Smallness and Infinity in Mathematics

An Intuitive Approach

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The widespread availability of pocket calculators has widened the domain of possible investigations at certain levels in mathematics, in the secondary school classroom no less than in the university lecture-room. Most school boys and girls use the calculator and many of them may have asked what is the meaning of the letter e which appears on the key e^x. This article is intended to explain the background which is necessary to understand the meaning of *e* and it is hoped that the numerical value of 2.718 assigned to *e* will no longer remain mysterious to readers who are non mathematicians. A knowledge of the realistic fractions e.g. that $1/6$ is less than $1/4$, is the kind of mathematics required to follow the discussion.

Section I

Adding the endless

Consider the sums S of the successive distances shown in Fig. 1. We use S_n to mean the sum of n successive distances starting with 1. Each distance is called a *term* of the series. We illustrate the first few series:-

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{4} \qquad = 1 + \frac{1}{2} + \frac{1}{2^2} = 1\frac{3}{4}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 1\frac{7}{8}$$

$$S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = 1\frac{15}{16}$$

What is the value of S_6? That 's easy. What is S_{10}? Now wait a minute. That's too long to write down. So we look for a shorter way of explaining S_n. Both from Fig. 1. and from the above illustrations of S_1, S_2, S_3, etc, it is clear that the increasing value of S_n is getting close to 2. In fact as one takes more and more terms (i.e. an endless number of them), S_n gets nearer and nearer to 2. See Fig. 2.

Also the difference between 2 and S_6 is:

$$2 - S_6 = \frac{1}{32} = \frac{1}{2^5}$$

The difference between 2 and S_7 is:

$$2 - S_7 = \frac{1}{64} = \frac{1}{2^6}$$

Therefore S_{10} is explained by the statement:

$$2 - S_{10} = \frac{1}{2^9} = \frac{1}{512} \qquad [1]$$

Without computing what is the difference between 2 and S_{15}?

$$2 - S_{15} = \frac{1}{2^{14}}$$

and we may leave it at that.

Next let's ask the followng questions and provide answers first in mathematical language and then in mathematical symbols.

(i) Do you think it is possible to find a large enough n so that the difference between 2 and S_n is less than $1/1000$?

Yes. We have almost done it in statement [1] because.

$$2 - S_n = \frac{1}{512}, \text{ where } n = 10.$$

Now, try n = 11,

$$2 - S_{11} = \frac{1}{2^{10}} = \frac{1}{1024}, \text{ which is less than } \frac{1}{1000}.$$

Therefore for all whole number values of n larger than 10 (i.e. n = 11, 12, 13, etc) the difference between 2 and S_n will be less than $1/1000$. In mathematical symbols this statement becomes.

$$2 - S_n < \frac{1}{1000} \text{ for } n > 10.$$

(ii) Consider

$$2 - S_n < \frac{1}{100{,}000}$$

This statement asks us to find a large enough n so that the difference between 2 and S_n is less than $1/1000\ 000$. Is it possible?

Yes. Try n = 21. Remember that $2 - S_{21} = \frac{1}{2^{20}}$

From the above two statements, we may conclude that:

"no matter how small we wish the difference between 2 and S_n to be, there is a whole number n that will do it".

Alternatively we may state the same conclusion in the following way:

"it is always possible to find a value for n large enough so that S_n is as near to 2 as we wish".

Since this is the case, then

"the *limiting value* of S_n as n *becomes very large* is 2" ----- [2]

In mathematical language statement [2] is written thus.

"the limit of S_n as n approaches infinity is 2" and using mathematical symbols, statement [2] is $\lim_{n\to\infty} S_n = 2$ ...... [*]

So we have established the important result that:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = 2,$$

or, what is the same thing:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots = 2$$

These three dots (and not more than three!) at the end of the series indicate that the number of terms is endless or infinite. As more and more terms are added the difference between 2 and S_n becomes smaller and smaller. Thus we have found the sum of an endless series of terms. Finally, to show more forcefully that an infinity of terms of the series will make the sum exactly 2, we write:

$$S_\infty = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = 2 \qquad [3]$$

The symbol ∞ stands for an infinite number of terms of the series S.

Section II

The Meaning and Value of *e*.

We consider another series S'_n given by:

$$S'_n = 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \cdots + \frac{1}{1.2.3\ldots n}\cdots$$

where the dots in the denominators imply multiplication. If we take an infinite number of terms, the above series is written thus:

$$S'_\infty = 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots$$

For simplicity (and economy in writing) we replace:

1.2 by 2! (read: 2 factorial)
1.2.3 by 3! (read: 3 factorial)

and so on, then:

1.2.3...n = n! (read: n factorial)

Therefore S'_n and S'_∞ now take the following forms:

$$S'_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \cdots \frac{1}{n!}\ .$$

$$\text{and } S'_\infty = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

Our next exercise is to compare the magnitude of S'_n given above with the magnitude of S_n of Section I. To examine the comparison more clearly we rewrite S_n and S'_n in their proper values:

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \text{ etc.}$$

$$S'_n = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \text{ etc.}$$

These two series begin to differ from the third term onwards. We observe that.

$^1/_6$ is less than $^1/_4$,
$^1/_{24}$ is less than $^1/_8$,
$^1/_{120}$ is less than $^1/_{16}$ etc.

We say that S'_n is less than S_n by comparing their terms. And if infinity of terms be taken we will have the result:

$S'_\infty < S_\infty$, the latter seen beeing equal to [*]. Thus.

$$S'_\infty = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < 2.$$

If we add 1 to S'_∞ we get

$$(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots) < 3$$

The endless sum in brackets is clearly greater than 2. It is denoted by the letter *e*.

Therefore:

$$2 < e < 3$$

My calculator gives *e* = 2.718 281828. The value 2.718 is sufficiently accurate for most purposes. The letter *e* stands for the word *exponential*. In mathematics beyond O Level a more general series is studied, namely.

$e^x = 1 + x + x^2/_2! + x^3/_3! + \ldots$ which is valid for all values of x. We have discussed only the particular case of e^x namely when x = 1,

i.e. $e^1 (= e) = 2.718$

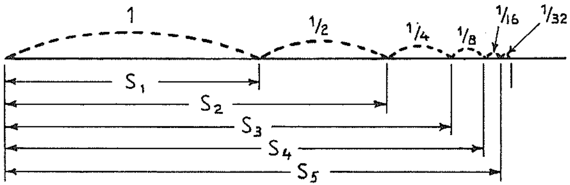

Fig. 1

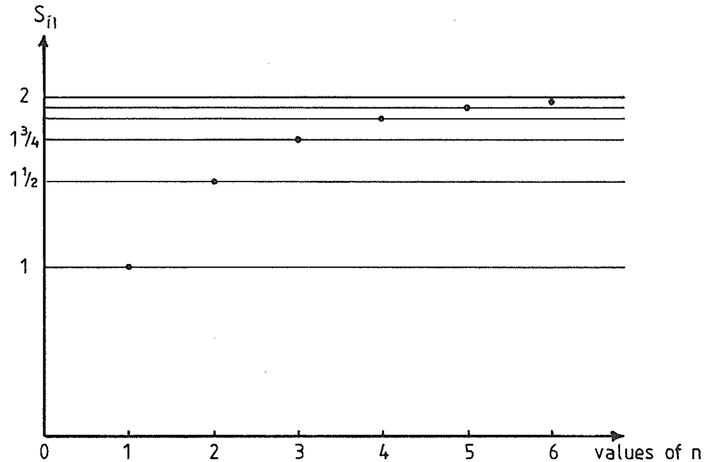

Fig. 2