorem 3.3.1, and vice-versa. □

From Theorem 3.3.1 we immediately obtain the next two results.

Proposition 3.3.3. *If G is a non-empty graph, then $\lambda_{\mathrm{e}}(G) \leq |M(G)|$, and equality holds if and only if $M_2(G) = M(G)$.*

Proof. For each $v \in M(G)$, let $e_v \in E_G(v)$. Since $\{e_v \colon v \in M(G)\}$ is a Δ-reducing edge set of G, $\lambda_{\mathrm{e}}(G) \leq |\{e_v \colon v \in M(G)\}| \leq |M(G)|$. By Theorem 3.3.1, $\lambda_{\mathrm{e}}(G) = |M(G)|$ if $M_2(G) = M(G)$. Suppose $M_2(G) \neq M(G)$. Then $M_1(G) \neq \emptyset$. Let $x \in M_1(G)$. By (3.6), $xy \in E(G[M_1(G)])$ for some $y \in M_1(G)\backslash\{x\}$. Also by (3.6), for each $v \in M_1(G)\backslash\{x, y\}$, there exists some $e'_v \in E_G(v)$ such that $e'_v \in E(G[M_1(G)])$. Let $L = \{xy\} \cup \{e'_v : v \in M_1(G)\backslash\{x, y\}\}$. Since L is an edge cover of $G[M_1(G)]$, $\beta'(G[M_1(G)]) \leq |L| \leq |M_1(G)| - 1$. Thus, by Theorem 3.3.1, $\lambda_e(G) \leq |M_2(G)| + |M_1(G)| - 1 < M(G)$. $\square$

Proposition 3.3.4. *If G is a graph with $M_2(G) \neq M(G)$, then $\Delta(G - M_2(G)) = \Delta(G)$ and $\lambda_e(G) = |M_2(G)| + \lambda_e(G - M_2(G))$.*

Proof. Let $H = G - M_2(G)$. Since $M_2(G) \neq M(G)$, $M_1(G) \neq \emptyset$. By (3.5), $E_G(M_1(G)) \subseteq E(H)$. Together with $M(G) = M_1(G) \cup M_2(G)$, this gives us $M(H) = M_1(G)$. Let K be an edge cover of $G[M_1(G)]$ of size $\beta'(G[M_1(G)])$ (K exists by (3.6)). Then K is a Δ-reducing edge set of H, and hence $\lambda_e(H) \leq \beta'(G[M_1(G)])$. By Theorem 3.3.1, $\lambda_e(G) \geq |M_2(G)| + \lambda_e(H)$. Now let L_1 be a Δ-reducing edge set of H of size $\lambda_e(H)$, and let L_2 be as in the proof of Theorem 3.3.1. Then $L_1 \cup L_2$ is a Δ-reducing edge set of G. Thus, $\lambda_e(G) \leq |L_1| + |L_2| = \lambda_e(H) + |M_2(G)|$. The result follows. $\square$

In the rest of the section, we take a look at how $\lambda_e(H)$ relates to $\lambda_e(G)$ for a subgraph H of G, or rather, how $\lambda_e(G)$ is affected by removing edges from G.

Lemma 3.3.5. *If G is a graph, H is a subgraph of G with $\Delta(H) = \Delta(G)$, and L is a Δ-reducing edge set of G, then $L \cap E(H)$ is a Δ-reducing edge set of H.*

Proof. Let $J = L \cap E(H)$. It is sufficient to show that for each $v \in M(H)$, $e \in E_H(v)$ for some $e \in J$. Let $v \in M(H)$. Since $\Delta(H) = \Delta(G)$, $v \in M(G)$

and $E_H(v) = E_G(v)$. Since $v \in M(G)$, $e \in E_G(v)$ for some $e \in L$. Since $E_G(v) = E_H(v)$, $e \in E(H)$. Therefore, $e \in J$. $\Box$

We point out that $|L| = \lambda_{\mathrm{e}}(G)$ does not guarantee that $|L \cap E(H)| = \lambda_{\mathrm{e}}(H)$. Indeed, let $k \geq 2$, let G_1 and G_2 be copies of $K_{1,k}$ with $V(G_1) \cap V(G_2) = \emptyset$, and let G be the union of G_1 and G_2. Let $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$. Let $e \in E(G_2) \backslash \{e_2\}$. Let $H = (V(G), E(G) \backslash \{e\})$. Let $L = \{e_1, e_2\}$. Then L is a Δ-reducing edge set of G of size $\lambda_{\mathrm{e}}(G)$, $L \cap E(H) = \{e_1, e_2\} = L$, but $\{e_1\}$ is a Δ-reducing edge set of H of size $\lambda_{\mathrm{e}}(H)$. Thus, $L \cap E(H)$ is not a smallest Δ-reducing edge set of H.

Corollary 3.3.6. *If H is a subgraph of G such that $\Delta(H) = \Delta(G)$, then $\lambda_{\mathrm{e}}(H) \leq \lambda_{\mathrm{e}}(G)$.*

Proof. Let L be a Δ-reducing edge set of G of size $\lambda_{\mathrm{e}}(G)$. Let $J = L \cap E(H)$. By Lemma 3.3.5, J is a Δ-reducing edge set of H. Therefore, $\lambda_{\mathrm{e}}(H) \leq |J| \leq |L| = \lambda_{\mathrm{e}}(G)$. $\Box$

Proposition 3.3.7. *If G is a graph and $G_1, \dots, G_r$ are the distinct components of G whose maximum degree is $\Delta(G)$, then $\lambda_{\mathrm{e}}(G) = \sum_{i=1}^{r} \lambda_{\mathrm{e}}(G_i)$.*

Proof. Let L be a Δ-reducing edge set of G of size $\lambda_{\mathrm{e}}(G)$. For each $i \in [r]$, let $L_i = L \cap E(G_i)$. Then $L_1, \dots, L_r$ partition L, so $|L| = \sum_{i=1}^{r} |L_i|$. By Lemma 3.3.5, for each $i \in [r]$, L_i is a Δ-reducing edge set of G_i, so $\lambda_{\mathrm{e}}(G_i) \leq |L_i|$. Suppose $\lambda_{\mathrm{e}}(G_j) < |L_j|$ for some $j \in [r]$. Let L'_j be a Δ-reducing edge set of G_j of size $\lambda_{\mathrm{e}}(G_j)$. Then $L'_j \cup \bigcup_{i \in [r] \backslash \{j\}} L_i$ is a Δ-reducing edge set of G that is smaller than L, a contradiction. Thus, $\lambda_{\mathrm{e}}(G_i) = |L_i|$ for each $i \in [r]$. We have $\lambda_{\mathrm{e}}(G) = |L| = \sum_{i=1}^{r} |L_i| = \sum_{i=1}^{r} \lambda_{\mathrm{e}}(G_i)$. $\Box$

Proposition 3.3.8. *If G is a graph, $u, v \in V(G) \backslash M(G)$, and $uv \in E(G)$, then $\lambda_e(G - uv) = \lambda_e(G)$.*

Proof. Let $e = uv$. Since $u, v \notin M(G)$, $\Delta(G - e) = \Delta(G)$. By Corollary 3.3.6, $\lambda_e(G - e) \leq \lambda_e(G)$. Let L be a Δ-reducing edge set of $G - e$ of size $\lambda_e(G - e)$. Since $u, v \notin M(G)$, $M(G - e) = M(G)$. Thus, L is a Δ-reducing edge set of G, and hence $\lambda_e(G) \leq \lambda_e(G - e)$. Since $\lambda_e(G - e) \leq \lambda_e(G)$, the result follows. $\square$

Proposition 3.3.9. *If G is a graph and $e \in E(G)$, then $\lambda_e(G) \leq 1 + \lambda_e(G - e)$.*

Proof. If $\Delta(G - e) < \Delta(G)$, then $\lambda_e(G) = 1$. Suppose $\Delta(G - e) = \Delta(G)$. Then $M(G - e) \subseteq M(G) \cup e$. Let L be a Δ-reducing edge set of $G - e$ of size $\lambda_e(G - e)$. Then $L \cup \{e\}$ is a Δ-reducing edge set of G. Thus, $\lambda_e(G) \leq |L \cup \{e\}| = 1 + \lambda_e(G - e)$. $\square$

Corollary 3.3.10. *If $e_1, \ldots, e_t$ are edges of a graph G, then $\lambda_e(G) \leq t + \lambda_e(G - \{e_1, \ldots, e_t\})$.*

Proof. The result follows by repeated application of Proposition 3.3.9. $\square$

3.4 Proofs of the main upper bounds

We now prove Lemma 3.2.2 and Theorems 3.2.3, 3.2.6, 3.2.7, and 3.2.9.

Lemma 3.2.2. *If G is a special k-star union, $m = |E(G)|$, and $t = |M(G)|$, then $m = kt$ and $\lambda_e(G) = t$.*

Proof of Lemma 3.2.2. Since G is a special k-star union, $\Delta(G) = k$ and $E(G) = E(G_1) \cup \cdots \cup E(G_r)$ for some k-stars $G_1, \dots, G_r$ that are pairwise edge-disjoint and k-wise vertex-disjoint. Thus, $m = kr$, and for $i \in [r]$, there exist $u_i, v_{i,1}, \dots, v_{i,k} \in V(G)$ such that $G_i = (\{u_i, v_{i,1}, \dots, v_{i,k}\}, \{u_i v_{i,1}, \dots, u_i v_{i,k}\})$. For $i \in [r]$, $|E_{G_i}(u_i)| = k = \Delta(G)$, so we have $E_G(u_i) = E_{G_i}(u_i) = E(G_i)$. Thus, since $E(G_1), \dots, E(G_r)$ are pairwise disjoint, $u_1, \dots, u_r$ are distinct. Consider any $w \in V(G) \backslash \{u_1, \dots, u_r\}$. For each $i \in [r]$ such that $w \in V(G_i)$, $E_G(w) \cap E(G_i) = \{u_i w\}$. Thus, $d_G(w) = |\{i \in [r] \colon w \in V(G_i)\}|$, and hence, since $G_1, \dots, G_r$ are k-wise vertex-disjoint, $d_G(w) < k$. Thus, $M(G) = \{u_1, \dots, u_r\}$, and hence $t = r$. Since $m = kr$, $m = kt$.

Now let L be a Δ-reducing edge set of G of size $\lambda_{\mathrm{e}}(G)$. For $i \in [r]$, there exists some $e_i \in E_G(u_i)$ such that $e_i \in L$. Let $L' = \{e_1, \dots, e_r\}$. For $i, j \in [r]$ with $i \neq j$, $E_G(u_i) \cap E_G(u_j) = E(G_i) \cap E(G_j) = \emptyset$, so $e_i \neq e_j$. Thus, $|L'| = r$. Now L' is a Δ-reducing edge set of G and $L' \subseteq L$, so $\lambda_{\mathrm{e}}(G) \leq |L'| \leq |L|$. Since $\lambda_{\mathrm{e}}(G) = |L|$, we obtain $L' = L$, so $\lambda_{\mathrm{e}}(G) = r$. Since $t = r$, the result is proved. $\square$

Theorem 3.2.3. *If G is a graph, $m = |E(G)|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\lambda_{\mathrm{e}}(G) \leq \frac{m + (k-1)t}{2k-1}.$$

Moreover, equality holds if and only if G is a special k-star union or each non-singleton component of G is a 2-star or a triangle.

Proof of Theorem 3.2.3. If G is a special k-star union, then, by Lemma 3.2.2, we have $m = kt$ and $\lambda_{\mathrm{e}}(G) = t = \frac{m+(k-1)t}{2k-1}$. If G has exactly $c_1 + c_2 + c_3$ components, c_1 components of G are singletons, c_2 components of G are 2-stars, and c_3 components of G are triangles, then $m = 2c_2 + 3c_3$, $k = 2$, $t = c_2 + 3c_3$, and, by Proposition 3.3.7, $\lambda_{\mathrm{e}}(G) = c_2\lambda_{\mathrm{e}}(P_2) + c_3\lambda_{\mathrm{e}}(C_3) = c_2 + 2c_3 = \frac{m+(k-1)t}{2k-1}$.

We now prove the bound in the theorem and show that it is attained only in the cases above. If $m = 1$, then $k = 1$, and the result follows immediately. We now proceed by induction on m. Thus, suppose $m \geq 2$. If $k = 1$, then the edges of G are pairwise disjoint, G is a special 1-star union, and $\lambda_{\mathrm{e}}(G) = m = \frac{m+(k-1)t}{2k-1}$. Suppose $k \geq 2$.

Suppose $M_2(G) = M(G)$. Let $v_1, \dots, v_t$ be the vertices in $M_2(G)$. By (3.5), $E_G(v_1)$, $\dots$, $E_G(v_t)$ are pairwise disjoint, therefore $|E_G(M_2(G))| = \sum_{i=1}^{t} |E_G(v_i)| = \sum_{i=1}^{t} k = kt$. Thus, $m \geq kt$, and equality holds only if $E(G) = \bigcup_{i=1}^{t} E_G(v_i)$. By Proposition 3.3.3, $\lambda_{\mathrm{e}}(G) = t = \frac{kt+(k-1)t}{2k-1} \leq \frac{m+(k-1)t}{2k-1}$. Suppose $\lambda_{\mathrm{e}}(G) = \frac{m+(k-1)t}{2k-1}$. Then $m = kt$, and hence $E(G) = \bigcup_{i=1}^{t} E_G(v_i)$. For $i \in [t]$, let G_i be the k-star $(N_G[v_i], E_G(v_i))$. Then $G_1, \dots, G_t$ are pairwise edge-disjoint. For $i \in [t]$, we have $d_{G_i}(v_i) = \Delta(G)$, so $v_i \notin V(G_j)$ for $j \in [t]\backslash\{i\}$. Consider any $w \in \bigcup_{i=1}^{t} V(G_i)\backslash\{v_1, \dots, v_t\}$. Then $w \notin M(G)$, and hence $d_G(w) < k$. For $i \in [t]$ such that $w \in V(G_i)$, $E_G(w) \cap E(G_i) = \{v_i w\}$. Thus, $|\{i \in [t] \colon w \in V(G_i)\}| = d_G(w) < k$. We have therefore shown that $G_1, \dots, G_t$ are k-wise vertex-disjoint. Since $E(G) = \bigcup_{i=1}^{t} E_G(v_i) = \bigcup_{i=1}^{t} E(G_i)$, G is a special k-star union.

Now suppose $M_2(G) \neq M(G)$. Then $xy \in E(G)$ for some $x, y \in M(G)$. Let $H = G - xy$. We have $m \geq |E_G(x) \cup E_G(y)| = |E_G(x)| + |E_G(y)| - |E_G(x) \cap E_G(y)| = 2k - |\{xy\}| = 2k - 1$. If $\Delta(H) < k$, then $M(G) = \{x, y\}$ and $\lambda_{\mathrm{e}}(G) = 1 < \frac{m+(k-1)t}{2k-1}$.

Suppose $\Delta(H) = k$. Then $M(H) = M(G) \backslash \{x, y\}$. By the induction hypothesis, $\lambda_{\rm e}(H) \leq \frac{(m-1)+(k-1)(t-2)}{2k-1}$. By Proposition 3.3.9,

$$\lambda_{\rm e}(G) \leq 1 + \lambda_{\rm e}(H) \leq 1 + \frac{(m-1)+(k-1)(t-2)}{2k-1} = \frac{m+(k-1)t}{2k-1}.$$

Suppose $\lambda_{\rm e}(G) = \frac{m+(k-1)t}{2k-1}$. Then $\lambda_{\rm e}(G) = 1 + \lambda_{\rm e}(H)$ and $\lambda_{\rm e}(H) = \frac{(m-1)+(k-1)(t-2)}{2k-1}$. By the induction hypothesis, H is a special k-star union or each non-singleton component of H is a 2-star or a triangle.

Suppose that H is a special k-star union. We have $|M(H)| = t - 2$. Let $u_1, \ldots, u_{t-2}$ be the distinct vertices in $M(H)$. By the proof of Lemma 3.2.2, $E_H(u_1), \ldots, E_H(u_{t-2})$ partition $E(H)$, and $\lambda_{\rm e}(H) = |M(H)|$. Since $d_H(x) = |E_G(x) \backslash \{xy\}| = k - 1 > 0$, $u_p x \in E(H)$ for some $p \in [t-2]$. Similarly, $u_q y \in E(H)$ for some $q \in [t-2]$. For each $i \in [t-2] \backslash \{p, q\}$, let $e_i \in E_H(u_i)$. Since $M(G) = \{u_1, \ldots, u_{t-2}\} \cup \{x, y\}$, $\{e_i \colon i \in [t-2] \backslash \{p, q\}\} \cup \{u_p x, u_q y\}$ is a Δ-reducing edge set of G. Together with $t - 2 = |M(H)| = \lambda_{\rm e}(H)$, this gives us $\lambda_{\rm e}(G) \leq \lambda_{\rm e}(H)$, which contradicts $\lambda_{\rm e}(G) = 1 + \lambda_{\rm e}(H)$.

Therefore, each non-singleton component of H is a 2-star or a triangle. Thus, $k = 2$. For $v \in \{x, y\}$, let H_v be the component of H such that $v \in V(H_v)$. Since $2 = k = d_G(x) = |E_{H_x}(x) \cup \{xy\}| = d_{H_x}(x) + 1$, we have $d_{H_x}(x) = 1$, so H_x is a 2-star and x is a leaf of H_x. Suppose $H_x \neq H_y$. Then there are 6 distinct vertices $a_1, \ldots, a_6$ of H such that $H_x = (\{a_1, a_2, a_3\}, \{a_1a_2, a_2a_3\})$, $H_y = (\{a_4, a_5, a_6\}, \{a_4a_5, a_5a_6\})$, $a_3 = x$, and $a_4 = y$. Let L be a smallest Δ-reducing edge set of H. Since H_x and H_y are components of H, we have $M(H) \cap (V(H_x) \cup V(H_y)) = \{a_2, a_5\}$ and $L \cap E(H_x) \neq \emptyset \neq L \cap E(H_y)$. Let $e_x \in L \cap E(H_x)$ and $e_y \in L \cap E(H_y)$. Let

$L' = (L \backslash \{e_x, e_y\}) \cup \{a_2a_3, a_4a_5\}$. Then L' is a Δ-reducing edge set of G. Thus, we have $\lambda_{\mathrm{e}}(G) \leq |L'| = |L| = \lambda_{\mathrm{e}}(H)$, which contradicts $\lambda_{\mathrm{e}}(G) = 1 + \lambda_{\mathrm{e}}(H)$. Therefore, $H_x = H_y$. Let $G_x = (V(H_x), E(H_x) \cup \{xy\})$. Then G_x is a component of G. Since x and y are the two leaves of the 2-star H_x, G_x is a triangle. Consequently, each non-singleton component of G is a 2-star or a triangle. □

Theorem 3.2.6. *If G is a graph, $m = |E(G_{\mathrm{e}})|$, $k = \Delta(G) \geq 2$, and $t = |M(G)|$, then*

$$\lambda_{\mathrm{e}}(G) \leq m \left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right).$$

Moreover, equality holds if G_{e} is a special k-star union.

Proof of Theorem 3.2.6. We may assume that $E_G(M(G)) = [m]$. By (3.2), $m \leq kt$. Let $p = 1 - \left(\frac{m}{kt}\right)^{\frac{1}{k-1}}$. We set up m independent random experiments, and in each experiment an edge is chosen with probability p. More formally, for $i \in [m]$, let (Ω_i, P_i) be given by $\Omega_i = \{0, 1\}$, $P_i(\{1\}) = p$, and $P_i(\{0\}) = 1 - p$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ and let $P\colon 2^{\Omega} \to [0, 1]$ (where $[0, 1]$ denotes $\{x \in \mathbb{R}\colon 0 \leq x \leq 1\}$) such that $P(\{\omega\}) = \prod_{i=1}^{m} P_i(\{\omega_i\})$ for each $\omega = (\omega_1, \dots, \omega_m) \in \Omega$, and $P(A) = \sum_{\omega \in A} P(\{\omega\})$ for each $A \subseteq \Omega$. Then (Ω, P) is a probability space.

For each $\omega = (\omega_1, \dots, \omega_m) \in \Omega$, let $S_\omega = \{i \in [m]\colon \omega_i = 1\}$ and $T_\omega = \{v \in M(G)\colon \text{no edge incident to } v \text{ is a member of } S_\omega\}$.

Let $X\colon \Omega \to \mathbb{R}$ be the random variable given by $X(\omega) = |S_\omega|$. For $i \in [m]$, let $X_i\colon \Omega \to \mathbb{R}$ such that, for $\omega = (\omega_1, \dots, \omega_m) \in \Omega$,

$$X_i(\omega) = \begin{cases} 1 & \text{if } i \in S_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then $X = \sum_{i=1}^{m} X_i$. For $i \in [m]$, $P(X_i = 1) = P_i(\{1\}) = p$.

Let $Y\colon \Omega \to \mathbb{R}$ be the random variable given by $Y(\omega) = |T_\omega|$. For $v \in M(G)$, let $Y_v\colon \Omega \to \mathbb{R}$ such that, for $\omega = (\omega_1, \dots, \omega_m) \in \Omega$,

$$Y_v(\omega) = \begin{cases} 1 & \text{if } v \in T_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y = \sum_{v \in M(G)} Y_v$. For $v \in M(G)$, $P(Y_v = 1) = (1-p)^k$.

For any random variable Z, let $\mathrm{E}[Z]$ denote the expected value of Z. By linearity of expectation,

$$\begin{aligned} \mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] &= \sum_{i=1}^{m} \mathrm{E}[X_i] + \sum_{v \in M(G)} \mathrm{E}[Y_v] \\ &= \sum_{i=1}^{m} P(X_i = 1) + \sum_{v \in M(G)} P(Y_v = 1) = mp + t(1-p)^k. \end{aligned}$$

Thus, by Proposition 2.4.3, there exists some $\omega^* \in \Omega$ such that $X(\omega^*) + Y(\omega^*) \le mp + t(1-p)^k$. For $v \in T_{\omega^*}$, let $e_v \in E_G(v)$. Let $L_{\omega^*} = S_{\omega^*} \cup \{e_v\colon v \in T_{\omega^*}\}$. Then L_{ω^*} is a Δ-reducing edge set of G. Thus, $\lambda_{\mathrm{e}}(G) \le |L_{\omega^*}| \le |S_{\omega^*}| + |T_{\omega^*}| = X(\omega^*) + Y(\omega^*) \le mp + t(1-p)^k = m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right)$. If G_{e} is a special k-star union, then, by Lemma 3.2.2, we have $m = kt$ and $\lambda_{\mathrm{e}}(G) = t$, and hence $\lambda_{\mathrm{e}}(G) = m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{\frac{1}{k-1}}\right)$. $\square$

Remark 3.4.1. Note that the minimum value of the function $f : [0,1] \to \mathbb{R}$ given by $f(p) = mp + t(1-p)^k$ occurs at $p = 1 - \left(\frac{m}{kt}\right)^{\frac{1}{k-1}}$, hence the choice of p in the proof above.

Theorem 3.2.7. *If G is a graph, $m = |E(G_{\mathrm{e}})|$, $k = \Delta(G) \geq 1$, and $t = |M(G)|$, then*

$$\lambda_{\mathrm{e}}(G) \leq \frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right).$$

Moreover, equality holds if G_{e} is a special k-star union.

Proof of Theorem 3.2.7. Let $p^* = 1 - \left(\frac{m}{kt}\right)^{\frac{1}{k-1}}$ and $q = \frac{1}{k}\ln\left(\frac{kt}{m}\right)$. By (3.2), $kt/2 \leq m \leq kt$. Thus, $0 \leq q \leq \frac{1}{k}\ln 2 < 1$. Let f be as in Remark 3.4.1. Thus, $f(p^*) \leq f(q)$. By the proof of Theorem 3.2.6, $\lambda_{\mathrm{e}}(G) \leq f(p^*) \leq f(q) = mq + t(1-q)^k$. Since $1 - q \leq e^{-q}$, we obtain $\lambda_{\mathrm{e}}(G) \leq mq + te^{-qk} = \frac{m}{k}\ln\left(\frac{kt}{m}\right) + te^{-\ln\left(\frac{kt}{m}\right)} = \frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right)$. If G_{e} is a special k-star union, then, by Lemma 3.2.2, we have $m = kt$ and $\lambda_{\mathrm{e}}(G) = t$, and hence $\lambda_{\mathrm{e}}(G) = \frac{m}{k}\left(1 + \ln\left(\frac{kt}{m}\right)\right)$. □

We now prove Theorem 3.2.9. Recall that if a vertex v of a graph G has only one neighbour in G, then v is called a *leaf of G*.

Theorem 3.2.9. *If T is a tree, $n = |V(T)|$, $m = |E(T)|$, and $k = \Delta(T) \geq 1$, then*

$$\lambda_{\mathrm{e}}(T) \leq \frac{n-1}{k} = \frac{m}{k}.$$

Moreover, equality holds if and only if T is an edge-disjoint k-star union.

Proof of Theorem 3.2.9. The result is trivial for $n \leq 2$. We now proceed by induction on n. Thus, consider $n \geq 3$. Since T is connected, $k \geq 2$.

Suppose that T has a leaf z whose neighbour is not in $M(T)$. Let w be the neighbour of z in T. Let $T' = T - z$. By (3.1), $\lambda_{\mathrm{e}}(T) = \lambda_{\mathrm{e}}(T')$ as $T_{\mathrm{e}} = T'_{\mathrm{e}}$. By the induction hypothesis, $\lambda_{\mathrm{e}}(T') \leq \frac{n-2}{k} < \frac{n-1}{k}$. Thus, $\lambda_{\mathrm{e}}(T) < \frac{n-1}{k}$. Suppose T is an edge-disjoint k-star union. Then T contains a k-star S such that $z \in V(S)$. Since $N_S(z) \subseteq N_T(z) = \{w\}$, z is a leaf of S and $S = (\{w, z'_1, \ldots, z'_k\}, \{wz'_1, \ldots, wz'_k\})$, where $z'_1 = z$ and $z'_2, \ldots, z'_k$ are distinct elements of $V(T) \backslash \{w, z\}$. Thus, we have $d_T(w) = k$, contradicting $w \notin M(T)$. Therefore, T is not an edge-disjoint k-star union.

Now suppose that each leaf of T has its neighbour in $M(T)$. Let x, m, and $D_0, \ldots, D_m$ be as in Lemma 2.4.1. Let $z \in V(T)$ such that $d_T(x, z) = m$. By Corollary 2.4.2, z is a leaf of T. Let w be the neighbour of z in T. By Lemma 2.4.1, $w \in D_{m-1}$.

Suppose $w = x$. Then $m = 1$ and $T = (\{x, z_1, \ldots, z_k\}, \{xz_1, \ldots, xz_k\})$ for some distinct vertices $z_1, \ldots, z_k$ in D_m. Thus, T is a k-star. Since xz_1 is a Δ-reducing edge set of T, $\lambda_{\mathrm{e}}(T) = 1 = \frac{n-1}{k}$.

Now suppose $w \neq x$. Together with Lemma 2.4.1, this implies that $N_T(w) = \{v, z_1, \ldots, z_{k-1}\}$ for some $v \in D_{m-2}$ and some distinct vertices $z_1, \ldots, z_{k-1}$ in D_m. By Corollary 2.4.2, $z_1, \ldots, z_{k-1}$ are leaves of T. Let $e = wv$. Let

$$T_1 = T - \{w, z_1, \ldots, z_{k-1}\} \quad \text{and} \quad T_2 = (\{w, z_1, \ldots, z_{k-1}\}, \{wz_1, \ldots, wz_{k-1}\}).$$

Clearly, T_1 and T_2 are the components of $T - e$, and they are trees. Let $T_2' = (\{v\} \cup V(T_2), \{e\} \cup E(T_2))$. If $T = T_2'$, then $\Delta(T - e) < k$, and hence $\lambda_{\mathrm{e}}(T) = 1 = \frac{n-1}{k}$. We have $\Delta(T_2) < k$.

Suppose $\Delta(T_1) < k$. Then $\Delta(T - e) < k$, and hence $\lambda_{\mathrm{e}}(T) = 1 \leq \frac{n-1}{k}$. Suppose $\lambda_{\mathrm{e}}(T) = \frac{n-1}{k}$. Then $n = k + 1 = |V(T_2)| + 1$. Since $n = |V(T_1)| + |V(T_2)|$, we obtain $|V(T_1)| = 1$, so $V(T_1) = \{v\}$. Thus, T is the k-star T_2'.

Finally, suppose $\Delta(T_1) = k$. By Proposition 3.3.7, $\lambda_{\mathrm{e}}(T - e) = \lambda_{\mathrm{e}}(T_1)$. By the induction hypothesis, $\lambda_{\mathrm{e}}(T_1) \leq \frac{n-k-1}{k}$, and equality holds if and only if T_1 is an edge-disjoint k-star union. By Proposition 3.3.9, $\lambda_{\mathrm{e}}(T) \leq 1 + \lambda_{\mathrm{e}}(T - e) \leq 1 + \frac{n-k-1}{k} = \frac{n-1}{k}$.

Suppose $\lambda_{\mathrm{e}}(T) = \frac{n-1}{k}$. Then $\lambda_{\mathrm{e}}(T_1) = \frac{n-k-1}{k}$, and hence T_1 is an edge-disjoint k-star union. Since T is the union of T_1 and T_2', T is an edge-disjoint k-star union.

We now prove the converse. Thus, suppose that T is an edge-disjoint k-star union. Then there exist pairwise edge-disjoint k-stars $G_1, \dots, G_r$ such that $z_1 \in V(G_r)$ and T is the union of $G_1, \dots, G_r$. Since $N_{G_r}(z_1) \subseteq N_T(z_1) = \{w\}$, $G_r = (\{w, z_1, y_1, \dots, y_{k-1}\}, \{wz_1, wy_1, \dots, w_{y_{k-1}}\})$ for some $y_1, \dots, y_{k-1} \in V(T)$. Since $d_{G_r}(w) = k = d_T(w)$, $N_{G_r}(w) = N_T(w)$. Thus, $\{z_1, y_1, \dots, y_{k-1}\} = \{z_1, \dots, z_{k-1}, v\}$, and hence $G_r = T_2'$. Consequently, T_1 is the union of $G_1, \dots, G_{r-1}$, and hence $\lambda_{\mathrm{e}}(T_1) = \frac{n-k-1}{k}$. Let L be a Δ-reducing edge set of T of size $\lambda_{\mathrm{e}}(T)$. Let $L_1 = L \cap E(T_1)$ and $L_2 = L \cap E(T_2')$. Since $E(T_1)$ and $E(T_2')$ partition $E(T)$, L_1 and L_2 partition L. Since $w \in M(T)$ and $E_T(w) = E(T_2')$, $L_2 \neq \emptyset$. Suppose that L_1 is not a Δ-reducing edge set of T_1. Then, since $\Delta(T_1) = k$, there exists some $u \in V(T_1)$ such that $d_{T_1}(u) = k$ and $E_T(u) \cap L \subseteq L_2$. Since $V(T_1) \cap V(T_2') = \{v\}$ and $L_2 \subseteq V(T_2')$,

$u = v$. Now $k \geq |E_T(v)| = |E_{T_1}(v) \cup \{e\}| > |E_{T_1}(v)| = d_{T_1}(v)$, which contradicts $d_{T_1}(v) = d_{T_1}(u) = k$. Thus, L_1 is a Δ-reducing edge set of T_1. We have $\frac{n-1}{k} \geq \lambda_{\mathrm{e}}(T) = |L| = |L_1| + |L_2| \geq \lambda_{\mathrm{e}}(T_1) + 1 = \frac{n-k-1}{k} + 1 = \frac{n-1}{k}$, so $\lambda_{\mathrm{e}}(T) = \frac{n-1}{k}$.

A basic result in the literature is that $|E(G)| = |V(G)| - 1$ if G is a tree. This completes the proof. $\square$

Chapter 4

A generalisation of Turán's problem

4.1 Introduction

In this chapter, we consider a generalisation of the classical problem of Turán [52]. We investigate the smallest number of edges that need to be removed from a non-empty graph G so that the resulting graph does not contain k-cliques. Let $\mathcal{C}_k(G)$ denote the set of distinct k-cliques of G. That is, $\mathcal{C}_k(G) = \{C \subseteq V(G) \colon C \text{ is a } k\text{-clique of } G\}$. We call $L \subseteq E(G)$ a *k-clique reducing edge set of* G if $\omega(G - L) < k$. We define $\lambda_{\mathrm{c}}(G, k)$ to be the size of a smallest k-clique reducing edge set of G; that is, $\lambda_{\mathrm{c}}(G, k) = \min\{|L| \colon L \subseteq E(G), \omega(G - L) < k\}$. We call $\lambda_{\mathrm{c}}(G, k)$ the *k-clique reducing edge number*.

We can now state our results, which are given in the next section. In Section 4.3, we investigate $\lambda_{\mathrm{c}}(G, k)$ from a structural point of view; that is, how the parameter changes with the removal of edges. Some of the structural results are then used in the proofs of the main results; these proofs are given in Section 4.4. Definitions and notation from Chapter 1 will be used.

4.2 Main results

In this section we present our results. We start by stating Turán's theorem, but first we define a special graph.

Let G be a graph such that $V(G)$ is partitioned into $k-1$ sets, $V_1, \dots, V_{k-1}$, $|V_i| = n_i$, for each $i \in [k-1]$, and $E(G) = \{\{u,v\}\colon u \in V_i, v \in V_j, i \neq j\}$. Let us denote the resulting graph by $K_{n_1,\dots,n_{k-1}}$. Note that $n = n_1 + \cdots + n_{k-1}$. The *Turán graph* $T(n,k)$ is the graph $K_{n_1,\dots,n_{k-1}}$, with $|n_i - n_j| \leq 1$, for every $i, j \in [k-1], i \neq j$.

Theorem 4.2.1 (Turán, [52])**.** *If G is a graph, $n = |V(G)|$, $m = |E(G)|$, and G does not contain k-cliques, then*

$$m \leq \left\lfloor \left(\frac{k-2}{k-1}\right)\frac{n^2}{2} \right\rfloor.$$

Moreover, the bound is attained if and only if G is the Turán graph $T(n,k)$.

Turán's theorem generalises a previous result [42] by Mantel which states that the maximum number of edges in a graph on n vertices and which does not contain a copy of K_3, is $\lfloor \frac{n^2}{4} \rfloor$.

Corollary 4.2.2. *If G is a graph, $n = |V(G)|$, $m = |E(G)|$, then*

$$\lambda_c(G,k) \geq m - \left\lfloor \left(\frac{k-2}{k-1}\right)\frac{n^2}{2} \right\rfloor.$$

Proof. Let G be a graph and let L be a k-clique reducing edge set of G of size $\lambda_{\mathrm{c}}(G,k)$. Thus, $\omega(G-L) < k$, and therefore, by Theorem 4.2.1, $|E(G-L)| \leq \lfloor (\frac{k-2}{k-1}) \frac{n^2}{2} \rfloor$. But $|E(G-L)| = |E(G)| - |L| = m - \lambda_{\mathrm{c}}(G,k)$. Thus, $\lambda_{\mathrm{c}}(G,k) = m - |E(G-L)| \geq m - \lfloor (\frac{k-2}{k-1}) \frac{n^2}{2} \rfloor$. $\square$

We point out that the bound in Corollary 4.2.2 is attained by any graph G (on n vertices) which contains the Turán graph $T(n,k)$ as a subgraph. Indeed, let $L = E(G) \setminus E(T(n,k))$. Note that $\omega(G-L) = \omega(T(n,k)) < k$. Thus, L is a k-clique reducing edge set of G. Therefore, $\lambda_{\mathrm{c}}(G,k) \leq |L| = |E(G) \setminus E(T(n,k))| = |E(G)| - |E(T(n,k))| = m - \lfloor (\frac{k-2}{k-1}) \frac{n^2}{2} \rfloor$. Thus, since $\lambda_{\mathrm{c}}(G,k) \geq m - \lfloor (\frac{k-2}{k-1}) \frac{n^2}{2} \rfloor$, then $\lambda_{\mathrm{c}}(G,k) = m - \lfloor (\frac{k-2}{k-1}) \frac{n^2}{2} \rfloor$. In particular, if G is the complete graph K_n, then $\lambda_{\mathrm{c}}(G,k) = \binom{n}{2} - \lfloor (\frac{k-2}{k-1}) \frac{n^2}{2} \rfloor$.

If we apply an approach similar to that used in the proof of Theorem 2.2.4 in chapter 2, we get the following bound; however, we prove this result by induction on the number of edges.

Theorem 4.2.3. *If G is a graph with $\delta(G) > 0$, $m = |E(G)|$ and $t = |\mathcal{C}_k(G)|$, then*

$$\lambda_{\mathrm{c}}(G,k) \leq \frac{m + \left(\binom{k}{2} - 1\right)t}{2\binom{k}{2} - 1}.$$

Moreover, the bound is attained if and only if G is a union of t pairwise edge-disjoint k-cliques.

By adapting an argument similar to that used by Alon in [4], we prove the following sharp bound.

Theorem 4.2.4. *If G is a graph, $m = |E(G)|$, $t = |\mathcal{C}_k(G)|$, and $\alpha = \binom{k}{2}$,* *then*

$$\lambda_c(G,k) \leq m\Big[1 - \Big(\frac{\alpha-1}{\alpha}\Big)\Big(\frac{m}{\alpha t}\Big)^{\frac{1}{\alpha-1}}\Big].$$

The bound in Theorem 4.2.4 is attained if G is a union of t pairwise edge-disjoint k-cliques.

4.3 Structural results

In this section we provide results on how $\lambda_c(G,k)$ is affected by removing edges from G. Some of these structural results are then used in the proofs of the main results.

Recall that $\mathcal{C}_k(G) = \{C \subseteq V(G) \colon C \text{ is a } k\text{-clique of } G\}$ is the set of all distinct k-cliques of G. Now for the rest of this chapter, for each $C \in \mathcal{C}_k(G)$, we will let $E(C) = E(G[C]) = \binom{C}{2}$.

Define $\mathcal{C}_k^1(G) = \{C \in \mathcal{C}_k(G) \colon E(C) \cap E(K) \neq \emptyset$ for some $K \in \mathcal{C}_k(G) \setminus C\}$. Let $\mathcal{C}_k^2(G) = \mathcal{C}(G) \setminus \mathcal{C}_k^1(G)$. That is, $\mathcal{C}_k^2(G) = \{C \in \mathcal{C}_k(G) \colon E(C) \cap E(K) = \emptyset$ for every $K \in \mathcal{C}_k(G) \setminus C\}$.

Lemma 4.3.1. *If G is a graph, H is a subgraph of G and L is a k-clique reducing edge set of G, then $L \cap E(H)$ is a k-clique reducing edge set of H.*

Proof. If $\omega(H) < k$, then the result is trivial. So suppose $\omega(H) \geq k$. Let $J = L \cap E(H)$. It is sufficient to show that for every $C \in \mathcal{C}_k(H)$, there exists $e \in E(C)$ for some $e \in J$. Let $C \in \mathcal{C}_k(H)$. Since H is a subgraph of G, then $C \in \mathcal{C}_k(G)$. Therefore, by definition of L, there exists $e \in L$ such that $e \in E(C)$. But since $C \in \mathcal{C}_k(H)$, then $e \in E(C) \subseteq E(H)$. Thus, $e \in L \cap E(H) = J$. Thus, the result follows. □

We point out that $|L| = \lambda_c(G, k)$ does not guarantee that $|L \cap E(H)| = \lambda_c(H, k)$. Indeed, let $k \geq 3$ and let G be a copy of K_k. Let $e_1 = \{1, 2\}$, $e_2 = \{k, 1\}$ and $H = G - e_2$. Then $L = \{e_1\}$ is a k-clique reducing edge set of G of size $\lambda_c(G, k)$, $L \cap E(H) = L$, but since $\omega(H) < k$, then we can take $\emptyset$ as a k-clique reducing edge set of H. Thus, $L \cap E(H)$ is not a smallest k-clique reducing edge set of H.

Corollary 4.3.2. *If H is a subgraph of G, then $\lambda_c(H, k) \leq \lambda_c(G, k)$.*

Proof. Let L be a k-clique edge reducing set of G of size $\lambda_c(G, k)$. Let $J = L \cap E(H)$. By Lemma 4.3.1, J is a k-clique reducing edge set of H. Therefore, $\lambda_c(H, k) \leq |J| \leq |L| = \lambda_c(G, k)$. □

Proposition 4.3.3. *If G is a graph and $G_1, \dots, G_r$ are the distinct components of G, then $\lambda_c(G, k) = \sum_{i=1}^{r} \lambda_c(G_i, k)$.*

Proof. Let L be a k-clique reducing edge set of G. For each $i \in [r]$, let $L_i = L \cap E(G_i)$. Then $L_1, \dots, L_r$ partition L, so $|L| = \sum_{i=1}^{r} |L_i|$. By Lemma 4.3.1, for each $i \in [r]$, L_i is a k-clique reducing edge set of G_i, so $\lambda_c(G_i, k) \leq |L_i|$. Suppose $\lambda_c(G_j, k) < |L_j|$ for some $j \in [r]$. Let L'_j be a k-clique reducing edge set of G'_j of size $\lambda_c(G_j, k)$. Then $L'_j \cup_{i \in [r] \setminus \{j\}} L_i$ is a k-clique reducing edge set of G that is smaller than L, a contradiction. Therefore, $\lambda_c(G_i, k) = |L_i|$ for each $i \in [r]$. Thus, $\lambda_c(G, k) = |L| = \sum_{i=1}^{r} |L_i| = \sum_{i=1}^{r} \lambda_c(G_i, k)$. □

Proposition 4.3.4. *If G is a graph and $e \in E(G) \setminus \cup_{C \in \mathcal{C}_k(G)} E(C)$, then $\lambda_c(G - e, k) = \lambda_c(G, k)$.*

Proof. Let $e \in E(G) \backslash \cup_{C \in \mathcal{C}_k(G)} E(C)$. Since $G - e$ is a subgraph of G, then by Corollary 4.3.2, $\lambda_c(G-e, k) \leq \lambda_c(G, k)$. Let L be a k-clique reducing edge set of $G-e$ of size $\lambda_c(G-e, k)$. Since $e \notin \cup_{C \in \mathcal{C}_k(G)} E(C)$, then $\mathcal{C}_k(G-e) = \mathcal{C}_k(G)$. Thus, L is a k-clique reducing edge set of G, so $\lambda_c(G, k) \leq |L| = \lambda_c(G - e, k)$. □

Proposition 4.3.5. *If G is a graph and $e \in E(G)$, then $\lambda_c(G, k) \leq 1 + \lambda_c(G - e, k)$.*

Proof. If $\mathcal{C}_k(G - e) = \emptyset$, then the result is trivial. Suppose $\mathcal{C}_k(G - e) \neq \emptyset$, so $\mathcal{C}_k(G - e) \subseteq \mathcal{C}_k(G)$. Let L be a k-clique reducing edge set of $G - e$ of size $\lambda_c(G - e, k)$. For any $C \in \mathcal{C}_k(G) \setminus \mathcal{C}_k(G - e)$, $e \in E(C)$. Thus, $L \cup \{e\}$ is a k-clique reducing edge set of G. Therefore, $\lambda_c(G, k) \leq |L \cup \{e\}| = |L| + 1 = \lambda_c(G - e, k) + 1$. □

Corollary 4.3.6. *If G is a graph and $e_1, \ldots, e_t$ are edges of G, then $\lambda_c(G, k) \leq t + \lambda_c(G - \{e_1, \ldots, e_t\}, k)$.*

Proof. The result follows by repeated application of Proposition 4.3.5. □

Proposition 4.3.7. *If G is a graph, $\lambda_c(G, k) \leq |\mathcal{C}_k(G)|$, and equality holds if and only if $\mathcal{C}_k(G) = \mathcal{C}_k^2(G)$.*

Proof. If $\mathcal{C}_k(G) = \emptyset$, then $\lambda_c(G, k) = 0$, and therefore the result follows. Suppose $\mathcal{C}_k(G) \neq \emptyset$. For each $C \in \mathcal{C}_k(G)$, choose a single edge $e_C \in E(C)$. Then $\{e_C \colon C \in \mathcal{C}_k(G)\}$ is a k-clique reducing edge set of G, and thus $\lambda_c(G, k) \leq |\{e_C \colon C \in \mathcal{C}_k(G)\}| \leq |\mathcal{C}_k(G)|$.

Suppose $\mathcal{C}_k(G) = \mathcal{C}_k^2(G)$. Then $\mathcal{C}_k^1(G) = \emptyset$. Suppose that G has a k-clique reducing edge set L of G such that $|L| < |\mathcal{C}_k(G)|$. By definition of

L, for every $C \in \mathcal{C}_k(G)$ there exists $e \in L$ such that $e \in E(C)$. But since $|L| < |\mathcal{C}_k(G)|$, then by the pigeonhole principle, there exists $e' \in L$, and $C_1, C_2 \in \mathcal{C}_k(G)$, $C_1 \neq C_2$, such that $e' \in E(C_1)$ and $e' \in E(C_2)$. Thus, $E(C_1) \cap E(C_2) \neq \emptyset$, which contradicts $\mathcal{C}_k(G) = \mathcal{C}_k^2(G)$. Suppose now that $\mathcal{C}_k(G) \neq \mathcal{C}_k^2(G)$. Then $\mathcal{C}_k^1(G) \neq \emptyset$. Let $C_1 \in \mathcal{C}_k^1(G)$, then $E(C_1) \cap E(C_2) \neq \emptyset$ for some $C_2 \in \mathcal{C}_k^1(G) \backslash \{C_1\}$. Let $e \in E(C_1) \cap E(C_2)$. Note that $L = \{e_C \colon C \in \mathcal{C}_k(G) \setminus \{C_1, C_2\}\} \cup \{e\}$ is a k-clique reducing edge set of G, and therefore, $\lambda_{\mathrm{c}}(G, k) \leq |L| \leq |\mathcal{C}_k(G)| - 1 < |\mathcal{C}_k(G)|$. $\square$

Proposition 4.3.8. *If G is a graph and $\mathcal{C}_k^2(G) \neq \mathcal{C}_k(G)$, then $\lambda_{\mathrm{c}}(G, k) = |\mathcal{C}_k^2(G)| + \lambda_{\mathrm{c}}(G - \cup_{C \in \mathcal{C}_k^2(G)} E(C), k)$.*

Proof. We use induction on $|\mathcal{C}_k^2(G)|$. The result is trivial if $|\mathcal{C}_k^2(G)| = 0$. Suppose $|\mathcal{C}_k^2(G)| \geq 1$. Let $C' \in \mathcal{C}_k^2(G)$. Since $\mathcal{C}_k^2(G) \neq \mathcal{C}_k(G)$, then $\mathcal{C}_k^1(G) \neq \emptyset$. Clearly, $\mathcal{C}_k^1(G - E(C')) = \mathcal{C}_k^1(G)$, and $\mathcal{C}_k^2(G - E(C')) = \mathcal{C}_k^2(G) \backslash \{C'\} \neq \mathcal{C}_k(G - E(C'))$. By the induction hypothesis, $\lambda_{\mathrm{c}}(G - E(C'), k) = |\mathcal{C}_k^2(G - E(C'))| + \lambda_{\mathrm{c}}((G - E(C')) - \cup_{C \in \mathcal{C}_k^2(G - E(C'))} E(C), k) = |\mathcal{C}_k^2(G)| - 1 + \lambda_{\mathrm{c}}((G - E(C')) - \cup_{C \in \mathcal{C}_k^2(G) \backslash \{C'\}} E(C), k) = |\mathcal{C}_k^2(G)| - 1 + \lambda_{\mathrm{c}}(G - (E(C') \cup_{C \in \mathcal{C}_k^2(G) \backslash \{C'\}} E(C)), k) = |\mathcal{C}_k^2(G)| - 1 + \lambda_{\mathrm{c}}(G - \cup_{C \in \mathcal{C}_k^2(G)} E(C), k)$. Let $e' \in E(C')$. Now since for every edge $e \in E(C') \setminus \{e'\}$, $e \notin \cup_{C \in \mathcal{C}_k(G - e')} E(C)$, then by repeated application of Proposition 4.3.4, $\lambda_{\mathrm{c}}(G - e', k) = \lambda_{\mathrm{c}}(G - E(C'), k)$. By Proposition 4.3.5, $\lambda_{\mathrm{c}}(G, k) \leq 1 + \lambda_{\mathrm{c}}(G - e', k) = 1 + \lambda_{\mathrm{c}}(G - E(C'), k)$. Suppose $\lambda_{\mathrm{c}}(G, k) \leq \lambda_{\mathrm{c}}(G - E(C'), k)$. Let L be a k-clique reducing edge set of G of size $\lambda_{\mathrm{c}}(G, k)$. Then there exists $e'' \in L$ such that $e'' \in E(C')$. Since $C' \in \mathcal{C}_k^2(G)$, $e'' \notin E(C)$ for some $C \in \mathcal{C}_k(G) \setminus \{C'\}$. We obtain that $L \setminus \{e''\}$ is a k-clique reducing edge set of $G - E(C')$ of size $\lambda_{\mathrm{c}}(G, k) - 1 \leq \lambda_{\mathrm{c}}(G - E(C'), k) - 1$, a contradiction. Therefore, $\lambda_c(G,k) = 1+\lambda_c(G-E(C'),k) = |\mathcal{C}_k^2(G)|+\lambda_c(G-\cup_{C\in\mathcal{C}_k^2(G)}E(C),k)$. $\square$

4.4 Proofs of the main results

In this section, we provide the proofs of Theorem 4.2.3 and Theorem 4.2.4.

Theorem 4.2.3. *If G is a graph with $\delta(G) > 0$, $m = |E(G)|$ and $t = |\mathcal{C}_k(G)|$, then*

$$\lambda_c(G,k) \le \frac{m + \left(\binom{k}{2} - 1\right)t}{2\binom{k}{2} - 1}.$$

Moreover, the bound is attained if and only if G is a union of t pairwise edge-disjoint k-cliques.

Proof of Theorem 4.2.3. If G is a union of t pairwise edge-disjoint k-cliques, then $m = \binom{k}{2}t$, and therefore, $\lambda_c(G,k) = t = \frac{m+\left(\binom{k}{2}-1\right)t}{2\binom{k}{2}-1}$. We now prove the bound and show it is attained only if G is a union of t pairwise edge-disjoint k-cliques. Suppose $m = 1$. If $k \ge 3$, then $\lambda_c(G,k) = 0 < \frac{m+\left(\binom{k}{2}-1\right)t}{2\binom{k}{2}-1}$. If $k = 2$, then $t = 1$ and thus, $\lambda_c(G,k) = 1 = \frac{m+\left(\binom{k}{2}-1\right)t}{2\binom{k}{2}-1}$. We now proceed by induction on m. Thus, suppose $m \ge 2$. If $k = 2$, then $t = m$ and $\lambda_c(G,k) = m = \frac{m+\left(\binom{k}{2}-1\right)t}{2\binom{k}{2}-1}$. In such a case, G is a union of t pairwise edge-disjoint 2-cliques. Suppose $k \ge 3$. If $\mathcal{C}_k(G) = \emptyset$, then the result is trivial. Thus, suppose $\mathcal{C}_k(G) \ne \emptyset$.

Suppose first that $\mathcal{C}_k^2(G) \ne \emptyset$. Let $K \in \mathcal{C}_k^2(G)$. Let $e' \in E(K)$. If $\mathcal{C}_k(G-e') = \emptyset$, then $\lambda_c(G,k) = 1 \le \frac{m+\left(\binom{k}{2}-1\right)t}{2\binom{k}{2}-1}$, which is true since $t = 1$ and

$\binom{k}{2} \leq m$ in such a case. If the bound is sharp, then $m = \binom{k}{2}$, and since $t = 1$, then G is a copy of K_k, and thus the result follows. If $\mathcal{C}_k(G - e') \neq \emptyset$, then for every $e \in E(K) \setminus \{e'\}$, $e \notin \cup_{C \in \mathcal{C}_k(G-e')} E(C)$. Therefore, $\mathcal{C}_k(G - E(K)) = \mathcal{C}_k(G - e') = \mathcal{C}_k(G) \setminus \{K\}$, and by repeated application of Proposition 4.3.4, $\lambda_{\mathrm{c}}(G - e', k) = \lambda_{\mathrm{c}}(G - E(K), k)$. Now let G' be the graph obtained from $G - E(K)$ by deleting all the isolated vertices (if any) of $G - E(K)$. Clearly, $\lambda_{\mathrm{c}}(G - E(K), k) = \lambda_{\mathrm{c}}(G', k)$. Therefore, by applying Proposition 4.3.5 and the induction hypothesis,

$$\begin{aligned}\lambda_{\mathrm{c}}(G, k) &\leq 1 + \lambda_{\mathrm{c}}(G - e', k) = 1 + \lambda_{\mathrm{c}}(G - E(K), k) = 1 + \lambda_{\mathrm{c}}(G', k) \\ &\leq 1 + \frac{(m - \binom{k}{2}) + (\binom{k}{2} - 1)(t - 1)}{2\binom{k}{2} - 1} = \frac{m + (\binom{k}{2} - 1)t}{2\binom{k}{2} - 1}.\end{aligned}$$

If the bound is sharp, then $\lambda_{\mathrm{c}}(G - E(K), k) = \frac{(m - \binom{k}{2}) + (\binom{k}{2} - 1)(t-1)}{2\binom{k}{2} - 1}$, and therefore by the induction hypothesis, $G - E(K)$ is a union of $(t-1)$ pairwise edge-disjoint k-cliques. Now since $K \in \mathcal{C}_k^2(G)$, then $E(K) \cap E(C) = \emptyset$, for every $C \in \mathcal{C}_k(G) \setminus \{K\} = \mathcal{C}_k(G - E(K))$. Therefore, G is a union of t pairwise edge-disjoint k-cliques.

Suppose now that $\mathcal{C}_k^2(G) = \emptyset$. (That is, $\mathcal{C}_k(G) = \mathcal{C}_k^1(G)$). Then there exists an edge $e' \in E(G)$ and $Q_1, Q_2 \in \mathcal{C}_k(G)$ such that $e' \in E(Q_1) \cap E(Q_2)$. Let $e' = \{vw\}$. Note that clearly $G - e'$ does not have any isolated vertices since $d_{G-e'}(v) = d_G(v) - 1 \geq 2 - 1 = 1$, and $d_{G-e'}(w) = d_G(w) - 1 \geq 2 - 1 = 1$.

If $\mathcal{C}_k(G - e') = \emptyset$, then $\lambda_{\mathrm{c}}(G, k) = 1 < \frac{m + (\binom{k}{2} - 1)t}{2\binom{k}{2} - 1}$, since we are assuming $k \geq 3$, and $t \geq 2$ and $m > \binom{k}{2}$ in such a case. Suppose $\mathcal{C}_k(G - e') \neq \emptyset$. Note that $\mathcal{C}_k(G - e') \subseteq \mathcal{C}_k(G) \setminus \{Q_1, Q_2\}$. Therefore, by applying Proposition 4.3.5

and the induction hypothesis,

$$\begin{aligned}\lambda_{\mathrm{c}}(G,k) \le 1+\lambda_{\mathrm{c}}(G-e',k) &\le 1+\frac{(m-1)+(\binom{k}{2}-1)(t-2)}{2\binom{k}{2}-1} \\ &= \frac{2\binom{k}{2}-1+m-1+(\binom{k}{2}-1)t-2\binom{k}{2}+2}{2\binom{k}{2}-1} = \frac{m+(\binom{k}{2}-1)t}{2\binom{k}{2}-1}.\end{aligned}$$

If the bound is sharp, then $\mathcal{C}_k(G-e') = \mathcal{C}_k(G)\setminus\{Q_1,Q_2\}$, and $\lambda_{\mathrm{c}}(G-e',k) = \frac{(m-1)+\left(\binom{k}{2}-1\right)(t-2)}{2\binom{k}{2}-1}$, and therefore by the induction hypothesis, $G-e'$ is a union of $(t-2)$ pairwise edge-disjoint k-cliques. Now since Q_1 and Q_2 are distinct, then there exists a vertex $x \in Q_1$ such that $x \notin Q_2$, and similarly, there exists a vertex $y \in Q_2$ such that $y \notin Q_1$. Thus, $\{xv\} \in E(Q_1)\setminus E(Q_2)$ and $\{yw\} \in E(Q_2)\setminus E(Q_1)$. Let $\mathcal{C}_k(G-e') = \{Q'_1,\ldots,Q'_{t-2}\}$. Now since $e' \notin \{xv, yw\}$, we have $\{xv\} \in E(Q'_i)$ for some $i \in [t-2]$, and $\{yw\} \in E(Q'_j)$ for some $j \in [t-2]$. Suppose $Q'_i = Q'_j$. Then $x,v,w,y \in V(Q'_i)$. Thus, we have $\{vw\} \in E(Q'_i)$, which contradicts $E(Q'_i) \subseteq E(G-e')$. Thus, $Q'_i \neq Q'_j$. Now for each $p \in [t-2]$, let $e_p \in E(Q'_p)$. Now since $\{xv\} \in E(Q_1)$ and $\{yw\} \in E(Q_2)$, then, if we let $L = (\{e_1,\ldots,e_{t-2}\}\setminus\{e_i,e_j\})\cup\{\{xv\},\{yw\}\}$, then $G-L$ contains none of the t k-cliques of G. Thus, L is a k-clique reducing edge set of G of size $|L| = t-2 < \lambda_{\mathrm{c}}(G,k)$. $\square$

Theorem 4.2.4. *If G is a graph, $m = |E(G)|$, $t = |\mathcal{C}_k(G)|$, and $\alpha = \binom{k}{2}$, then*

$$\lambda_{\mathrm{c}}(G,k) \le m\Big[1-\Big(\frac{\alpha-1}{\alpha}\Big)\Big(\frac{m}{\alpha t}\Big)^{\frac{1}{\alpha-1}}\Big].$$

Proof of Theorem 4.2.4. We may assume that $E(G) = [m]$. Let $p = 1 - \left(\frac{m}{\alpha t}\right)^{\frac{1}{\alpha - 1}}$. We set up m independent random experiments, and in each experiment an edge is chosen with probability p (and hence not chosen with probability $1 - p$). More formally, for $i \in E(G)$, let (Ω_i, P_i) be given by $\Omega_i \in \{0, 1\}$, $P_i(\{1\}) = p$ and $P_i(\{0\}) = 1 - p$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ and let $P\colon 2^{\Omega} \to [0, 1]$ such that $P(\{\omega\}) = \prod_{i=1}^{m} P_i(\{\omega_i\})$ for each $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$, and $P(A) = \sum_{\omega \in A} P(\{\omega\})$ for each $A \subseteq \Omega$. Then (Ω, P) is a probability space.

For each $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$, let S_ω be the subset of $E(G)$ such that ω is the characteristic vector of S_ω (that is, $S_\omega = \{i \in [m] \colon \omega_i = 1\}$). Let $\mathcal{T}_\omega$ be the set of k-cliques in $\mathcal{C}_k(G)$ such that if $C \in \mathcal{T}_\omega$, then $E(C) \cap S_\omega = \emptyset$. That is, $\mathcal{T}_\omega = \{C \in \mathcal{C}_k(G) \colon E(C) \cap S_\omega = \emptyset\}$. For each $C \in \mathcal{T}_\omega$, let $e_C \in E(C)$. Let $T'_\omega = \{e_C \colon C \in \mathcal{T}_\omega\}$. Note that T'_ω may contain multiple elements and thus is a multiset. Obtain the set T''_ω from the multiset T'_ω. Note that $|T''_\omega| \leq |T'_\omega| = |\mathcal{T}_\omega|$.

Define $D_\omega = S_\omega \cup T''_\omega$. Then D_ω is a k-clique reducing edge set of G.

Let $Y \colon \Omega \to \mathbb{R}$ be the random variable given by $Y(\omega) = |\mathcal{T}_\omega|$. For each $C \in \mathcal{C}_k(G)$ and for each $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$,

$$Y_C(\omega) = \begin{cases} 1 & \text{if } E(C) \cap S_\omega = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y = \sum_{C \in \mathcal{C}_k(G)} Y_C$.

Let $X \colon \Omega \to \mathbb{R}$ be the random variable given by $X(\omega) = |S_\omega|$. For each $i \in [m]$, let $X_i \colon \Omega \to \mathbb{R}$ be the indicator random variable for whether edge i

is in S_ω, that is, for each $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$,

$$X_i(\omega) = \begin{cases} 1 & \text{if } i \in S_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then $X = \sum_{i=1}^m X_i$.

For each $i \in [m]$, $P(X_i = 1) = P_i(\{1\}) = p$ and for each $C \in \mathcal{C}_k(G)$, $P(Y_C = 1) = (1-p)^\alpha$.

For any random variable Z, let $\mathrm{E}[Z]$ denote the expected value of Z. By linearity of expectation,

$$\begin{aligned} \mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] &= \sum_{i=1}^m \mathrm{E}[X_i] + \sum_{C \in \mathcal{C}_k(G)} \mathrm{E}[Y_C] \\ &= \sum_{i=1}^m P(X_i = 1) + \sum_{C \in \mathcal{C}_k(G)} P(Y_C = 1) = mp + t(1-p)^\alpha. \end{aligned}$$

By Proposition 2.4.3, there exists $\omega^* \in \Omega$ such that $X(\omega^*) + Y(\omega^*) \leq mp + t(1-p)^\alpha$. Now $|D_{\omega^*}| = |S_{\omega^*}| + |T''_{\omega^*}| \leq |S_{\omega^*}| + |T_{\omega^*}| \leq mp + t(1-p)^\alpha = m[1 - (\frac{\alpha-1}{\alpha})(\frac{m}{\alpha t})^{\frac{1}{\alpha-1}}]$. $\square$

Chapter 5

Isolation of k-cliques

5.1 Introduction

In this chapter, we investigate the size of a smallest set of vertices that when removed together with its closed neighbourhood from a graph, we obtain a subgraph with no k-cliques.

A more natural problem is to investigate the size of a smallest set of vertices whose deletion from a graph induces a subgraph with no k-cliques. However, as we shall now explain (see below), this problem is equivalent to that of finding the size of a smallest transversal of a uniform hypergraph, given by Alon in [4]. We first make way for the following definitions.

A *hypergraph* H is a pair (X, Y), where X is a set, called the *vertex set of* H, and Y is a subset of $\mathcal{P}(X)$ and is called the *edge set of* H. The vertex set of H and the edge set of H are denoted by $V(H)$ and $E(H)$, respectively. An element of $V(H)$ is called a *vertex of* H, and an element of $E(H)$ is called a *hyperedge of* H (or simply an *edge of* H). A hypergraph H is said to be *k-uniform* if all hyperedges of H have size k. A graph is a 2-uniform hypergraph.

Let $\mathcal{F}$ be a finite family of subsets of a finite set X. Let S be a subset of X. If S intersects each member of $\mathcal{F}$, then S is called a *transversal of $\mathcal{F}$*. Let H be a hypergraph. Then $T \subseteq V(H)$ is said to be a *transversal of H* if T intersects each member of $E(H)$. The *transversal number of H*, denoted by $\tau(H)$, is the size of a smallest transversal of H.

Recall that $\mathcal{C}_k(G)$ denotes the set of distinct k-cliques of G. That is, $\mathcal{C}_k(G) = \{C \subseteq V(G) \colon C \text{ is a } k\text{-clique of } G\}$.

We will now show that the problem of finding a smallest set of vertices whose deletion from a graph induces a subgraph with no k-cliques is equivalent to finding a smallest transversal of a uniform hypergraph. Indeed, let G be a graph and suppose $\mathcal{C}_k(G) \neq \emptyset$. Let R be a smallest set of vertices of G such that $\omega(G - R) < k$. Then its not difficult to see that for each $C \in \mathcal{C}_k(G)$, $R \cap C \neq \emptyset$. We construct a k-uniform hypergraph H from G. Let $H = (V(H), E(H))$, where $V(H) = V(G)$ and $E(H) = \mathcal{C}_k(G)$. Then H is a k-uniform hypergraph and R is a transversal set of H. Suppose $\tau(H) < |R|$. Let $R^* \subseteq V(H)$ be a set which realizes $\tau(H)$. Then R^* intersects all the edges of H. Since $V(H) = V(G)$ and $E(H) = \mathcal{C}_k(G)$, then $R^* \subseteq V(G)$, and $R^* \cap C \neq \emptyset$, for every $C \in \mathcal{C}_k(G)$. Thus, R^* is a set of vertices of G such that $\omega(G - R^*) < k$, and $|R^*| = \tau(H) < |R|$, contradicting the minimality of R.

If $\mathcal{F}$ is a set of graphs and F is a copy of a graph in $\mathcal{F}$, then we call F an *$\mathcal{F}$-graph*. If G is a graph and $D \subseteq V(G)$ such that $G - N[D]$ contains no $\mathcal{F}$-graph, then D is called an *$\mathcal{F}$-isolating set of G*. Let $\iota(G, \mathcal{F})$ denote the size of a smallest $\mathcal{F}$-isolating set of G. The study of isolating sets was introduced recently by Caro and Hansberg [14, 15]. It is an appealing and natural generalization of the classical domination problem. Indeed, D is a $\{K_1\}$-isolating set of G if and only if D is a *dominating set of* G (that is, $N[D] = V(G)$), so $\iota(G, \{K_1\})$ is the *domination number of* G (the size of a smallest dominating set of G). In this paper, we obtain a sharp upper bound for $\iota(G, \{K_k\})$, and consequently we solve a problem of Caro and Hansberg [14].

We call a subset D of $V(G)$ a *k-clique isolating set of* G if $G - N[D]$ contains no k-clique. We denote the size of a smallest k-clique isolating set of G by $\iota(G, k)$. That is, $\iota(G, k) = \min\{|D| \colon D \subseteq V(G), \omega(G - N[D]) < k\}$. Thus, $\iota(G, k) = \iota(G, \{K_k\})$.

We are now ready to state our main results; which are given in the next section. The proofs of the main results are given in Section 5.3. Definitions and notation from Chapter 1 will be used.

5.2 Main results

Before stating our first result, we require the following definitions.

For $n, k \in \mathbb{N}$, let $a_{n,k} = \left\lfloor \frac{n}{k+1} \right\rfloor$ and $b_{n,k} = n - ka_{n,k}$. We have $a_{n,k} \leq b_{n,k} \leq a_{n,k} + k$. If $n \leq k$, then let $B_{n,k} = P_n$. If $n \geq k + 1$, then let $F_1, \dots, F_{a_{n,k}}$ be copies of K_k such that $P_{b_{n,k}}, F_1, \dots, F_{a_{n,k}}$ are vertex-disjoint, and let $B_{n,k}$ be the connected n-vertex graph given by $V(B_{n,k}) = V(P_{b_{n,k}}) \cup \bigcup_{i=1}^{a_{n,k}} V(F_i)$ and $E(B_{n,k}) = E(P_{b_{n,k}}) \cup \{iv \colon i \in [a_{n,k}], v \in V(F_i)\} \cup \bigcup_{i=1}^{a_{n,k}} E(F_i)$. Thus, $B_{n,k}$ is the graph obtained by taking $P_{b_{n,k}}, F_1, \dots, F_{a_{n,k}}$ and joining i (a vertex of $P_{b_{n,k}}$) to each vertex of F_i for each $i \in [a_{n,k}]$.

For $n, k \in \mathbb{N}$ with $k \neq 2$, let

$$\iota(n,k) = \max\{\iota(G,k) \colon G \text{ is a connected graph}, V(G) = [n], G \not\simeq K_k\}.$$

For $n \in \mathbb{N}$, let

$$\iota(n,2) = \max\{\iota(G,2) \colon G \text{ is a connected graph}, V(G) = [n], G \not\simeq K_2, G \not\simeq C_5\}.$$

The following is our primary result.

Theorem 5.2.1. *If G is a connected n-vertex graph, then, unless G is a k-clique or $k = 2$ and G is a 5-cycle,*

$$\iota(G,k) \leq \frac{n}{k+1}.$$

Consequently, for any $k \geq 1$ and $n \geq 3$,

$$\iota(n,k) = \iota(B_{n,k},k) = \left\lfloor \frac{n}{k+1} \right\rfloor.$$

A classical result of Ore [47] is that the domination number of a graph G with $\min\{d(v) \colon v \in V(G)\} \geq 1$ is at most $\frac{n}{2}$ (see [29]). Since the domination number is $\iota(G,1)$, it follows by Lemma 5.3.2 in Section 5.3 that Ore's result is equivalent to the bound in Theorem 5.2.1 for $k = 1$. The case $k = 2$ is also particularly interesting; while deleting the closed neighbourhood of a $\{K_1\}$-isolating set yields the graph with no vertices, deleting the closed neighbourhood of a $\{K_2\}$-isolating set yields a graph with no edges. In [14], Caro and Hansberg proved Theorem 5.2.1 for $k = 2$, using a different argument. Consequently, they established that $\frac{1}{k+1} \leq \limsup_{n\to\infty} \frac{\iota(n,k)}{n} \leq \frac{1}{3}$. In the same paper, they asked for the value of $\limsup_{n\to\infty} \frac{\iota(n,k)}{n}$. The answer is given by Theorem 5.2.1.

Corollary 5.2.2. *For any $k \geq 1$,*

$$\limsup_{n\to\infty} \frac{\iota(n,k)}{n} = \frac{1}{k+1}.$$

Proof. By Theorem 5.2.1, for any $n \geq 3$, $\frac{1}{k+1} - \frac{k}{(k+1)n} = \frac{1}{n}\left(\frac{n-k}{k+1}\right) \leq \frac{\iota(n,k)}{n} \leq \frac{1}{k+1}$, and, if n is a multiple of $k+1$, then $\frac{\iota(n,k)}{n} = \frac{1}{k+1}$. Thus, $\lim_{n\to\infty} \sup\left\{\frac{\iota(p,k)}{p} : p \geq n\right\} = \lim_{n\to\infty} \frac{1}{k+1} = \frac{1}{k+1}$. $\square$

We will now exhibit graphs which attain the bound in Theorem 5.2.1. We start by defining a special graph which attains the bound in Theorem 5.2.1 when $k = 2$.

Let $r_1 \in \mathbb{N} \cup \{0\}$. If $r_1 \geq 1$, then let $C_1, \ldots, C_{r_1}$ be distinct copies of C_5. For each $i \in [r_1]$, let $V(C_i) = \{v_1^i, \ldots, v_5^i\}$ and $E(C_i) = \{\{v_1^i, v_2^i\}, \ldots, \{v_5^i, v_1^i\}\}$. Let $V_1 = \{v_1, \ldots, v_{r_1}\}$ be such that $V_1 \cap V(C_i) = \emptyset$, for each $i \in [r_1]$. For each $i \in [r_1]$, let $G_i = (V(G_i), E(G_i))$ where $V(G_i) = V(C_i) \cup \{v_i\}$ and $E(G_i)$ is one of the following;

(i) $E(G_i) = E(C_i) \cup \{v_1^i, v_i\}$

(ii) $E(G_i) = E(C_i) \cup \{\{v_1^i, v_i\}, \{v_i, v_2^i\}\}$

(iii) $E(G_i) = E(C_i) \cup \{\{v_1^i, v_i\}, \{v_i, v_3^i\}\}$

(iv) $E(G_i) = E(C_i) \cup \{\{v_1^i, v_i\}, \{v_i, v_2^i\}, \{v_i, v_3^i\}\}$

(v) $E(G_i) = E(C_i) \cup \{\{v_1^i, v_i\}, \{v_i, v_2^i\}, \{v_i, v_5^i\}\}$

Let $r_2 \in \mathbb{N} \cup \{0\}$. If $r_2 \geq 1$, then let $Q_1, \ldots, Q_{r_2}$ be distinct copies of K_2. For each $i \in [r_2]$, let $V(Q_i) = \{u_1^i, u_2^i\}$ and $E(Q_i) = \{\{u_1^i, u_2^i\}\}$. Let $V_2 = \{u_1, \ldots, u_{r_2}\}$ be such that $V_2 \cap V(Q_i) = \emptyset$, for each $i \in [r_2]$. For each $i \in [r_2]$. Let $G_i' = (V(G_i'), E(G_i'))$ where $V(G_i') = V(Q_i) \cup \{u_i\}$ and $\{E(Q_i) \cup \{u_1^i, u_i\}\} \subseteq E(G_i') \subseteq \binom{V(G_i')}{2}$. Then $G = (V(G), E(G))$ where $V(G) = (\cup_{i=1}^{r_1} V(G_i)) \cup (\cup_{i=1}^{r_2} V(G_i'))$ and $(\cup_{i=1}^{r_1} E(G_i)) \cup (\cup_{i=1}^{r_2} E(G_i')) \subseteq E(G) \subseteq (\cup_{i=1}^{r_1} E(G_i)) \cup (\cup_{i=1}^{r_2} E(G_i')) \cup \binom{V_1 \cup V_2}{2}$. If H is a copy of G, then we say that *H is a 2-clique special graph*. We call $G_1, \ldots, G_{r_1}$ the 5-cycle constituents of G and $G_1', \ldots, G_{r_2}'$ the 2-clique constituents of G. We call $v_1, \ldots, v_{r_1}$ the 5-cycle connections of $G_1, \ldots, G_{r_1}$ in G (respectively), and $u_1, \ldots, u_{r_2}$ the 2-clique connections of $G_1', \ldots, G_{r_2}'$ in G (respectively).

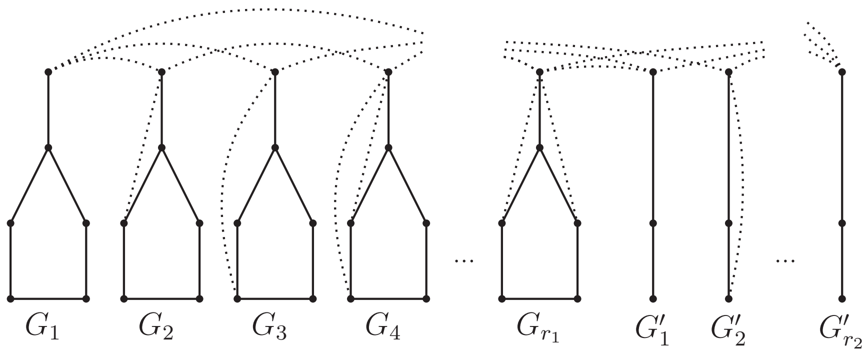

Figure 6: An illustration of a 2-clique special graph.

Proposition 5.2.3. *If G is a 2-clique special graph on n vertices and has r_1 5-cycle constituents and r_2 2-clique constituents, then $n = 6r_1 + 3r_2$ and $\iota(G, 2) = 2r_1 + r_2$.*

We now define a special graph which attains the bound in Theorem 5.2.1 when $k \geq 3$.

Let $Q_1, \dots, Q_r$ be distinct sets of vertices, each of size k. For each $i \in [r]$, let $Q_i = \{u_1^i, \dots, u_k^i\}$. Let $V = \{v_1, \dots, v_r\}$ be such that $V \cap Q_i = \emptyset$, for each $i \in [r]$. Let $G_i = (V(G_i), E(G_i))$ where $V(G_i) = Q_i \cup \{v_i\}$, and $\{\binom{Q_i}{2} \cup \{u_1^i, v_i\}\} \subseteq E(G_i) \subseteq \binom{V(G_i)}{2}$. Then $G = (V(G), E(G))$, where $V(G) = \cup_{i=1}^r V(G_i)$, and $\cup_{i=1}^r E(G_i) \subseteq E(G) \subseteq \cup_{i=1}^r E(G_i) \cup \binom{V}{2}$. We say that G *is a k-clique special graph* and we call $G_1, \dots, G_r$ *the k-clique constituents of* G and $v_1, \dots, v_r$ *the k-clique connections of* $G_1, \dots, G_r$ *in* G (respectively).

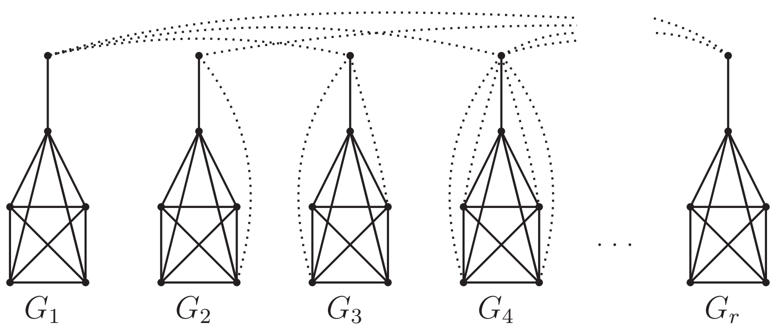

Figure 7: An illustration of a 5-clique special graph.

Proposition 5.2.4. *If G is a k-clique special graph on n vertices and has r k-clique constituents, then $n = (k+1)r$ and $\iota(G, k) = r$.*

We propose the following conjecture.

Conjecture 5.2.5. *If G is a graph, then G attains the bound in Theorem 5.2.1 if and only if either $k = 2$ and G is a 2-clique special graph or $k \geq 3$ and G is a k-clique special graph.*

We can now move on to state our second main result. But first we define a special graph.

Let $k \geq 2$. Let $Q_1, \ldots, Q_r$ be distinct sets of vertices, each of size k. For each $i \in [r]$, let $Q_i = \{u_1^i, \ldots, u_k^i\}$. Let $V = \{v_1, \ldots, v_r\}$ be such that $V \cap Q_i = \emptyset$, for each $i \in [r]$. Let $G_i = (V(G_i), E(G_i))$, where $V(G_i) = Q_i \cup \{v_i\}$. and $E(G_i) = \{\binom{Q_i}{2} \cup \{u_1^i, v_i\}\}$. Then $G = (V(G), E(G))$, where $V(G) = \cup_{i=1}^r V(G_i)$, and $E(G) = \cup_{i=1}^r E(G_i) \cup E(T_V)$, where T_V is a tree induced by the vertices of V. We say that G is a k-clique edge-special graph and we call $G_1, \ldots, G_r$ the k-clique constituents of G and $v_1, \ldots, v_r$ the k-clique connections of $G_1, \ldots, G_r$ in G (respectively).

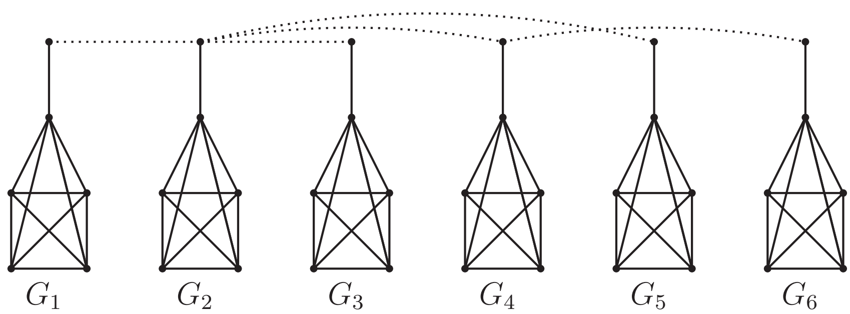

Figure 8: An illustration of a 5-clique edge-special graph.

Proposition 5.2.6. *If G is a k-clique edge-special graph with r k-clique constituents, $n = |V(G)|$ and $m = |E(G)|$, then $n = (k+1)r$, $m = (\binom{k}{2} + 2)r - 1$ and $\iota(G, k) = r$.*

We now state our second main result.

Theorem 5.2.7. *If G is a connected graph, $m = |E(G)|$, and G is not a k-clique, then for $k \geq 2$,*

$$\iota(G, k) \leq \frac{m+1}{\binom{k}{2} + 2}.$$

Moreover, equality holds if and only if G is a k-clique edge-special graph or $k = 2$ and G is a 5-cycle.

We now consider a slight variant of $\iota(G, k)$. We define $\iota'(G, k)$ to be the size of a smallest independent k-clique isolating set of G; that is, $\iota'(G, k) = \min\{|D| \colon D \subseteq V(G), \omega(G - N_G[D]) < k, D$ is an independent set of $G\}$.

We provide the following sharp bound.

Theorem 5.2.8. *If G is a connected graph, $n = |V(G)|$, and $\Delta = \Delta(G)$, then*

$$\iota'(G, k) \leq \frac{n - \Delta - 1 + k}{k}.$$

The above bound is attained, for example, when G is a union of t pairwise vertex-disjoint k-cliques.

5.3 Proofs of the main results

We start by proving Proposition 5.2.3, Proposition 5.2.4 and Proposition 5.2.6. We then prove the main results; that is, Theorem 5.2.1, Theorem 5.2.7 and Theorem 5.2.8.

Proposition 5.2.3. *If G is a 2-clique special graph on n vertices and has r_1 5-cycle constituents and r_2 2-clique constituents, then $n = 6r_1 + 3r_2$ and $\iota(G, 2) = 2r_1 + r_2$.*

Proof of Proposition 5.2.3. Let G be a 2-clique special graph as described in Section 5.2. By construction, $n = |V(G)| = |(\cup_{i=1}^{r_1} V(G_i)) \cup (\cup_{i=1}^{r_2} V(G'_i))| = \sum_{i=1}^{r_1} |V(G_i)| + \sum_{i=1}^{r_2} |V(G'_i)| = 6r_1 + 3r_2$. We now show that $\iota(G, 2) = 2r_1 + r_2$. It is not difficult to see that $X = V_1 \cup V_2 \cup \{v_3^1, \ldots, v_3^{r_1}\}$ is a 2-clique isolating set of G. Therefore, $\iota(G,2) \leq |X| = r_1 + r_2 + r_1 = 2r_1 + r_2$. Suppose that $\iota(G,2) < 2r_1 + r_2$, and let X' be a set which realizes $\iota(G,2)$. Then we must have one of the following cases.

Case 1: there exists $x \in X'$ such that $x \in N_G[C_i] \cap N_G[Q_j]$, for some $i \in [r_1]$, $j \in [r_2]$. Note that $C_i \cap Q_j = \emptyset$. If there exists a vertex in C_i which is adjacent to some vertex in Q_j, then this contradicts the construction of G. If $x \in V_1 \cup V_2$, then this also contradicts the construction of G.

Case 2: there exists $x \in X'$ such that $x \in N_G[C_i] \cap N_G[C_j]$, for some $i, j \in [r_1]$, $i \neq j$. Note that $C_i \cap C_j = \emptyset$. If there exists a vertex in C_i which is adjacent to some vertex in C_j, then this contradicts the construction of G. If $x \in V_1$, then this contradicts the construction of G.

Case 3: there exists $x \in X'$ such that $x \in N_G[Q_i] \cap N_G[Q_j]$, for some $i, j \in [r_2]$, $i \neq j$. Note that $Q_i \cap Q_j = \emptyset$. If there exists a vertex in Q_i which is adjacent to some vertex in Q_j, then this contradicts the construction of G. If $x \in V_2$, then this contradicts the construction of G. This completes the proof. $\square$

Proposition 5.2.4. *If G is a k-clique special graph on n vertices and has r k-clique constituents, then $n = (k+1)r$ and $\iota(G,k) = r$.*

Proof of Proposition 5.2.4. Let G be a k-clique special graph as described in Section 5.2. By construction, $n = |V(G)| = |\cup_{i=1}^{r} V(G_i)| = \sum_{i=1}^{r} |V(G_i)| = \sum_{i=1}^{r} |Q_i \cup \{v_i\}| = r(k+1)$. We now show that $\iota(G,k) = r$. It is not difficult to see that V is a k-clique isolating set of G. Therefore, $\iota(G,k) \leq |V| = r$. Suppose $\iota(G,k) < r$, and let V' be a set which realizes

$\iota(G,k)$. Since there are r distinct cliques $Q_1,\dots,Q_r$, then by the pigeonhole principle, there exists $v \in V'$ such that $v \in N_G[Q_i] \cap N_G[Q_j]$, for some $i,j \in [r], i \neq j$. Since $Q_1,\dots,Q_r$ are distinct, then $Q_i \cap Q_j = \emptyset$. If there exists a vertex of Q_i which is adjacent to a vertex of Q_j, then this contradicts the construction of G. If $v \in V$, then this again contradicts the construction of G. This completes the proof. $\square$

Proposition 5.2.6. *If G is a k-clique edge-special graph with r k-clique constituents, $n = |V(G)|$ and $m = |E(G)|$, then $n = (k+1)r$, $m = (\binom{k}{2}+2)r-1$ and $\iota(G,k) = r$.*

Proof of Proposition 5.2.6. Let G be a k-clique edge-special graph as described in Section 5.2. By construction, $n = |V(G)| = |\cup_{i=1}^r V(G_i)| = \sum_{i=1}^r |V(G_i)| = \sum_{i=1}^r |Q_i \cup \{v_i\}| = r(k+1)$. Also, $m = |E(G)| = |\cup_{i=1}^r E(G_i) \cup E(T_V)| = |\cup_{i=1}^r E(G_i)| + |E(T_V)| = \sum_{i=1}^r |E(G_i)| + |E(T_V)| = (\binom{k}{2}+1)r + (r-1) = (\binom{k}{2}+2)r - 1$. We now show that $\iota(G,k) = r$. It is not difficult to see that V is a k-clique reducing closed neighbourhood set of G. Therefore, $\iota(G,k) \leq |V| = r$. Suppose $\iota(G,k) < r$, and let V' be a set which realizes $\iota(G,k)$. Since there are r distinct cliques $Q_1,\dots,Q_r$, then by the pigeonhole principle, there exists $v \in V'$ such that $v \in N_G[Q_i] \cap N_G[Q_j]$, for some $i,j \in [r], i \neq j$. Since $Q_1,\dots,Q_r$ are distinct, then $Q_i \cap Q_j = \emptyset$. If there exists a vertex of Q_i which is adjacent to a vertex of Q_j, then this contradicts the construction of G. If $v \in V$, then this again contradicts the construction of G. This completes the proof. $\square$

We now prove the main results in the previous section. We start with two lemmas that will be used repeatedly.

Lemma 5.3.1. *If v is a vertex of a graph G, then $\iota(G,k) \leq 1 + \iota(G - N_G[v], k)$.*

Proof. Let D be a k-clique isolating set of $G - N_G[v]$ of size $\iota(G - N_G[v], k)$. Clearly, $C \cap N_G[v] \neq \emptyset$ for each $C \in \mathcal{C}_k(G) \backslash \mathcal{C}_k(G - N_G[v])$. Thus, $D \cup \{v\}$ is a k-clique isolating set of G. The result follows. $\square$

Lemma 5.3.2. *If $G_1, \ldots, G_r$ are the distinct components of a graph G, then $\iota(G,k) = \sum_{i=1}^{r} \iota(G_i, k)$.*

Proof. For each $i \in [r]$, let D_i be a smallest k-clique isolating set of G_i. Then, $\bigcup_{i=1}^{r} D_i$ is a k-clique isolating set of G. Thus, $\iota(G,k) \leq \sum_{i=1}^{r} |D_i| = \sum_{i=1}^{r} \iota(G_i, k)$. Let D be a smallest k-clique isolating set of G. For each $i \in [r]$, $D \cap V(G_i)$ is a k-clique isolating set of G_i, so $|D_i| \leq |D \cap V(G_i)|$. We have $\sum_{i=1}^{r} \iota(G_i, k) = \sum_{i=1}^{r} |D_i| \leq \sum_{i=1}^{r} |D \cap V(G_i)| = |D| = \iota(G,k)$. The result follows. $\square$

Theorem 5.2.1. *If G is a connected n-vertex graph, then, unless G is a k-clique or $k = 2$ and G is a 5-cycle,*

$$\iota(G,k) \leq \frac{n}{k+1}.$$

Consequently, for any $k \geq 1$ and $n \geq 3$,

$$\iota(n,k) = \iota(B_{n,k}, k) = \left\lfloor \frac{n}{k+1} \right\rfloor.$$

Proof of Theorem 5.2.1. We use induction on n. If G is a k-clique, then $\iota(G,k) = 1 = \frac{n+1}{k+1}$. If $k = 2$ and G is a 5-cycle, then $\iota(G,2) = 2 = \frac{n+1}{k+1}$. Suppose that G is not a k-clique and that, if $k = 2$, then G is not a 5-cycle. Suppose $n \leq 2$. If $k \geq 3$, then $\iota(G,k) = 0$. If $k = 2$, then $G \simeq K_1$, so $\iota(G,2) = 0$. If $k = 1$, then $G \simeq K_2$, so $\iota(G,1) = 1 = \frac{n}{k+1}$. Now suppose $n \geq 3$. If $\mathcal{C}_k(G) = \emptyset$, then $\iota(G,k) = 0$. Suppose $\mathcal{C}_k(G) \neq \emptyset$. Let $C \in \mathcal{C}_k(G)$. Since G is connected and G is not a k-clique, there exists some $v \in C$ such that $N[v] \backslash C \neq \emptyset$. Thus, $|N[v]| \geq k+1$ as $C \subset N[v]$. If $V(G) = N[v]$, then $\{v\}$ is a k-clique isolating set of G, so $\iota(G,k) = 1 \leq \frac{n}{k+1}$. Suppose $V(G) \neq N[v]$. Let $G' = G - N[v]$ and $n' = |V(G')|$. Then,

$$n \geq n' + k + 1$$

and $V(G') \neq \emptyset$. Let $\mathcal{H}$ be the set of components of G'. If $k \neq 2$, then let $\mathcal{H}' = \{H \in \mathcal{H} \colon H \simeq K_k\}$. If $k = 2$, then let $\mathcal{H}' = \{H \in \mathcal{H} \colon H \simeq K_k \text{ or } H \simeq C_5\}$. By the induction hypothesis, $\iota(H,k) \leq \frac{|V(H)|}{k+1}$ for each $H \in \mathcal{H} \backslash \mathcal{H}'$. If $\mathcal{H}' = \emptyset$, then, by Lemmas 5.3.1 and 5.3.2,

$$\begin{aligned} \iota(G,k) &\leq 1 + \iota(G',k) = 1 + \sum_{H \in \mathcal{H}} \iota(H,k) \\ &\leq 1 + \sum_{H \in \mathcal{H}} \frac{|V(H)|}{k+1} = \frac{k+1+n'}{k+1} \leq \frac{n}{k+1}. \end{aligned} \tag{5.1}$$

Suppose $\mathcal{H}' \neq \emptyset$. For any $H \in \mathcal{H}$ and any $x \in N(v)$, we say that H is *linked to x* if $xy \in E(G)$ for some $y \in V(H)$. Since G is connected, each member of $\mathcal{H}$ is linked to at least one member of $N(v)$. One of Case 1 and Case 2 below holds.

Case 1: For each $H \in \mathcal{H}'$, H is linked to at least two members of $N(v)$. Let $H' \in \mathcal{H}'$ and $x \in N(v)$ such that H' is linked to x. Let $\mathcal{H}_x$ be the set of members of $\mathcal{H}$ that are linked to x only. Then,

$$\mathcal{H}_x \subseteq \mathcal{H} \backslash \mathcal{H}',$$

and hence, by the induction hypothesis, each member H of $\mathcal{H}_x$ has a k-clique isolating set D_H with $|D_H| \leq \frac{|V(H)|}{k+1}$.

Let $X = \{x\} \cup V(H')$ and $G^* = G - X$. Then, G^* has a component G_v^* with $N[v]\backslash\{x\} \subseteq V(G_v^*)$, and the other components of G^* are the members of $\mathcal{H}_x$. Let D_v^* be a k-clique isolating set of G_v^* of size $\iota(G_v^*, k)$. Since H' is linked to x, $xy \in E(G)$ for some $y \in V(H')$. If H' is a k-clique, then let $D' = \{y\}$. If $k = 2$ and H' is a 5-cycle, then let y' be one of the two vertices in $V(H')\backslash N_{H'}[y]$, and let $D' = \{y, y'\}$. We have $X \subseteq N[D']$ and $|D'| = \frac{|X|}{k+1}$. Let $D = D' \cup D_v^* \cup \bigcup_{H \in \mathcal{H}_x} D_H$. Since the components of G^* are G_v^* and the members of $\mathcal{H}_x$, we have $V(G) = X \cup V(G_v^*) \cup \bigcup_{H \in \mathcal{H}_x} V(H)$, and, since $X \subseteq N[D']$, D is a k-clique isolating set of G. Thus,

$$\iota(G, k) \leq |D| = |D_v^*| + |D'| + \sum_{H \in \mathcal{H}_x} |D_H| \leq |D_v^*| + \frac{|X|}{k+1} + \sum_{H \in \mathcal{H}_x} \frac{|V(H)|}{k+1}. \quad (5.2)$$

Subcase 1.1: G_v^* is neither a k-clique nor a 5-cycle.

Then, $|D_v^*| \leq \frac{|V(G_v^*)|}{k+1}$ by the induction hypothesis. By (5.2),

$$\iota(G,k) \leq \frac{1}{k+1}\left(|V(G_v^*)| + |X| + \sum_{H\in\mathcal{H}_x} |V(H)|\right) = \frac{n}{k+1}.$$

Subcase 1.2: G_v^* is a k-clique.

Since $|N[v]| \geq k+1$ and $N[v]\backslash\{x\} \subseteq V(G_v^*)$, we have $V(G_v^*) = N[v]\backslash\{x\}$. If H' is a k-clique, then let $X' = \{y\}$ and $D'' = \{x\}$. If $k = 2$ and H' is a 5-cycle, then let X' be the set whose members are y, y', and the two neighbours of y' in H', and let $D'' = \{x, y'\}$. Let $Y = (X \cup V(G_v^*))\backslash(\{v,x\} \cup X')$. Let $G_Y = G - (\{v,x\} \cup X')$. Then, the components of G_Y are the components of $G[Y]$ and the members of $\mathcal{H}_x$.

If $G[Y]$ has no k-clique, then, since $\{v,x\} \cup X' \subseteq N[D'']$, $D'' \cup \bigcup_{H\in\mathcal{H}_x} D_H$ is a k-clique isolating set of G, and hence

$$\iota(G,k) \leq |D''| + \sum_{H\in\mathcal{H}_x} |D_H| < \frac{|X \cup V(G_v^*)|}{k+1} + \sum_{H\in\mathcal{H}_x} \frac{|V(H)|}{k+1} = \frac{n}{k+1}.$$

This is the case if $k = 1$ as we then have $Y = \emptyset$.

Suppose that $k \geq 2$ and $G[Y]$ has a k-clique C_Y. We have

$$V(C_Y) \subseteq (V(G_v^*)\backslash\{v\}) \cup (V(H')\backslash X'). \tag{5.3}$$

Thus, $|V(C_Y) \cap V(G_v^*)| = |V(C_Y)\backslash(V(H')\backslash X')| \geq k - (k-1) = 1$ and $|V(C_Y) \cap V(H')| = |V(C_Y)\backslash(V(G_v^*)\backslash\{v\})| \geq k - (k-1) = 1$. Let $z \in V(C_Y) \cap V(G_v^*)$ and $Z = V(G_v^*) \cup V(C_Y)$. Since z is a vertex of each of the

k-cliques G_v^* and C_Y,

$$Z \subseteq N[z]. \tag{5.4}$$

We have

$$|Z| = |V(G_v^*)| + |V(C_Y) \backslash V(G_v^*)| = k + |V(C_Y) \cap V(H')| \geq k + 1. \tag{5.5}$$

Let $G_Z = G - Z$. Then, $V(G_Z) = \{x\} \cup (V(H') \backslash V(C_Y)) \cup \bigcup_{H \in \mathcal{H}_x} V(H)$. We have that the components of $G_Z - x$ are $G_Z[V(H') \backslash V(C_Y)]$ (which is a clique or a path, depending on whether H' a k-clique or a 5-cycle) and the members of $\mathcal{H}_x$, $y \in V(H') \backslash V(C_Y)$ (by (5.3)), $y \in N_{G_Z}[x]$, and, by the definition of $\mathcal{H}_x$, $N_{G_Z}(x) \cap V(H) \neq \emptyset$ for each $H \in \mathcal{H}_x$. Thus, G_Z is connected, and, if $\mathcal{H}_x \neq \emptyset$, then G_Z is neither a clique nor a 5-cycle.

Suppose $\mathcal{H}_x \neq \emptyset$. By the induction hypothesis, $\iota(G_Z, k) \leq \frac{|V(G_Z)|}{k+1}$. Let D_{G_Z} be a k-clique isolating set of G_Z of size $\iota(G_Z, k)$. By (5.4), $\{z\} \cup D_{G_Z}$ is a k-clique isolating set of G. Thus, $\iota(G, k) \leq 1 + \iota(G_Z, k) \leq 1 + \frac{|V(G_Z)|}{k+1}$, and hence, by (5.5), $\iota(G, k) \leq \frac{|Z|}{k+1} + \frac{|V(G_Z)|}{k+1} = \frac{n}{k+1}$.

Now suppose $\mathcal{H}_x = \emptyset$. Then, $G^* = G_v^*$, so $V(G) = V(G_v^*) \cup \{x\} \cup V(H')$. Recall that either H' is a k-clique or $k = 2$ and H' is a 5-cycle.

Suppose that H' is a k-clique. Then, $n = 2k + 1$. By (5.4), $|V(G - N[z])| \leq |V(G - Z)| = n - |Z| = 2k + 1 - |Z|$. Suppose $|Z| \geq k + 2$. Then, $|V(G - N[z])| \leq k - 1$, and hence $\{z\}$ is a k-clique isolating set of G. Thus, $\iota(G, k) = 1 < \frac{n}{k+1}$. Now suppose $|Z| \leq k + 1$. Then, by (5.5), $|Z| = k + 1$ and $|V(C_Y) \cap V(H')| = 1$. Let z' be the element of $V(C_Y) \cap V(H')$, and let $Z' = V(C_Y) \cup V(H')$. Since z' is a vertex of each of the k-cliques C_Y and H', $Z' \subseteq N[z']$. We have $|Z'| = |V(C_Y)| + |V(H')| - |V(C_Y) \cap V(H')| = 2k - 1$

and $|V(G - N[z'])| \leq |V(G - Z')| = n - |Z'| = (2k+1) - (2k-1) = 2$. If $k \geq 3$, then $\{z'\}$ is a k-clique isolating set of G, and hence $\iota(G,k) = 1 < \frac{n}{k+1}$. Suppose $k = 2$. Then, H', G_v^*, and C_Y are the 2-cliques with vertex sets $\{y, z'\}$, $\{v, z\}$, and $\{z, z'\}$, respectively. Thus, $V(G) = \{v, z, z', y, x\}$, and G contains the 5-cycle with edge set $\{xv, vz, zz', z'y, yx\}$. Since G is not a 5-cycle, $d(w) \geq 3$ for some $w \in V(G)$. Since $|V(G - N[w])| = 5 - |N[w]| \leq 1$, $\{w\}$ is a k-clique isolating set of G, and hence $\iota(G,k) = 1 < \frac{5}{3} = \frac{n}{k+1}$.

Now suppose that $k = 2$ and H' is a 5-cycle. Then, $V(G_v^*) = \{v, z\}$ and $E(H') = \{yy_1, y_1y_2, y_2y_3, y_3y_4, y_4y\}$ for some $y_1, y_2, y_3, y_4 \in V(G)$. Recall that $|V(C_Y) \cap V(H')| \geq 1$. Let $z' \in V(C_Y) \cap V(H')$. Since z and z' are vertices of C_Y, $zz' \in E(G)$. We have $V(G) = \{v, z, x, y, y_1, y_2, y_3, y_4\}$, $N(v) = \{x, z\}$, $z' \in \{y_1, y_2, y_3, y_4\}$ (as $y \notin V(C_Y)$ by (5.3)), and $\{vx, vz, xy, zz'\} \cup E(H') \subseteq E(G)$. If z' is y_1 or y_4, then $V(G - N[\{y, z'\}])$ is $\{v, y_3\}$ or $\{v, y_2\}$. If z' is y_2 or y_3, then $V(G - N[\{y, z'\}]) = \{v\}$. Thus, $\{y, z'\}$ is a k-clique reducing set of G, and hence $\iota(G,k) = 2 < \frac{8}{3} = \frac{n}{k+1}$.

Subcase 1.3: G_v^* is a 5-cycle.

If $k \neq 2$, then the result follows as in Subcase 1.1. Suppose $k = 2$. We have $E(G_v^*) = \{vv_1, v_1v_2, v_2v_3, v_3v_4, v_4v\}$ for some $v_1, v_2, v_3, v_4 \in V(G)$. Let $Y = \{v_2, v_3, v_4\}$. Recall that the components of G^* are G_v^* and the members of $\mathcal{H}_x$. Thus, $G - Y$ is connected and $V(G - Y) = \{v, v_1, x\} \cup V(H') \cup \bigcup_{H \in \mathcal{H}_x} V(H)$.

Suppose that $G - Y$ is not a 5-cycle. By the induction hypothesis, $G - Y$ has a k-clique isolating set D with $|D| \leq \frac{|V(G-Y)|}{k+1} = \frac{n-3}{3} = \frac{n}{3} - 1$. Since $Y \subseteq N[v_3]$, $\{v_3\} \cup D$ is a k-clique isolating set of G, so $\iota(G,k) \leq \frac{n}{3} = \frac{n}{k+1}$.

Now suppose that $G - Y$ is a 5-cycle. Then, H' is a 2-clique and $V(G - Y) = \{v, v_1, x, y, z\}$, where $\{z\} = V(H') \backslash \{y\}$. Since $v_1v, vx, xy, yz \in$

$E(G-Y)$ and $G-Y$ is a 5-cycle, $E(G-Y)=\{v_1v, vx, xy, yz, zv_1\}$. We have $V(G-N[\{v,v_1\}])\subseteq\{v_3,y\}$. If $v_3y\notin E(G)$, then $\{v,v_1\}$ is a k-clique isolating set of G. If $v_3y\in E(G)$, then $V(G-(N[v]\cup N[v_3]))\subseteq\{z\}$, so $\{v,v_3\}$ is a k-clique isolating set of G. Therefore, $\iota(G,k)=2<\frac{8}{3}=\frac{n}{k+1}$.

Case 2: For some $x\in N(v)$ and some $H'\in\mathcal{H}'$, H' is linked to x only. Let $\mathcal{H}_1=\{H\in\mathcal{H}'\colon H$ is linked to x only$\}$ and $\mathcal{H}_2=\{H\in\mathcal{H}\backslash\mathcal{H}'\colon H$ is linked to x only$\}$. Let $h_1=|\mathcal{H}_1|$ and $h_2=|\mathcal{H}_2|$. Since $H'\in\mathcal{H}_1$, $h_1\geq 1$. For each $H\in\mathcal{H}_1$, $y_H\in N(x)$ for some $y_H\in V(H)$. Let $X=\{x\}\cup\bigcup_{H\in\mathcal{H}_1}V(H)$.

For each k-clique $H\in\mathcal{H}_1$, let $D_H=\{x\}$. If $k=2$, then, for each 5-cycle $H\in\mathcal{H}_1$, let y'_H be one of the two vertices in $V(H)\backslash N_H[y_H]$, and let $D_H=\{x,y'_H\}$. Let $D_X=\bigcup_{H\in\mathcal{H}_1}D_H$. Then, D_X is a k-clique isolating set of $G[X]$. If $k\neq 2$, then $D_X=\{x\}$, so $|D_X|=1\leq\frac{1+k|\mathcal{H}_1|}{k+1}=\frac{|X|}{k+1}$. If $k=2$ and we let $h'_1=|\{H\in\mathcal{H}_1\colon H\simeq C_5\}|$, then $|D_X|=1+h'_1\leq\frac{1+5h'_1+2(h_1-h'_1)}{3}=\frac{|X|}{k+1}$.

Let $G^*=G-X$. Then, G^* has a component G^*_v with $N[v]\backslash\{x\}\subseteq V(G^*_v)$, and the other components of G^* are the members of $\mathcal{H}_2$. By the induction hypothesis, $\iota(H,k)\leq\frac{|V(H)|}{k+1}$ for each $H\in\mathcal{H}_2$. For each $H\in\mathcal{H}_2$, let D_H be a k-clique isolating set of H of size $\iota(H,k)$.

If G^*_v is a k-clique, then let $D^*_v=\{x\}$. If $k=2$ and G^*_v is a 5-cycle, then let v' be one of the two vertices in $V(G^*_v)\backslash N_{G^*_v}[v]$, and let $D^*_v=\{x,v'\}$. If neither G^*_v is a k-clique nor $k=2$ and G^*_v is a 5-cycle, then, by the induction hypothesis, G^*_v has a k-clique isolating set D^*_v with $|D^*_v|\leq\frac{|V(G^*_v)|}{k+1}$.

Let $D=D^*_v\cup D_X\cup\bigcup_{H\in\mathcal{H}_2}D_H$. By the definition of $\mathcal{H}_1$ and $\mathcal{H}_2$, the components of $G-x$ are G^*_v and the members of $\mathcal{H}_1\cup\mathcal{H}_2$. Thus, D is a k-clique isolating set of G since $x\in D$, $v\in V(G^*_v)\cap N[x]$, and D_X is a

k-clique isolating set of $G[X]$. Let $D' = D_X \cup \bigcup_{H \in \mathcal{H}_2} D_H$ and $n^* = |V(G_v^*)|$. We have

$$|D'| = |D_X| + \sum_{H \in \mathcal{H}_2} |D_H| \le \frac{|X|}{k+1} + \sum_{H \in \mathcal{H}_2} \frac{|V(H)|}{k+1} = \frac{n - n^*}{k+1}.$$

If G_v^* is a k-clique, then $|D| = |D'| < \frac{n}{k+1}$. If $k = 2$ and G_v^* is a 5-cycle, then

$$|D| = 1 + |D'| \le 1 + \frac{n - n^*}{k+1} = 1 + \frac{n-5}{3} < \frac{n}{k+1}.$$

If neither G_v^* is a k-clique nor $k = 2$ and G_v^* is a 5-cycle, then

$$|D| = |D_v^*| + |D'| \le \frac{n^*}{k+1} + \frac{n - n^*}{k+1} = \frac{n}{k+1}. \tag{5.6}$$

This completes the proof. □

Recall that for every $v \in V(G)$, let $E_G(v) = \{e \in E(G) \colon e$ is incident to v in $G\}$. Recall also that for $X \subseteq V(G)$, $E_G(X)$ denotes $\cup_{v \in X} E_G(v)$. Now for the remaining of this chapter, for $X \subseteq V(G)$, we will let $E(X) = E(G[X])$.

Theorem 5.2.7. *If G is a connected graph, $m = |E(G)|$, and G is not a k-clique, then for $k \ge 2$,*

$$\iota(G,k) \le \frac{m+1}{\binom{k}{2}+2}.$$

Moreover, equality holds if and only if G is a k-clique edge-special graph or $k = 2$ and G is a 5-cycle.

Proof of Theorem 5.2.7. If G is a k-clique edge-special graph with r k-clique constituents, then by Proposition 5.2.6, $\iota(G,k) = r = \frac{m+1}{\binom{k}{2}+2}$. Note also that if G is a 5-cycle, then $\iota(G,2) = 2 = \frac{m+1}{\binom{k}{2}+2}$. We now prove the bound and show that it is attained if G is a k-clique edge-special graph or $k = 2$ and G is a 5-cycle. We use induction on m. Suppose G is a connected graph on m edges different from K_k. We first point out that if $k = 2$, then the bound is given by Theorem 5.2.1. Indeed, we first note that $\mathcal{C}_2(G) = E(G)$. We note also that if G is a connected graph and $n = |V(G)|$ and $m = |E(G)|$, then $m \geq n - 1$, and thus, $n \leq m + 1$. Therefore, by Theorem 5.2.1, $\iota(G,2) \leq \frac{n}{k+1} = \frac{n}{3} \leq \frac{m+1}{3} = \frac{m+1}{\binom{k}{2}+2}$.

Suppose the bound is sharp. Then $\iota(G,2) = \frac{n}{3} = \frac{m+1}{3}$. Thus, $m = n - 1$, and therefore, since G is connected, G is a tree. Since $\iota(G,2) = \frac{n}{3}$, then the bound in Theorem 5.2.1 is sharp. Following along the lines of the proof of Theorem 5.2.1, if $V(G) = N[v]$ and $\iota(G,2) = 1 = \frac{n}{3}$, then $n = 3$ and $m = n - 1 = 2$, thus G is a 2-clique edge-special graph with 1 2-clique constituent. Continuing along the lines of the proof of Theorem 5.2.1, if $\iota(G,2) = \frac{n}{3} = \frac{m+1}{3}$, then (5.1) is sharp and $n = n' + k + 1 = n' + 3$. Since $n = n' + 3$, then $|N[v]| = 3$. Let $N[v] = C \cup \{x\}$, for some $x \in V(G)$, and let $C = \{u, v\}$, for some $u \in V(G)$. Note that $d_G(v) = 2$. Since (5.1) is sharp, then for each $H \in \mathcal{H}$, $\iota(H,2) = \frac{|V(H)|}{3}$. Now for each component $H \in \mathcal{H}$, since G is a tree and H is a subgraph of G, then H is a tree. Thus, for each $H \in \mathcal{H}$, $|E(H)| = |V(H)| - 1$. Therefore, for each $H \in \mathcal{H}$, $\iota(H,2) = \frac{|V(H)|}{3} = \frac{|E(H)|+1}{3}$. Thus, by the induction hypothesis, for each $H \in \mathcal{H}$, H is a 2-clique edge-special graph. Let $\mathcal{H} = \{H_1, \dots, H_p\}$. For each $i \in [p]$, let H_i have r_i 2-clique constituents $G_1^i, \dots, G_{r_i}^i$. For each $i \in [p]$, let $V_i = \{v_1^i, \dots, v_{r_i}^i\}$ be the set of the 2-clique connections of $G^i_1, \dots, G^i_{r_i}$ in H_i. Also, for each $i \in [p]$ and for each $j \in [r_i]$, let $V(G^i_j) \setminus \{v^i_j\} = \{{}_iu^j_1, {}_iu^j_2\}$. We can assume that for each $i \in [p]$ and for each $j \in [r_i]$, ${}_iu^j_1$ is adjacent to v^i_j. Note that by Proposition 5.2.6, $\iota(H_i, 2) = r_i = \frac{|E(H_i)|+1}{3}$, for each $i \in [p]$. Let $V = \cup_{i=1}^p V_i$. Note that $|V| = \sum_{i=1}^p r_i = \sum_{i=1}^p \frac{|E(H_i)|+1}{3} = \sum_{i=1}^p \frac{|V(H_i)|}{3} = \frac{n}{3} - 1 = \frac{m+1}{3} - 1$.

For any $H \in \mathcal{H}$ and any $w \in N(v)$, we say that H is *linked to* w if $wy \in E(G)$ for some $y \in V(H)$. If there exists a component H_i, for some $i \in [p]$, such that H_i is linked to both u and x, then G contains a cycle, and thus, this contradicts that G is a tree. Thus, for each $H_i \in \mathcal{H}$, H_i is linked to only one neighbour of v; call this neighbour w_i. If there exists more than one vertex in H_i which is adjacent to w_i, then G contains a cycle, and thus, this contradicts that G is a tree. Thus for each $H_i \in \mathcal{H}$, there exists just one edge which links H_i to $w_i \in N(v)$. Note that u and x cannot be adjacent as this would create a cycle in G and thus, contradicts that G is a tree. Suppose $d_G(u) \geq 2$ and $d_G(x) \geq 2$. Let $y_1 \in N_G(x) \setminus \{v\}$ and let $y_2 \in N_G(u) \setminus \{v\}$. Suppose $y_1 \in V(G^j_s)$, for some $s \in [r_j]$, $j \in [p]$, and suppose $y_2 \in V(G^q_t)$, for some $t \in [r_q]$, $q \in [p]$. Then $V \setminus \{v^j_s, v^q_t\} \cup \{y_1, y_2\}$ is a k-clique isolating set of G of size less than $\frac{m+1}{3}$, a contradiction. Thus, without loss of generality, assume that $d_G(u) = 1$. Then for each $i \in [p]$, H_i is linked to x by just one edge. Let $D = V \cup \{x\}$. Then $|D| = |V| + 1 = \frac{m+1}{3}$. Consider H_j, for some $j \in [p]$. Suppose H_j is linked to x by some edge $\{xy\}$, where $y \in V(G^j_s)$, for some $s \in [r_j]$. If $y = {}_ju^s_1$, then $D \setminus \{v^j_s\}$ is a k-clique isolating set of G of size less than $\frac{m+1}{3}$, a contradiction. Suppose $y = {}_ju^s_2$. If $r_j > 1$, then $D \setminus \{v^j_s\}$ is a k-clique isolating set of G of size less than $\frac{m+1}{3}$, a contradiction. If $r_j = 1$, then $H_j = (V(H_j), E(H_j))$, where $V(H_j) = \{v^j_1, {}_ju^1_1, {}_ju^1_2\}$, and

$E(H_j) = \{\{v_1^j {}_j u_1^1\}, \{{}_j u_1^1 {}_j u_2^1\}\}$. In such a case, relabel the vertices of H_j such that v_1^j is relabelled to ${}_j u_2^1$, ${}_j u_1^1$ is relabelled to ${}_j u_1^1$, and ${}_j u_2^1$ is relabelled to v_1^j.

Therefore, for each $i \in [p]$, H_i is linked to x by some edge $\{xy\}$, where $y = v_s^i$, for some $s \in [r_i]$. Thus, $G[V \cup \{x\}]$ is a tree, and therefore, it is not difficult to see that G is a 2-clique edge-special graph with $1 + \sum_{i=1}^{p} r_i$ constituents $G[N[v]], G_1^1, \ldots, G_{r_1}^1, \ldots, G_1^p, \ldots, G_{r_p}^p$, and x is the 2-clique connection of $G[N[v]]$ in G.

Following along the lines of the proof of Theorem 5.2.1, *case 1* cannot occur since otherwise this would create a cycle in G, and thus, contradicts that G is a tree.

Following along the lines of the proof of Theorem 5.2.1, if the bound is sharp, then (5.6) is sharp. If (5.6) is sharp, then $|D_v^*| = \frac{n^*}{k+1} = \frac{n^*}{3}$, and $|D'| = \frac{n-n^*}{k+1} = \frac{n-n^*}{3}$. Since $|D'| = \frac{n-n^*}{k+1} = \frac{n-n^*}{3}$, then $|D_X| = \frac{|X|}{k+1} = \frac{|X|}{3}$, and for each $H \in \mathcal{H}_2$, $|D_H| = \frac{|V(H)|}{k+1} = \frac{|V(H)|}{3}$. Note that for each $H \in \mathcal{H}_1 \cup \mathcal{H}_2$, H is linked to x by just one edge as otherwise, this would create a cycle in G, which contradicts the fact that G is a tree. Now for each component $H \in \mathcal{H}_1 \cup \mathcal{H}_2$, since G is a tree and H is a subgraph of G, then H is also a tree. Note that since G is a tree, $\{H \in \mathcal{H}_1 \colon H \simeq C_5\} = \emptyset$, and thus, $h_1' = 0$. Since $|D_X| = \frac{|X|}{k+1} = \frac{|X|}{3}$, then $|D_X| = 1 + h_1' = \frac{1+5h_1'+2(h_1-h_1')}{3} = \frac{|X|}{k+1} = \frac{|X|}{3}$, but since $h_1' = 0$, then $|D_X| = 1 = \frac{1+2h_1}{3}$, and therefore, $h_1 = 1$. Therefore, $G[X]$ is a 2-clique edge-special graph with 1 2-clique constituent. Note therefore, that $|D_X| = 1 = \frac{|X|}{3} = \frac{|V(G[X])|}{3} = \frac{|E(G[X])|+1}{3}$. Since for each $H \in \mathcal{H}_2$, H is a tree, then $|E(H)| = |V(H)| - 1$ for each $H \in \mathcal{H}_2$. Thus, for each $H \in \mathcal{H}_2$, $|D_H| = \frac{|V(H)|}{3} = \frac{|E(H)|+1}{3}$, and therefore, by the induction hypothesis, for each $H \in \mathcal{H}_2$, H is a 2-clique edge-special graph. Since the component G_v^* is a subgraph of G and G is a tree, then G_v^* is also a tree, and thus, $|E(G_v^*)| = |V(G_v^*)| - 1$. Therefore, $|D_v^*| = \frac{n^*}{3} = \frac{|V(G_v^*)|}{3} = \frac{|E(G_v^*)|+1}{3}$, and thus, by the induction hypothesis, G_v^* is a 2-clique edge-special graph.

Let $\mathcal{H}_2 = \{H_1, \ldots, H_p\}$. For each $i \in [p]$, let H_i have r_i 2-clique constituents $G_1^i, \ldots, G_{r_i}^i$. For each $i \in [p]$, let $V_i = \{v_1^i, \ldots, v_{r_i}^i\}$ be the set of the 2-clique connections of $G_1^i, \ldots, G_{r_i}^i$ in H_i. Also, for each $i \in [p]$ and for each $j \in [r_i]$, let $V(G_j^i) \setminus \{v_j^i\} = \{{}_iu_1^j, {}_iu_2^j\}$. We can assume that for each $i \in [p]$ and for each $j \in [r_i]$, ${}_iu_1^j$ is adjacent to v_j^i. Note that by Proposition 5.2.6, $\iota(H_i, 2) = r_i = \frac{|E(H_i)|+1}{3}$, for each $i \in [p]$. Let $V = \cup_{i=1}^p V_i$. Let $G[X] = (V(G[X]), E(G[X]))$ where $V(G[X]) = \{x, y_1, y_2\}$, for some $y_1, y_2 \in V(G)$, and $E(G[X]) = \{\{xy_1\}, \{y_1y_2\}\}$. Let G_v^* have r^* 2-clique constituents $G_1, \ldots, G_{r^*}$. Let $\{v_1, \ldots, v_{r^*}\}$ be the set of the 2-clique connections of $G_1, \ldots, G_{r^*}$ in G_v^*. For each $j \in [r^*]$, let $V(G_j) \setminus \{v_j\} = \{u_1^j, u_2^j\}$. We can assume that for each $j \in [r^*]$, v_j is adjacent to u_1^j. Note that by Proposition 5.2.6, $\iota(G_v^*, 2) = r^* = \frac{|E(G_v^*)|+1}{3}$. Let $D'' = \{x\} \cup \{v_1, \ldots, v_{r^*}\} \cup V$. Since (5.6) is sharp, then $|D''| = 1 + r^* = \sum_{i=1}^p r_i = \frac{m+1}{3}$. Recall that $d_G(v) = 2$, thus we can let $N_G(v) = \{u, x\}$, for some $u \in V(G)$. Note that since $N[v] \setminus \{x\} \subseteq V(G_v^*)$, then $u \in V(G_v^*)$. Note also that $d_{G_v^*}(v) = 1$, and thus, v is only adjacent to u in G_v^*. Suppose $r^* > 1$, then since $d_{G_v^*}(v) = 1$, then $v = u_2^j$ for some $j \in [r^*]$. Thus, $D'' \setminus \{v_j\}$ is a k-clique isolating set of G of size less than $\frac{m+1}{3}$, a contradiction. Therefore, $r^* = 1$. That is, $G_v^* = G_1$. Then $G_v^* = (V(G_v^*), E(G_v^*))$, where $V(G_v^*) = \{v_1, u_1^1, u_2^1\}$, and $E(G_v^*) = \{\{v_1u_1^1\}, \{u_1^1u_2^1\}\}$. We can assume that $v = v_1$, (if $v = u_2^1$, then relabel the vertices of G_v^* such that v_1 is relabelled to u_2^1, u_1^1 is relabelled to

u_1^1, and u_2^1 is relabelled to v_1). Recall that for each $H \in \mathcal{H}_2$, H is linked to x by just one edge. Consider $H_j \in \mathcal{H}_2$. Suppose H_j is linked to x by some edge $\{xy\}$, for some $y \in V(H_j)$. Then $y \in V(G_s^j)$, for some $s \in [r_j]$. Suppose $y \in V(G_s^j) \setminus \{v_s^j\}$. If $y = {}_ju_1^s$, then $D'' \setminus \{v_s^j\}$ is a k-clique isolating set of G of size less than $\frac{m+1}{3}$, a contradiction. Suppose $y = {}_ju_2^s$. If $r_j > 1$, then $D'' \setminus \{v_s^j\}$ is a k-clique isolating set of G of size less than $\frac{m+1}{3}$. If $r_j = 1$, then $H_j = (V(H_j), E(H_j))$, where $V(H_j) = \{v_1^j, {}_ju_1^1, {}_ju_2^1\}$, and $E(H_j) = \{\{v_1^j {}_ju_1^1\}, \{{}_ju_1^1 {}_ju_2^1\}\}$. In such a case, relabel the vertices of H_j such that v_1^j is relabelled to ${}_ju_2^1$, ${}_ju_1^1$ is relabelled to ${}_ju_1^1$, and ${}_ju_2^1$ is relabelled to v_1^j. Therefore, for each $i \in [p]$, H_i is linked to x by some edge $\{xy\}$, where $y = v_s^j$, for some $s \in [r_i]$. Thus, $G[D'']$ is a tree, and therefore, it is not difficult to see that G is a 2-clique edge-special graph with $1+r^*+\sum_{i=1}^{p} r_i = 2+\sum_{i=1}^{p} r_i$ constituents $G[X], G_v^*, G_1^1, \ldots, G_{r_1}^1, \ldots, G_1^p, \ldots, G_{r_p}^p$. Note that x is the 2-clique connection of $G[X]$ in G, v_1 is the 2-clique connection of G_v^* in G, and for each $j \in [p]$, $s \in [r_j]$, v_s^j is the 2-clique connection of G_s^j in G.

So suppose $k \geq 3$. Suppose $m \leq 4$. If $m \leq 3$, then since $k \geq 3$ and G is different from K_k, then $\iota(G, k) = 0$. Suppose $m = 4$. If $k \geq 4$, then $\iota(G, k) = 0$. If $k = 3$ and $\mathcal{C}_3(G) = \emptyset$, then $\iota(G, 3) = 0 < \frac{m+1}{\binom{k}{2}+2}$. If $k = 3$ and $\mathcal{C}_3(G) \neq \emptyset$, then since G is connected, $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4\}$. Thus, $\iota(G, 3) = 1 = \frac{4+1}{\binom{3}{2}+2} = \frac{m+1}{\binom{k}{2}+2}$. Note that in such a case, G is a 3-clique edge-special graph with 1 3-clique constituent. We proceed by induction on m. If $\mathcal{C}_k(G) = \emptyset$, then $\iota(G, k) = 0$. Suppose $\mathcal{C}_k(G) \neq \emptyset$. Let $C \in \mathcal{C}_k(G)$. Then since G is connected and G is not a k-clique, there exists a vertex $v \in C$ such that $N_G[v] \setminus C \neq \emptyset$. Let $u \in N_G[v] \setminus C$. Note that $E(C) \cup \{uv\} \subseteq E_G(N_G[v])$. If $V(G) = N_G[v]$, then

$\{v\}$ is a k-clique isolating set of G, so $\iota(G,k) = 1 \leq \frac{m+1}{\binom{k}{2}+2}$. If the bound is sharp, then $m = \binom{k}{2} + 1$, and thus, G is a k-clique edge-special graph with 1 k-clique constituent. Suppose $V(G) \neq N_G[v]$. Let $G' = G - N_G[v]$ and $m' = |E(G')|$. Then,

$$m \geq m' + \binom{k}{2} + 1$$

and $V(G') \neq \emptyset$. Let $\mathcal{H}$ be the set of components of G' and let $\mathcal{H}' = \{H \in \mathcal{H} \colon H \simeq K_k\}$. By the induction hypothesis,

$$\iota(H,k) \leq \frac{|E(H)|+1}{\binom{k}{2}+2} \quad \text{for each } H \in \mathcal{H}\backslash\mathcal{H}'.$$

Now for each component $H \in \mathcal{H}$, there exists at least one edge $e_H \in E(G)$ such that e_H is incident to a vertex in $N_G(v)$ and incident to a vertex in the component. Trivially, $\{e_H \colon H \in \mathcal{H}\} \cap (E(C) \cup \{uv\}) = \emptyset$, and for each $H \in \mathcal{H}$, $e_H \notin E(H)$. Therefore, we have

$$m \geq |E(C) \cup \{uv\}| + \sum_{H \in \mathcal{H}} |E(H) \cup e_H| = \binom{k}{2} + 1 + \sum_{H \in \mathcal{H}} (|E(H)| + 1). \quad (5.7)$$

Thus, we get

$$\sum_{H \in \mathcal{H}} (|E(H)| + 1) \leq m - \binom{k}{2} - 1. \quad (5.8)$$

If $\mathcal{H}' = \emptyset$, then, by Lemmas 5.3.1 and 5.3.2,

$$\begin{aligned}
\iota(G,k) &\leq 1 + \iota(G',k) = 1 + \sum_{H \in \mathcal{H}} \iota(H,k) \\
&\leq 1 + \sum_{H \in \mathcal{H}} \frac{|E(H)|+1}{\binom{k}{2}+2} = \frac{\binom{k}{2} + 2 + \sum_{H \in \mathcal{H}}(|E(H)|+1)}{\binom{k}{2}+2} \\
&\leq \frac{\binom{k}{2} + 2 + (m - \binom{k}{2} - 1)}{\binom{k}{2}+2} = \frac{m+1}{\binom{k}{2}+2}. \qquad (5.9)
\end{aligned}$$

If the bound is sharp, then (5.7), (5.8), and (5.9) are sharp. Since (5.7) is sharp, for each $H \in \mathcal{H}$, there exists only one edge e_H which is incident to a vertex in $N_G(v)$ and incident to a vertex in $V(H)$. Since (5.9) is sharp, then for each $H \in \mathcal{H}$, $\iota(H,k) = \frac{|E(H)|+1}{\binom{k}{2}+2}$. Thus, by the induction hypothesis, for each $H \in \mathcal{H}$, H is a k-clique edge-special graph. Let $\mathcal{H} = \{H_1, \ldots, H_p\}$. For each $i \in [p]$, let H_i have r_i k-clique constituents $G_1^i, \ldots, G_{r_i}^i$. For each $i \in [p]$, let $V_i = \{v_1^i, \ldots, v_{r_i}^i\}$ be the set of the k-clique connections of $G_1^i, \ldots, G_{r_i}^i$ in H_i. Also, for each $i \in [p]$ and for each $j \in [r_i]$, let $V(G_j^i) \setminus \{v_j^i\} = \{{}_iu_1^j, \ldots, {}_iu_k^j\}$. We can assume that for each $i \in [p]$ and for each $j \in [r_i]$, ${}_iu_1^j$ is adjacent to v_j^i. Note that by Proposition 5.2.6, $\iota(H_i,k) = r_i = \frac{|E(H_i)|+1}{\binom{k}{2}+2}$, for each $i \in [p]$. Note also that since (5.7) is sharp, then $N_G[v] = C \cup \{u\}$ and u is only adjacent to v in C. Let $N_G(v) = \{v_1, \ldots, v_{k-1}, u\}$ where $\{v_1, \ldots, v_{k-1}\} \cup \{v\} = C$. Let $V = \cup_{i \in [p]} V_i$, and let $D = V \cup \{u\}$. Then it is not difficult to see that D is a k-clique isolating set of G and since (5.9) is sharp, $|D| = 1 + |V| = 1 + \sum_{i=1}^p |V_i| = 1 + \sum_{i=1}^p r_i = 1 + \sum_{i=1}^p \frac{|E(H_i)|+1}{\binom{k}{2}+2} = \frac{m+1}{\binom{k}{2}+2} = \iota(G,k)$.

Let $H_i \in \mathcal{H}$ and consider e_{H_i}. Suppose $e_{H_i} = \{v_s, v_j^i\}$, for some $s \in [k-1]$, $j \in [r_i]$. Then V is a k-clique isolating set of G of size less than $\iota(G,k)$, a contradiction. Suppose $e_{H_i} = \{v_s, {}_iu_l^j\}$, for some $s \in [k-1]$, $j \in [r_i]$, $l \in [k]$. Then $D \setminus \{v_j^i, u\} \cup \{v_s\}$ is a k-clique isolating set of G of size less than $\iota(G,k)$, a contradiction. Thus, the vertex in $N_G(v)$ which e_{H_i} is incident to, is u. Since H_i is arbitrary, this is true for all $i \in [p]$. Suppose now that $e_{H_i} = \{u, {}_iu_l^j\}$, for some $j \in [r_i]$, $l \in [k]$. Then $D \setminus \{v_j^i\}$ is a k-clique isolating set of G of size less than $\iota(G,k)$, a contradiction. Therefore, for each $i \in [p]$, $e_{H_i} = \{u, v_j^i\}$, for some $j \in [r_i]$. Thus, $G[V \cup \{u\}]$ is a tree, and therefore, it is not difficult to see that G is a k-clique edge-special graph with $1 + \sum_{i=1}^{p} r_i$ constituents $G[N[v]], G_1^1, \ldots, G_{r_1}^1, \ldots, G_1^p, \ldots, G_{r_p}^p$, and u is the k-clique connection of $G[N[v]]$ in G.

Suppose $\mathcal{H}' \neq \emptyset$. Since G is connected, each member of $\mathcal{H}$ is linked to at least one member of $N_G(v)$. One of Case 1 and Case 2 below holds.

Case 1: For each $H \in \mathcal{H}'$, H is linked to at least two members of $N(v)$. Let $H' \in \mathcal{H}'$ and $x \in N(v)$ such that H' is linked to x. Let $\mathcal{H}_x$ be the set of members of $\mathcal{H}$ that are linked to x only. Then,

$$\mathcal{H}_x \subseteq \mathcal{H} \backslash \mathcal{H}',$$

and hence, by the induction hypothesis, each member H of $\mathcal{H}_x$ has a k-clique isolating set D_H with $|D_H| \leq \frac{|E(H)|+1}{\binom{k}{2}+2}$.

Let $X = \{x\} \cup V(H')$ and $G^* = G - X$. Then, G^* has a component G_v^* with $N[v] \backslash \{x\} \subseteq V(G_v^*)$, and the other components of G^* are the members of $\mathcal{H}_x$. Let D_v^* be a k-clique isolating set of G_v^* with $|D_v^*| = \iota(G_v^*, k)$. Since H' is linked to x, $xy \in E(G)$ for some $y \in V(H')$. Since H' is linked to at least two members of $N(v)$, then there exists an edge $x'y' \neq xy$ such that $x' \in N(v)$ and $y' \in V(H')$. Let $D' = \{y\}$. Then $X \subseteq N_G[D']$, $E(H') \cup \{xy\} \subseteq E_G(N_G[y])$, and $|D'| = 1 = \frac{|E(H') \cup \{xy\}|+1}{\binom{k}{2}+2}$. Let $G_X = G[X]$, then since $V(G_X) \subseteq N_G[D']$, D' is a k-clique isolating set of G_X. Let $D = D' \cup D_v^* \cup \bigcup_{H \in \mathcal{H}_x} D_H$. Since the components of G^* are G_v^* and the members of $\mathcal{H}_x$, we have $V(G) = X \cup V(G_v^*) \cup \bigcup_{H \in \mathcal{H}_x} V(H)$, and, since $X \subseteq N_G[D']$, D is a k-clique isolating set of G. For each component H of $\mathcal{H}_x$, let $e_H \in E(G)$ be one edge which links the component to x. Thus, we have

$$\begin{aligned} m &\geq |E(G_v^*) \cup \{vx\}| + |E(H') \cup \{xy\} \cup \{x'y'\}| + \sum_{H \in \mathcal{H}_x} |E(H) \cup e_H| \\ &= |E(G_v^*)| + 1 + \binom{k}{2} + 2 + \sum_{H \in \mathcal{H}_x} (|E(H)| + 1). \end{aligned} \tag{5.10}$$

Therefore,

$$\begin{aligned} \iota(G,k) \leq |D| &= |D_v^*| + |D'| + \sum_{H \in \mathcal{H}_x} |D_H| \\ &\leq |D_v^*| + \frac{|E(H') \cup \{xy\}| + 1}{\binom{k}{2} + 2} + \sum_{H \in \mathcal{H}_x} \frac{|E(H)| + 1}{\binom{k}{2} + 2}. \end{aligned} \tag{5.11}$$

Subcase 1.1: G_v^* is not a k-clique.

Then $|D_v^*| \leq \frac{|E(G_v^*)|+1}{\binom{k}{2}+2}$, by the induction hypothesis. Therefore, by (5.11) and

(5.10),

$$\begin{aligned}
\iota(G,k) &\leq \frac{|E(G_v^*)|+1}{\binom{k}{2}+2} + \frac{|E(H')\cup\{xy\}|+1}{\binom{k}{2}+2} + \sum_{H\in\mathcal{H}_x}\frac{|E(H)|+1}{\binom{k}{2}+2} \\
&= \frac{|E(G_v^*)|+1+|E(H')\cup\{xy\}|+1+\sum_{H\in\mathcal{H}_x}(|E(H)|+1)}{\binom{k}{2}+2} \\
&= \frac{|E(G_v^*)|+1+(\binom{k}{2}+1)+1+\sum_{H\in\mathcal{H}_x}(|E(H)|+1)}{\binom{k}{2}+2} \\
&\leq \frac{m}{\binom{k}{2}+2} < \frac{m+1}{\binom{k}{2}+2}.
\end{aligned}$$

Subcase 1.2: G_v^* is a k-clique.

Since $|N[v]| \geq k+1$ and $N[v]\backslash\{x\} \subseteq V(G_v^*)$, we have $V(G_v^*) = N[v]\backslash\{x\}$. Let $Y = (X \cup V(G_v^*))\backslash\{v,x,y\}$. Let $G_Y = G - \{v,x,y\}$. Then, the components of G_Y are the components of $G[Y]$ and the members of $\mathcal{H}_x$.

If $G[Y]$ has no k-clique, then, since $v, y \in N[x]$, $\{x\} \cup \bigcup_{H\in\mathcal{H}_x} D_H$ is a k-clique isolating set of G, by (5.10),

$$m \geq 2(\binom{k}{2}+1)+1+\sum_{H\in\mathcal{H}_x}(|E(H)|+1),$$

and thus,

$$\begin{aligned}
\iota(G,k) &\leq 1+\sum_{H\in\mathcal{H}_x}|D_H| < \frac{2(\binom{k}{2}+1)}{\binom{k}{2}+2} + \sum_{H\in\mathcal{H}_x}\frac{|E(H)|+1}{\binom{k}{2}+2} \\
&= \frac{2(\binom{k}{2}+1)+\sum_{H\in\mathcal{H}_x}(|E(H)|+1)}{\binom{k}{2}+2} \leq \frac{m-1}{\binom{k}{2}+2} < \frac{m+1}{\binom{k}{2}+2}.
\end{aligned}$$

Suppose that $G[Y]$ has a k-clique C_Y. We have

$$V(C_Y) \subseteq (V(G_v^*)\backslash\{v\}) \cup (V(H')\backslash\{y\}). \tag{5.12}$$

Thus, $|V(C_Y) \cap V(G_v^*)| = |V(C_Y) \backslash (V(H') \backslash \{y\})| \geq k - (k-1) = 1$ and $|V(C_Y) \cap V(H')| = |V(C_Y) \backslash (V(G_v^*) \backslash \{v\})| \geq k - (k-1) = 1$. Let $z \in V(C_Y) \cap V(G_v^*)$ and $Z = V(G_v^*) \cup V(C_Y)$. Since z is a vertex of each of the k-cliques G_v^* and C_Y,

$$Z \subseteq N[z]. \tag{5.13}$$

We have

$$|Z| = |V(G_v^*)| + |V(C_Y) \backslash V(G_v^*)| = k + |V(C_Y) \cap V(H')| \geq k + 1. \tag{5.14}$$

Let $G_Z = G - Z$. Then, $V(G_Z) = \{x\} \cup (V(H') \backslash V(C_Y)) \cup \bigcup_{H \in \mathcal{H}_x} V(H)$. We have that the components of $G_Z - x$ are $G_Z[V(H') \backslash V(C_Y)]$ (which is a clique) and the members of $\mathcal{H}_x$, $y \in V(H') \backslash V(C_Y)$ (by (5.12)), $y \in N_{G_Z}[x]$, and, by the definition of $\mathcal{H}_x$, $N_{G_Z}(x) \cap V(H) \neq \emptyset$ for each $H \in \mathcal{H}_x$. Thus, G_Z is connected, and, if $\mathcal{H}_x \neq \emptyset$, then G_Z is not a clique.

Suppose $\mathcal{H}_x \neq \emptyset$. By the induction hypothesis, $\iota(G_Z, k) \leq \frac{|E(G_Z)|+1}{\binom{k}{2}+2}$. Let D_{G_Z} be a k-clique isolating set of G_Z of size $\iota(G_Z, k)$. Since $Z \subseteq N[z]$, $\{z\} \cup D_{G_Z}$ is a k-clique isolating set of G. Now since the k-cliques G_v^* and C_Y can intersect on at most $k-1$ vertices, we have that $|E(G_v^*) \cap E(C_Y)| \leq \binom{k-1}{2}$, and thus, $|E(G_v^*) \cup E(C_Y)| = |E(G_v^*)| + |E(C_Y)| - |E(G_v^*) \cap E(C_Y)| \geq 2\binom{k}{2} - \binom{k-1}{2} = \binom{k}{2} + k - 1$. (By applying the well known fact $\binom{p+1}{q} = \binom{p}{q} + \binom{p}{q-1}$ with $p = k-1$ and $q = 2$). Therefore, we have

$$m \geq |E(G_v^*) \cup E(C_Y)| + |\{vx\}| + |E(G_Z)| \geq \binom{k}{2} + k - 1 + 1 + |E(G_Z)|.$$

Thus, we have

$$\iota(G,k) \le 1 + \iota(G_Z,k) \le \frac{\binom{k}{2}+k-1}{\binom{k}{2}+2} + \frac{|E(G_Z)|+1}{\binom{k}{2}+2} \le \frac{m}{\binom{k}{2}+2} < \frac{m+1}{\binom{k}{2}+2}.$$

Now suppose $\mathcal{H}_x = \emptyset$. Then, $G^* = G_v^*$, so $V(G) = V(G_v^*) \cup \{x\} \cup V(H')$. Recall that H' is a k-clique. Then, $n = 2k+1$. By (5.13), $|V(G - N[z])| \le |V(G - Z)| = n - |Z| = 2k + 1 - |Z|$. Suppose $|Z| \ge k+2$. Then, $|V(G - N[z])| \le k-1$, and hence $\{z\}$ is k-clique isolating set of G. Note that in this case $m \ge 2\binom{k}{2} + 3$ since H' is linked to at least 2 neighbours of $N_G(v)$. Thus, $\iota(G,k) = 1 < 2 = \frac{(2\binom{k}{2}+3)+1}{\binom{k}{2}+2} \le \frac{m+1}{\binom{k}{2}+2}$. Now suppose $|Z| \le k+1$. Then, by (5.14), $|Z| = k+1$ and $|V(C_Y) \cap V(H')| = 1$. Let z' be the element of $V(C_Y) \cap V(H')$, and let $Z' = V(C_Y) \cup V(H')$. Since z' is a vertex of each of the k-cliques C_Y and H', $Z' \subseteq N[z']$. We have $|Z'| = |V(C_Y)| + |V(H')| - |V(C_Y) \cap V(H')| = 2k - 1$ and $|V(G - N[z'])| \le |V(G - Z')| = n - |Z'| = (2k+1) - (2k-1) = 2$. Therefore, $\{z'\}$ is a k-clique isolating set of G, and since $m \ge 2\binom{k}{2} + 3$, $\iota(G,k) = 1 < 2 = \frac{(2\binom{k}{2}+3)+1}{\binom{k}{2}+2} \le \frac{m+1}{\binom{k}{2}+2}$.

Case 2: For some $x \in N_G(v)$ and some $H' \in \mathcal{H}'$, H' is linked to x only. Let $\mathcal{H}_1 = \{H \in \mathcal{H}' \colon H \text{ is linked to } x \text{ only}\}$ and $\mathcal{H}_2 = \{H \in \mathcal{H} \backslash \mathcal{H}' \colon H \text{ is linked to } x \text{ only}\}$. Let $h_1 = |\mathcal{H}_1|$ and $h_2 = |\mathcal{H}_2|$. Since $H' \in \mathcal{H}_1$, $h_1 \ge 1$. For each $H \in \mathcal{H}_1 \cup \mathcal{H}_2$, $y_H \in N(x)$ for some $y_H \in V(H)$. Let $X = \{x\} \cup \bigcup_{H \in \mathcal{H}_1} V(H)$. Let $D_X = \{x\}$. Then D_X is a k-clique isolating set of $G[X]$. So,

$$|D_X| = 1 \le \frac{|\bigcup_{H \in \mathcal{H}_1} (E(H) \cup \{xy_H\})| + 1}{\binom{k}{2}+2} = \frac{h_1(\binom{k}{2}+1)+1}{\binom{k}{2}+2}.$$

Let $G^* = G - X$. Then, G^* has a component G_v^* with $N[v] \backslash \{x\} \subseteq V(G_v^*)$, and the other components of G^* are the members of $\mathcal{H}_2$. By the induction hypothesis, $\iota(H,k) \leq \frac{|E(H)|+1}{\binom{k}{2}+2}$ for each $H \in \mathcal{H}_2$. For each $H \in \mathcal{H}_2$, let D_H be a k-clique isolating set of size $\iota(H,k)$.

If G_v^* is a k-clique, then let $D_v^* = \{x\}$. If G_v^* is not a k-clique , then, by the induction hypothesis, G_v^* has a k-clique isolating set D_v^* with $|D_v^*| \leq \frac{|E(G_v^*)|+1}{\binom{k}{2}+2}$.

Let $D = D_v^* \cup D_X \cup \bigcup_{H \in \mathcal{H}_2} D_H$. By the definition of $\mathcal{H}_1$ and $\mathcal{H}_2$, the components of $G - x$ are G_v^* and the members of $\mathcal{H}_1 \cup \mathcal{H}_2$. Thus, D is a k-clique isolating set of G since $x \in D$, $v \in V(G_v^*) \cap N_G[x]$, and D_X is a k-clique isolating set of $G[X]$. Note that

$$\begin{aligned} m &\geq |E(G_v^*) \cup \{vx\}| + |\bigcup_{H \in \mathcal{H}_1} (E(H) \cup \{xy_H\})| + |\bigcup_{H \in \mathcal{H}_2} (E(H) \cup \{xy_H\})| \\ &= |E(G_v^*)| + 1 + h_1(\binom{k}{2} + 1) + \sum_{H \in \mathcal{H}_2} (|E(H)| + 1). \end{aligned} \tag{5.15}$$

If G_v^* is a k-clique, then $D = D_v^* \cup D_X \cup \bigcup_{H \in \mathcal{H}_2} D_H = \{x\} \cup \bigcup_{H \in \mathcal{H}_2} D_H$, and thus, from (5.15),

$$m \geq \binom{k}{2} + 1 + h_1(\binom{k}{2} + 1) + \sum_{H \in \mathcal{H}_2} (|E(H)| + 1).$$

Therefore,

$$\begin{aligned}
\iota(G,k) \le |D| = 1 + \sum_{H\in\mathcal{H}_2} |D_H| &\le 1 + \sum_{H\in\mathcal{H}_2} \frac{|E(H)|+1}{\binom{k}{2}+2} \\
&< \frac{(h_1+1)(\binom{k}{2}+1)}{\binom{k}{2}+2} + \sum_{H\in\mathcal{H}_2} \frac{|E(H)|+1}{\binom{k}{2}+2} \\
&= \frac{\binom{k}{2}+1+h_1(\binom{k}{2}+1)+\sum_{H\in\mathcal{H}_2}(|E(H)|+1)}{\binom{k}{2}+2} \\
&\le \frac{m}{\binom{k}{2}+2} < \frac{m+1}{\binom{k}{2}+2}.
\end{aligned}$$

If G_v^* is not a k-clique, then $D = D_v^* \cup D_X \cup \bigcup_{H\in\mathcal{H}_2} D_H$, and by (5.15),

$$\begin{aligned}
\iota(G,k) \le |D| &= |D_v^*| + |D_X| + \sum_{H\in\mathcal{H}_2} |D_H| \\
&\le \frac{|E(G_v^*)|+1}{\binom{k}{2}+2} + 1 + \sum_{H\in\mathcal{H}_2} \frac{|E(H)|+1}{\binom{k}{2}+2} \\
&\le \frac{|E(G_v^*)|+1}{\binom{k}{2}+2} + \frac{h_1(\binom{k}{2}+1)+1}{\binom{k}{2}+2} + \sum_{H\in\mathcal{H}_2} \frac{|E(H)|+1}{\binom{k}{2}+2} \\
&= \frac{|E(G_v^*)|+1+h_1(\binom{k}{2}+1)+1+\sum_{H\in\mathcal{H}_2}(|E(H)|+1)}{\binom{k}{2}+2} \\
&\le \frac{m+1}{\binom{k}{2}+2}. \qquad (5.16)
\end{aligned}$$

If the bound is sharp, then (5.15) and (5.16) are sharp, and thus, we have that for each $H \in \mathcal{H}_2$, $|D_H| = \frac{|E(H)|+1}{\binom{k}{2}+2}$, $|D_v^*| = \frac{|E(G_v^*)|+1}{\binom{k}{2}+2}$, and since $|D_X| = 1$, $h_1 = 1$.

Since for each $H \in \mathcal{H}_2$, $\iota(H,k) = \frac{|E(H)|+1}{\binom{k}{2}+2}$, then by the induction hypothesis, for each $H \in \mathcal{H}_2$, H is a k-clique edge-special graph. Let $\mathcal{H}_2 =$

$\{H_1, \dots, H_p\}$. For each $i \in [p]$, let H_i have r_i k-clique constituents $G_1^i, \dots, G_{r_i}^i$. For each $i \in [p]$, let $V_i = \{v_1^i, \dots, v_{r_i}^i\}$ be the set of the k-clique connections of $G_1^i, \dots, G_{r_i}^i$ in H_i. Also, for each $i \in [p]$ and for each $j \in [r_i]$, let $V(G_j^i) \setminus \{v_j^i\} = \{{}_iu_1^j, \dots, {}_iu_k^j\}$. We can assume that for each $i \in [p]$ and for each $j \in [r_i]$, ${}_iu_1^j$ is adjacent to v_j^i. Note that by Proposition 5.2.6, $\iota(H_i, k) = r_i = \frac{|E(H_i)|+1}{\binom{k}{2}+2}$, for each $i \in [p]$. Let $V = \cup_{i\in[p]} V_i$. Since, $|D_v^*| = \frac{|E(G_v^*)|+1}{\binom{k}{2}+2}$, then by the induction hypothesis, G_v^* is a k-clique edge-special graph. Let G_v^* have r^* k-clique constituents $G_1, \dots, G_{r^*}$. Let $\{v_1, \dots, v_{r^*}\}$ be the set of the k-clique connections of $G_1, \dots, G_{r^*}$ in G_v^*. For each $j \in [r^*]$, let $V(G_j) \setminus \{v_j\} = \{u_1^j, \dots, u_k^j\}$. We can assume that for each $j \in [r^*]$, v_j is adjacent to u_1^j. Note that by Proposition 5.2.6, $\iota(G_v^*, k) = r^* = \frac{|E(G_v^*)|+1}{\binom{k}{2}+2}$. Now since $h_1 = 1$, let $\mathcal{H}_1 = \{H'\}$. Let $V(H') = \{y_1, \dots, y_k\}$ and without loss of generality, assume x is adjacent to y_1, that is, $y_{H'} = y_1$. Let $D' = V \cup \{v_1, \dots, v_{r^*}\} \cup \{x\}$. Since (5.16) is sharp, then $|D'| = \sum_{i=1}^{p} r_i + r^* + 1 = \frac{m+1}{\binom{k}{2}+2}$.

Let $H_i \in \mathcal{H}_2$ and consider y_{H_i}. If $y_{H_i} = {}_iu_l^j$ for some $l \in [k]$, $j \in [r_i]$, then $D' \setminus \{v_j^i\}$ is a k-clique isolating set of G of size less than $\frac{m+1}{\binom{k}{2}+2}$, a contradiction. Therefore, for each $H_i \in \mathcal{H}_2$, $y_{H_i} = v_j^i$, for some $j \in [r_i]$.

Now consider G_v^*. Since (5.15) is sharp, then G_v^* is linked to x only via the edge $\{vx\}$. Consider the vertex v. If $v = u_l^j$ for some $l \in [k]$, $j \in [r^*]$, then $D' \setminus \{v_j\}$ is a k-clique isolating set of G of size less than $\frac{m+1}{\binom{k}{2}+2}$. Thus, $v = v_j$ for some $j \in [r^*]$.

Finally note that $G[D]$ is a tree. Therefore, G is a k-clique edge-special graph with $1 + r^* + \sum_{i=1}^{p} r_i$ constituents $G[X], G_1, \dots, G_{r^*}, G_1^1, \dots, G_{r_1}^1, \dots, G_1^p, \dots, G_{r_p}^p$. This completes the proof. $\square$

Theorem 5.2.8. *If G is a connected graph, $n = |V(G)|$, and $\Delta = \Delta(G)$, then*

$$\iota'(G,k) \leq \frac{n - \Delta - 1 + k}{k}.$$

Proof of Theorem 5.2.8. Clearly, $n \geq \Delta + 1$. If G does not contain a clique of k vertices, then $\iota'(G,k) = 0 \leq \frac{n-\Delta-1+k}{k}$. Thus, suppose that G has a clique of k vertices. We let $G_1 = G$. Choose a vertex x_1 which has maximum degree in G_1. Then, delete its closed neighbourhood from the graph G_1 and denote the resulting graph $G_1 - N_{G_1}[x_1]$ by G_2. If G_2 does not contain k-cliques, then $\{x_1\}$ is an independent k-clique isolating set of G. Thus, $\iota'(G,k) = 1 \leq \frac{n-\Delta-1+k}{k}$. If G_2 has a clique of k vertices, then choose a vertex x_2 which is in a clique of k vertices of G_2. Clearly, $|N_{G_2}[x_2]| \geq k$. Then, delete its closed neighbourhood from the graph G_2 and denote the resulting graph $G_2 - N_{G_2}[x_2]$ by G_3. Note that $N_{G_1}[x_1] \cap N_{G_2}[x_2] = \emptyset$. Thus, $\{x_1, x_2\}$ is an independent set. Moreover, $n \geq |N_{G_1}[x_1]| + |N_{G_2}[x_2]| \geq \Delta + 1 + k$. Thus, if G_3 does not contain k-cliques, then $\{x_1, x_2\}$ is an independent k-clique isolating set of G. Thus, $\iota'(G,k) \leq 2 \leq \frac{(\Delta+1+k)-\Delta-1+k}{k} \leq \frac{n-\Delta-1+k}{k}$. If G_3 has a clique of k vertices, then choose a vertex x_3 which is in a clique of k vertices of G_3. Then, delete its closed neighbourhood from the graph G_3 and denote the resulting graph $G_3 - N_{G_3}[x_3]$ by G_4. Continuing this way, we obtain $x_1, \dots, x_r$ and $G_1, \dots, G_{r+1}$ such that $G_i = G_{i-1} - N_{G_{i-1}}[x_{i-1}]$ for each $i \in [r+1] \setminus \{1\}$, and G_{r+1} does not contain k-cliques. Note also that $N_{G_i}[x_i] \cap N_{G_j}[x_j] = \emptyset$ for every $i, j \in [r]$ with $i < j$. Thus, $\{x_1, ..., x_r\}$ is an independent k-clique isolating set of G. Now since x_1 was chosen to be a vertex of maximum degree in $G_1 = G$, then $|N_{G_1}[x_1]| = \Delta + 1$. Also, since x_i is in a clique of k vertices of G_i for $i \in [r] \setminus \{1\}$, we have that $|N_{G_i}[x_i]| \geq k$ for each $i \in [r] \setminus \{1\}$. Thus, we have

$$n \geq |N_{G_1}[x_1]| + |N_{G_2}[x_2]| + \cdots + |N_{G_r}[x_r]| \geq \Delta + 1 + (r-1)k,$$

and therefore,

$$\iota'(G,k) \leq r \leq \frac{n - \Delta - 1 + k}{k}.$$

This completes the proof. □

Chapter 6

Irregular independence

6.1 Introduction

In this chapter and the next, we will consider the notions of irregular independence and irregular domination (respectively) as counterparts of the notions of regular independence and regular domination (also referred to as fair domination), which were recently introduced in [17, 18]. In this chapter and the next, we present our work from our paper in [12]. Definitions and notation from Chapter 1 will be used.

If A is an independent set of a graph G such that the vertices in A have pairwise different degrees, then we call A an *irregular independent set of G*. The size of a largest irregular independent set of G will be called the *irregular independence number of G* and will be denoted by $\alpha_{ir}(G)$. If A is an independent set of a graph G such that the vertices in A have the same degree, then A is called a *regular independent set of G*. The size of a largest regular independent set of G is called the *regular independence number of G*

and is denoted by $\alpha_{reg}(G)$.

The regular independence number was first introduced by Albertson and Boutin in [3]. They proved lower bounds for planar graphs, maximal planar graphs, bounded-degree graphs and trees. Recently, Caro, Hansberg and Pepper [18] generalised the regular independence number by introducing the regular k-independence number $\alpha_{k-reg}(G)$ of a graph G, and they generalized the results in [3] and found lower bounds for the regular k-independence numbers of trees, forests, planar graphs, k-trees and k-degenerate graphs. Guo, Zhao, Lai and Mao [28] obtained the exact values of the regular k-independence numbers of some special classes of graphs, and they established some lower bounds and upper bounds for line graphs and trees with a given diameter. They also obtained results of Nordhaus–Gaddum [45] type.

Unless specified otherwise, we make use of the following notation: $n = |V(G)|$, $m = |E(G)|$, $d(v) = |N(v)|$, $\delta(G) = \min\{d(v) \colon v \in V(G)\}$, $\Delta(G) = \max\{d(v) \colon v \in V(G)\}$.

For a graph G and a subset A of $V(G)$, $E(A, V(G)\backslash A)$ denotes the set of edges of G which have one vertex in A and the other in $V(G)\backslash A$. We denote by $e(A, V(G)\backslash A)$ the size of $E(A, V(G)\backslash A)$. We define the *max-cut of G*, denoted by $\beta(G)$, as $\beta(G) = \max\{e(A, V(G)\backslash A) \colon A \subseteq V(G)\}$.

We provide several sharp bounds for $\alpha_{ir}(G)$. Our results are given in the next two sections. In Section 6.3, we study the particularly interesting case when $\alpha_{ir}(G) = 1$. In Section 6.4, we obtain the Nordhaus-Gaddum type results for the irregular independence number.

6.2 Results

In this section, we provide various bounds for $\alpha_{ir}(G)$. We start with bounds in terms of basic graph parameters.

For any graph G, we denote by $span(G)$ the number of distinct values in the degree sequence of G. More formally, $span(G) = |\{d(v)\colon v \in V(G)\}|$. Clearly, $span(G) \leq \Delta - \delta + 1$.

Theorem 6.2.1. *If G is a graph, $n = |V(G)|$, $m = |E(G)|$, $\delta = \delta(G)$ and $\Delta = \Delta(G)$, then*

$$1 \leq \alpha_{ir}(G) \leq \min\left\{\Delta - \delta + 1,\ \left\lfloor \frac{n-\delta+1}{2} \right\rfloor,\ \left\lfloor \frac{1+\sqrt{2n^2-2n-4m+1}}{2} \right\rfloor\right\}.$$

Moreover, the bounds are sharp.

Proof. We have $\alpha_{ir}(G) \geq 1$ as $\{v\}$ is an irregular independent set for each $v \in V(G)$. Clearly, $\alpha_{ir}(G) \leq span(G) \leq \Delta - \delta + 1$. Let A be a largest irregular independent set. Let $v_1, \dots, v_t$ be the distinct vertices of A with $\delta \leq d(v_1) < \cdots < d(v_t)$. Thus, $\delta + t - 1 \leq d(v_t) \leq |V(G)\backslash A| = n - t$, from which we get $t \leq \left\lfloor \frac{n-\delta+1}{2} \right\rfloor$. Let $B = V(G)\backslash A$. We have

$$\begin{aligned} m = |E(G[B])| + \sum_{v \in A} d(v) &\leq \frac{1}{2}(n-t)(n-t-1) + \sum_{i=1}^{t}(n-2t+i) \\ &= \frac{1}{2}(n-t)(n-t-1) + \frac{t}{2}(2n-3t+1), \end{aligned}$$

so $2t^2-2t+(n+2m-n^2) \leq 0$, and hence $\alpha_{ir}(G) \leq \frac{1}{2}\left(1+\sqrt{2n^2-2n-4m+1}\right)$. This establishes the bound in the theorem.

The lower bound is attained if G is regular. We now show that the upper bound is sharp. Let r and t be positive integers.

If G is the union of t vertex-disjoint graphs $G_1, \dots, G_t$ such that G_i is a copy of K_{r+i-1} for each $i \in [t]$, then $\alpha_{ir}(G) = \Delta - \delta + 1$.

Let $k = r + t - 1$. Suppose that G is constructed as follows: let $v_1, \dots, v_t$, $w_1, \dots, w_k$ be the distinct vertices of G, and, for each $i \in [t]$, form exactly $r + i - 1$ distinct edges of the form $\{v_i, w_j\}$. Let $A = \{v_1, \dots, v_t\}$ and $B = \{w_1, \dots, w_k\}$. Since A is an irregular independent set of G, $\alpha_{ir}(G) \geq t$. But $\alpha_{ir}(G) \leq \lfloor \frac{n-\delta+1}{2} \rfloor = \lfloor \frac{(\delta+2t-1)-\delta+1}{2} \rfloor = t$. Thus, $\alpha_{ir}(G) = \lfloor \frac{n-\delta+1}{2} \rfloor$.

Let $r \geq t$. Suppose that G is constructed as follows: let $v_1, \dots, v_t, w_1, \dots,$ w_r be the distinct vertices of G, form a complete graph on the vertices $w_1, \dots, w_r$, and, for each $i \in [t]$, form exactly $r-t+i$ distinct edges of the form $\{v_i, w_j\}$. Let $A = \{v_1, \dots, v_t\}$. Since A is an irregular independent set of G, $t \leq \alpha_{ir}(G)$. We have $m = \frac{1}{2}r(r-1) + \sum_{i=1}^{t}(r-t+i) = \frac{1}{2}r(r-1) + \frac{1}{2}t(2r-t+1)$. Since $n = r + t$, $2m = (n-t)(n-t-1) + t(2n-3t+1) = n^2 - n - 2t^2 + 2t$. By the established bound, $\alpha_{ir}(G) \leq \frac{1}{2}\left(1 + \sqrt{2n^2 - 2n - 4m + 1}\right) \leq t$. Since $\alpha_{ir}(G) \geq t$, $\alpha_{ir}(G) = \frac{1}{2}\left(1 + \sqrt{2n^2 - 2n - 4m + 1}\right)$. $\square$

We also have

$$\alpha_{ir}(G) \leq \frac{-2\delta + 1 + \sqrt{(2\delta - 1)^2 + 8m}}{2}. \tag{6.1}$$

This is immediate from our next result, the proof of which also shows that (6.1) is sharp.

Theorem 6.2.2. *If G is a graph, $\delta = \delta(G)$ and $\beta = \beta(G)$, then*

$$\alpha_{ir}(G) \leq \frac{-2\delta + 1 + \sqrt{(2\delta - 1)^2 + 8\beta}}{2}.$$

Moreover, the bound is sharp.

Proof. Let $t = \alpha_{ir}(G)$. Let A be an irregular independent set of G of size t, and let $v_1, \dots, v_t$ be the distinct vertices in A. We have $\beta \geq e(A, V(G)\backslash A) = \sum_{i=1}^{t} d(v_i) \geq \sum_{i=0}^{t-1}(\delta + i) = \frac{1}{2}t(2\delta + t - 1)$, so $0 \geq t^2 + (2\delta - 1)t - 2\beta$. Solving the quadratic inequality, we obtain $t \leq \frac{1}{2}\left(-2\delta + 1 + \sqrt{(2\delta - 1)^2 + 8\beta}\right)$.

We now prove that the bound is sharp. Let r and t be positive integers such that $t(t-1) \geq 2r(r-1)$. Let $k = r+t-1$. Let mod* be the usual modulo operation with the exception that, for every two positive integers a and b, $ba \bmod^* a$ is a instead of 0. Let $s_0 = 0$, and let $s_i = \sum_{j=0}^{i-1}(r + j)$ for each $i \in [t]$. Suppose that G is constructed as follows: let $v_1, \dots, v_t, w_1, \dots, w_k$ be the distinct vertices of G, and, for each $i \in [t]$, let v_i be adjacent to the vertices in $\{w_{j \bmod^* k} \colon j \in [s_{i-1}+1, s_i]\}$. Thus, v_1 is adjacent to $w_1, \dots, w_r$, v_2 is adjacent to $w_{r+1}, \dots, w_{2r+1}$, v_3 is adjacent to $w_{2r+2}, \dots, w_{3r+3}$, and so on, where the indices are taken mod* k. By construction, $d(w_k) = \min\{d(w_j) \colon j \in [k]\}$. Let $A = \{v_1, \dots, v_t\}$ and $B = \{w_1, \dots, w_k\}$. Since G is a bipartite graph with partite sets A and B, we have $\beta = m = e(A, B) = \sum_{i=1}^{t} d(v_t) = s_t = \frac{1}{2}t(2r + t - 1)$. We also have $m = \sum_{j=1}^{k} d(w_j) \geq d(w_k)k$, so $\frac{1}{2}t(2r + t - 1) \leq d(w_k)k$, and hence $d(w_k) \geq \frac{t(2r+t-1)}{2k} = \frac{t(2r+t-1)}{2(r+t-1)}$. If we assume that $\frac{t(2r+t-1)}{2(r+t-1)} < r$, then we get a contradiction to the condition $t(t-1) \geq 2r(r-1)$. Thus, $d(w_k) \geq r$. Since $\min\{d(v_i) \colon i \in [t]\} = d(v_1) = r \leq d(w_k) = \min\{d(w_j) \colon j \in [k]\}$, $\delta = d(v_1) = r$. Now A is an irregular independent set of G, so $\alpha_{ir}(G) \geq t$.

By the bound in the theorem,

$$
\begin{aligned}
\alpha_{ir}(G) &\le \frac{-2\delta + 1 + \sqrt{(2\delta-1)^2 + 8\beta}}{2} \\
&= \frac{-2r + 1 + \sqrt{(2r-1)^2 + 4t(2r+t-1)}}{2} \\
&= \frac{-2r + 1 + \sqrt{(2r+2t-1)^2}}{2} = t.
\end{aligned}
$$

Since $\alpha_{ir}(G) \ge t$, $\alpha_{ir}(G) = \frac{1}{2}\left(-2\delta + 1 + \sqrt{(2\delta-1)^2 + 8\beta}\right)$. □

Our next result provides inequalities relating $\alpha_{ir}(G)$ to $\alpha_{reg}(G)$.

Theorem 6.2.3. *For any graph G on n vertices,*

(i) $2 \le \alpha_{ir}(G) + \alpha_{reg}(G) \le n + 1$,

(ii) $\alpha(G) \le \alpha_{ir}(G)\alpha_{reg}(G) \le (\alpha(G))^2$,

(iii) if $n \ge 4$, then $1 \le \alpha_{ir}(G)\alpha_{reg}(G) \le \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.

Moreover, the following assertions hold:

(a) The bounds are sharp.

(b) The upper bound in (i) is attained if and only if G is empty. Also, for any integer k with $2 \le k \le n+1$, $\alpha_{ir}(G) + \alpha_{reg}(G) = k$ if $G = E_{k-2} \cup K_{n-k+2}$.

Proof. Let A be an irregular independent set of G of size $\alpha_{ir}(G)$. Let B be a regular independent set of G of size $\alpha_{reg}(G)$. Let I be a largest independent set of G.

(i) Trivially, $\alpha_{ir}(G) \geq 1$, $\alpha_{reg}(G) \geq 1$, and hence the lower bound is clear. Clearly, $|A \cap B| \leq 1$. We have $n \geq |A \cup B| = |A| + |B| - |A \cap B| \geq \alpha_{ir}(G) + \alpha_{reg}(G) - 1$, so $\alpha_{ir}(G) + \alpha_{reg}(G) \leq n + 1$.

(ii) Let $d_1, \dots, d_r$ be the distinct degrees of the vertices in I. For each $i \in [r]$, let D_i be the set of vertices in I of degree d_i. Let $s = \max\{|D_i| \colon i \in [r]\}$. We have $r \leq \alpha_{ir}(G)$, $s \leq \alpha_{reg}(G)$ and $\alpha(G) = |I| = |D_1| + \cdots + |D_r| \leq rs \leq \alpha_{ir}(G)\alpha_{reg}(G)$. Trivially, $\alpha_{ir}(G) \leq \alpha(G)$, $\alpha_{reg}(G) \leq \alpha(G)$, and hence the upper bound.

(iii) As in (i), the lower bound is trivial. By (i), $|A| + |B| \leq n + 1$. Suppose equality holds. Then $G = E_n$ by (b), which is proved below. Thus, $|A||B| = n \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$ if $n \geq 4$. Now suppose $|A| + |B| \leq n$. Then $|A||B| \leq |A|(n - |A|)$. By differentiating the function $f(r) = r(n - r)$, we see that f increases as r increases from 0 to $\frac{n}{2}$. Thus, $|A||B| \leq \lfloor \frac{n}{2} \rfloor (n - \lfloor \frac{n}{2} \rfloor) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. Hence the upper bound.

(a) The lower bounds in (i)–(iii) and the upper bound in (ii) are attained if $G = K_n$. The upper bound in (i) is attained if $G = E_n$.

We now show that the upper bound in (iii) is sharp. For each of Cases 1–4 below, we construct a graph that attains the bound. Let $v_1, \dots, v_n$ be its distinct vertices. If $n \bmod 4 = 0$, then let $X = \{v_1, \dots, v_{\frac{n}{2}}\}$, let $Y = \{v_{\frac{n}{2}+1}, \dots, v_n\}$, and, for each $j \in [n/4]$, let v_j be adjacent to exactly $j - 1$ vertices in Y, and let $v_{\frac{n}{2}-j+1}$ be adjacent to the remaining vertices in Y. If $n \bmod 4 = 1$, then let $X = \{v_1, \dots, v_{\frac{n-1}{2}}\}$, let $Y = \{v_{\frac{n+1}{2}+1}, \dots, v_n\}$, and, for each $j \in [(n-1)/4]$, let v_j be adjacent to exactly j vertices in Y, and let $v_{\frac{n-1}{2}-j+1}$ be adjacent to the remaining vertices in Y. If $n \bmod 4 = 2$, then let $X = \{v_1, \dots, v_{\frac{n}{2}}\}$, let $Y = \{v_{\frac{n}{2}+1}, \dots, v_n\}$, let $v_{\frac{n}{2}}$ be adjacent to each vertex in Y, and, for each $j \in [(n-2)/4]$, let v_j be adjacent to exactly j vertices in Y, and let $v_{\frac{n}{2}-j}$ be adjacent to the remaining vertices in Y. If $n \bmod 4 = 3$, then let $X = \{v_1, \ldots, v_{\frac{n+1}{2}}\}$, let $Y = \{v_{\frac{n+3}{2}}, \ldots, v_n\}$, and, for each $j \in [(n+1)/4]$, let v_j be adjacent to exactly $j-1$ vertices in Y, and let $v_{\frac{n+1}{2}-j+1}$ be adjacent to the remaining vertices in Y. Suppose that the resulting graph is G. Then X is an irregular independent set of G, Y is a regular independent set of G, and $|X||Y| = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. By the bound in (iii), $\alpha_{ir}(G)\alpha_{reg}(G) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.

(b) As stated in (a), the upper bound in (i) is attained in $G = E_n$. We now prove the converse. Thus, suppose $\alpha_{ir}(G) + \alpha_{reg}(G) = n + 1$. Thus, $|A| + |B| = n+1$. Recall that $|A \cap B| \leq 1$. Thus, $n \leq |A| + |B| - |A \cap B| = |A \cup B| \leq n$, giving $|A \cup B| = n$ and $|A \cap B| = 1$. Thus, for some $v \in V(G)$, $A \cap B = \{v\}$ and $A = (V(G) \backslash B) \cup \{v\}$. If $d(v) = 0$, then since $v \in B$, all the vertices of B must have degree 0. Since A and B are independent sets containing v, v has no neighbours in $A \cup B$. Thus, $d(v) = 0$ as $A \cup B = V(G)$. Hence $d(w) = 0$ for each $w \in B$. Now consider any $x \in V(G) \backslash B$. We have $x \in A$. Since A is independent, $N(x) \subseteq B$. Since the vertices in B have no neighbours, $N(x) = \emptyset$. Thus, G is empty, as required.

It is easy to check that $\alpha_{ir}(G) + \alpha_{reg}(G) = k$ if $G = E_{k-2} \cup K_{n-k+2}$ with $2 \leq k \leq n+1$. $\square$

Corollary 6.2.4. *For any graph G on $n \geq 4$ vertices,*

$$\alpha_{ir}(G)\alpha_{reg}(G) \leq \min\{(\alpha(G))^2, \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor\}.$$

6.3 Graphs with irregular independence number 1

We now investigate the particularly interesting case $\alpha_{ir}(G) = 1$.

6.3.1 A general characterization

Let G be a graph. Let $n = |V(G)|$ and $\delta = \delta(G)$. Let $D(G)$ denote the set of degrees of vertices of G. For any $i \in D(G)$, let N_i denote the set of vertices of G of degree i. Let $n_i = |N_i|$. For any two disjoint subsets X and Y of $V(G)$, let $<X, Y>$ denote the subgraph of G given by $(X \cup Y, \{\{x, y\} \in E(G) \colon x \in X, y \in Y\})$.

Lemma 6.3.1. *If $\alpha_{ir}(G) = 1$, then*

(i) $<N_i, N_j>$ is a complete bipartite graph for any $i, j \in D(G)$ with $i \neq j$,

(ii) the subgraph of G induced by N_k is $(k+n_k-n)$-regular for any $k \in D(G)$.

Proof. (i) Suppose $\{v, w\} \notin E(G)$ for some $v \in N_i$ and some $w \in N_j$ with $i \neq j$. Then $\{v, w\}$ is an irregular independent set of G of size 2. This contradicts $\alpha_{ir}(G) = 1$.

(ii) Let $v \in N_k$. By (i), for any $j \in D(G)\backslash\{k\}$, v is adjacent to each $w \in N_j$. Thus, v is adjacent to each vertex in $V(G)\backslash N_k$. By definition of N_k, the degree of v in the subgraph of G induced by N_k is $k - (n - n_k)$. $\Box$

Theorem 6.3.2. *If $\alpha_{ir}(G) = 1$, then*

(i) $n_k \geq n - k$ for any $k \in D(G)$,

(ii) $span(G) \leq \frac{1}{2}(1 + \sqrt{1 + 8\delta})$.

Moreover, the bound in (ii) is sharp.

Proof. (i) By Lemma 6.3.1(ii), $k + n_k - n \geq 0$.

(ii) Let $t = span(G)$. If $t = 1$, then the result is immediate. Suppose $t \geq 2$. Then $D(G) = \{d_1, \dots, d_t\}$ for some integers $d_1, \dots, d_t$ with $0 \leq d_1 < \cdots < d_t \leq n - 1$. For $i \in [t]\backslash\{1\}$, we have $d_1 \leq d_2 - 1 \leq \cdots \leq d_i - (i-1) \leq \cdots \leq d_t - (t-1) \leq n - 1 - (t-1) = n - t$, so $d_i \leq n - t + (i-1)$. By (i), $n_{d_i} \geq n - d_i$ for $i \in [t]$. We have

$$\begin{aligned}
n &= \sum_{i=1}^{t} n_{d_i} \geq \sum_{i=1}^{t}(n - d_i) = (n - d_1) + \sum_{i=2}^{t}(n - d_i) \\
&= (n - \delta) + \sum_{i=2}^{t} n - \sum_{i=2}^{t} d_i \geq (n - \delta) + (t-1)n - \sum_{i=2}^{t}(n - t + i - 1) \\
&= tn - \delta - \frac{(t-1)}{2}(2n - t).
\end{aligned}$$

Therefore, $0 \geq t^2 - t - 2\delta$, and the bound follows. The bound is attained if, for example, G is the complete k-partite graph $K_{1,\dots,k}$. Indeed, we then have $\alpha_{ir}(G) = 1$, $\delta = n - k$, $n = 1 + \cdots + k = \frac{k}{2}(k+1)$ and

$$\begin{aligned}
k = span(G) &\leq \frac{1 + \sqrt{1 + 8\delta}}{2} = \frac{1 + \sqrt{1 + 8(n-k)}}{2} \\
&= \frac{1 + \sqrt{1 + 8(\frac{k}{2}(k+1)) - 8k}}{2} = \frac{1 + \sqrt{(2k-1)^2}}{2} = k,
\end{aligned}$$

so $span(G) = \frac{1+\sqrt{1+8\delta}}{2}$. $\square$

6.3.2 Planar graphs and outerplanar graphs

We now determine the planar graphs and outerplanar graphs whose irregular independence number is 1.

Suppose that G and H are vertex-disjoint graphs. The *join of G and H*, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{\{x, y\} \colon x \in V(G), y \in V(H)\}$. If $k \geq 2$, $r \geq 2$, $G = K_1$, and H is the union of r vertex-disjoint copies of K_{k-1}, then $G + H$ is called a *k-windmill graph* and is denoted by $Wd(k, r)$. Note that $Wd(k, r)$ is merely the union of r copies of K_k that have exactly one common vertex.

Theorem 6.3.3. *A graph G is planar and $\alpha_{ir}(G) = 1$ if and only if G is a regular planar graph or a copy of one of the graphs $K_{1,n-1}$, $K_{2,n-2}$, $K_2 + E_{n-2}$, $K_2 + \frac{n-2}{2}K_2$, $E_2 + \frac{n-2}{2}K_2$, $E_2 + C_{n-2}$, $Wd(3, \frac{n-1}{2})$ and $K_1 + H$, where H is a union of vertex-disjoint cycles.*

Before giving the proof of the theorem above, we need the following lemmas.

Lemma 6.3.4. *If a planar graph G has a vertex v that is adjacent to all the other vertices of G, then $G - v$ is outerplanar.*

Proof. Indeed, by deleting v (and all edges incident to it) from a plane drawing of G, we obtain a plane drawing of $G - v$ that has all the vertices on the same face. This means that $G - v$ is outerplanar because, for any face F of a plane drawing φ of a planar graph, φ can be transformed to another plane drawing of the same graph in such a way that F becomes the unbounded face, for example, by using stereographic projection (see [54, Remark 6.1.27]). □

Lemma 6.3.5. *If φ is a plane drawing of $E_2 + C_k$ ($k \geq 3$), then a vertex v of E_2 is mapped by φ into the interior I of the drawing of C_k, and the other vertex w of E_2 is mapped by φ into the exterior E of the drawing of C_k.*

Proof. Let $G = E_2 + C_k$. Let $F \in \{I, E\}$ such that v is mapped by φ into F. Since v is adjacent to each vertex of C_k, each face of F in the drawing of $G - w$ has exactly 3 vertices on its boundary, one of which is v. Thus, if we assume that w is mapped into F, then we obtain that w lies in the interior of one of these faces, and hence that w is adjacent to at most two vertices of C_k, a contradiction. $\square$

Proof of Theorem 6.3.3. It is easy to check that if G is one of the explicit graphs in Theorem 6.3.3, then G is planar and $\alpha_{ir}(G) = 1$. We now prove the converse.

Let G be a planar graph with $\alpha_{ir}(G) = 1$. Since K_5 and $K_{3,3}$ are non-planar, G does not contain any copies of these. It is well known that having G planar implies that $m \leq 3n - 6$. Suppose that G is not regular. Setting $t = span(G)$, we then have $t \geq 2$ (and $n \geq 3$). We have $D(G) = \{d_1, \dots, d_t\}$ for some integers $d_1, \dots, d_t$ with $0 \leq d_1 < \cdots < d_t$. We will often use Lemma 6.3.1(i), which tells us that, for any $i, j \in D(G)$ with $i \neq j$, each vertex of N_{d_i} is adjacent to each vertex of N_{d_j}. The first immediate deduction from this is that $d_1 \geq 1$ as $t \geq 2$.

Suppose $t \geq 3$. Let $\{a_1, \dots, a_t\} = \{d_1, \dots, d_t\}$ such that $n_{a_1} \leq \cdots \leq n_{a_t}$. If we assume that $n_{a_1} = n_{a_2} = 1$, then Lemma 6.3.1(i) gives us $a_1 = a_2 = n - 1$, a contradiction (as $a_1, \dots, a_t$ are distinct). Thus, $n_{a_i} \geq 2$ for each $i \in [2, t]$. If we assume that $\sum_{i=3}^{t} n_{a_i} \geq 3$, then, by Lemma 6.3.1(i), we obtain that $<N_{a_1} \cup N_{a_2}, \bigcup_{i=3}^{t} N_{a_i}>$ contains a copy of $K_{3,3}$, a contradiction. Thus, $t = 3$ and $n_{a_2} = n_{a_3} = 2$. Let $\{u_1, u_2\} = N_{a_2}$ and $\{v_1, v_2\} = N_{a_3}$. We cannot have $\{u_1, u_2\}, \{v_1, v_2\} \in E(G)$, because otherwise Lemma 6.3.1(i) gives us $a_2 = n_{a_1} + n_{a_3} + 1 = n_{a_1} + 3 = n_{a_1} + n_{a_2} + 1 = a_3$, a contradiction. Similarly, we cannot have $\{u_1, u_2\}, \{v_1, v_2\} \notin E(G)$. Thus, for some $i \in \{2, 3\}$, $a_i = n_{a_1} + 2$ and $a_{5-i} = n_{a_1} + 3$. We cannot have $n_{a_1} = 1$, because otherwise $a_1 = n_{a_2} + n_{a_3} = 4 = a_{5-i}$. Thus, $n_{a_1} = 2$. Let $\{w_1, w_2\} = N_{a_1}$. We cannot have $\{w_1, w_2\} \in E(G)$, because otherwise $a_1 = 5 = a_{5-i}$. Thus, we have $\{w_1, w_2\} \notin E(G)$, which gives us $a_1 = 4 = a_i$, a contradiction.

Therefore, $t = 2$. If we assume that $n_{d_1} \geq 3$ and $n_{d_2} \geq 3$, then, by Lemma 6.3.1(i), we obtain that G contains a copy of $K_{3,3}$, a contradiction. Thus, $n_{d_i} \leq 2$ for some $i \in \{1, 2\}$. Let $j = 3 - i$. By Lemma 6.3.1(i), $G = G[N_{d_i}] + G[N_{d_j}]$. By Lemma 6.3.1(ii), $G[N_{d_j}]$ is k-regular, where $k = d_j + n_{d_j} - n$.

Suppose $n_{d_i} = 1$. Let $\{v\} = N_{d_i}$. Thus, $G = (\{v\}, \emptyset) + G[N_{d_j}]$. By Lemma 6.3.4, $G[N_{d_j}]$ is outerplanar. Since the minimum degree of an outerplanar graph is at most 2 (see [54, Proposition 6.1.20]), $k \leq 2$. If $k = 0$, then G is a copy of $K_{1,n-1}$. If $k = 1$, then $G[N_{d_j}]$ is a copy of $\frac{n-1}{2}K_2$, so G is a copy of $Wd(3, \frac{n-1}{2})$. If $k = 2$, then $G[N_{d_j}]$ is a cycle or a union of vertex-disjoint cycles.

Now suppose $n_{d_i} = 2$. Let $\{v, w\} = N_{d_i}$ and let $\{u_1, \dots, u_{n-2}\} = N_{d_j}$. By the handshaking lemma, $|E(G[N_{d_j}])| = \frac{k(n-2)}{2}$. By Lemma 6.3.1(i), $|E(<N_{d_i}, N_{d_j}>)| = 2(n-2)$. Now

$$m = |E(G[N_{d_i}])| + |E(G[N_{d_j}])| + |E(<N_{d_i}, N_{d_j}>)| \geq \frac{k(n-2)}{2} + 2(n-2).$$

Since $m \leq 3n - 6$, we obtain $k \leq 2$.

If $k = 0$ and $\{v, w\} \in E(G)$, then G is a copy of $K_2 + E_{n-2}$. If $k = 0$ and $\{v, w\} \notin E(G)$, then G is a copy of $E_2 + E_{n-2} = K_{2,n-2}$. If $k = 1$ and $\{v, w\} \in E(G)$, then G is a copy of $K_2 + \frac{n-2}{2}K_2$. If $k = 1$ and $\{v, w\} \notin E(G)$, then G is a copy of $E_2 + \frac{n-2}{2}K_2$.

Finally, suppose $k = 2$. We cannot have v adjacent to w, because otherwise $m = 1 + \frac{2(n-2)}{2} + 2(n-2) > 3n - 6$. Since $k = 2$, $G[N_{d_j}]$ is a union of vertex-disjoint cycles $G_1, \ldots, G_r$. Suppose $r \geq 2$. Let θ be a plane drawing of G. Let φ be the drawing obtained by restricting θ to the subgraph $G' = (\{v, w\}, \emptyset) + G_1$ of G. By Lemma 6.3.5, no face of φ has both v and w on its boundary. Since G' and G_2 are vertex-disjoint, the drawing of G_2 in θ lies in the interior of one of the faces of φ. Thus, no vertex of G_2 is adjacent to both v and w. This contradicts $G = G[N_{d_i}] + G[N_{d_j}]$. Therefore, $r = 1$. Thus, G is $G[N_{d_i}] + G_1$, which is a copy of $E_2 + C_{n-2}$. □

Corollary 6.3.6. *A graph G is outerplanar and $\alpha_{ir}(G) = 1$ if and only if G is a union of vertex-disjoint cycles or a copy of one of the graphs E_n, $\frac{n}{2}K_2$, $K_{1,n-1}$, $K_2 + E_2$ and $Wd(3, \frac{n-1}{2})$.*

Proof. It is trivial that if G is one of the explicit graphs in the statement of Corollary 6.3.6, then G is outerplanar and $\alpha_{ir}(G) = 1$.

We now prove the converse. Let G be an outerplanar graph with $\alpha_{ir}(G) = 1$. This means that $\delta \leq 2$, as mentioned in the proof of Theorem 6.3.3. If G is k-regular, then $k \leq 2$, and hence G is a copy of E_n (if $k = 0$) or a copy of $\frac{n}{2}K_2$ (if $k = 1$) or a union of vertex-disjoint cycles (if $k = 2$). Suppose that G is not regular. Since $\delta \leq 2$, it follows by Theorem 6.3.3 that G is a copy of one of $K_{1,n-1}$, $K_{2,n-2}$, $K_2 + E_{n-2}$, $E_2 + \frac{n-2}{2}K_2$ and $Wd(3, \frac{n-1}{2})$. It is well known that $K_{2,3}$ is not outerplanar. Thus, $K_{2,n-2}$ is outerplanar only if $n \leq 4$; note that $K_{2,n-2}$ is the cycle C_4 if $n = 4$. Also, for $n \geq 5$, $K_2 + E_{n-2}$ is not outerplanar as it contains $K_{2,3}$. Similarly, $E_2 + \frac{n-2}{2}K_2$ is planar only if $\frac{n-2}{2} \leq 1$. $\square$

6.4 Nordhaus–Gaddum-type results

In this section, we provide results of Nordhaus–Gaddum type [45] for the irregular independence number. In the proof, we need to use the following more precise notation (see Chapter 1). For a vertex v of a graph G, we will denote the set of neighbours of v in G by $N_G(v)$, and the degree of v in G by $d_G(v)$. Formally, $N_G(v) = \{w \in V(G) \colon vw \in E(G)\}$ and $d_G(v) = |N_G(v)|$.

Theorem 6.4.1. *If G is a graph on $n \geq 2$ vertices, then*

(i) $2 \leq \alpha_{ir}(G) + \alpha_{ir}(\bar{G}) \leq n$,

(ii) $1 \leq \alpha_{ir}(G)\alpha_{ir}(\bar{G}) \leq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$.

Moreover, the bounds are sharp.

Proof. By Theorem 6.2.1, $1 \leq \alpha_{ir}(G) \leq \lfloor \frac{n-\delta(G)+1}{2} \rfloor$ and $1 \leq \alpha_{ir}(\bar{G}) \leq \lfloor \frac{n-\delta(\bar{G})+1}{2} \rfloor$. The lower bounds follow immediately, and they are attained if G is regular. If $\delta(G) = 0$, then G has a vertex v with no neighbours, so $\delta(\bar{G}) \geq 1$ (as $v \in N_{\bar{G}}(u)$ for each $u \in V(\bar{G}) \backslash \{v\}$). Thus, $\delta(G) \geq 1$ or $\delta(\bar{G}) \geq 1$. Hence $\alpha_{ir}(G) + \alpha_{ir}(\bar{G}) \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor \leq n$ and $\alpha_{ir}(G)\alpha_{ir}(\bar{G}) \leq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$.

We now show that the upper bounds are sharp. Let $k = \lceil \frac{n}{2} \rceil$ and $l = \lfloor \frac{n}{2} \rfloor$. Suppose that G is constructed as follows: let $u_1, \dots, u_k, v_1, \dots, v_l$ be the distinct vertices of G, let every two distinct vertices in $\{v_1, \dots, v_l\}$ be adjacent, and, for each $i \in [k]$, let u_i be adjacent to the vertices in $\{v_j : j \in [i-1]\}$. Clearly, $\{u_1, \dots, u_k\}$ is an irregular independent set of G, and $\{v_1, \dots, v_l\}$ is an irregular independent set of $\bar{G}$. Therefore, $\alpha_{ir}(G)+\alpha_{ir}(\bar{G}) \geq k+l = n$ and $\alpha_{ir}(G)\alpha_{ir}(\bar{G}) \geq kl$. By (i) and (ii), we actually have $\alpha_{ir}(G) + \alpha_{ir}(\bar{G}) = n$ and $\alpha_{ir}(G)\alpha_{ir}(\bar{G}) = kl$. Finally, note that $k = \lfloor \frac{n+1}{2} \rfloor$. $\square$

Chapter 7

Irregular domination

7.1 Introduction

In this chapter, we will consider the notion of irregular domination as a counterpart of the notion of regular domination. Definitions and notation from Chapter 1 will be used.

If D is a dominating set of G such that $|N(u) \cap D| \neq |N(v) \cap D|$ for every two distinct vertices u and v in $V(G)\backslash D$, then we call D an *irregular dominating set of* G. The size of a smallest irregular dominating set of G will be called the *irregular domination number of* G and will be denoted by $\gamma_{ir}(G)$. If D is a dominating set of G such that $|N(u) \cap D| = |N(v) \cap D|$ for every two vertices u and v in $V(G)\backslash D$, then D is called a *regular dominating set of* G. The size of a smallest regular dominating set of G is called the *regular domination number of* G and is denoted by $\gamma_{reg}(G)$. Observe that the notion of irregular domination is an extreme case of the well-studied notion of location-domination [7]: a set D is called a *locating-dominating set of* G

if D is a dominating set of G such that $N(u) \cap D \neq N(v) \cap D$ for every two distinct vertices u and v in $V(G) \backslash D$.

The regular domination number was first introduced and studied by Caro, Hansberg and Henning [17]. They referred to the regular domination number as the fair domination number. Das and Desormeaux [23] considered the problem of minimizing the size of a regular dominating set that induces a connected subgraph. Further results on fair domination are obtained in [20, 43].

Unless specified otherwise, we make use of the following notation: $n = |V(G)|$, $m = |E(G)|$, $d(v) = |N(v)|$, $\delta(G) = \min\{d(v) \colon v \in V(G)\}$, $\Delta(G) = \max\{d(v) \colon v \in V(G)\}$.

Recall from the previous chapter, that for a graph G and a subset A of $V(G)$, $E(A, V(G) \backslash A)$ denotes the set of edges of G which have one vertex in A and the other in $V(G) \backslash A$. We denote by $e(A, V(G) \backslash A)$ the size of $E(A, V(G) \backslash A)$. We define the *max-cut of* G, denoted by $\beta(G)$, as $\beta(G) = \max\{e(A, V(G) \backslash A) \colon A \subseteq V(G)\}$.

We obtain several sharp bounds for $\gamma_{ir}(G)$. Our results are given in the following sections.

In the next section, we provide a number of sharp results on $\gamma_{ir}(G)$. In Section 7.3, we obtain a set of inequalities relating the irregular independence number to the irregular domination number. In Section 7.4, we obtain the Nordhaus-Gaddum type results for the irregular domination number.

7.2 Results

We will start with lower bounds for $\gamma_{ir}(G)$.

Theorem 7.2.1. *If G is a graph, $n = |V(G)|$ and $\Delta = \Delta(G)$, then*

$$\gamma_{ir}(G) \geq \max\left\{\left\lceil \frac{n}{2} \right\rceil, n - \Delta\right\}.$$

Moreover, the bound is sharp.

Proof. Let $t = \gamma_{ir}(G)$. Let D be an irregular dominating set of G of size t. Let $v_1, \dots, v_{n-t}$ be the vertices in $V(G)\backslash D$. For each $i \in [n-t]$, let $w_i = |N(v_i) \cap D|$; since D is a dominating set, $w_i \geq 1$. We may assume that $w_1 < \dots < w_{n-t}$. We have $t = |D| \geq w_{n-t} \geq n - t$, and hence $t \geq \left\lceil \frac{n}{2} \right\rceil$. Since $n - t \leq w_{n-t} \leq \Delta$, $t \geq n - \Delta$.

We now show that the bound is sharp. Let $k = \left\lceil \frac{n}{2} \right\rceil$ and $n' = n - k$. Suppose that G is constructed as follows: let $u_1, \dots, u_k, v_1, \dots, v_{n'}$ be the distinct vertices of G, and, for each $i \in [n']$, let v_i be adjacent to exactly i of the vertices $u_1, \dots, u_k$. Since $\max\{d(u_i) \colon i \in [k]\} \leq n' = d(v_{n'}) = \max\{d(v_i) \colon i \in [n']\}$, $\Delta = n'$. Clearly, $\{u_1, \dots, u_k\}$ is an irregular dominating set of G of size $\left\lceil \frac{n}{2} \right\rceil = n - n' = n - \Delta$. $\square$

Theorem 7.2.2. *If G is a graph, $n = |V(G)|$ and $\beta = \beta(G)$, then*

$$\gamma_{ir}(G) \geq n + \frac{1 - \sqrt{1 + 8\beta}}{2}.$$

Moreover, the bound is sharp.

Proof. Let t, D, $v_1, \dots, v_{n-t}$, $w_1, \dots, w_{n-t}$ be as in the proof of Theorem 7.2.1. We have $\beta \geq e(D, V(G)\backslash D) = \sum_{i=1}^{n-t} w_i \geq \sum_{i=1}^{n-t} i = \frac{1}{2}(n-t)(n-t+1)$, so $0 \geq t^2-(2n+1)t+(n^2+n-2\beta)$, and hence $t \geq n+\frac{1}{2}(1-\sqrt{1+8\beta})$.

We now show that the bound is sharp. Let $n/2 \leq k \leq n-1$ and $n' = n-k$. Suppose that G is constructed as follows: let $u_1, \dots, u_k, v_1, \dots, v_{n'}$ be the distinct vertices of G, and, for each $i \in [n']$, let v_i be adjacent to exactly i of the vertices $u_1, \dots, u_k$. Let $D = \{u_1, \dots, u_k\}$. Since D is an irregular dominating set of G, $\gamma_{ir}(G) \leq k$. Since $m = e(D, V(G)\backslash D)$, we have $\beta = e(D, V(G)\backslash D) = \frac{1}{2}(n')(n'+1)$. By the established bound,

$$\begin{aligned}\gamma_{ir}(G) &\geq n + \frac{1-\sqrt{1+8\beta}}{2} = n + \frac{1-\sqrt{1+4(n')(n'+1)}}{2} \\ &= n + \frac{1-\sqrt{(2n-2k+1)^2}}{2} = k.\end{aligned}$$

Since $\gamma_{ir}(G) \leq k$, $\gamma_{ir}(G) = n + \frac{1-\sqrt{1+8\beta}}{2}$. □

Corollary 7.2.3. *If G is an n-vertex graph with average degree d, then*

$$\gamma_{ir}(G) \geq n - \sqrt{dn}.$$

Moreover, equality holds if and only if G is empty.

Proof. Since $\beta \leq m$, $\gamma_{ir}(G) \geq n + \frac{1}{2}(1-\sqrt{1+8m})$ by Theorem 7.2.2. Now $dn = \sum_{v \in V(G)} d(v) = 2m$ (by the handshaking lemma), so $4dn = 8m$. Thus, $\gamma_{ir}(G) \geq n + \frac{1}{2}(1-\sqrt{1+4dn}) \geq n + \frac{1}{2}(-\sqrt{4dn}) = n - \sqrt{dn}$. Note that equality holds throughout only if $d = 0$, in which case G is empty.

If G is empty, then $d = 0$ and $\gamma_{ir}(G) = n = n - \sqrt{dn}$. □

Next, we give a full characterization of the cases $\gamma_{ir}(G) = n$ and $\gamma_{ir}(G) = n - 1$. For two graphs G and H, we write $G \simeq H$ if G is a copy of H.

Theorem 7.2.4. *For any graph G on n vertices, the following assertions hold:*

(i) $\gamma_{ir}(G) = n$ if and only if $G \simeq E_n$.

(ii) $\gamma_{ir}(G) = n - 1$ if and only if, for some $t \geq 0$ and some $r \geq 1$, $G \simeq tK_1 \cup K_{1,r}$ or $G \simeq tK_1 \cup H$ for some r-regular graph H.

Proof. (i) If G has an edge $\{v, w\}$, then $V(G)\backslash\{v\}$ is an irregular dominating set of G, so $\gamma_{ir}(G) \leq n-1$. Therefore, $\gamma_{ir}(G) = n$ only if $G \simeq E_n$. If $G \simeq E_n$, then $V(G)$ is the only dominating set of G, so $\gamma_{ir}(G) = n$.

(ii) It is easy to see that $\gamma_{ir}(G) = n-1$ if $G \simeq tK_1 \cup K_{1,r}$ or $G \simeq tK_1 \cup H$ for some r-regular graph H. We now prove the converse. Thus, suppose $\gamma_{ir}(G) = n - 1$. By (i), $E(G) \neq \emptyset$.

Suppose that G has two vertices u and v such that $2 \leq d(u) < d(v)$. Then $V(G)\backslash\{u, v\}$ is an irregular dominating set of G (independently of whether u and v are adjacent or not). Thus, we have $\gamma_{ir}(G) \leq n-2$, a contradiction. Therefore,

$$d(u) \leq 1 \text{ for any } u, v \in V(G) \text{ with } d(u) < d(v). \tag{7.1}$$

Suppose $span(G) \geq 4$. Then there exist $v_1, v_2, v_3, v_4 \in V(G)$ such that $d(v_1) < d(v_2) < d(v_3) < d(v_4)$. Thus, we have $2 \leq d(v_3) < d(v_4)$, which contradicts (7.1). Therefore, $span(G) \leq 3$.

If $span(G) = 1$, then G is an r-regular graph for some $r \geq 1$ ($r \neq 0$ as $E(G) \neq \emptyset$), and we are done.

Suppose $span(G) = 2$. Then $\{d(v) \colon v \in V(G)\} = \{p, r\}$ with $0 \leq p < r$. By (7.1), $p \leq 1$. If $p = 0$, then $G \simeq tK_1 \cup H$ for some $t \geq 1$ and some r-regular graph H. Suppose $p = 1$. Then $r \geq 2$. If we assume that there exists a pair of non-adjacent vertices u and v of degrees 1 and r, respectively, then we obtain that $V(G)\backslash\{u, v\}$ is an irregular dominating set of G of size $n - 2$, which contradicts $\gamma_{ir}(G) = n - 1$. Thus, each vertex x of degree 1 is adjacent to each vertex of degree r. Since x has only one neighbour, there is only one vertex of degree r. Consequently, $G = K_{1,r}$.

Finally, suppose $span(G) = 3$. Then there exist $v_1, v_2, v_3 \in V(G)$ such that $d(v_1) < d(v_2) < d(v_3)$. If we assume that G has no vertex of degree 0 or no vertex of degree 1, then we obtain $2 \leq d(v_2) < d(v_3)$, which contradicts (7.1). Thus, since $span(G) = 3$, $\{d(v) \colon v \in V(G)\} = \{0, 1, r\}$ for some $r \geq 2$. Let G' be the graph obtained by removing from G the set I of vertices of G of degree 0. Then $\{d(v) \colon v \in V(G')\} = \{1, r\}$. As in the case $span(G) = 2$ above, this yields $G' \simeq K_{1,r}$, so $G = tK_1 \cup K_{1,r}$, where $t = |I|$. □

The *Ramsey number* $R(p, q)$ is the smallest number n such that every graph on n vertices contains a clique of order p or an independent set of order q.

Theorem 7.2.5. *For any graph G on n vertices, the following assertions hold:*

(i) If $span(G) \geq R(k, k)$ and $\delta(G) \geq k$, then $\gamma_{ir}(G) \leq n - k$.

(ii) If $span(G) \geq 5$ and $\delta(G) \geq 3$, then $\gamma_{ir}(G) \leq n - 3$.

Proof. (i) Suppose $span(G) \geq R(k,k)$ and $\delta \geq k$. Let B be a set of $R(k,k)$ vertices of G of distinct degrees. Then $G[B]$ has an independent set of size k or a clique of size k. If $G[B]$ has an independent set I of size k, then $V(G)\backslash I$ is an irregular dominating set of G of size $n-k$. If $G[B]$ has a clique K of size k, then, since $\delta \geq k$, $V(G)\backslash K$ is an irregular dominating set of G of size $n-k$.

(ii) Suppose $span(G) \geq 5$ and $\delta \geq 3$. Let B be a set of 5 vertices of G of distinct degrees. It is easy to see that if a 5-vertex graph does not have an independent set of size 3, then it is a copy of C_5 or has a clique of size 3. If $G[B]$ is a copy of C_5, then each vertex in B has a distinct number of neighbours in $V(G)\backslash B$, and hence, since $\delta \geq 3$, $V(G)\backslash B$ is an irregular dominating set of G of size $n-5$. As in the proof of (i), $\gamma_{ir}(G) \leq n-3$ if $G[B]$ has an independent set of size 3 or a clique of size 3. □

7.3 Relations between irregular independence and irregular domination

We now establish a set of inequalities relating the irregular independence number to the irregular domination number. These are gathered in the theorem below. In the proof, we need to use the following more precise notation (see Chapter 1). Recall that for a vertex v of a graph G, we will denote the set of neighbours of v in G by $N_G(v)$, and the degree of v in G by $d_G(v)$. Formally, $N_G(v) = \{w \in V(G) \colon vw \in E(G)\}$ and $d_G(v) = |N_G(v)|$.

Theorem 7.3.1. *For any graph G on n vertices, the following assertions hold:*

(i) $\alpha_{ir}(G) + \gamma_{ir}(G) \leq n + 1$ *if* $\delta(G) = 0$, *and* $\alpha_{ir}(G) + \gamma_{ir}(G) \leq n$ *if* $\delta(G) \geq 1$.

(ii) $\alpha_{ir}(G)\gamma_{ir}(G) \leq \lfloor \frac{n+1}{2} \rfloor \lceil \frac{n+1}{2} \rceil$ *if* $\delta(G) = 0$, *and* $\alpha_{ir}(G)\gamma_{ir}(G) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ *if* $\delta(G) \geq 1$.

(iii) $\alpha_{ir}(G) + \gamma_{ir}(\bar{G}) \leq n + 1$.

(iv) $\alpha_{ir}(G)\gamma_{ir}(\bar{G}) \leq \lfloor \frac{n+1}{2} \rfloor \lceil \frac{n+1}{2} \rceil$.

Moreover, the bounds are sharp.

Proof. Let A be an irregular independent set of G of size $\alpha_{ir}(G)$, and let $D = V(G)\backslash A$. Let $\delta = \delta(G)$.

Suppose $\delta \geq 1$. Then D is an irregular dominating set of G, so $\alpha_{ir}(G) + \gamma_{ir}(G) \leq |A| + |D| \leq n$ and $\alpha_{ir}(G)\gamma_{ir}(G) \leq |A||D| = |A|(n - |A|) \leq \lfloor \frac{n}{2} \rfloor (n - \lfloor \frac{n}{2} \rfloor) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ (as in the proof of Theorem 6.2.3(iii)). Now suppose $\delta = 0$. Let V_0 be the set of vertices of G of degree 0, and let V_1 be the set of vertices of G of degree at least 1. Clearly, A has exactly one element x of V_0, and $D \cup \{x\}$ is an irregular dominating set of G. As in the case $\delta \geq 1$, $\alpha_{ir}(G[V_1]) + \gamma_{ir}(G[V_1]) \leq |V_1|$. We have $\alpha_{ir}(G) + \gamma_{ir}(G) = (\alpha_{ir}(G[V_1]) + 1) + (\gamma_{ir}(G[V_1]) + |V_0|) \leq |V_0| + |V_1| + 1 = n + 1$ and $\alpha_{ir}(G)\gamma_{ir}(G) \leq |A|(|D| + 1) \leq |A|(n + 1 - |A|) \leq \lfloor \frac{n+1}{2} \rfloor (n + 1 - \lfloor \frac{n+1}{2} \rfloor) = \lfloor \frac{n+1}{2} \rfloor \lceil \frac{n+1}{2} \rceil$. Hence (i) and (ii).

Let $v_1, \dots, v_t$ be the distinct vertices in A, where $d_G(v_1) < \cdots < d_G(v_t)$. We have $d_G(v_t) \leq |V(G)\backslash A| = n - t$. For each $i \in [t]$, let $a_i = |N_{\bar{G}}(v_i) \cap D|$. For each $i \in [t]$, $a_i = n - t - d_G(v_i) \geq n - t - d_G(v_t)$. Thus, if $d_G(v_t) \leq n - t - 1$, then D is an irregular dominating set of $\bar{G}$, and hence $\alpha_{ir}(G) + \gamma_{ir}(\bar{G}) \leq |A| + |D| = t + (n-t) = n$. Suppose $d_G(v_t) = n - t$. We have $a_i \geq 1$ for each $i \in [t-1]$. Let $A' = A \backslash \{v_t\}$. Let $D' = D \cup \{v_t\}$. For each $i \in [t-1]$, let $b_i = |N_{\bar{G}}(v_i) \cap D'|$. For each $i \in [t-1]$, we have $N_{\bar{G}}(v_i) \cap D' = (N_{\bar{G}}(v_i) \cap D) \cup \{v_t\}$, so $b_i = a_i + 1 = n - t - d_G(v_i) + 1$. Thus, D' is an irregular dominating set of $\bar{G}$. Consequently, $\alpha_{ir}(G) + \gamma_{ir}(\bar{G}) \leq |A| + |D'| = t + (n-t+1) = n+1$ and $\alpha_{ir}(G)\gamma_{ir}(\bar{G}) \leq |A||D'| = t(n+1-t) \leq \lfloor \frac{n+1}{2} \rfloor (n+1-\lfloor \frac{n+1}{2} \rfloor) = \lfloor \frac{n+1}{2} \rfloor \lceil \frac{n+1}{2} \rceil$. Hence (iii) and (iv).

We now show that the bounds are sharp. We use constructions similar to that in the proof of Theorem 7.2.1.

Let $k = \lceil \frac{n}{2} \rceil$ and $n' = n - k$. Suppose that G is constructed as follows: let $u_1, \dots, u_k, v_1, \dots, v_{n'}$ be the distinct vertices of G, and, for each $i \in [n']$, let v_i be adjacent to exactly $k - i + 1$ of the vertices $u_1, \dots, u_k$. Clearly, $\delta \geq 1$. Also, $\{v_1, \dots, v_{n'}\}$ is an irregular independent set, and, by Theorem 6.2.1, it is of maximum size. Moreover, $\{u_1, \dots, u_k\}$ is an irregular dominating set of G, and, by Theorem 7.2.1, it is of minimum size. Thus, $\alpha_{ir}(G) + \gamma_{ir}(G) = n' + k = n$ and $\alpha_{ir}(G)\gamma_{ir}(G) = n'k = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. Now suppose that we instead have that $k = \lceil \frac{n-1}{2} \rceil$, $n' = n - k$, and, for each $i \in [n']$, v_i is adjacent to exactly $i - 1$ of $u_1, \dots, u_k$. Since $d(v_1) = 0$, $\delta = 0$. Similarly to the above, $\{u_1, \dots, u_k, v_1\}$ is an irregular dominating set of G of minimum size as $\{u_1, \dots, u_k\}$ is an irregular dominating set of $G - v_1$ of minimum size. Also, $\{v_1, \dots, v_{n'}\}$ is an irregular independent set of maximum size. Thus, $\alpha_{ir}(G) + \gamma_{ir}(G) = n' + k + 1 = n + 1$ and $\alpha_{ir}(G)\gamma_{ir}(G) = n'(k+1) = \lfloor \frac{n+1}{2} \rfloor \lceil \frac{n+1}{2} \rceil$. We have established that (i) and (ii) are sharp.

Let $k = \lceil \frac{n-1}{2} \rceil$ and $n' = n - k$. Suppose that G is constructed as follows:

let $u_1, \ldots, u_k, v_1, \ldots, v_{n'}$ be the distinct vertices of G, and, for each $i \in [n']$, let v_i be adjacent to exactly $k - i + 1$ of the vertices $u_1, \ldots, u_k$. Thus, $\{v_1, \ldots, v_{n'}\}$ is an irregular independent set, and, by Theorem 6.2.1, it is of maximum size (note that δ is $d(v_{n'})$, which is 0 if n is odd, and 1 if n is even). Also, we clearly have that $\{u_1, \ldots, u_k, v_1\}$ is an irregular dominating set of $\bar{G}$, and it is of minimum size because $d_{\bar{G}}(v_1) = 0$ and, by Theorem 7.2.1, $\{u_1, \ldots, u_k\}$ is an irregular dominating set of $\bar{G} - v_1$ of minimum size. Thus, $\alpha_{ir}(G) + \gamma_{ir}(\bar{G}) = n' + k + 1 = n + 1$ and $\alpha_{ir}(G)\gamma_{ir}(\bar{G}) = n'(k+1) = \lfloor \frac{n+1}{2} \rfloor \lceil \frac{n+1}{2} \rceil$. □

7.4 Nordhaus–Gaddum-type results

In this section, we provide results of Nordhaus–Gaddum type [45] for the irregular domination number. We shall use the notation introduced in the preceding section.

Theorem 7.4.1. *If G is a graph on $n \geq 2$ vertices, then*

(i) $2\lceil \frac{n}{2} \rceil \leq \gamma_{ir}(G) + \gamma_{ir}(\bar{G}) \leq 2n - 1$,

(ii) $(\lceil \frac{n}{2} \rceil)^2 \leq \gamma_{ir}(G)\gamma_{ir}(\bar{G}) \leq n(n-1)$.

Moreover, the following assertions hold:

(a) The bounds are attainable for any $n \geq 3$.

(b) For each of (i) and (ii), the upper bound is attained if and only if G is empty or complete.

Proof. By Theorem 7.2.1, $\gamma_{ir}(G) \geq \lceil \frac{n}{2} \rceil$ and $\gamma_{ir}(\bar{G}) \geq \lceil \frac{n}{2} \rceil$. The lower bounds in (i) and (ii) follow immediately. If G is empty, then $\bar{G}$ is complete, so $\gamma_{ir}(G) + \gamma_{ir}(\bar{G}) = n + n - 1 = 2n - 1$ and $\gamma_{ir}(G)\gamma_{ir}(\bar{G}) = n(n-1)$. If G is complete, then $\bar{G}$ is empty, so $\gamma_{ir}(G) + \gamma_{ir}(\bar{G}) = 2n - 1$ and $\gamma_{ir}(G)\gamma_{ir}(\bar{G}) = (n-1)n$. If G is neither empty nor complete, then $\bar{G}$ is non-empty, and hence, by Theorem 7.2.4, $\gamma_{ir}(G) + \gamma_{ir}(\bar{G}) \leq 2(n-1) < 2n - 1$ and $\gamma_{ir}(G)\gamma_{ir}(\bar{G}) \leq (n-1)^2 < n(n-1)$.

It remains to show that the lower bounds in (i) and (ii) are attainable for any $n \geq 3$.

Suppose first that n is odd. Let $k = \frac{n-1}{2}$. Suppose that G is constructed as follows: let $u_1, \dots, u_k, v_1, \dots, v_{k+1}$ be the distinct vertices of G, and, for each $i \in [k]$, let u_i be adjacent to $v_1, \dots, v_i$. Clearly, $\{v_1, \dots, v_{k+1}\}$ is an irregular dominating set of G and of $\bar{G}$. Thus, $\gamma_{ir}(G) + \gamma_{ir}(\bar{G}) \geq 2(k+1) = 2\lceil \frac{n}{2} \rceil$ and $\gamma_{ir}(G)\gamma_{ir}(\bar{G}) \geq (k+1)^2 = \lceil \frac{n}{2} \rceil^2$. By (i) and (ii), we actually have $\gamma_{ir}(G) + \gamma_{ir}(\bar{G}) = 2\lceil \frac{n}{2} \rceil$ and $\gamma_{ir}(G)\gamma_{ir}(\bar{G}) = \lceil \frac{n}{2} \rceil^2$.

Now suppose that n is even and $n \geq 8$. Let $k = \frac{n}{2}$. Suppose that $V(G) = \{u_1, \dots, u_k, v_1, \dots, v_k\}$ and that, for each $i \in [k] \backslash \{2\}$, u_i is adjacent to $v_1, \dots, v_i$, u_2 is adjacent to v_2 and v_3, v_2 is adjacent to $v_4, \dots, v_k$, v_3 is adjacent to $v_4, \dots, v_k$, and there are no other adjacencies. Let $A = \{v_1, \dots, v_k\}$ and $B = \{u_1, u_k, v_1, v_4, \dots, v_k\}$. Clearly, A is an irregular dominating set of G. Let $w_1 = v_3$, $w_2 = v_2$, $w_3 = u_{k-1}, w_4 = u_{k-2}, \dots, w_k = u_2$. Thus, $V(G) \backslash B = \{w_1, \dots, w_k\}$. Note that $|N_{\bar{G}}(w_i) \cap B| = i$ for each $i \in [k]$. Thus, B is an irregular dominating set of $\bar{G}$. Therefore, we have $\gamma_{ir}(G) \geq |A| = k$ and $\gamma_{ir}(\bar{G}) \geq |B| = k$, and hence the lower bounds in (i) and (ii) are attained.

Suppose that $n = 6$, $u_1, u_2, u_3, v_1, v_2, v_3$ are the vertices of G, and $\{u_1, v_1\}$,

$\{u_2, v_2\}$, $\{u_2, v_3\}, \{u_3, v_1\}, \{u_3, v_2\}, \{u_3, v_3\}$ are the edges of G. Clearly, $\{v_1, v_2, v_3\}$ is an irregular dominating set of G, and $\{u_1, v_1, v_3\}$ is an irregular dominating set of $\bar{G}$. Thus, the lower bounds in (i) and (ii) are attained.

Finally, suppose that $n = 4$ and G is the path $P_4 = ([4], \{\{1, 2\}, \{2, 3\}, \{3, 4\}\})$. Then $\{1, 3\}$ is an irregular dominating set of G, and $\{1, 2\}$ is an irregular dominating set of $\bar{G} = ([4], \{\{2, 4\}, \{4, 1\}, \{1, 3\}\})$. Thus, the lower bounds in (i) and (ii) are attained. $\square$

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