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Subspaces of ordinals

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A B S T R A C T

It was recently proved that each subspace of an ordinal space is also orderable. The present note aims to give a simple proof of this fact.

1. Introduction

A space X is *orderable* (or, *linearly ordered*) if the topology of X coincides with the open interval topology on X generated by a linear ordering on X. In particular, X is an *ordinal space* if it is an ordinal equipped with the open interval topology. Subspaces of orderable spaces are not necessarily orderable, they are called *suborderable* (or, *generalised ordered*). However, in case of ordinal spaces the following result was obtained by Hirata and Kemoto.

Theorem 1.1. *([4]) Each subspace of an ordinal space is orderable.*

It was remarked by the reviewer of [4] that Theorem 1.1 is a special case of a more general result of Purisch [5,6] for orderability of scattered spaces. Subsequently, a shorter proof was offered in [1]. At the end of their paper, the authors of [1] conjectured that there might be even a shorter proof utilising van Dalen and Wattel's characterisation of orderable spaces [7]. While this remains open, the aim of the present note is to give another simple proof of Theorem 1.1 that is fairly straightforward; its idea is briefly explained below. Let X be a subspace of an ordinal space and $\preccurlyeq$ be the linear ordering on X inherited from the ordinal. Then each point of X is contained in an *order-convex* set of the ordered set $(X, \preccurlyeq)$ that is *maximal* with respect to the property of being an orderable space with respect to $\preccurlyeq$ (as a subspace of X), see Proposition 2.2. Such a set is not as arbitrary as it might seem at first. On the one hand, it must have a first element because X is well-ordered with respect to $\preccurlyeq$. On the other hand, if it has a last element, then that element must be also the last element of X. All these order-convex sets form a partition $\mathscr{P}$ of X consisting of orderable

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subspaces of X; all sets of this partition have first elements and possibly only one of them may have a last element. The proof now can be accomplished along the lines of the idea of [3, Theorem 4.2] for rearranging partitions of orderable subspaces, see Theorem 3.1.

2. Orderable subspaces

For a linearly ordered set $(X \preccurlyeq)$ and points $x, y \in X$, we will use the standard notation for the intervals of $(X, \preccurlyeq)$, namely $(x, \rightarrow)_{\preccurlyeq}$, $(\leftarrow, y)_{\preccurlyeq}$, $(x, y)_{\preccurlyeq}$, $(x, y]_{\preccurlyeq}$, etc. For subsets $S, T \subset X$ we will write $S \prec T$ if $x \prec y$ for every $x \in S$ and $y \in T$; similarly, $S \preccurlyeq T$ means that $x \preccurlyeq y$ for every $x \in S$ and $y \in T$. If one of these sets is a singleton, say $T = \{y\}$, we will simply write $S \prec y$ or $S \preccurlyeq y$. A subset $S \subset X$ is called *order-convex* if $[x, y]_{\preccurlyeq} \subset S$ for every $x, y \in S$ with $x \preccurlyeq y$. E. Čech [2] was the first to prove that a space X is suborderable if and only if it admits a linear ordering such that the corresponding open interval topology is coarser than the topology of X and X has a base of order-convex sets with respect to this order. In the sequel, every such linear ordering $\preccurlyeq$ on X will be called *compatible* and we will often say that X is suborderable with respect to $\preccurlyeq$.

Recall that a pair (B, A) of subsets of an ordered set $(X, \preccurlyeq)$ is a *cut* if $X = B \cup A$, $B \neq \varnothing \neq A$ and $B \prec A$. A cut (B, A) is a *jump* if B has a last element and A has a first one; (B, A) is a *gap* if B has no last element and A has no first one. It is evident that in an orderable space, each cut consisting of clopen sets (i.e., a *clopen cut*) must be either a jump or a gap. Since each nonempty proper clopen set of a suborderable space generates a clopen cut, the following characterisation of orderability of suborderable spaces holds; it was explicitly stated in [3, Lemma 6.4], and also follows from the Čech construction of embedding a suborderable space into an orderable one [2].

Proposition 2.1. *A suborderable space X is orderable with respect to a compatible linear ordering on it if and only if each of its clopen cuts is either a gap or a jump.*

Let X be suborderable with respect to a linear ordering $\preccurlyeq$ on it. Then each subspace of X is also suborderable with respect to $\preccurlyeq$, but some subspaces may happen to be orderable with respect to $\preccurlyeq$. In the sequel, every such subspace will be called $\preccurlyeq$*-orderable*. If $Q, R \subset X$ are order-convex sets that are $\preccurlyeq$-orderable and $Q \cap R \neq \varnothing$, then $T = Q \cup R$ is also $\preccurlyeq$-orderable. Namely, given a clopen cut (B, A) of T, it defines a clopen cut of one of the sets Q or R. Since both sets are order-convex, the type of the cut is preserved. Since both are $\preccurlyeq$-orderable, the statement follows by Proposition 2.1. More generally, suppose that $\mathscr{Q}$ is a family of order-convex sets, with $\bigcap \mathscr{Q} \neq \varnothing$, and each member of $\mathscr{Q}$ is $\preccurlyeq$-orderable. Then each cut of $\bigcup \mathscr{Q}$ defines a cut of the union of two members of $\mathscr{Q}$. Hence, the following result holds.

Proposition 2.2. *If X is a suborderable space with respect to a linear ordering $\preccurlyeq$ on it and $\mathscr{Q}$ is a family of order-convex sets of X such that $\bigcap \mathscr{Q} \neq \varnothing$ and each member of $\mathscr{Q}$ is $\preccurlyeq$-orderable, then $\bigcup \mathscr{Q}$ is also $\preccurlyeq$-orderable.*

We conclude this section with the following crucial observation related to subspaces of ordinal spaces.

Proposition 2.3. *Let X be a subspace of an ordinal space, and $\preccurlyeq$ be the order on X inherited from the ordinal. Then X has a partition $\mathscr{P} = \{P_\alpha\colon\ \alpha < \tau\}$ consisting of closed $\preccurlyeq$-orderable subspaces of X such that*

(a) *$P_\alpha \prec P_\beta$ whenever $\alpha < \beta < \tau$,*
(b) *each P_α, $\alpha + 1 < \tau$, has a first element but no last one,*
(c) *P_β is clopen whenever $\beta < \tau$ is a successor ordinal.*

Proof. Each point $x \in X$ is contained in an order-convex set of $(X, \preccurlyeq)$ which is $\preccurlyeq$-orderable, for instance the singleton $\{x\}$. Let P_x be the maximal order-convex set with this property, which exists by Proposition 2.2. The maximality of P_x implies that it is also closed. Indeed, if $y \in \overline{P_x} \setminus P_x$, then $P_x \prec y$ because P_x is order-convex and has a first element being well-ordered with respect to $\preccurlyeq$. Since $P_x \cup \{y\}$ is not $\preccurlyeq$-orderable, by Proposition 2.1, it must have a clopen cut that is neither a jump nor a gap. Since P_x is orderable, this cut must be $(P_x, \{y\})$. In particular, y will be an isolated point of $P_x \cup \{y\}$ which contradicts the assumption that $y \in \overline{P_x}$. Thus, P_x is closed. The resulting partition $\mathscr{P} = \{P_x\colon x \in X\}$ of X is as required. Since $P \prec Q$ or $Q \prec P$ for every pair of distinct elements $P, Q \in \mathscr{P}$, it follows that $\mathscr{P}$ is itself well-ordered by $\preccurlyeq$. Hence, it can be represented as in (a), namely $\mathscr{P} = \{P_\alpha\colon \alpha < \tau\}$ for some ordinal τ such that $P_\alpha \prec P_\beta$ whenever $\alpha < \beta < \tau$. Each P_α has a first element. If $\alpha + 1 < \tau$ and P_α has a last element, then the clopen cut $(P_\alpha, P_{\alpha+1})$ of $P_\alpha \cup P_{\alpha+1}$ will be a jump. Hence, $P_\alpha \cup P_{\alpha+1}$ will be an orderable space with respect to $\preccurlyeq$ (by Proposition 2.1), but it will contradict the maximality of these sets. Thus, (b) holds as well. Finally, suppose that $\beta = \alpha + 1$ is a successor ordinal. Just like before, (P_α, P_β) is a clopen cut of $P_\alpha \cup P_\beta$ because $P_\alpha \cup P_\beta$ is not $\preccurlyeq$-orderable, and so is the cut (B, A) of X where $B = \{x \in X\colon x \prec P_\beta\}$ and $A = \{x \in X\colon P_\alpha \prec x\}$. If $\beta + 1 = \tau$, then $P_\beta = A$. If $\beta + 1 < \tau$ and $x \in P_\beta$, then $P_\beta \cup (\leftarrow, x)_\preccurlyeq$ is open because P_β has no last element (by (b)), and so is $P_\beta = A \cap (P_\beta \cup (\leftarrow, x)_\preccurlyeq)$. This is (c), and the proof is completed. $\square$

3. Proof of Theorem 1.1

In this section, we finalise the proof of Theorem 1.1 by showing how to rearrange partitions of the type described in Proposition 2.3.

Theorem 3.1. *Let X be a suborderable space with respect to a linear ordering $\preccurlyeq$, and $\mathscr{P} = \{P_\alpha\colon \alpha < \tau\}$ be a closed partition of X such that*

(a) *$P_\alpha \prec P_\beta$ whenever $\alpha < \beta < \tau$,*
(b) *each P_α, $\alpha < \tau$, has a first element but no last one,*
(c) *P_β is clopen whenever $\beta < \tau$ is a successor ordinal.*

Then, X is orderable provided that each P_α, $\alpha < \tau$, is $\preccurlyeq$-orderable.

First of all, let us see that Theorem 1.1 follows by Theorem 3.1. Indeed, suppose that X is a subspace of an ordinal space and $\preccurlyeq$ is the order on X inherited from the ordinal. The case of interest is when X is not orderable with respect to $\preccurlyeq$. In this case, X has a partition $\mathscr{P} = \{P_\alpha\colon \alpha < \tau\}$ as in Proposition 2.3 which clearly consists of at least two sets. In fact, it is the same as the one in Theorem 3.1 with the only difference that P_α may have a last element in case $\alpha + 1 = \tau$. If this may occur, then one can replace P_α with $P_\alpha \cup P_0$ making $P_\alpha \prec P_0$, in particular ensuring that $P_\alpha \cup P_0$ remains orderable by Proposition 2.1. Thus, one can always assume that no member of the partition has a last element. According to Theorem 3.1, X must be orderable.

Turning to the proof of Theorem 3.1, we proceed with a couple of observations the first of which deals with the non-clopen members P_α of $\mathscr{P}$; in particular, showing that the topology in the first elements of such members is uniquely determined by all previous members of the partition. To this end, for an ordered set $(X, \preccurlyeq)$ and a subset $A \subset X$, we will use the intervals

$$(\leftarrow, A)_\preccurlyeq = \{x \in X\colon x \prec A\} \quad \text{and} \quad (A, \rightarrow)_\preccurlyeq = \{x \in X\colon A \prec x\}.$$

If X is suborderable with respect to $\preccurlyeq$ and $A \subset X$ is closed and order-convex, then both intervals $(\leftarrow, A)_\preccurlyeq$ and $(A, \rightarrow)_\preccurlyeq$ are open as can be seen from the fact that $(\leftarrow, A)_\preccurlyeq = (\leftarrow, x)_\preccurlyeq \setminus A$ and $(A, \rightarrow)_\preccurlyeq = (x, \rightarrow)_\preccurlyeq \setminus A$ for any $x \in A$.

Proposition 3.2. *Let Z be a suborderable space, $\preccurlyeq$ be a compatible linear ordering on it, $z \in Z$ be the last element of Z, and $\mathscr{P}$ be a family of order-convex closed subsets of $Z \setminus \{z\}$ such that for every $y \prec z$ there is $P \in \mathscr{P}$, with $y \prec P$. If z is a non-isolated point of Z, then the family $(P, \to)_{\preccurlyeq}$, $P \in \mathscr{P}$, is a local base at z.*

Proof. If $y \prec z$, then $y \prec P$ for some $P \in \mathscr{P}$, and $z \in (P, \to)_{\preccurlyeq} \subset (y, z]_{\preccurlyeq}$. Since $(P, \to)_{\preccurlyeq}$ is open as remarked above, the proof is completed. $\square$

The next statement represents the main idea to rearrange the clopen elements of the partition by reversing the order of each "even" element, for instance; this can be illustrated by the following diagram:

$$\underbrace{\to\to\to\to\cdots}_{\mathscr{P}} \quad \Rightarrow \quad \underbrace{\leftarrow\to\leftarrow\to\cdots}_{\mathscr{P}}$$

Proposition 3.3. *Let τ be an ordinal, $\mathscr{L}(\tau)$ be the set of all limit ordinals of τ, and $t : \mathscr{L}(\tau) \to \{-1, 1\}$ be a function. Then, t can be extended to a function $s : \tau \to \{-1, 1\}$ such that $s(\alpha + 1) = -s(\alpha)$, for every $\alpha < \tau$.*

Proof. Define $s(0)$ in an arbitrary way, say $s(0) = -1$, and proceed by transfinite induction. If $s(\beta)$ is defined for every $\beta < \alpha$, then set $s(\alpha) = t(\alpha)$ provided that α is a limit ordinal. If $\alpha = \beta + 1$, then set $s(\alpha) = -s(\beta)$. $\square$

We are now ready for the proof of Theorem 3.1. In this proof, for a linear ordering $\preccurlyeq$ on a set, we will write $\preccurlyeq^1$ for the order $\preccurlyeq$, and $\preccurlyeq^{-1}$ – for the reverse one.

Proof of Theorem 3.1. Suppose that each P_α, $\alpha < \tau$, is $\preccurlyeq$-orderable and $\preccurlyeq_\alpha$ is the restriction of the order $\preccurlyeq$ on P_α. For the limit ordinals $\mathscr{L}(\tau)$ of τ, define a function $t : \mathscr{L}(\tau) \to \{-1, 1\}$ by letting

$$t(\lambda) = \begin{cases} -1 & \text{if } P_\lambda \text{ is clopen} \\ 1 & \text{if } P_\lambda \text{ is not clopen.} \end{cases} \tag{3.1}$$

Next, let $s : \tau \to \{-1, 1\}$ be as in Proposition 3.3 applied to this particular t. Finally, let $\preccurlyeq_*$ be the lexicographical-like order on X obtained from the order "$<$" on τ and the orders "$\preccurlyeq_\alpha^{s(\alpha)}$", $\alpha < \tau$, on each P_α. Namely, for points on different elements of the partition $\mathscr{P}$, $\preccurlyeq_*$ preserves the order $<$, while on each element P_α, $\alpha < \tau$, it is the same as $\preccurlyeq_\alpha^{s(\alpha)}$. Thus, each P_α, $\alpha < \tau$, remains orderable with respect $\preccurlyeq_*$, and an order-convex set as well. Moreover, the partition $\mathscr{P}$ remains ordered in the same way because $P_\alpha \prec P_\beta$ if and only if $P_\alpha \prec_* P_\beta$. If P_α is not $\preccurlyeq$-clopen, then, by (c), α is a limit ordinal and $s(\alpha) = t(\alpha) = 1$, by (3.1). Thus, P_α has a first element $z = p_\alpha$ with respect to both $\preccurlyeq$ and $\preccurlyeq_*$ which is a non-isolated point of $Z = (\leftarrow, z]_{\preccurlyeq_*} = (\leftarrow, z]_{\preccurlyeq}$. By Proposition 3.2, the family $(P_{\beta+1}, \to)_{\preccurlyeq_*} \cap Z = (P_{\beta+1}, \to)_{\preccurlyeq} \cap Z$, $\beta < \alpha$, is a local base at z in Z because all sets $P_{\beta+1}$, $\beta < \alpha$, are clopen with respect to $\preccurlyeq$, see (c). Thus, keeping clopen with respect to $\preccurlyeq_*$ all those sets P_α, $\alpha < \tau$, that are clopen with respect to $\preccurlyeq$, we get that X is also suborderable with respect to $\preccurlyeq_*$. We are going to show that X is, in fact, orderable with respect to $\preccurlyeq_*$. To this end, take a clopen (in X) cut (B, A) of $(X, \preccurlyeq_*)$. If there exists an $\alpha < \tau$ such that $B \cap P_\alpha \neq \varnothing \neq P_\alpha \cap A$, then (B, A) defines a clopen cut $(B \cap P_\alpha, P_\alpha \cap A)$ of P_α. Since P_α is a $\preccurlyeq_\alpha^{s(\alpha)}$-orderable subset of X, by Proposition 2.1, the cut must be either a jump or a gap, hence so is (B, A). Otherwise, if $P_\alpha \subset B$ or $P_\alpha \subset A$ for every $\alpha < \tau$, let $\gamma = \min\{\alpha < \tau\colon\ P_\alpha \subset A\}$. If $\gamma = \beta + 1$ is a successor ordinal, then $B = \bigcup\{P_\alpha\colon\ \alpha \leqslant \beta\}$, and the type of the cut (B, A) is determined by P_β and P_γ. Since $s(\gamma) = s(\beta + 1) = -s(\beta)$, by (b), P_β has a last element with respect to $\preccurlyeq_* = \preccurlyeq_\beta^{s(\beta)}$ if and only if P_γ has a first element with respect to $\preccurlyeq_* = \preccurlyeq_\gamma^{s(\gamma)}$. Thus, (B, A) is either a jump or a gap. If γ is a limit ordinal, then P_γ is clopen because so is A. Hence, by (3.1), $s(\gamma) = t(\gamma) = -1$ and, by (b), P_γ has no first element with respect to $\preccurlyeq_*$. Consequently, (B, A) forms a gap in $(X, \preccurlyeq_*)$. Thus, in each of these cases, (B, A) is either a jump or a gap and, by Proposition 2.1, X is orderable with respect to $\preccurlyeq_*$. $\square$

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