The Determination of the Mechanical Parameters of the General Uniform Polygonal Lamina

Victor CILIA-VINCENTI, B.Sc.(Hons)
Mathematics Department, University of Malta Junior College, Malta
E-mail:victor.cilia-vincenti@um.edu.mt

Abstract: *The motion of rigid bodies in fields of force depends on their mass, mass-centre location and moments of inertia. These fundamental quantities, here referred to as* mechanical parameters, *are determined for the general uniform polygonal lamina. The precise description of polygons through* border vectors *is followed by the determination of the mechanical parameters of elemental triangular laminae. Introducing a general function of area,* m-area, *effectively integrates the separate parameters so that the study's central theorem, extending the results of triangular laminae, involves only m-area. As the worked example illustrates, the theorem permits easy evaluation of numerical results and lends itself well for computerization purposes.*

Keywords: *applied mathematics, mechanics, area, centre of mass, moment of inertia, polygon, lamina.*

1. Defining and Describing the General Polygonal Lamina

The shape of the **general polygonal lamina** considered here is that of *a plane figure with 3 or more straight sides which form the edges of its outer boundary together with a number of straight sides which form the edges of its hole or holes if any*. The **mass per unit area** of the lamina is taken to be 1 so that *mass and area are numerically equal.*

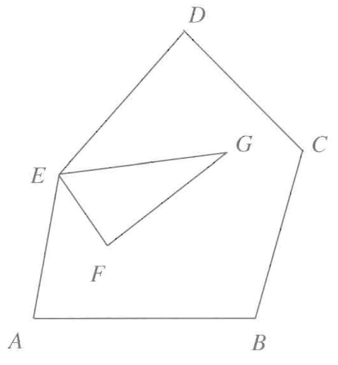


Fig. 1. *A, B, C, D, F, G* are *double vertices* while *E*, the intersection of the outer boundary and hole *EFG*, is a *multiple vertex.*

When only 2 straight sides or edges of the lamina meet at a point we may refer to the point as a **double vertex**. In more complex shapes more than two sides may meet at a point. Each of these points will be referred to as a **multiple vertex**. Closer examination reveals that a multiple vertex is a point where 2 or more double vertices meet and occurs, say, when a hole meets the outer boundary of the polygon. It also arises at a vertex where two or more holes meet which may also be a point on the outer boundary of the polygon.

Clearly an adequate description of the shape of the polygon has to go beyond giving the position of its vertices as a number of polygons may be drawn from a given set of coplanar points. If, however, besides giving the position of the vertices, say A, $B, \ldots \ldots, G$, a description of the sides of the figure is given in terms of its vertices the description should define the shape of the lamina completely. Thus for the polygon in the adjoining diagram, (Fig. 1.) the edges

$$AB,\ BC,\ CD,\ DE,\ EF,\ FG,\ GE$$

define the polygon completely once the position of the vertices is known.

When the pairs of endpoints of each side are given, where possible in order, they must necessarily consist of a number of loops of the form

$$\begin{array}{l} A_1A_2,\ A_2A_3, \ldots,\ A_{i-1}A_i\,; \\ A_{h+1}A_{h+2},\ A_{h+2}A_{h+3}, \ldots,\ A_{j-1}A_j\,; \\ \vdots \\ \vdots \\ A_{m+1}A_{m+2},\ A_{m+3}A_{m+4}, \ldots,\ A_{n-1}A_n \end{array}$$

where one of the loops corresponds to the contour of the outer boundary of the polygon and each of the remaining loops refer to the inner boundaries of the polygon or the edges of the holes. Of course it is not immediately obvious, especially in cases where there are multiple vertices, which one of the loops refers to the outer boundary of the polygon and which to the holes. The method used here to describe the shape of a given polygon should make things a little simpler to interpret.

Border Vectors and Border Vector Sets

A **border vector** *of the general polygon is a vector along an external or internal edge of the polygon, from one end point of a straight edge to its other end, whose sense is always* ***anti-clockwise*** *relative to an internal point of the polygon close to the vector.*

Thus if A and B are the end points of a straight edge of the polygon and O is any point inside the polygon close to the edge AB such that the sense of OAB is anti-clockwise (*see* Fig. 2) then vector **AB** is the algebraic representation of the corresponding border vector.

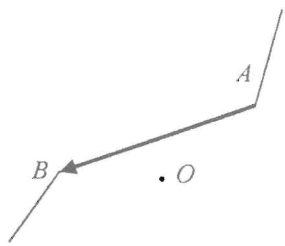


Fig. 2. The direction of the Border Vector **AB** is such that the sense of OAB, where O is an internal point of the lamina, is anti-clockwise.

The **border vector set** *of the general polygonal lamina is the set of the border vectors which correspond to each outer and inner side or edge of the polygon.* The following **properties concerning border vectors and border vector sets** follow almost immediately from the definitions.

Properties concerning Border Vectors and Border Vector Sets

1. If Q is a double vertex where the edges PQ and QR meet, then if **PQ** is the border vector for edge PQ the border vector for edge QR must be **QR** **(1.1)**

2. If Q is a multiple vertex then
(a) the number of edges having Q as one of the terminal points must be **even**.
(b) adjoining border vectors, partitioning the small region around Q, must have Q alternating as an initial point and an end point. **(1.2)**

3. The border vector set of the general polygonal lamina with h straight-edged holes consists of $h+1$ subsets with each subset consisting of a number of border vectors which may be expressed in order algebraically with the terminal point of one border vector being the initial point of the next and the terminal point of the last border vector being the initial point of the first. **(1.3)**

Property 1 is a direct consequence of the definition of a border vector. For any internal point O on the lamina close to Q, the fact that the sense of OPQ is anti-clockwise (*see* Fig. 3(a)) implies that the sense of OQR must also be anti-clockwise. Alternatively, if **QP** is the border vector for edge PQ then for the same reason as before the border vector for edge QR must be **RQ**. (*see* Fig. 3(b))

In the case of **Property 2** consider the small region enclosing multiple vertex Q (*see* Fig. 4.). This region neighbouring Q is partitioned into a number of sections equal in number to the edges meeting at Q. Now, obviously, no 2 adjoining sections can both be part of the polygon as otherwise the dividing line of the 2 adjoining sections will not be an edge of the polygon. Likewise no 2 adjoining sections may both be regions outside the polygon or part of the holes. This implies property 2(a).

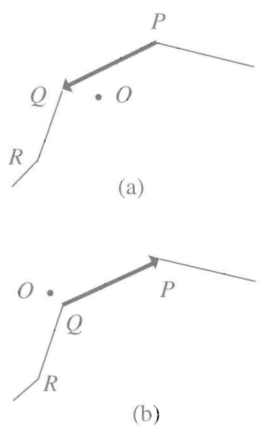


Fig. 3 (a). If the Border Vector for PQ is **PQ** then the Border Vector for QR must be **QR**.

(b) If the Border Vector for PQ is **QP** then the Border Vector for QR must be **RQ**.

Also if PQ and QR are any 2 adjoining edges and O is a point close to Q in the section with the edges PQ and QR then if O is an internal point of the polygon the respective border vectors will be **PQ** and **QR**. On the other hand if O happens to be outside the polygon the respective border vectors will be **RQ** and **QP**. This leads to property 2(b). Combining properties 1 and 2 in turn leads to property 3.

Geometrically one of the $(h+1)$ subsets of border vectors forms a directed loop which traces in order the outline of the outer boundary of the lamina and each of the remaining h subsets forms a directed loop which traces, in order, the outline of one of the polygon holes.

Border vector sets may be expressed algebraically with the different subsets of vectors within curly brackets and separated from each other by semi-colons. The first subset consists of a number of border vectors in order, separated by commas, and describing the outer boundary of the polygon. Similarly for the subsequent subsets which describe each of the polygon holes.

The geometrical description, by border vectors, of the polygonal lamina P (Fig. 5(a)) is shown in the adjoining figure (Fig. 5(b)). The border vectors give an indication of the inside and outside regions of the polygon.

To describe P algebraically let the points with coordinates (2,1), (8,1), (6,2), (6,4), (5,6), (5,2) and (5,4) be A, B, C, D, E, F and G respectively. Then the polygonal lamina P may be described by the border vector set:

$$\{\mathbf{AB}, \mathbf{BC}, \mathbf{CD}, \mathbf{DE}, \mathbf{EA}; \mathbf{CF}, \mathbf{FG}, \mathbf{GC}\}$$

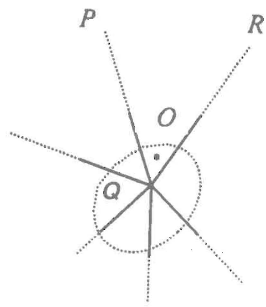


Fig. 4. The number of edges meeting at Q must be even and adjacent border vectors must have Q alternating between an initial point and an end point

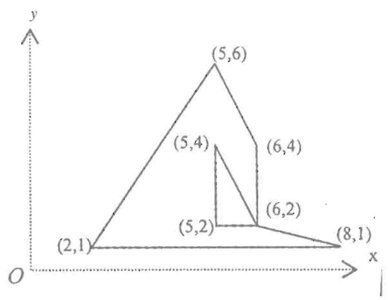


Fig. 5(a). Polygonal Lamina P which has a triangular hole with vertices at the points (5,4) (5,2) and (6,2).

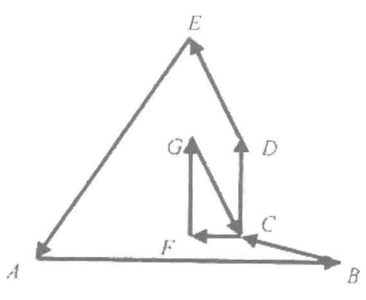


Fig. 5(b). The polygonal lamina P described by border vectors.

2. Mechanical Parameters – Definitions, Relations and Values for the General Uniform Triangular Lamina

The motion of the general uniform polygonal lamina under the action of any given force system will depend on its mass, position of its centre of gravity (or the value of its related first moment), and its relevant moments of inertia. These basic attributes of any rigid body, determining not only the physical nature but also its dynamics under the action of a force system, are here collectively referred to as **mechanical parameters**. Before finding the values of these parameters in the case of the simplest polygonal lamina, the triangle, and then later extrapolating the result to the general case, we will first consider the definitions and some basic results involving mechanical parameters.

Mass

Mass may intuitively be described as *quantity of matter*. More to the point, mass may be said to be *a measure of a body's resistance to acceleration* or simply a *measure of the body's inertia*. For our purposes, since the laminae we consider always have unit mass per unit area, the quantities of mass and area will always have the same numerical values.

Centre of Mass and First Moment

By definition of **centre of mass** we have:

> The **position** vector, $\mathbf{R}_0$[particles], of the **centre of mass** of a number of particles referred to an origin O with the **general particle** having mass m and **position** vector $\mathbf{r}$ is given by:
>
> $$\Big(\sum_{particles} m\Big)\,\boldsymbol{R}_0[\text{particles}] \overset{def}{=} \sum_{particles} (m\,\mathbf{r}) \qquad \textbf{(2.1)}$$

The summation on the right hand side of the last equation will be referred to as the vector **first moment** of the particles with respect to O as origin. That is, using symbols,

> The vector **first moment** of a number of particles with respect to origin O is :
>
> $$\mathbf{F}_0[\text{particles}] \overset{def}{=} \sum_{particles} (m\,\mathbf{r}) \qquad \textbf{(2.2)}$$

Composite Body Results for Centre of Mass

(a) The **position** vector of the **centre of mass,** $\mathbf{R}[12]$, of a **composite body** with mass $M[12]$ consisting of two parts having masses $M[1]$ and $M[2]$ and centres of mass at positions $\mathbf{R}[1]$ and $\mathbf{R}[2]$ respectively, is, on using the definition, given by:

$$M[12]\ \mathbf{R}[12] = \sum_{12} (m\,\mathbf{r}) = \sum_{1} (m\,\mathbf{r}) + \sum_{2} (m\,\mathbf{r})$$

leading to:

$$M[12]\ \mathbf{R}[12] = M[1]\ \mathbf{R}[1] + M[2]\ \mathbf{R}[2] \qquad \textbf{(2.3)}$$

(b) The **position** vector of the **centre of mass**, $\mathbf{R}[1...n]$, of a composite body of mass $M[1...n]$ consisting of n parts having masses $M[1], \ldots, M[n]$ and centres of mass at positions $\mathbf{R}[1], \ldots \mathbf{R}[n]$ respectively, is given, in virtue of a simple extension of (2.3) by:

$$M[1...n]\ \mathbf{R}[1...n] = \sum_{i=1}^{i=n} (M[i]\ \mathbf{R}[i]) \qquad \textbf{(2.4)}$$

Moment of Inertia

By definition,

the **moment of inertia,** $I_L[\text{body}]$, of a rigid body, **about an axis** L, made up of particles with the **general particle** having mass m and **distant** r from L is given by:

$$I_L[\text{body}] \overset{def}{=} \sum_{body} (m\,r^2) \qquad \textbf{(2.5)}$$

The above definition leads immediately to the following result.

Composite Body Result for Moment of Inertia
The moment of inertia of a composite body, $I[1...n]$, consisting of n parts, about any axis is the sum of the moments of inertia $I[1], \ldots, I[n]$ of each part about the same axis since:

$$I[1...n] = \sum_{1...n} (m\,r^2) = \sum_{1} (m\,r^2) + \ldots + \sum_{n} (m\,r^2)$$

$$I[1...n] = I[1] + \ldots + I[n] \qquad \textbf{(2.6)}$$

Two very useful theorems follow.

Perpendicular Axes Theorem

If the moment of inertia of a lamina about any **two perpendicular axes in the plane of the lamina** and passing through point *O* on the lamina are I_{OX} [lamina] and I_{OY} [lamina] respectively, then I_{OZ} [lamina], the moment of inertia of the lamina about an axis passing through *O* and **perpendicular to the lamina**, (Fig. 6 (a)) is given by:

$$I_{OZ}\text{ [lamina]} = I_{OX}\text{ [lamina]} + I_{OY}\text{ [lamina]} \qquad \textbf{(2.7)}$$

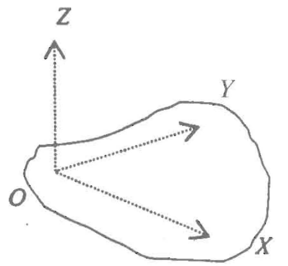


Fig. 6(a) *X* and *Y* axes are in the plane of the lamina.

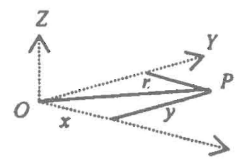


Fig. 6(b). *P*, the general point is distant $r = (x^2 + y^2)^{1/2}$ from the origin *O*.

Proof

By definition (*see* Fig. 6 (b))

$$\begin{aligned} I_{OZ}\text{ [lamina]} &= \sum m\, r^2 = \sum m\,(y^2 + x^2) \\ &= \sum m\, y^2 + \sum m\, x^2 \\ &= I_{OX}\text{ [lamina]} + I_{OY}\text{ [lamina]} \end{aligned}$$

Parallel Axes Theorem

The moment of inertia of a body, of mass *M*, about an axis L_O passing through a point *O*, I_O, is the same as that of the moment of inertia, I_G, of the same body about a **parallel axis through the centre of mass**, *G*, of the body plus the product of *M* and the square of the distance, *a*, between the two parallel axes (*see* Fig. 7). That is:

$$I_O = I_G + M\,a^2 \qquad \textbf{(2.8)}$$

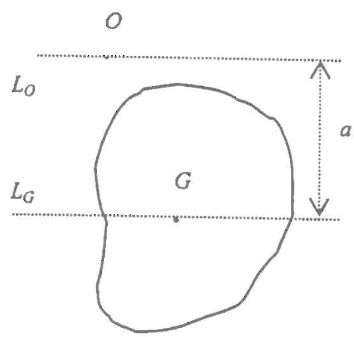


Fig. 7. In this side view of the body L_G is any axis passing through *G*, the centre of mass. L_O is the parallel axis through point *O*.

Proof

Consider a general particle *P*, of the body distant *r* from the axis L_O, through *O*, and distant *x* from the axis L_G through the mass-centre G. Let its mass be m (*see* Fig.8 and Fig.9).

Then P is one of the vertices of the triangle whose plane is perpendicular to that of the parallel axes and has its other 2 vertices one on the axis through O and the other vertex on the parallel axis L_G. This triangle is shown in the adjoining diagram (Fig. 9) with the angle θ as indicated.

Applying the Cosine Formula to this triangle gives:

$$r^2 = x^2 + a^2 - 2ax\cos\theta$$

Taking G as origin and y-axis in the direction at right angles to the parallel axes and in their plane with its sense towards O, we have:

$$y_P = x\cos\theta \qquad \text{and}$$

$$y\text{-component of } \mathbf{R}_G[\text{body}] = \mathrm{Y}_G[\text{body}] = \Big(\sum_{body} m\, y_P\Big)/M$$

which is zero since the origin is taken at G itself.

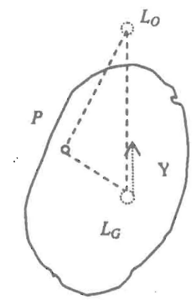


Fig. 8. Frontal view of the same body showing the general particle P.

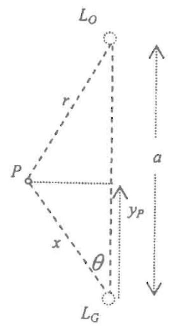


Fig. 9. Frontal view showing pertinent distances.

From the definition (2.5),

$$\mathbf{I}_O = \sum_{body} m\, r^2 = \sum_{body} m\,(x^2 + a^2 - 2ax\cos\theta)$$

$$= \sum_{body} m\, x^2 + \sum_{body} m\, a^2 - 2a\sum_{body} m\, x\cos\theta$$

$$= \mathrm{I}_G + M a^2 - 2a\, M\, \mathrm{Y}_G[\text{body}]$$

and the result follows since $\mathrm{Y}_G[\text{body}] = 0$

Lemma 1

Let O, A, B be the vertices of **any uniform triangular lamina** of **unit mass per unit area** in the x-y plane such that the sense of OAB is **anti-clockwise** (*see* Fig.10). Take O as origin and **i, j, k** to be the unit vectors in the x, y, z directions respectively. Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2) respectively. Then the **mass of lamina** OAB is:

$$M[OAB] = \{(x_1 y_2 - x_2 y_1)/2\} \quad \textbf{(2.9)}$$

Proof

From the definition of the **vector product** of any 2 vectors **OA** and **OB** we have:

$$\mathbf{OA} \times \mathbf{OB} = ((OA)(OB)(\sin\theta))\ \mathbf{n}$$

where θ is the angle between **OA** and **OB** and **n** is the unit vector perpendicular to both **OA** and **OB** such that **OA**, **OB** and **n** form a **right-handed system**. (*This implies that for persons standing with their head in the direction of* **n** *and facing the direction* **OB**, *the direction* **OA** *would be to their right*).

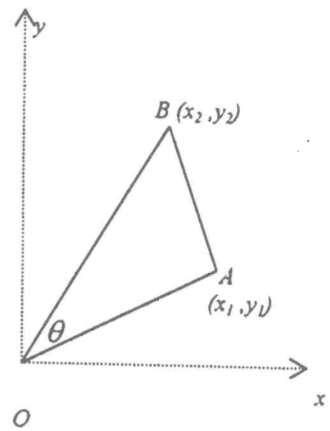


Fig. 10. A and B are any 2 points in the x–y plane such that OAB is anticlockwise.

Now since the sense of OAB is anti-clockwise, the unit vector **n** must be **k** as the 3 mutually perpendicular axes Ox, Oy, Oz, by definition, likewise, form a right-handed system (*see* Fig. 11). Hence:

$$\mathbf{OA} \times \mathbf{OB} = ((OA)(OB)(\sin\theta))\ \mathbf{k}$$

$$= (2(\text{area of } OAB))\ \mathbf{k} \quad \textbf{(2.10)}$$

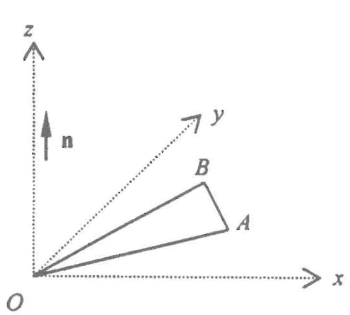


Fig. 11. **OA**, **OB** and **n** form a right handed system and so must **OA**, **OB** and **k**.

But it can be shown that the definition of the vector product also implies that the operation itself is distributive. Hence writing the vectors **OA** and **OB** in component form in terms of the unit vectors **i** and **j**, we have:

$$\begin{aligned}\mathbf{OA} \times \mathbf{OB} &= (x_1\mathbf{i} + y_1\mathbf{j}) \times (x_2\mathbf{i} + y_2\mathbf{j}) \\ &= ((x_1\mathbf{i}) \times (x_2\mathbf{i})) + ((y_1\mathbf{j}) \times (x_2\mathbf{i})) \\ &\quad + ((x_1\mathbf{i}) \times (y_2\mathbf{j})) + ((y_1\mathbf{j}) \times (y_2\mathbf{j})) \\ &= (\mathbf{0}) + (y_1 x_2 (-\mathbf{k})) + (x_1 y_2 \mathbf{k}) + (\mathbf{0}) \\ &= (x_1 y_2 - x_2 y_1)\,\mathbf{k} \qquad \textbf{(2.11)}\end{aligned}$$

The required result then follows from equations (2.10) and (2.11) and the equivalence of mass and area.

Lemma 2

Let O, A, B be the vertices of **any uniform triangular lamina** of **unit mass per unit area** in the x-y plane such that the sense of OAB is **anti-clockwise**. Take O as origin and **i**, **j**, **k** to be the unit vectors in the x, y, z directions respectively. Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2) respectively. Then the **vector first moment** of **lamina** OAB referred to origin O is:

$$\mathbf{F}_O[OAB] = \{(x_1 y_2 - x_2 y_1)((x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j})/6\} \qquad \textbf{(2.12)}$$

Proof

Assume the lamina OAB to be made up of elemental thin rods parallel to the side AB. The centre of mass of each of these thin rods must be on some point of the median joining O to C (*see* Fig.12) where C is the midpoint of AB.

Symmetry of the vertices implies that the centre of mass must also lie on the other medians. This, in turn, implies that medians must concur at some point and that this point is the centre of mass of the lamina. But, from geometry, the point of intersection of the medians of a triangle trisects each median so that if G is this point then:

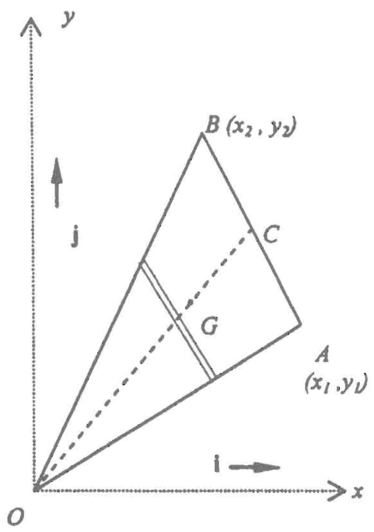


Fig.12. The centre of mass of each thin rod must lie on the median OC.

$$\begin{aligned}\mathbf{OG} &= (2/3)\,\mathbf{OC} = (2/3)\{(x_1 + x_2)\,\mathbf{i} + (y_1 + y_2)\mathbf{j}\}/2 \\ &= ((x_1 + x_2)\,\mathbf{i} + (y_1 + y_2)\mathbf{j})/3\end{aligned}$$

Now, from the definitions of first moment and centre of mass, (equations (2.2) and (2.1)) we have:

$$\mathbf{F}_O[OAB] = \sum_{OAB} m\,\mathbf{r} = M[OAB]\,\mathbf{R}_O[OAB]$$

$$= \{(x_1 y_2 - x_2 y_1)/2\}\ \mathbf{OG}$$

$$= \{(x_1 y_2 - x_2 y_1)\,((x_1 + x_2)\,\mathbf{i} + (y_1 + y_2)\,\mathbf{j})/6\}$$

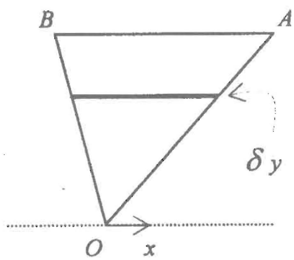
Lemma 3

Let O, A, B be the vertices of **any uniform triangular lamina** of **unit mass per unit area** in the x-y plane such that the sense of OAB is **anti-clockwise**. Take O as origin and the coordinates of A and B to be (x_1, y_1) and (x_2, y_2) respectively. Then the **moments of inertia** of **triangular lamina** OAB about the x, y, z axes are respectively

(a) $\mathrm{I}_{OX}[OAB] = \{(x_1 y_2 - x_2 y_1)(y_1^2 + y_1 y_2 + y_2^2)/12\}$ **(2.13)**
(b) $\mathrm{I}_{OY}[OAB] = \{(x_1 y_2 - x_2 y_1)(x_1^2 + x_1 x_2 + x_2^2)/12\}$ **(2.14)**
(c) $\mathrm{I}_{OZ}[OAB] = \{(x_1 y_2 - x_2 y_1)(x_1^2 + x_1 x_2 + x_2^2 + y_1^2 + y_1 y_2 + y_2^2)/12\}$ **(2.15)**

Proof of (a)

Three cases arise which will be considered separately.



Case (1) AB is parallel to the x-axis.
Case (2) Not both of the points A and B are above the x-axis or both below the x- axis.
Case (3) A and B are either both above or both below the x-axis, AB is not parallel to the x-axis.

Case (1) (*see* Fig. 13)

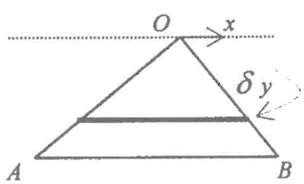


Fig. 13. Lamina OAB with AB parallel to the x-axis.

In this case we have $y_1 = y_2$. Assume the lamina OAB to be made up of thin rods parallel to the side AB. Let the y-value of the general elemental thin rod be y and its thickness be δy. Then we have:

$$\textit{length of elemental thin rod} = (AB)(y)/(y_1)$$

as y and y_1 are either both positive or both negative. So

mass (or area) *of elemental thin rod* $(AB)(y) \quad (\delta y)/(y_1)$

Hence the moment of inertia of the thin rod about the x-axis is:

$$\mathrm{I}_{OX}\,[\text{thin rod}] \;=\; \sum_{\textit{thin rod}} (m\,y^2) \;=\; y^2 \sum_{\textit{thin rod}} m$$

$$= \; (AB)(y^3)\,(\delta y)/(y_1)$$

By the Composite Body Result (2.6), the moment of inertia of lamina OAB about the x-axis is:

$$\mathrm{I}_{OX}[OAB] \;=\; \lim \sum_{OAB} \left[\, (AB)(y^3)\,/\,(y_1)\,\right] \delta y$$

$$= \; (AB/y_1) \int_0^{y_1} y^3\,\mathrm{d}y \;=\; (AB)(y_1^{\,3})\,/\,4$$

and since $M[OAB] = (AB)\,(y_1)\,/2$ we have:

$$\mathrm{I}_{OX}[OAB] \;=\; M[OAB]\;(\,y_1^{\,2}\;/\,2\,) \tag{2.16}$$

which in virtue of equation (2.9) and the equality of y_1 and y_2 is compatible with equation (2.13).

Case (2)

In this case either AB cuts or it touches the x-axis. Let C be the point where AB cuts or touches the x-axis. This means that C may only be a point between A and B with both A and B included.

Going back to case(1), if G is the centre of mass of lamina OAB, by the Parallel Axes Theorem (2.8) we have that the moments of inertia of lamina OAB about an axis through G parallel to the x-axis and about an axis through A also parallel to the x-axis are (*see* Fig. 14) given by:

$$\mathrm{I}_{GX}[OAB] \;=\; M[OAB]\;\{(\,y_1^{\,2}\,/2) - (2y_1/3)^2\}$$
$$= \; M[OAB]\;(y_1^{\,2}\,/18) \qquad \text{and}$$

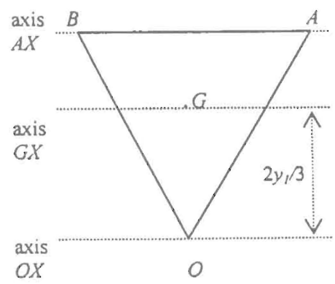


Fig. 14. The Parallel Axis Theorem relates the moments of inertia about the 3 parallel axes.

$$I_{AX}[OAB] = M[OAB]\ \{(y_1^2/18) + (y_1/3)^2\}$$
$$= M[OAB]\,(y_1^{\,2}/6) \qquad \textbf{(2.17)}$$

Now suppose $y_1 > y_2$ where $y_2 \leq 0$ (Fig. 15)

Then by the Composite Body Result (2.6)

$$I_{OX}[OAB] = I_{OX}[OAC] + I_{OX}[OCB]$$

Applying equation (2.17) to both terms on the right hand side of the above equation we get:

$$I_{OX}[OAB] = M[OAC]\ (y_1^2/6) + M[OCB]\,(y_2^2/6)$$

As $M[OAC] \,/\, M[OAB] = (y_1) \,/\, (y_1 - y_2)$ and $M[OCB] \,/\, M[OAB] = (-y_2) \,/\, (y_1 - y_2)$ we have

$$I_{OX}[OAB] = M[OAB]\ ((y_1) \,/\, (y_1 - y_2))(y_1^2/6)$$
$$+\ M[OAB]\,((-y_2) \,/\, (y_1 - y_2))(y_2^{\,2}/6)$$

which in virtue of Lemma 1 (2.9) leads to the required result (2.13).

Consider now $y_1 < y_2$ where $y_1 \leq 0$

Following the same argument as before the equation corresponding to the last one above would read:

$$I_{OX}[OAB] = M[OAB]\,((-y_1)/(y_2 - y_1))(y_1^2/6)$$
$$+ M[OAB]\ ((y_2)/(y_2 - y_1))(y_2^2/6)$$

which again leads to the required result (2.13).

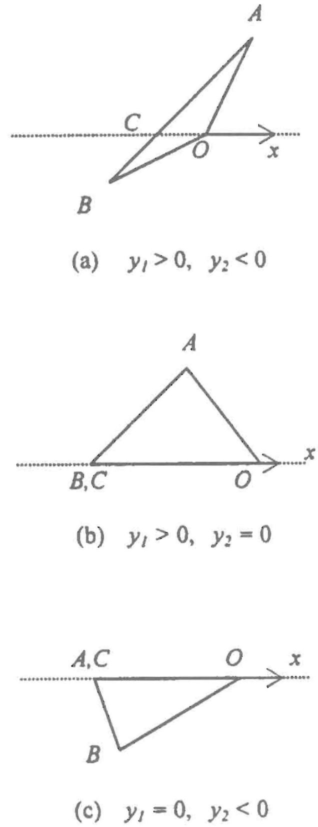


Fig. 15. Case 2 with $y_1 > y_2$ and $y_2 \leq 0$.

Case (3)

Let C be the point where the line AB produced cuts the x-axis. Consider

modulus (y_1) > **modulus** (y_2) (*see* Fig.16)

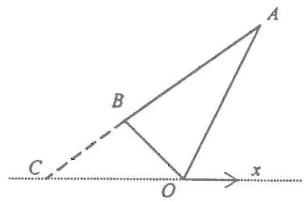


Fig. 16 (a). A and B both have x-axis modulus (y_1) > modulus (y_2).

By Result (2.6) and equation (2.17)

$$I_{OX}[OAB] = I_{OX}[OAC] - I_{OX}[OBC]$$
$$= M[OAC]\,(y_1^2/6) - M[OBC]\,(y_2^2/6)$$

Now geometry and the equivalence of mass and area give:

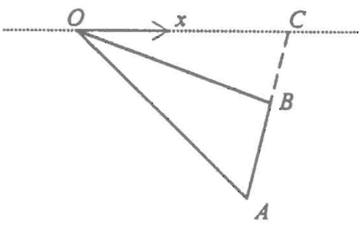


Fig. 16 (b). A and B both below x-axis modulus (y_1) > modulus (y_2).

$$M[OAC] - M[OBC] = M[OAB]$$

and $$M[OAC]\,/\,M[OBC] = (y_1/y_2)$$

implying $$M[OAC] = M[OAB]\,(y_1/(y_1 - y_2))$$

and $$M[OBC] = M[OAB]\,(y_2/(y_1 - y_2))$$

so that:

$$I_{OX}[OAB] = M[OAB]\,(y_1/(y_1 - y_2))\,(y_1^2/6) - M[OAB]\,(y_2/(y_1 - y_2))\,(y_2^2/6)$$

which by equation (2.9) leads to the required (2.13).

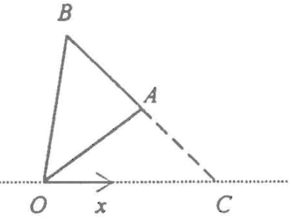


Fig. 17 (a). A and B both above x-axis modulus (y_1) < modulus (y_2).

Consider now **modulus** (y_1) < **modulus** (y_2)

In this case (Fig.17 (a) and (b))

$$M[OCB] - M[OCA] = M[OAB]$$

and $$M[OCB] \;/\; M[OCA] = (y_2 / y_1)$$

give $$M[OCB] = M[OAB]\ (y_2 /(y_2 - y_1))$$

and $$M[OCA] = M[OAB]\ (y_1 /(y_2 - y_1))$$

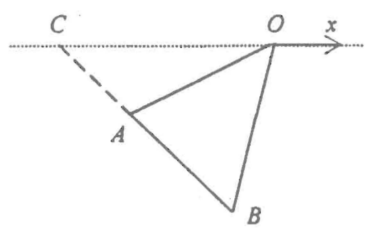


Fig. 17 (b). *A* and *B* both below *x*-axis modulus (y_1) < modulus (y_2).

Hence as before we have:

$$\begin{aligned} \mathrm{I}_{ox}[OAB] &= \mathrm{I}_{ox}[OCB] - \mathrm{I}_{ox}[OCA] \\ &= M[OAB]\ (y_2 /(y_2 - y_1))\ (y_2^2 /6) \\ &\quad - M[OAB]\ (y_1 /(y_2 - y_1))\ (y_1^2 /6) \end{aligned}$$

which again leads to (2.13).

Proof of (b)

The proof of equation (2.14) follows directly from the above result and the symmetry of the x and y axes.

Proof of (c)

This follows directly from the above results (2.13) and (2.14) on applying the Perpendicular Axes Theorem (2.7).

3. Unifying the Mechanical Parameters

m-area

We define a function of the area of the general polygonal lamina, which may both be scalar or vector, called **mechanical area** or simply **m-area**, by its following basic property.

Composite Body Property of m-area

For laminar regions of **area** [1] and [2] and the **composite area** [12] consisting of a simple addition of regions [1] and [2],

m-area [1] + **m-area** [2] = **m-area** [12] scalar/**vector (3.1)**

The mechanical parameters of mass, first moment (both scalar and vector) and moment of inertia about any axis when referred to uniform polygonal laminae, are all functions of the laminar area besides obeying the additive property of the Composite Body Result.

In fact, mass, M, is simply the value of the area since the density of the lamina is assumed to be unit. Thus if δa is a small portion of area of the lamina,

$$M[1] + M[2] = \sum_{1} \delta a + \sum_{2} \delta a = \sum_{12} \delta a = M[12]$$

Similar equations hold for the x-component of the first moment referred to origin O, $F_O X$, and the moment of inertia of the polygonal lamina, I_{OY}, about the y-axis.

$$F_O X[1] + F_O X[2] = \sum_{1} x\delta a + \sum_{2} x\delta a = \sum_{12} x\delta a = F_O X[12]$$

$$I_{OX}[1] + I_{OX}[2] = \sum x^2\delta a + \sum_{2} x^2\delta a = \sum_{12} x^2\delta a = I_{OX}[12]$$

On the other hand, when m-area is referred to the vector first moment, it is of course a vector function,

Thus: $$\mathbf{F}_0[1] + \mathbf{F}_0[2] = \sum_{1} \mathbf{r}\delta a + \sum_{2} \mathbf{r}\delta a = \sum_{12} \mathbf{r}\delta a = \mathbf{F}_O[12]$$

However the value of the closely related mechanical parameter centre of mass, $\mathbf{R}$, is not an m-area since

$$\mathbf{R}_0[1] + \mathbf{R}_0[2] \neq \mathbf{R}_0[12]$$

For convenience

the **m-area** of the **elemental uniform triangular lamina** *OAB* with vertices *O, A, B* at the origin and the points $(x_1, y_1), (x_2, y_2)$ respectively such that the sense of *OAB* is **anti-clockwise** will be designated as follows:
m-area $[OAB] \overset{def}{=} \mathbf{A \wedge B} \overset{def}{=} (x_1, y_1) \wedge (x_2, y_2)$ scalar/**vector (3.2)**

Summarizing the results of Section 2 which referred to the mechanical area values of lamina *OAB*, we have:

mechanical parameter		**m-area** $[OAB] = \mathbf{A \wedge B} = (x_1, y_1) \wedge (x_2, y_2)$ O (origin), $A(x_1, y_1)$, $B(x_2, y_2)$ are the vertices of a uniform triangular lamina of unit mass per unit area with the sense *OAB* being anti-clockwise
mass		$M[OAB] = \{(x_1 y_2 - x_2 y_1)/2\}$
vector first moment		$\mathbf{F}_O[OAB] = \{(x_1 y_2 - x_2 y_1)((x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j})/6\}$
moment of inertia	**about x-axis**	$I_{OX}[OAB] = \{(x_1 y_2 - x_2 y_1)(y_1^2 + y_1 y_2 + y_2^2)/12\}$
	about y-axis	$I_{OY}[OAB] = \{(x_1 y_2 - x_2 y_1)(x_1^2 + x_1 x_2 + x_2^2)/12\}$
	about z-axis	$I_{OZ}[OAB] = \{(x_1 y_2 - x_2 y_1)(x_1^2 + x_1 x_2 + x_2^2 + y_1^2 + y_1 y_2 + y_2^2)/12\}$

centre of mass	$\mathbf{R}_O[OAB] = (\mathbf{F}_O[OAB])/(M[OAB]) = \{(x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j})/3\}$

scalar/**vector (3.3)**

It follows from the above equations (3.2) and (3.3) that:

the **m-area** of the **uniform triangular lamina *OAB*** with vertices *O, A, B* at the origin and the points $(x_1, y_1), (x_2, y_2)$ respectively, such that the sense of *OAB* is **clockwise,** is:
m-area $[OBA] = \mathbf{B \wedge A} = (x_2, y_2) \wedge (x_1, y_1) = -((x_1, y_1) \wedge (x_2, y_2)) = -(\mathbf{A \wedge B})$ scalar/**vector (3.4)**

It is now time to extrapolate these results to the general polygonal lamina.

4. The m-area of the General Uniform Polygonal Lamina

THEOREM

Let the set of vectors $\{\mathbf{A_1A'_1},\ \mathbf{A_2A'_2},\ .\ .\ .\ .\ .\ ,\mathbf{A_nA'_n}\}$ be the border vector set of the general uniform polygonal lamina. Then the mechanical area of the polygonal lamina, **m-area** [polygon], is given by:

$$\textbf{m-area}\ [\text{polygon}] = \sum_{r=1}^{r=n} \mathbf{A_r \wedge A'_r} \qquad \textbf{(4.1)}$$

Proof

Partitioning the polygon

From the origin O draw lines to pass through each vertex of the polygon (Fig.18). We call these lines **radial lines**. Let the number of radial lines be m. Choose L_1 to be the radial line through any vertex of the polygon. Then:

$$L_1,\ L_2,\ .\ .\ .\ .\ .\ .\ ,\ L_m$$

are chosen to turn in *anti-clockwise* sense. Also L_{m+1} is chosen to be L_1.

(Since each of the $2n$ end-points A_1, A'_1, A_2, A'_2, . . . ,A_n, A'_n, of the n border vectors of the polygon coincides with at least one other end-point at a vertex – with exactly 1 other in the case of a double vertex, more than 1 in the case of a multiple vertex – then the number, v, of vertices is less or equal to n. Also a radial line may pass through more than one vertex so that $m \leq v \leq n$).

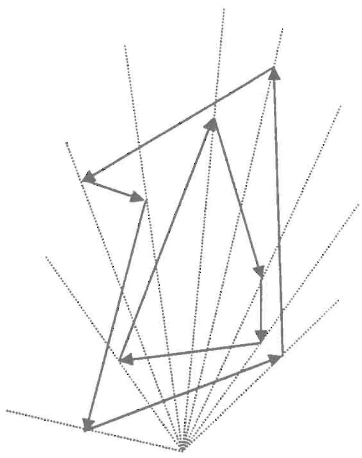


Fig. 18. Polygon described by its border vectors and partitioned in sectors by radial lines.

The m radial lines partition the entire area of the x-y plane into m regions bounded by adjacent radial lines. We call these regions **sectors**. The area of the polygon within the sector bounded by radial lines L_S and L_{S+1} we describe by [sector$_{(S,\ S+1)}$].

Both the area of the polygon as well as each of the sides or edges of the polygon will also be partitioned by a number of these sectors. If the origin O is an internal point of the polygon or a point inside one of its holes, portions of the polygon area will be found in each sector. On the other hand, if O is outside the polygon there may be just one sector containing no area. In either case by the Composite Body Property of m-area (3.1) we have:

$$\textbf{m-area}\ [\text{polygon}] = \sum_{s=1}^{s=m} \textbf{m-area}\ [\text{sector}_{(s,s+1)}] \qquad \textbf{(4.2)}$$

Partitioning the border edges

Let the point where the general radial line L_S meets the general polygon edge $A_rA'_r$ internally or externally be A_{rs}. If A_{re} and A_{rf}, $f > e$, happen to be the *internal* points where the radial line L_e and L_f meet the endpoints of the side $A_rA'_r$, then clearly the other internal points where the other radial lines cut $A_rA'_r$ are:

$$A_{r(e+1)},\ A_{r(e+2)},\ \ldots,\ A_{r(f-1)} \qquad (\text{Fig.19})$$

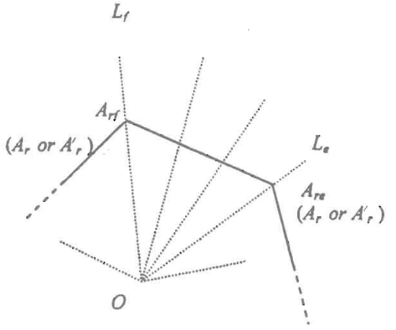


Fig. 19. Radial lines L_e ,...., L_f partitioning side $A_rA'_r$ in border segments.

The portions of the polygon edge:

$$A_{re}A_{r(e+1)},\ A_{r(e+1)}A_{r(e+2)},\ \ldots\ \ldots\ \ldots,\ A_{r(f-1)}A_{rf}$$

we call **border segments** and the portions of the border vectors corresponding to these border segments we call **border vector segments**.

Now the direction of the border vector segment corresponding to $A_{re}A_{r(e+1)}$ must be the same as that of the border vector $\mathbf{A_r A'_r}$. Hence the border vector segment of $A_{re}A_{r(e+1)}$ is $(A_{re}A_{r(e+1)}/A_rA'_r)\ \mathbf{A_rA'_r}$ and the mechanical area of triangle $OA_{re}A_{r(e+1)}$ is $(A_{re}A_{r(e+1)}/A_rA'_r)\ (\mathbf{A_r{\wedge}A'_r})$. Hence by the Composite Body Property (3.1) we have **m-area** $[OA_rA'_r]$ $= \mathbf{A_r{\wedge}A'_r}$

$$= \sum_{s=e}^{s=f-1} (A_{rs}A_{r(s+1)}/A_rA'_r)\ (\mathbf{A_r{\wedge}A'_r})$$

We will now make use of the following definition of **inclusion** to write an alternative form of the last summation above.

Definition of inclusion ( )*

$$(A_{rs}A_{r(s+1)})^* \overset{def}{=} \begin{cases} \text{(i) } A_{rs}A_{r(s+1)} \ \textbf{if both } A_{rs} \textbf{ and } A_{r(s+1)} \textbf{ are on } A_rA'_r \\ \text{(ii) } 0 \qquad \textbf{otherwise} \end{cases} \qquad \textbf{(4.3)}$$

Thus we may write:

$$\textbf{m-area } [OA_rA'_r] = \sum_{s=e}^{s=f-1} (A_{rs}A_{r(s+1)}/A_rA'_r)\,(\mathbf{A_r}\wedge\mathbf{A'_r}) = \sum_{s=1}^{s=m} ((A_{rs}A_{r(s+1)})^*/(A_rA'_r))\,(\mathbf{A_r}\wedge\mathbf{A'_r}) \qquad \textbf{(4.4)}$$

Finding the mechanical area of the general sector

Consider the area of the polygon in the general sector flanked by the radial lines L_S and L_{S+1} where L_{S+1} is anti-clockwise relative to L_S (Fig.20). This region is crossed by portions of the sides of the polygon, border segments, of the form $A_{rs}A_{r(s+1)}$ where $n \geq r \geq 1$ and $m \geq s \geq 1$.

Let us, temporarily, call the border segment furthest away from the origin $B_1B'_1$, the next one $B_2B'_2$ and so on with the points B_1, B_2, ... being on line L_S and the points B'_1, B'_2, ... on line L_{S+1}. Naturally, since any of the points B_i or B'_j may be a vertex of the polygon two or more adjacent B or B' points may coincide. Now we may consider mechanical area within a sector in the simplest cases.

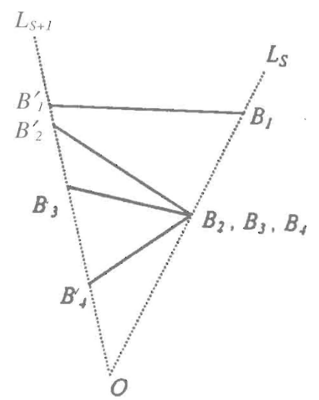


Fig. 20. The radial lines in the general segment crossed by border segments with B_2, B_3, B_4 coincident.

(a) *no border segments*

If there are no border segments within the radial lines then obviously the m-area of the polygon in the sector is zero.

(b) *one border segment*

If there is only one border segment (*see* Fig. 21) then this must be part of the outer edge furthest away from O so that the internal points of the lamina in the sector must lie within the triangle $OB_1B'_1$. From the

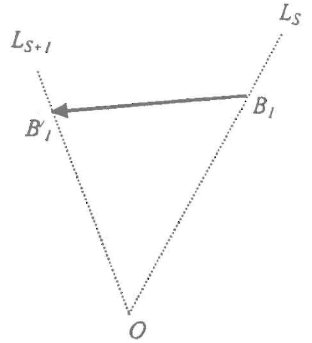


Fig. 21. The general sector crossed by one border vector segment.

definition of border vector it follows that the corresponding border vector segment is $\mathbf{B_1B'_1}$. Thus from definition (3.2):

$$\textbf{m-area}\ [OB_1B'_1] = \mathbf{B_1{}^\wedge B'_1}$$

(c) *two border segments*

In this case if $B_1B'_1$ and $B_2B'_2$ are the only border segments of the lamina in the general sector, the internal points indicated by the dot 'o' in the adjoining Fig. 22 must lie within the quadrilateral $B_1B'_1B'_2B_2$ which reduces to a triangle in the case of a pair of coincident B points. Thus it follows that $\mathbf{B_1B'_1}$ and $\mathbf{B'_2B_2}$ must be the border vector segments in this case. Hence in virtue of the Composite Body Property (3.1), and equations (3.2) and (3.4) we have:

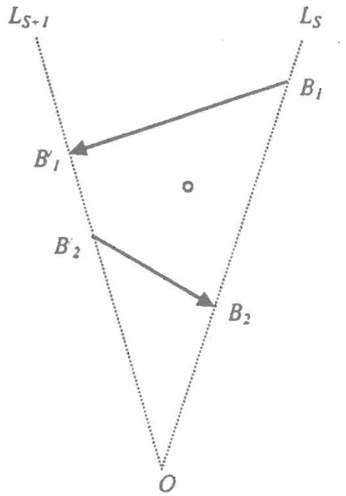


Fig. 22. The general sector with two border vector segments with the internal points in the region indicated by 'o'.

$$\begin{aligned} &\textbf{m-area}\ [\text{sector}_{(S,\,S+1)}] \\ &= \textbf{m-area}\ [OB_1B'_1] - \textbf{m-area}\ [OB_2B'_2] \\ &= \mathbf{B_1{}^\wedge B'_1} - \mathbf{B_2{}^\wedge B'_2} \\ &= \mathbf{B_1{}^\wedge B'_1} + \mathbf{B'_2{}^\wedge B_2} \end{aligned}$$

(d) *in general*

Lemma 4

In the general polygonal lamina partitioned in sectors, and with the general sector flanked by radial lines L_S and L_{S+1} (with L_{S+1} being anti-clockwise relative to L_S) and having b border segments of the form $B_1B'_1$, $B_2B'_2$, ... $B_bB'_b$ given in the order of their distance from O with $B_1B'_1$ being the furthest (where points B_1, B_2, ... are on L_S and points B'_1, B'_2, ... on L_{S+1})

(i) the corresponding border vector segments are respectively:

	$\mathbf{B_1B'_1}$, $\mathbf{B'_2B_2}$, $\mathbf{B_3B'_3}$, ... , $\mathbf{B_bB'_b}$	if b is **odd**
and	$\mathbf{B_1B'_1}$, $\mathbf{B'_2B_2}$, $\mathbf{B_3B'_3}$, ... , $\mathbf{B'_b\,B_b}$	if b is **even**

(ii) **m-area** [general sector] = $\mathbf{B_1{}^\wedge B'_1} + \mathbf{B'_2{}^\wedge B_2} + \mathbf{B_3{}^\wedge B'_3} + \ldots\ldots + \mathbf{B_b{}^\wedge B'_b}$ (or $\mathbf{B'_b{}^\wedge B_b}$)

(4.5)

Proof

In general the furthest border vector segment from O must be anti-clockwise oriented relative to O as there are no internal points of the polygon beyond the furthest border segment. This, in turn, implies that the next border vector segment is clockwise oriented relative to O as the internal points of the lamina must lie between the two border vector segments. The argument repeats itself indefinitely with the orientation of any border vector segment always being different from its previous. This proves part (i) of the Lemma.

We have already established the result (ii) in the case when $b = 1$ and $b = 2$ in cases (b) and (c) above. Assuming that the result (ii) holds for $b = k$ we consider separately the two cases when k is odd and when it is even.

(Case 1) k is **odd**

In this case, since k is odd, the last border vector segment corresponding to border segment $B_kB'_k$ must be $\mathbf{B_kB'_k}$ rather than $\mathbf{B'_kB_k}$ (*see* Fig. 24). So we have:

m-area [general sector]

$$= \mathbf{B_1 \wedge B'_1 + B'_2 \wedge B_2 + B_3 \wedge B'_3 + \ldots\ldots\ldots + B_k \wedge B'_k}$$

Consider now the case when $b = k + 1$. The addition of the last border segment means that the portion of area previously pertaining to triangle $OB_kB'_k$ has now become reduced to that of the quadrilateral (which reduces to a triangle in the case of coincident adjacent B or B' points) $B_kB'_kB'_{k+1}B_{k+1}$ (Fig. 25). Hence the total sector m-area has been reduced by an amount equal to the m-area of triangle $OB'_{k+1}B_{k+1}$. So for $b = k + 1$ using (3.1), (3.2) and (3.4)

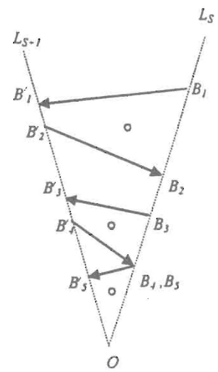


Fig. 23. The general sector of the lamina with the regions of its internal points indicated by the dots 'o'.

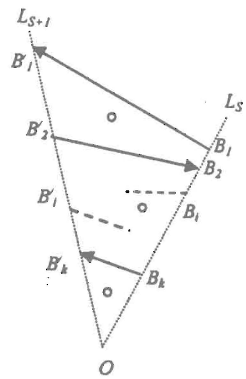


Fig. 24. The general sector with $b = k$ k is ODD.

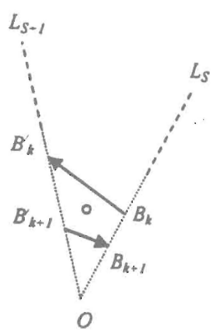


Fig. 25. Adding $\mathbf{B_{k+1}B'_{k+1}}$ reduces sector area by area $OB_{k+1}B_{k+1}$.

new sector m-area

$= $ previous sector m-area – m-area of triangle $OB'_{k+1}B_{k+1}$
$= \mathbf{B_1 \wedge B'_1 + B'_2 \wedge B_2 + B_3 \wedge B'_3} + \ldots\ldots + \mathbf{B_k \wedge B'_k} - \mathbf{B_{k+1} \wedge B'_{k+1}}$
$= \mathbf{B_1 \wedge B'_1 + B'_2 \wedge B_2 + B_3 \wedge B'_3} + \ldots\ldots + \mathbf{B_k \wedge B'_k} + \mathbf{B'_{k+1} \wedge B_{k+1}}$

and since with $b = k + 1$ the additional border vector introduced is $\mathbf{B'_{k+1} \wedge B_{k+1}}$ (as $k + 1$ is even) the result (ii) is true also for $b = k + 1$ if it is true for $b = k$.

(Case 2) k is **even**

In this case since k is even the last border vector segment corresponding to border segment $B_kB'_k$ must be $\mathbf{B'_kB_k}$ rather than $\mathbf{B_kB'_k}$ (*see* Fig. 26). So we have:

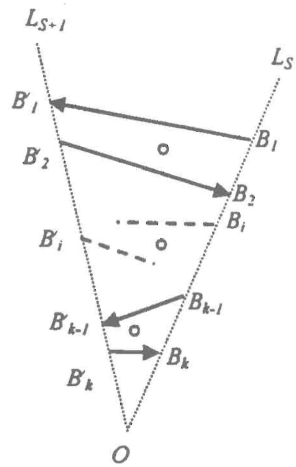


Fig. 26. The general sector with $b = k$.
k is EVEN

m-area [general sector]

$= \mathbf{B_1 \wedge B'_1 + B'_2 \wedge B_2 + B_3 \wedge B'_3} + \ldots\ldots\ldots + \mathbf{B'_kB_k}$

Considering now the case $b = k + 1$ the addition of the border vector segment $\mathbf{B_{k+1}B'_{k+1}}$ means that the sector m-area has increased by an amount equal to the m-area of triangle $OB_{k+1}B'_{k+1}$ that is $\mathbf{B_{k+1} \wedge B'_{k+1}}$ (*see* Fig. 27). Hence by (3.1), (3.2) and (3.4).

new sector m-area

$= $ previous sector m-area + m-area of triangle $OB_{k+1}B'_{k+1}$
$= \mathbf{B_1 \wedge B'_1 + B'_2 \wedge B_2 + B_3 \wedge B'_3} + \ldots\ldots + \mathbf{B_{k+1} \wedge B_k} + \mathbf{B_{k+1} \wedge B'_{k+1}}$

and since with $b = k + 1$ the additional border vector introduced is $\mathbf{B_{k+1} \wedge B'_{k+1}}$ (as $k + 1$ is odd) the result (ii) is true also for $b = k + 1$ if it is true for $b = k$.

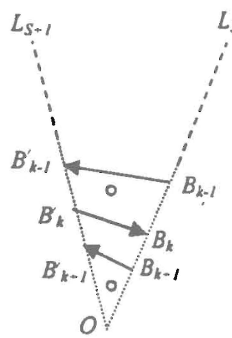


Fig. 27. Adding $\mathbf{B_{k+1}B'_{k+1}}$ increases sector area by an amount equal to the area of triangle $Ob_{k+1}B'_{k+1}$.

Hence since the result is valid for $b = 1$ and $b = 2$, it is valid for all non-negative integers.

Summing it up

The collection of the border segments in the sector flanked by radial lines L_S and L_{S+1} that is:

$$B_1B'_1,\ B_2B'_2, \ldots\ldots B_bB'_b$$

is in fact the same as the set of all border segments of the form:

$$A_{rs}A_{r(s+1)} \quad (1 \leq r \leq m)$$

where A_{rs} and $A_{r(s+1)}$ are *both internal points* of $A_rA'_r$ (and that includes also the end-points).

Moreover the direction of each of the border vector segments of the border segments $B_1B'_1,\ B_2B'_2, \ldots\ldots B_bB'_b$ must obviously be the same as the border vector segment of the corresponding border segment of the form $A_{rs}A_{r(s+1)}$. Hence using the definition of inclusion (4.3).

$$\mathbf{B_1{}^\wedge B'_1 + B'_2{}^\wedge B_2 + B_3{}^\wedge B'_3 + \ldots\ldots + B'_b{}^\wedge B_b} \text{ (or } \mathbf{B_3{}^\wedge B'_b}\text{)}$$
$$= \sum_{r=1}^{r=n} (A_{rs}A_{r(s+1)})^* / (A_rA'_r))\ (\mathbf{A_r{}^\wedge A'_r}) \qquad \textbf{(4.6)}$$

Recalling results (4.2), (4.5), (4.6) and (4.4) we conclude

$$\textbf{m-area} \text{ [polygon]}$$

$$= \sum_{s=1}^{s=m} \textbf{m-area} \text{ [sector}(_{S,S+1})\text{]}$$

$$= \sum_{s=1}^{s=m} \sum_{r=1}^{r=n} (A_{rs}A_{r(s+1)})^* / (A_rA_r))\ (\mathbf{A_r{}^\wedge A'_r})$$

$$= \sum_{r=1}^{r=n} \sum_{s=1}^{s=m} (A_{rs}A_{r(s+1)})^* / (A_rA_r))\ (\mathbf{A_r{}^\wedge A'_r})$$

$$= \sum_{r=1}^{r=n} (\mathbf{A_r{}^\wedge A'_r})$$

5. Finding the Numerical Values

We will now determine the values of the relevant mechanical parameters of a concrete polygonal lamina. As an example we take the lamina P already considered at the beginning of this study (*see* Fig. 28).

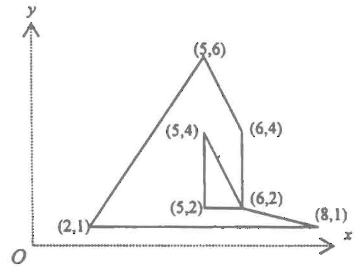


Fig. 28. Polygonal Lamina P with original origin.

Referred to an origin O, lamina P has vertices at:

$$A(2,1),\ B(8,1),\ C(6,2),\ D(6,4),\ E(5,6),\ F(5,2) \text{ and } G(5,4)$$

and border vector set given by:

$$\{\mathbf{AB},\ \mathbf{BC},\ \mathbf{CD},\ \mathbf{DE},\ \mathbf{EA};\ \mathbf{CF},\ \mathbf{FG},\ \mathbf{GC}\}$$

Shifting the origin to vertex A by moving all three axes parallel to themselves to concur at A the position of the vertices will be now given by:

$$A(0,0),\ B(6,0),\ C(4,1),\ D(4,3),\ E(3,5),\ F(3,1) \text{ and } G(3,3)$$

while the border vector set remains the same (*see* Fig. 29).

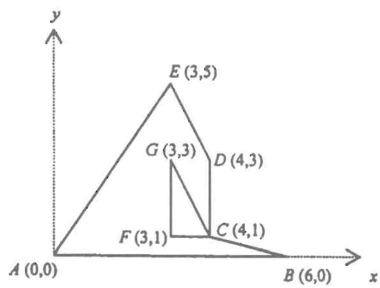


Fig. 29. Polygonal lamina P with shifted origin.

Applying the theorem (4.1) the mechanical area of the polygonal lamina P will be given by:

$$\textbf{m-area}\ [\text{polygon } P] = \sum_{r=1}^{r=m} (\mathbf{A}_r \wedge \mathbf{A}'_r)$$

$$= \mathbf{A} \wedge \mathbf{B} + \mathbf{B} \wedge \mathbf{C} + \mathbf{C} \wedge \mathbf{D} + \mathbf{D} \wedge \mathbf{E} + \mathbf{E} \wedge \mathbf{A} + \mathbf{C} \wedge \mathbf{F} + \mathbf{F} \wedge \mathbf{G} + \mathbf{G} \wedge \mathbf{C}$$

$$= \mathbf{B} \wedge \mathbf{C} + \mathbf{C} \wedge \mathbf{D} + \mathbf{D} \wedge \mathbf{E} + \mathbf{C} \wedge \mathbf{F} + \mathbf{F} \wedge \mathbf{G} + \mathbf{G} \wedge \mathbf{C}$$

since $\mathbf{A} \wedge \mathbf{B}$ and $\mathbf{E} \wedge \mathbf{A}$ are both zero as A is the origin. Hence we can arrive at the following numerical values.

Determining the values of the mechanical parameters of uniform polygonal lamina *P*										
mechanical parameter		**m-area** $[OAB]$ **A^B** $= (x_1,y_1)$^(x_2,y_2)	(6,0)^(4,1)	(4,1)^(4,3)	(4,3)^(3,5)	(4,1)^(3,1)	(3,1)^(3,3)	(3,3)^(4,1)	**m-area** $[P]$ $\sum \mathbf{A}_r$^$\mathbf{A'}_r$
mass		$M\,[OAB]$ $\{(x_1y_2 - x_2y_1)/2\}$	(3)	(4)	(11/2)	(1/2)	(3)	(–9/2)	$\underline{23/2}$
vector first moment		$\mathbf{F}_A[OAB]$ $\{(x_1y_2 - x_2y_1)((x_1 + x_2)\mathrm{i} + (y_1 + y_2)\mathrm{j})/6)\}$	(3)(10i+j)/3	(4)(8i+4j)/3	(11/2)(7i+8j)/3	(1/2)(7i+2j)/3	(3)(6i+4j)/3	(-9/2)(7i+4j)/3	$\underline{(181\mathrm{i}+116\mathrm{j})/6}$
moment of inertia	**about** ***x*-axis**	$\mathrm{I}_{AX}[OAB]$ $\{(x_1y_2 - x_2y_1)\,(y_1^2+y_1y_2+y_2^2)/12\}$	(3)(1)/6	(4)(13)/6	(11/2)(3)/6	(1/2)(3)/6	(3)(13)/6	(-9/2)(13)/6	$\underline{613/12}$
	about ***y*-axis**	$\mathrm{I}_{AX}[OAB]$ $\{(x_1y_2 - x_2\,y_1)(x_1^2+x_1x_2+x_2^2)/12\}$	(3)(76)/6	(4)(48)/6	(11/2)(37)/6	(1/2)(37)/6	(3)(27)/6	(-9/2)(37)/6	$\underline{371/4}$

centre of mass	$\mathbf{R}_A[P] = \mathbf{F}_A[P]\,/\,M[P] = \underline{(181\mathrm{i} + 116\mathrm{j})\,/\,69}$
	$\mathbf{R}_O[P] = \mathbf{R}_A[P] + \mathbf{OA} = \underline{(319\mathrm{i} + 185\mathrm{j})\,/\,69}$
moment of inertia	$\mathrm{I}_{OX}[P] = \mathrm{I}_{GX}[P] + M[P]\,(185/69)^2 = \mathrm{I}_{AX}[P] - M[P]\,116/69)^2 + M[P]\,(185/69)^2 = (613/12) + (23/2)(185^2 - 116^2)/69^2) = \underline{405/4}$
	$\mathrm{I}_{OY}[P] = \mathrm{I}_{GY}[P] + M[P]\,(319/69)^2 = \mathrm{I}_{AY}[P] - M[P]\,181/69)^2 + M[P]\,(319/69)^2 = (371/12) + (23/2)(319^2 - 181^2)/69+2) = \underline{3113/12}$
	$\mathrm{I}_{AZ}[P] = \mathrm{I}_{AX}[P] + \mathrm{I}_{AY}[P] = \underline{863/6},\quad \mathrm{I}_{OZ}[P] = \mathrm{I}_{OX}[P] + \mathrm{I}_{OY}[P] = \underline{1082/3}$

NOTE: In the expressions above, such as $\mathbf{F}_A[OAB]$, *OAB* in the square brackets refers to the general triangular lamina as defined previously while the subscript *A* of **F** refers to the specific vertex of the polygonal lamina *P*.

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