Minimax Theorem and Nash Equilibrium

Manuel Alberto M. Ferreira[#1], Marina Andrade[#2], Maria Cristina Peixoto Matos[+3], José António Filipe[#4], Manuel Pacheco Coelho[*5]

[#]Department of Quantitative Methods
Instituto Universitário de Lisboa (ISCTE-IUL), BRU – IUL
Lisboa Portugal

[+]Department of Mathematics IPV-ESTV - Portugal

[*]SOCIUS & ISEG/UTL – Portugal

[1]manuel.ferreira@iscte.pt
[2]marina.andrade@iscte.pt
[3]cristina.peixoto@math.estv.ipv.pt
[4]jose.filipe@iscte.pt
[5]coelho@iseg.utl.pt

Abstract— **Two important results in Economics, the Minimax Theorem and the Nash Equilibrium are presented together with their mathematical fundaments. The results are obtained in the field of Functional Analysis.**

***Keywords*—** *Minimax Theorem, Nash Equilibrium.*

1. Introduction

In this work it will be seen as the convex sets strict separation result allows obtaining a fundamental result in Game Theory: The Minimax Theorem. The mathematical structure considered is the real Hilbert spaces, see Ferreira *et. al* (2010).

Then the same will be done for Nash Equilibrium using mainly Kakutani's Theorem, see Kakutani (1941), Matos and Ferreira (2006) and Ferreira *et. al* (2010).

2. Minimax Theorem

The context considered is the one of the Games of two players with null sum:

- Be $\Phi(\boldsymbol{x},\boldsymbol{y})$ a two variables real function, $\boldsymbol{x},\boldsymbol{y} \in H$, being H a real Hilbert space.
- Be A and B two convex sets in H.
- One of the players chooses strategies (points) in A in order to maximize $\Phi(\boldsymbol{x},\boldsymbol{y})$ (or minimize $-\Phi(\boldsymbol{x},\boldsymbol{y})$): it is a maximizing player.
- The other player chooses strategies (points) in B in order to minimize $\Phi(\boldsymbol{x},\boldsymbol{y})$ (or maximize $-\Phi(\boldsymbol{x},\boldsymbol{y})$): it is the minimizing player.

The function $\Phi(\boldsymbol{x},\boldsymbol{y})$ is the *payoff function*. $\Phi(\boldsymbol{x}_0,\boldsymbol{y}_0)$ represents, simultaneously, the maximizing player gain and the minimizing player loss in a move where they choose, respectively, the strategies $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$. So the gain of one of the players is identical to the loss of the other. Because of it the game is said of null sum.

In these conditions the game has value *C* if

$$\sup_{\boldsymbol{x}\in A}\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) = C = \inf_{\boldsymbol{y}\in B}\sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y}). \quad (2.1)$$

If, for any $(\boldsymbol{x}_0,\boldsymbol{y}_0), \Phi(\boldsymbol{x}_0,\boldsymbol{y}_0) = C$, $(\boldsymbol{x}_0,\boldsymbol{y}_0)$ is said to be a pair of *optimal strategies*. It will be also a saddle point if it verifies in addition

$$\Phi(\boldsymbol{x},\boldsymbol{y}_0) \leq \Phi(\boldsymbol{x}_0,\boldsymbol{y}_0) \leq \Phi(\boldsymbol{x}_0,\boldsymbol{y}), \boldsymbol{x} \in A, \boldsymbol{y} \in B. \quad (2.2)$$

It is conceptually easy to generalize this situation to a n players null sum game, although algebraically fastidious.

The fundamental result in this section is:

Theorem 2.1 (Minimax Theorem)

A and B are closed convex sets in H and A also limited. $\Phi(\boldsymbol{x},\boldsymbol{y})$ is a real function defined for $\boldsymbol{x}$ in A and $\boldsymbol{y}$ in B such that:

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- $\Phi(\boldsymbol{x},(1-\theta)\boldsymbol{y}_1+\theta\boldsymbol{y}_2) \le (1-\theta)\Phi(\boldsymbol{x},\boldsymbol{y}_1)+\theta\Phi(\boldsymbol{x},\boldsymbol{y}_2)$ for $\boldsymbol{x}$ in A and $\boldsymbol{y}_1,\boldsymbol{y}_2$ in B, $0 \le \theta \le 1$ (that is: $\Phi(\boldsymbol{x},\boldsymbol{y})$ is convex in $\boldsymbol{y}$ for each $\boldsymbol{x}$),

- $\Phi\big((1-\theta)\boldsymbol{x}_1+\theta\boldsymbol{x}_2,\boldsymbol{y}\big) \ge (1-\theta)\Phi(\boldsymbol{x}_1,\boldsymbol{y})+\theta\Phi(\boldsymbol{x}_2,\boldsymbol{y})$ for $\boldsymbol{y}$ in B and $\boldsymbol{x}_1,\boldsymbol{x}_2$ in A, $0 \le \theta \le 1$ (that is: $\Phi(\boldsymbol{x},\boldsymbol{y})$ is concave in $\boldsymbol{x}$ for each $\boldsymbol{y}$),

- $\Phi(\boldsymbol{x},\boldsymbol{y})$ is continuous in $\boldsymbol{x}$ for each $\boldsymbol{y}$.

So (2.1) holds, that is the game has a value.

Demonstration:

Beginning by the most trivial part of the demonstration:

$$\inf_{y\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \le \Phi(\boldsymbol{x},\boldsymbol{y}) \le \sup_{x\in A}\Phi(\boldsymbol{x},\boldsymbol{y})$$

and so

$$\sup_{x\in A}\inf_{y\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \le \inf_{y\in B}\sup_{x\in A}\Phi(\boldsymbol{x},\boldsymbol{y}).$$

Then, as $\Phi(\boldsymbol{x},\boldsymbol{y})$ is concave and continuous in $\boldsymbol{x}\in A$, A convex, closed and limited, it follows that $\sup_{x\in A}\Phi(\boldsymbol{x},\boldsymbol{y}) < \infty$.

Be $C = \inf_{y\in B}\sup_{x\in A}\Phi(\boldsymbol{x},\boldsymbol{y})$.

Suppose now that there is $\boldsymbol{x}_0 \in A$ such that $\Phi(\boldsymbol{x}_0,\boldsymbol{y}) \ge C$, for any $\boldsymbol{y}$ in B. In this case, $\inf_{y\in B}\Phi(\boldsymbol{x}_0,\boldsymbol{y}) \ge C$ or $\sup_{x\in A}\inf_{y\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \ge C$ as it is convenient. Then the existence of such a $\boldsymbol{x}_0$ will be proved.

For any $\boldsymbol{y}$ in B, be $A_{\boldsymbol{y}} = \{\boldsymbol{x}\in A: \Phi(\boldsymbol{x},\boldsymbol{y}) \ge C\}$.
$A_{\boldsymbol{y}}$ is closed, limited and convex. Suppose that, for a finite set $(\boldsymbol{y}_1,\boldsymbol{y}_2,\dots,\boldsymbol{y}_n)$, $\bigcap_{i=1}^{n} A_{\boldsymbol{y}_i} = \emptyset$. Consider the transformation from A to E_n defined by

$$f(\boldsymbol{x}) = (\Phi(\boldsymbol{x},\boldsymbol{y}_1)-C,\Phi(\boldsymbol{x},\boldsymbol{y}_2)-C,\dots,\Phi(\boldsymbol{x},\boldsymbol{y}_n)-C).$$

Call G the $f(A)$ convex hull closure. Be P the E_n closed positive cone. Now it is shown $P\cap G = \emptyset$: in fact, being $\Phi(\boldsymbol{x},\boldsymbol{y})$ concave in $\boldsymbol{x}$, for any $\boldsymbol{x}_k$ in A, $k = 1,2,\dots,n, 0 \le \theta_k \le 1, \sum_{k=1}^{n}\theta_k = 1$,

$$\sum_{k=1}^{n}\theta_k(\Phi(\boldsymbol{x}_k,\boldsymbol{y})-C) \le \Phi\left(\sum_{k=1}^{n}\theta_k\boldsymbol{x}_k,\boldsymbol{y}\right)-C$$

and so the convex extension of $f(A)$ does not intersect P.

Consider now a sequence $\boldsymbol{x}_n$ of elements of A, such that $f(\boldsymbol{x}_n)$ converges for $\boldsymbol{v},\boldsymbol{v}\in E_n$. As A is closed, limited and convex, it is possible to define a subsequence, designated $\boldsymbol{x}_m$ such that $\boldsymbol{x}_m$ converges weakly for an element of A (call it $\boldsymbol{x}_0$). And, for any $\boldsymbol{y}_i$ as $\Phi(\boldsymbol{x},\boldsymbol{y}_i)$ is concave in $\boldsymbol{x}$,

$$\overline{lim}\Phi(\boldsymbol{x}_m,\boldsymbol{y}_i) \le \Phi(\boldsymbol{x}_0,\boldsymbol{y}_i), \text{or } f(\boldsymbol{x}_0) \ge \overline{lim} f(\boldsymbol{x}_m = \boldsymbol{v}).$$

So $P\cap G = \emptyset$. Then, G and P may be strictly separated, and it is possible to find a vector in E_n with coordinates a_k, such that

$$\sup_{x\in A}\sum_{i=1}^{n}a_i(\Phi(\boldsymbol{x},\boldsymbol{y}_i)-C) < \sum_{i=1}^{n}a_i\boldsymbol{e}_i,$$

with the whole a_i greater or equal than zero.

Obviously, the a_i cannot be simultaneously null. So dividing for $\sum_{i=1}^{n}a_i$ and taking in account the convexity of $\Phi(\boldsymbol{x},\boldsymbol{y})$ in $\boldsymbol{y}$

$$\sup_{x\in A}\Phi(\boldsymbol{x},\overline{\boldsymbol{y}})-C < 0, \text{where } \overline{\boldsymbol{y}} = \frac{\sum_{k=1}^{n}a_k y_k}{\sum_{k=1}^{n}a_k}.$$

And, evidently, or $\overline{\boldsymbol{y}}\in B$ or $\inf_{y\in B}\sup_{x\in A}\Phi(\boldsymbol{x},\boldsymbol{y}) < C$. This contradicts the definition of C. So,

$$\bigcap_{i=1}^{n}A_{y_i} \ne \emptyset.$$

In fact,

$$\bigcap_{y\in B}A_{y} \ne \emptyset$$

as it will be seen in the sequence using that result and proceeding by absurd. Note that $A_{\boldsymbol{y}}$ is a closed and convex set and so it is also weakly closed. And being bounded it is compact in the weak topology[1], as A. Calling $G_{\boldsymbol{y}}$ the complement of $A_{\boldsymbol{y}}$ it results that $G_{\boldsymbol{y}}$ is open in the weak topology. So, if

[1] See, for instance, Kantorovich and Akilov (1982).

$\bigcap_{y\in B} A_{\boldsymbol{y}}$ is empty, $\bigcap_{y\in B} G_{\boldsymbol{y}} \supset H \supset A$. But, being A compact, a finite number of $G_{\boldsymbol{y}_i}$ is enough to cover A:

$$\bigcup_{i=1}^{n} G_{y_i} \supset A;$$

that is: $\bigcap_{i=1}^{n} A_i$ is in the complement of A and so it must be $\bigcap_{i=1}^{n} A_{y_i} = \emptyset$, leading to a contradiction.

Suppose then that $\boldsymbol{x}_0 \in \bigcap_{y\in B} A_{\boldsymbol{y}}$. So, in fact $\boldsymbol{x}_0$ satisfies $\Phi(\boldsymbol{x}_0, \boldsymbol{y}) \geq C$, as requested.

Then it follows a Corollary of Theorem 2.1, obtained strengthening its hypothesis.

Corollary 2.1

Suppose that the functional $\Phi(\boldsymbol{x}, \boldsymbol{y})$ defined in Theorem 2.1 is continuous in both variables, separately, and that B is also limited. So, there is an optimal pair of strategies, with the property of being a saddle point.

Demonstration:

It was already seen that exists $\boldsymbol{x}_0$ such that

$$\Phi(\boldsymbol{x}_0, \boldsymbol{y}) \geq C \qquad (2.3)$$

for each $\boldsymbol{y}$. As $\Phi(\boldsymbol{x}_0, \boldsymbol{y})$ is continuous in $\boldsymbol{y}$ and B is limited

$$\inf_{y\in B} \Phi(\boldsymbol{x}_0, \boldsymbol{y}) = \Phi(\boldsymbol{x}_0, \boldsymbol{y}_0) \geq C \qquad (2.4)$$

for any $\boldsymbol{y}_0$ in B [2]. But $\inf_{y\in B} \Phi(\boldsymbol{x}_0, \boldsymbol{y}) \leq \sup_{x\in A} \inf_{y\in B} \Phi(\boldsymbol{x}, \boldsymbol{y}) = C$ and, so

$$\Phi(\boldsymbol{x}_0, \boldsymbol{y}_0) = C. \qquad (2.5)$$

The saddle point property follows immediately from (2.3), (2.4) and (2.5)■.

3. Nash Equilibrium

The formulation and resolution of a game is very important in Game Theory. There are several game solution concepts. But some of these concepts are restrict to a certain kind of games. The most important solution concept was defined by John Nash (Nash, 1950). It will be seen that the Nash equilibrium existence is guaranteed for a large class of games.

E_n is the finite set of available strategies for a player. The Cartesian product of these sets is denoted by E. A typical element of this set is $\boldsymbol{e} = (e_1, e_2, \dots, e_N)$, called a pure strategy profile, where each e_n is a pure strategy for player n.

Definition 3.1

A mixed strategy of a player n is a lottery over the pure strategies of player n.

Observation:

- One of player n's mixed strategies is denoted σ_n and the set of all player n's mixed strategies is denoted Σ_n.
- Thus $\boldsymbol{\sigma}_n = \left(\sigma_n(e_n^1), \sigma_n(e_n^2), \dots, \sigma_n\left(e_n^{k_n}\right)\right)$ where k_n is the number of pure strategies of player n and $\sigma_n(e_n^i) \geq 0, i = 1, 2, \dots, k_n$ and $\sum_{i=1}^{k_n} \sigma_n(e_n^i) = 1$.
- The Cartesian product $\Sigma = \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_N$ is the set of all mixed strategy profiles.
- So, the mixed strategy set for each player is the probability distribution set over its pure strategy set.

Definition 3.2

A n-dimensional simplex defined by the $n+1$ points $\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ in $\mathbb{R}^p, p \geq n$, is denoted $\langle \boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n \rangle$ and is defined by the set

$$\left\{\mathbb{R}^p : \boldsymbol{x} = \sum_{j=0}^{n} \theta_j \boldsymbol{x}_j, \sum_{j=0}^{n} \theta_j = 1, \theta_j \geq 0\right\}.$$

Observation:

- The simplex is non degenerate if the n vectors $\boldsymbol{x}_1 - \boldsymbol{x}_0, \dots, \boldsymbol{x}_n - \boldsymbol{x}_0$ are linearly independent.
- If $\boldsymbol{x} = \sum_{j=0}^{n} \theta_j \boldsymbol{x}_j$, the numbers $\theta_0, \theta_1, \dots, \theta_n$ are called the barycenter coordinates of $\boldsymbol{x}$.

[2] A continuous convex functional in a Hilbert space has minimum in any limited closed convex set.

- The barycentre of the simplex $\langle \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \rangle$ is the point having the whole barycenter coordinates equal to $(n+1)^{-1}$.

Definition 3.3

Call $u_n(\sigma)$ the expected payoff function of player n associated to the mixed strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$.

Definition 3.4

A Nash equilibrium of a game is a profile of mixed strategies $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ such that for each $n = 1, 2, \ldots, N$ for each e_n and $e_n^{'}$ in E_n, if $\sigma_n(e_n) > 0$ then

$$u_n(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, e_n, \sigma_{n+1}, \ldots, \sigma_N) \geq u_n(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, e_n^{'}, \sigma_{n+1}, \ldots, \sigma_N).$$

Observation:

- So an equilibrium is a profile of mixed strategies such that a player knows what strategies the other players will go to choose, and no player has incentive to deviate from the equilibrium since that it cannot improve its payoff through an unilateral change of its strategy.

- A Nash equilibrium induces a necessary condition of strategic stability.

For the sequence it is necessary the following result:

Theorem 3.1 (Kakutani)

Let $M \subset \mathbb{R}^n$ be a compact convex set. Let $F \rightarrow\rightarrow M$ an upper hemi-continuous convex valued correspondence. Then the correspondence F has a fixed point.

Theorem 3.2 (Nash)

The mixed extension of every finite game has, at least, one strategic equilibrium.

Demonstration:

Consider the set-valued mapping that maps each strategy profile, $\boldsymbol{x}$, to all strategy profiles in which each player's component strategy is a best response to $\boldsymbol{x}$. That is, maximizes the player's payoff given that the others are adopting their components of $\boldsymbol{x}$**.** If a strategy profile is contained in the set to which it is mapped (is a fixed point) then it is an equilibrium. This is so because a strategic equilibrium is, in effect, defined as profile that is a best response it itself.

Thus the proof of existence of equilibrium amounts to a demonstration that the best response correspondence has a fixed point. The fixed – point theorem of Kakutani asserts the existence of a fixed point for every correspondence from a convex and compact subset of Euclidean space into itself, provided two conditions hold. One, the image of every set must be convex. And two, the graph of the correspondence (the set of pairs ($\boldsymbol{x}$,$\boldsymbol{y}$) where $\boldsymbol{y}$ is the image of $\boldsymbol{x}$) must be closed.

Now, in the mixed extension of a finite game, the strategies set of each player consists of all vectors (with as many components as there are pure strategies) of non negative numbers that sum to 1; that is, it is a simplex. Thus, the set of all strategy profiles is a product of simplexes. In particular, it is a convex and compact subset of Euclidean space. Given a particular choice of strategies by the other players, a player's best responses consist of all (mixed) strategies that put positive weight on those pure strategies that highest expected payoff among all the pure strategies. Thus, the set of best responses is a sub simplex. In particular, it is convex.

Finally, note that the conditions that must be met for a given strategy to be a best response to a given profile are all weak polynomial inequalities, so the graph of the best response correspondence is closed.

Thus, all the conditions of Kakutani's theorem hold, and this completes the proof of Theorem 3.2.

4. Conclusions

Minimax Theorem, see Neumann and Morgenstern (1947), and Nash Equilibrium, see Nash (1951), were two main achievements that give raise to a great spread of the Game Theory Applications namely in the Economic Domain.

Both concepts were not developed initially in a pure mathematical context. Only latter the problem of rigorous mathematic application to develop these results was considered. A simple and clear way to develop mathematically the Minimax Theorem may be seen in Brézis (1983). For the Nash Equilibrium see for instance Matos and Ferreira (2006) and Ferreira *et. al* (2010).

The due value in practical applications was recognized to Minimax Theorem first than to the Nash Equilibrium. This one had in recent times finally the deserved recognition with the award of the Economics Nobel Prize.

It may be said that the Minimax Theorem is more considered in domains like Operations Research than in Economics. The opposite happens with the Nash Equilibrium. In particular in the famous Cournot-Nash Model, among others.

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