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Modified Gravity: Cosmological Tests

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Publications

During the course of my studies, this gave rise to the following publications:

• Farrugia, G. and Said, J. L., Growth factor in $f(T, \mathcal{T})$ gravity, Phys. Rev. D 94, no. 12, 124004 (2016).

• Farrugia, G. and Said, J. L., Stability of the flat FLRW metric in $f(T)$ gravity, Phys. Rev. D 94, no. 12, 124054 (2016).

• de la Cruz-Dombriz, A., Farrugia, G., Said, J. L. and Saez-Gomez, D., Cosmological reconstructed solutions in extended teleparallel gravity theories with non-vanishing boundary terms, Class. Quant. Grav. 34 no. 23, 235011 (2017).

• Chakrabarti, S., Said, J. L. and Farrugia, G., Some aspects of reconstruction using a scalar field in $f(T)$ gravity, Eur. Phys. J. C77 no. 12, 815 (2017).

• de la Cruz-Dombriz, A., Farrugia, G., Said, J. L. and Saez-Gomez, D., Cosmological bouncing solutions in extended teleparallel gravity theories, Phys. Rev. D 97, no. 10, 104040 (2018).

• Farrugia, G., Said, J. L., Gakis, V. and Saridakis, E.N., Gravitational Waves in Modified Teleparallel Theories, Phys. Rev. D 97, no. 12, 124064 (2018).

• Soudi, I., Farrugia, G., Gakis, V., Said, J. L. and Saridakis, E.N., Polarization of gravitational waves in symmetric teleparallel theories of gravity and their modifications (accepted for publication) (2019).

The first publication discusses the formation of large scale structure in the context of $f(T, \mathcal{T})$ gravity where it shown that, in general, the models exhibit a strong spatial dependence which could be incompatible with observations. This work shall be presented in Chapter 7.

In the second publication, the stability of the Friedmann-Lemaître-Robertson-Walker geometry against homogeneous and isotropic perturbations in the context of $f(T)$ gravity has been investigated. Here, general expressions for the perturbation variables are derived and has been shown that $f(T)$ models can exhibit stable behaviour. The results are reproduced in Chapter 3.

The third and fifth publications, in collaboration with Dr Álvaro de la Cruz-Dombriz and Dr Diego Sáez-Chillón Gómez, discuss the method of reconstruction for the teleparallel Gauss-Bonnet extension, $f(T, T_G)$ gravity. The reconstruction procedure has been applied for various cosmologies, including those known as bouncing cosmologies. By demanding the models to satisfy vacuum constraints, different gravitational Lagrangians were derived. These two works are presented in Chapters 4 and 5.

Next, the work in collaboration with Dr Soumya Chakrabarti discusses the role of a minimally coupled scalar field in the context of $f(T)$ gravity. By considering various different ansatz choices for the scalar field potential, various cosmological solutions are obtained for which a corresponding $f(T)$ gravitational Lagrangian is reconstructed.

Lastly, the recent work in collaboration with Prof. Emmanuel N. Saridakis, Viktor Gakis and Ismail Soudi discuss the so called symmetric teleparallel theories of gravity in the context of gravitational waves. Throughout this work, it has been shown that overall, the models exhibit tensor modes while certain configurations give rise to massive scalar modes.

Contents

List of Figures

List of Tables

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Abstract

The formulation of General Relativity revolutionised the principles to understand the mechanics of gravitation. Under this viewpoint, gravity was no longer a force as postulated by Newton but a manifestation of curvature of the unification of space and time, spacetime. Despite its success, General Relativity started to be incompatible with observations as it required the introduction of dark matter and dark energy. Furthermore, General Relativity is a classical theory and is not renormalisable. Modifications to General Relativity have therefore been proposed. In this work, an alternative proposal, known as Teleparallel gravity, is investigated. This formulation reintroduces gravity as a force by replacing curvature of spacetime through torsion. Furthermore, the theory can reproduce the field equations of General Relativity, referred to as Teleparallel Equivalent of General Relativity, and hence can be deemed as an alternative description of gravitation. Throughout this work, various teleparallel gravity models are studied under different topics: (*i*) stability of the Friedmann-Lemaître-Robertson-Walker geometry through homogeneous and isotropic perturbations, (*ii*) reconstruction of the gravitational Lagrangian, (*iii*) gravitational waves, and (*iv*) large scale structure. It is shown that this formulation is capable of accounting for the accelerating universe without invoking dark energy, while also being stable under homogeneous perturbations, hosting various expansion histories, agreeing with the observed predictions of gravitational waves, while also being able to correctly generate the observed large scale structure.

Acronyms

CDM Cold Dark Matter. 5, 147, 148, 154, 163–166, 172, 182, 184

CMB Cosmic Microwave Background. 3, 5, 7, 27, 30, 32, 84, 146, 147, 150, 151, 193, 194

CPL Chevallier-Polarski-Linder. 46

EoS Equation of State. 28, 41, 42, 46–50, 54–57, 61, 62, 71–75, 79, 82, 85, 91, 95, 157, 173–177

FLRW Friedmann-Lemaître-Robert-Walker. 2, 9, 10, 27, 28, 30, 33, 34, 61, 66, 107, 141, 143, 145–150, 152, 156, 157, 188, 191, 192

GR General Relativity. 2, 3, 5, 7–10, 16–20, 23, 24, 26, 28, 32, 35, 83, 84, 105–107, 113, 114, 117, 127, 128, 130, 139–141, 144, 145, 148, 151, 161, 171, 178, 185, 187, 190, 191, 193

GW Gravitational Wave. 2, 105–108, 112–114, 119–125, 127–130, 132, 133, 135–137, 140–145, 148, 190–192, 194

KAGRA Kamioka Gravitational Wave Detector. 106

LHS Left-hand Side. 68, 76, 77

LIGO Laser Interferometer Gravitational-Wave Observatory. 2, 106

LQC Loop Quantum Cosmology. 89, 93, 103

NP Newman-Penrose. 120–123, 132

ODE Ordinary Differential Equation. 66, 68

PDE Partial Differential Equation. 66, 76, 77, 81, 172

RHS Right-hand Side. 68, 77, 157

Chapter 1

Introduction

The major technological advancements since the end of the 20^{th} century paved the way for more precise measurements about different aspects of the vast universe, hailing the term "the era of precision cosmology". Thanks to these advances, more observational resources are available to probe towards a more fundamental understanding of the forces of nature, primarily that of gravity. The critical understanding of how the fundamental force of gravity works has been investigated throughout the ages, with major breakthroughs such as those by Galileo Galilei regarding free fall and Kepler in his laws of motion of planetary bodies [1]. However, there had not yet been a fully consistent mathematical formulation of gravity.

This was until Isaac Newton in 1686, who, in his famous publication *Philosophiae naturalis principia mathematica*, had formulated the first mathematical treatment of gravity acting as an inverse square law force, now referred to as Newtonian gravity [2]. Newton's theory of gravitation proved to be quite a successful theory as it managed to derive Kepler's laws of motion, while also leading to the discovery of Neptune [1, 3]. However, the theory was far from perfect. Observations of the perihelion of Mercury showed that Newton's gravitational theory was insufficient to correctly account for its observed precession, questioning its validity. This problem persisted until Einstein completely revolutionised the notion of gravity [4].

Albert Einstein reformulated gravity as simply being a manifestation of curvature of spacetime, a unification of space and time, caused by the mass (or equivalently, energy) of objects. Therefore, gravity was no longer a force but a realisation of actual spacetime deformation. The success of Einstein's theory of gravity, General Relativity (GR), was prominent even shortly after its publication in 1915 [5], due to its success of predicting Mercury's perihelion precession, as well as correctly predicting the deflection angle of light according to the gravitational field of the Sun. Furthermore, the theory had long predicted the existence of ripples of spacetime. These ripples are waves propagating throughout this spacetime medium caused by gravity, adequately named gravitational waves (GWs) [6, 7]. However, due to their extremely weak strength, it took almost a century since their postulated existence to be first observed by the Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2015 [8].

Following seminal works by Friedmann [9] and Lemaître [10], and later on by Robertson [11–13] and Walker [14], a geometry describing the expansion of a homogeneous and isotropic universe referred to as the Friedmann-Lemaître-Robert-Walker (FLRW) metric, the idea of an expanding non-static universe started being proposed, an idea which Einstein initially refused to accept, since he instead opted for a static universe [15]. However, the first measurements of Hubble in Ref. [16], and much later through a compilation of measurements between the Supernova Cosmology Project and High-z Supernova Search Team [17], have shown not only the existence of cosmological expansion but also of its accelerated expansion. The latter observation proved to be a substantial problem for GR since such an acceleration can only be generated by some fluid which constitutes negative pressure, a fluid referred to as dark energy.

This was not the only problem. GR predicts that stars orbiting around the centre of a galaxy (its galactic centre) have an orbital speed which decreases with distance, in a similar way to that predicted by Newtonian gravity. However, observations carried out by Rubin, Thonnard and Ford [18,19] had shown that this was not the case. In fact, the velocity turned out to approach a constant value. These velocity-distance profiles, known as galactic rotation curves, could only be described by requiring the galaxy mass to be much larger. However, this is not observed, leaving this missing mass to be termed as dark matter.

Issues with GR do not stop there. GR is a classical theory and hence the theory breaks down for high energies (short distances). This requires quantisation, a property which is impossible to carry out in GR because the theory is non-renormalisable [20, 21].

Going back to the dark energy problem, this unknown fluid exhibiting negative pressure has raised important questions regarding its possible source and nature, which are properties that have not been yet answered. Various proposals have been considered to describe these two properties, a few of which are discussed below.

One of the first proposals was originally considered by Einstein himself (albeit for a different purpose), the (in)famous cosmological constant Λ (a brief historical account is given in Refs. [22, 23]). Einstein considered the cosmological constant only to maintain a static universe, an idea that was abandoned after Hubble's discovery [15]. Despite this initial consideration by Einstein, this cosmological constant has instead taken a pivotal role to not only describe the universe's accelerated expansion (in contrast to a static one) but also the observed anisotropies in the Cosmic Microwave Background (CMB).

In its most basic form, the cosmological constant acts as a source of "repulsive" gravity, a fluid having a constant negative pressure p and energy density ρ satisfying a perfect fluid equation of state $\rho = -p$ [24]. In absence of other sources, the cosmological constant causes the universe to expand at an accelerating, exponential rate.

Despite its success, the cosmological constant still suffers from several conceptual problems. The first major problem, known as the fine-tuning problem, is centred around its very small value. Due to its unknown physical nature, various possible physical mechanisms have been proposed to account for its smallness but to no avail. Some of these different proposed mechanisms are summarised below:

1. **Vacuum energy:** A possible source for Λ is that it represents vacuum energy. However, the theoretical prediction of such energy is extraordinarily larger than the one observed by the cosmological constant, with a difference of about 10^{120} orders of magnitude in GeV^4 units [15, 23, 25–27].[1]

2. **Anthropic Principle:** Anthropic arguments have been proposed as a means to set bounds on the cosmological constant to justify its observed small value. This principle mainly falls under three different versions [15, 23, 25]:

   (a) **Very Weak:** In the weakest version, the existence of humankind is treated as an experimental data point. Such an argument would explain why the half-life of protons is much larger than the lifetime of humankind.

   (b) **Weak:** The fundamental constants appear to be as they are within specific regions in the universe with different sub-universe regions exhibiting different fundamental constants. This is due to this region of space allowing for the existence of life.

   (c) **Strong:** The strong anthropic principle treats the fundamental constants of nature to be what they are observed to be because otherwise, there would be no existence of life to make such measurements in the first place. Thus, everything revolves on the basis of the existence of humankind.

   From these anthropic principles, especially from the weak version, different mechanisms have been proposed in order to set bounds on the magnitude of the cosmological constant. However, they have been mostly unsuccessful

[1] A more detailed account of vacuum energy through quantum mechanics is provided in Ref. [28].

in that regard. For a more detailed review on these mechanisms, see Refs. [23, 25, 27, 29, 30].

3. **Nature of source:** The cosmological constant may be described to originate from two possible sources, either gravitational through modification of the gravitational interaction, or due to some unknown fluid source (or possibly, a combination of both) [20, 23, 24, 27, 31–34]. This also brings forth a second fundamental question of whether the behaviour of dark energy is constant with time, which poses the question whether the same source for dark energy is responsible for both the late time acceleration and the early inflationary phase.

A second problem is known as the coincidence problem, which questions why the observed density magnitudes of matter (including dark matter) and dark energy are roughly of the same order at present times despite their different evolutionary mechanisms [20, 27, 34].

Lastly, recent local and cosmological observations have posed serious tension regarding the role of the cosmological constant, commonly referred to as the H_0 tension, where H_0 represents the present value of the Hubble parameter. Based on the Planck 2018 CMB data, GR with the presence of a cosmological constant and cold dark matter (CDM) (referred to as the ΛCDM model), an observed value of $H_0 = (67.4 \pm 0.5)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [35] is obtained, whereas local measurements using Supernovae Type Ia (SNe Ia) data and Cepheids have resulted in a continuous deviation from the CMB value, with the latest value being $H_0 = (74.03 \pm 1.42)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, meaning a deviation of 4.4σ [36–40].

In light of these observations and problems, one therefore questions the validity of dark energy being sourced by a cosmological constant and hence one should instead consider other possible alternatives. As previously mentioned, the source of dark energy can either be stemming from some matter field or is an artefact of gravitation.

Starting with the former, one alternative is to consider dark energy to be sourced by some scalar field Q. Such an approach can be developed in various different and distinct ways; here, only the model known as quintessence is discussed. For other considerations such as phantom, tachyonic and k-essence, see Refs. [23, 24, 26, 27, 33] and references therein.

Quintessence defines a dark energy fluid source having energy density ρ and pressure p in the form

$$\rho = \frac{1}{2}\dot{Q}^2 + V(Q), \qquad\qquad p = \frac{1}{2}\dot{Q}^2 - V(Q), \tag{1.1}$$

where $V(\phi)$ represents the potential of the scalar field. Through this formulation, the dark energy fluid becomes dynamical in nature with different values of the scalar field at different epochs leading to different behaviours throughout the cosmological history. For quintessence to achieve a cosmological constant-like state, the kinetic energy must be sufficiently smaller than the potential energy i.e. $\frac{1}{2}\dot{Q}^2 \ll V(Q)$ [26, 30, 34].

As a means to tackle the fine-tuning and coincidence problems, a special class of scalar field potentials known as tracker solutions are considered. The role of such models is to construct an evolutionary behaviour which, independent of initial conditions, their final resulting state is always a quasi-cosmological constant [26,30,34,41]. In this way, the coincidence problem would be effectively resolved. However, the model still needs to be fine-tuned in order to obtain such features and the relatively small value of the cosmological constant. Quintessence also suffers from other fundamental questions, including the fundamental role of the scalar field, whether the field couples to other matter components or with gravity, why must it necessarily dominate now, and why must the field be necessarily homogeneous [20, 42].

A second proposal revolves around the possible unification of dark energy and dark matter through a single fluid which behaves as dark matter in high density regimes, and as dark energy in low density regimes. This approach could resolve the coincidence problem [20]. Alternatively, one may also consider an interaction between dark matter and dark energy, a proposal which may resolve the present H_0 tension [43, 44].

The next instance, which investigates dark energy as a manifestation of gravitation, will be the main focus of this work. For instance, introducing more curvature invariants in the action may provide a more natural mechanism to explain the given observations, where certain terms become more dominant during earlier epochs (responsible for inflation and CMB) while others become more dominant at later times (responsible for the observed acceleration) [21, 31, 32].

Overall, for any modification of gravitation to be viable, the theory must agree with the different observations and limits: [20, 27]

1. Recover Newtonian gravity in the weak-field limit;
2. Be able to reproduce galactic dynamics;
3. Correctly predict the formation of large scale structure;
4. Be able to predict the cosmological dynamics (CMB, accelerated expansion, and so forth).

Most modified theories considered in literature end up being severely constrained in the weak-field regime, leading to difficulties in obtaining feasible models which reconcile both local and cosmological observations. Furthermore, extensions beyond GR usually involve higher order field equations which are more difficult to solve both analytically as well as numerically. Nonetheless, an enormous amount of work has been carried out in this field with the aim of reconciling all of the above given criteria.

Throughout this work, an alternative viewpoint in describing the mechanics of gravity is explored. Instead of gravity being described through the manifestation of curvature of spacetime (without torsional effects), this notion is now reversed. Spacetime is now twisted (i.e. has non-zero torsion) but does not manifest any curvature effects. After all, there is no fundamental principle which requires spacetime to be strictly curved without any torsion. This approach is known as teleparallel gravity, one which had originally been considered by Einstein as means to unify gravitation and electromagnetism [45, 46]. Following other seminal works, notably by Møller [47–49], Pellegrini and Plebanski [50], and Cho [51] (a more detailed historical account is given in Ref. [52]), it has been shown that describing the effects of gravity through torsion is equivalent to GR at the level of equations of motion, and hence this theory is referred to as the Teleparallel Equivalent of General Relativity (TEGR). Thus, in essence, most problems encountered in GR are transferred to its teleparallel equivalent.

Nonetheless, the theories are not exactly equivalent as they differ at the level of the gravitational action, despite being equivalent at the level of the field equations. Moreover, gravity is now a gauge theory. Thus, gravity is again being considered a force, not a geometric effect. This can be explained through the motion of free falling particles. In GR, the particles move along a geometric trajectory of curved space (called geodesics) while in teleparallel gravity, the motion of particles moves according to force equations similar to electromagnetism. This leads to an important consequence in the notion of universality (also known as Weak Equivalence Principle, WEP) which states that inertial effects are locally indistinguishable from gravity (or in simpler terms, the inertial mass is equivalent to the gravitational mass). In GR (and other curvature extensions), university is important since if it is violated, particles will no longer move along a geodesic, leading to a breakdown of the theory. On the other hand, teleparallel theories do not need to satisfy this requirement. Teleparallel gravity allows for a separation between the effects of inertia and gravitation, a property which curvature based theories lack [21, 52, 53].

The role of universality puts forth further consequences in explaining the energy of the gravitational field. If universality holds, the energy of gravity cannot be localised as one cannot distinguish between inertia and gravitation (in other words, there always exists a point where the effect of gravity is locally absent) [54–56]. Therefore, GR and other curvature based theories cannot locally define the magnitude of the energy of the gravitational field. On the other hand, since universality is not a requirement in teleparallel theory, this energy can be better defined through a gauge gravitational current [21, 52, 53].

Therefore, even at the level of the Einstein field equations of GR, teleparallel gravity appears to provide a better approach in handling certain phenomenological issues encountered in GR. Nonetheless, extensions of the theory still need be explored to account for the effects of dark energy and dark matter.

Throughout this work, different extensions of teleparallel gravity shall be explored, these being $f(T)$ gravity, $f(T, B)$ gravity, $f(T, T_G)$ gravity and $f(T, \mathcal{T})$ gravity, where B, T_G and $\mathcal{T}$ represent the boundary term, the Teleparallel Equivalent Gauss-Bonnet (TEGB) scalar and the trace of the energy-momentum tensor respectively. These theories, as well as their curvature analogues, have been examined in great detail in literature, both in local and cosmological applications. The work is presented as follows.

In Chapter 2, a brief introduction about the notion of spacetime, curvature and torsion is presented, together with the necessary ingredients to construct the mathematical formulation of GR and TEGR as well as the teleparallel extensions explored throughout this work. A short overview about the homogeneous and isotropic FLRW universe is also presented.

Chapter 3 discusses the notion of stability against homogeneous and isotropic perturbations of the FLRW universe in the context of $f(T)$ gravity. This determines whether the FLRW cosmology remains a viable background description of the universe throughout its expansion history notably during late times. For the two considered viable $f(T)$ models, it will be shown that these satisfy the stability requirement and hence are deemed stable.

Throughout Chapters 4 and 5, the reconstruction procedure is presented and applied for various cosmological behaviours in $f(T, T_G)$ gravity. This method allows for the investigation of whether such teleparallel theory of gravity is able to reproduce specific cosmological histories without invoking any cosmological constant while also being able to recover vacuum solutions (such as Minkowski space). Throughout Chapter 4, simple models, namely power law scale factor, de Sitter phase and ΛCDM cosmology are investigated, whereas in Chapter 5, the so called bouncing cosmologies are explored. The latter offer a possible alternative to the Big Bang and inflation scenario to relieve some of the issues encountered in both formulations. Furthermore, these also offer a possible description of the future final state of the universe. Overall, it is observed that under certain conditions, the investigated models can describe the different cosmological behaviours while recovering the vacuum solutions.

The gravitational wave properties for $f(T)$, $f(T, B)$ and $f(T, T_G)$ gravity in the weak-field regime are then investigated in Chapter 6. For each model, the propagation speed and polarisation states of the gravitational waves are explored. It is found that $f(T)$ and $f(T, T_G)$ gravity exhibit similar features to GR (two tensor polarisations propagating at the speed of light) while $f(T, B)$ gravity exhibits an extra massive (propagating at sub-light speeds) scalar mode similar to that encountered in $f(R)$ gravity.

The formation of large scale structure is then investigated in Chapter 7 under $f(T, \mathcal{T})$ gravity. A brief description of cosmological perturbation theory is presented together with the scalar-vector-tensor (SVT) decomposition of the background FLRW metric. Through the use of scalar perturbations, the growth of structure under sub-horizon regimes is explored. Despite the fact that the considered models are able to match with the observed accelerated expansion, these do not match with the observed structure formation at different sub-horizon scales.

A final summary of the results together with possible future work is presented in Chapter 8. Throughout this work, the reduced Planck unit system $8\pi G = c = \hbar = 1$ shall be used.

Chapter 2

Curvature, Torsion and the FLRW Universe

2.1 Notion of Spacetime: Curvature, Torsion and Metricity

The first step in understanding gravity through geometry is to define a mathematical object representing spacetime which describes the effects of gravitation through its curvature and torsion. A simple consideration is through what is known as a manifold, a surface which locally appears flat (i.e. the geometry appears as $\mathbb{R}^n$ for some dimension n) [57]. This manifold encompasses all the relevant information regarding time and space, however it does not define how these are related. This is achieved through the metric tensor $g_{\mu\nu}$, defined as

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{2.1}$$

where ds^2 represents the line element, a measure of distance between any two points, and $\mathrm{d}x^\mu$ are the differentials of the coordinates [58].[2] The metric tensor, besides describing the geometry of spacetime, provides a notion of past, present and future (hence causality), and replaces the role of the gravitational potential [57]. The term tensor stems from objects defined by the same name, generally represented by $T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}$, which have the property to transform covariantly under a coordinate transformation, meaning that there is no preferred coordinate system [1, 57, 58]. Another property of the metric tensor is that it is non-degenerate, meaning an inverse $g^{\mu\nu}$ can be defined, leading to the property

$$g^{\mu\nu} g_{\mu\rho} = \delta^\nu_\rho, \tag{2.2}$$

where δ^ν_ρ represents the Kronecker delta.

Another important property, which shall prove useful in describing teleparallelism, is the distinction between global and local spacetime. In the former, global refers to the spacetime under the influence of gravitation. On the other hand, local spacetime refers to when the effects of gravity are absent (namely representing the spacetime of Special Relativity). In the latter, the metric tensor is the Minkowski metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. A link between these two spacetimes is achieved through tetrads (or veirbeins), denoted by $e^a{}_\mu$, via the relation [52, 57]

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu. \tag{2.3}$$

Here, the Latin indices correspond to the local spacetime (governed by the Minkowski metric) while Greek indices correspond to the global spacetime (governed by the metric tensor). These tetrads obey the simple relations

$$e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu, \qquad e^a{}_\mu e_b{}^\mu = \delta^a_b. \tag{2.4}$$

[2]Here, Einstein's summation convention is used, where repeated indices are summed over. In this example, $\mathrm{d}s^2 = \sum_{\mu,\nu} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$.

Furthermore, these can be used to exchange between local and global indices, as shown in the following examples

$$e^a{}_\mu V^\mu = V^a, \qquad\qquad e^a{}_\mu V_a = V_\mu. \tag{2.5}$$

Similar relations hold for the inverse tetrad $e_a{}^\mu$.

The next step is to generalise the concept of a derivative for tensors. Although the notion of partial derivatives is well defined, it does not transform as a tensor, making it susceptible to a change of coordinates. In order to preserve covariance, a more general differential operator which transforms as a tensor is defined. This new derivative is called the covariant derivative. Due to the distinction between the local and global spacetime, two variants of the derivative are defined. For the local spacetime, the covariant derivative (denoted by $\mathcal{D}$) for a given arbitrary tensor $A^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ is defined to be [52]

$$\begin{aligned}\mathcal{D}_\sigma A^{a_1\ldots a_k}{}_{b_1\ldots b_l} = \partial_\sigma A^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \omega^{a_1}{}_{c\sigma} A^{ca_2\ldots a_k}{}_{b_1\ldots b_l} + \ldots + \omega^{a_k}{}_{c\sigma} A^{a_1\ldots a_{k-1}c}{}_{b_1\ldots b_l} \\ - \omega^c{}_{b_1\sigma} A^{a_1\ldots a_k}{}_{cb_2\ldots b_l} - \ldots - \omega^c{}_{b_l\sigma} A^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1}c},\end{aligned} \tag{2.6}$$

where $\omega^a{}_{b\mu}$ are called spin connections whose role is to represent the inertial effects of the tetrad frame. The spin connection has the further property that it is anti-symmetric in the first two indices ($\omega_{ab\mu} = -\omega_{ba\mu}$) and it can be expressed in the form [52, 59]

$$\omega^a{}_{b\mu} = -\Lambda_b{}^c \partial_\mu \Lambda^a{}_c, \tag{2.7}$$

where $\Lambda_b{}^c$ represents the local Lorentz matrix. On the other hand, for the global spacetime, the covariant derivative (denoted by ∇) for a given arbitrary tensor $A^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l}$ is defined to be [1, 57, 58]

$$\nabla_\sigma A^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \partial_\sigma A^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \Gamma^{\mu_1}_{\lambda\sigma} A^{\lambda \mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \dots + \Gamma^{\mu_k}_{\lambda\sigma} A^{\mu_1 \dots \mu_{k-1}\lambda}{}_{\nu_1 \dots \nu_l} - \Gamma^{\lambda}_{\nu_1\sigma} A^{\mu_1 \dots \mu_k}{}_{\lambda \nu_2 \dots \nu_l} - \dots - \Gamma^{\lambda}_{\nu_l\sigma} A^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_{l-1}\lambda}, \tag{2.8}$$

where $\Gamma^{\rho}_{\nu\sigma}$ are called connection coefficients.

These connection coefficients, while introduced as means to preserve covariance of the covariant derivative, also describe the geometrical properties of spacetime [57]. In fact, the connection is said to describe the motion of free falling particles while the metric determines the causal structure [20, 21].

To distinguish between the possible different geometries, the Riemann tensor, torsion tensor and metric compatibility are defined, all of which are dependent on the form of the connection. Starting off from the Riemann tensor, which is defined to be [1, 57, 58, 60, 61]

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{2.9}$$

it provides a notion of curvature. This can be interpreted as a measurement of the orientation difference once a vector is parallel transported[3] around a closed curve, as shown in Fig. 2.1a. Therefore, a connection is considered flat (absent of curvature) when $R^{\rho}{}_{\sigma\mu\nu} = 0$.

The torsion tensor is defined as

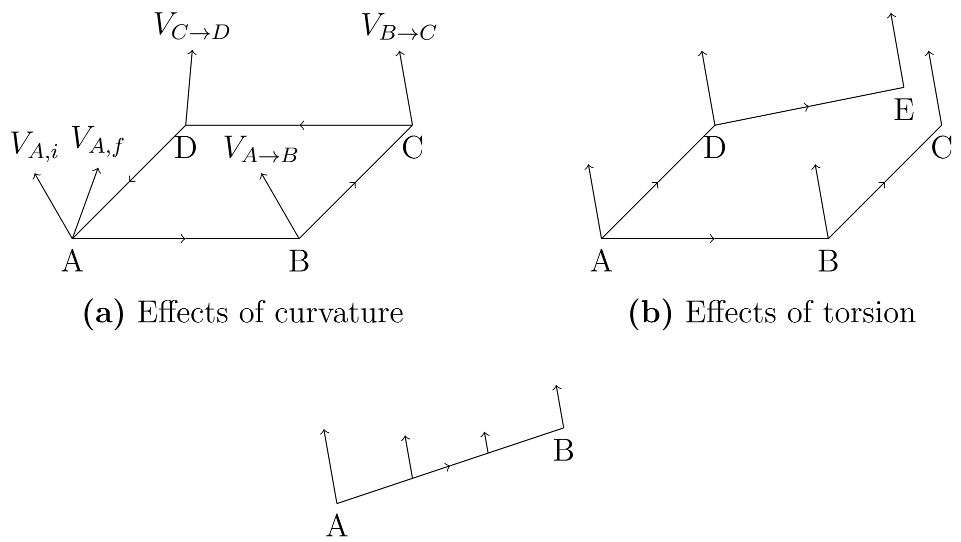
$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu} - \Gamma^{\rho}_{\mu\nu} \tag{2.10}$$

and provides a measure of the non-closure of parallelograms, as vectors are parallel transported [60]. This feature is illustrated in Fig. 2.1b. Thus, a connection is said to be torsionless if $T^{\rho}{}_{\mu\nu} = 0$ [57, 60].

[3]Parallel transport is the process by which a vector preserves its (parallel) orientation as it is being transported along a curve [57, 58, 61].

Lastly, metricity provides a measure of the length variation of the vector as it is being parallel transported between two point. In other words, it determines whether the norm of the vector is preserved along the path [60, 61]. This property is illustrated in Fig. 2.1c. A metric which preserves lengths is said to be metric compatible and obeys the property $\nabla_\mu g_{\nu\rho} = 0$.

In the context of GR, the connection is assumed to be torsionless and obeys metric compatibility, meaning spacetime is fully described through curvature. This yields a unique connection, called the Levi-Civita connection, which is purely expressed in



(a) Effects of curvature

(b) Effects of torsion

(c) Effects of non-metricity

Figure 2.1: Representation of the possible different geometrical effects through the choice of the connection. In **(a)**, the effects of curvature are presented. By taking an initial vector $V_{A,i}$ at A which is then parallel transported around the loop ABCDA, the vector ends up in a different final orientation $V_{A,f}$ caused by the curvature of spacetime. **(b)** represents torsion. Take a vector which is parallel transported along the path AB followed by BC. Now, starting again from A, parallel transport by taking a parallel path to BC, AD, followed by DE which is parallel to AB. The final point E does not coincide with C leading to the non-closure of the parallelogram. This difference accounts to torsion. Finally, **(c)** represents non-metricity. As the vector is being parallel transported from A to B, the magnitude of the vector changes. Illustrations based on Refs. [57, 58, 60].

terms of the metric tensor [57]

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\alpha} \left(\partial_{\nu} g_{\alpha\sigma} + \partial_{\sigma} g_{\alpha\nu} - \partial_{\alpha} g_{\nu\sigma} \right). \tag{2.11}$$

This intrinsic relationship shows that once a metric is defined, the geometry, curvature and motion of particles is straightforwardly determined.

Teleparallel gravity however demands the connection to be flat and that metric compatibility holds, leaving torsion as the remaining non-zero quantity. A connection with these properties is the Weitzenböck connection given to be

$$\widehat{\Gamma}^{\rho}_{\nu\mu} \equiv e_a{}^{\rho} \partial_{\mu} e^a{}_{\nu} + e_a{}^{\rho} \omega^a{}_{b\mu} e^b{}_{\nu}. \tag{2.12}$$

Before progressing further, it is also useful to define the contorsion tensor, which computes the difference between the torsional and curvature parts of the connection [52]

$$K^{\mu}{}_{\alpha\beta} = \widehat{\Gamma}^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\alpha\beta}. \tag{2.13}$$

It can be shown that this can also be expressed in terms of torsion tensors to be

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2} \left(T_{\mu}{}^{\lambda}{}_{\nu} + T_{\nu}{}^{\lambda}{}_{\mu} - T^{\lambda}{}_{\mu\nu} \right). \tag{2.14}$$

2.2 The Gravitational Action

The relationship between curvature (gravity) and mass (energy) is described through a set of field equations. Through the notion of an action of some Lagrangian density and the use of the principle of least action, these field equations can be derived, an approach first devised by Hilbert [62]. The Lagrangian is a scalar, and as gravity is described through a manifestation of curvature, this scalar must be constructed from the Riemann tensor as this encodes this property. As for GR, this is given by the Ricci scalar $R = R^{\mu}{}_{\mu}$, where $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ is the Ricci tensor. To introduce the contributions from matter, the matter Lagrangian, $\mathcal{L}_{\rm m}$, which encodes all the information regarding the matter fields, is added to the total Lagrangian. In this way, the complete gravitational and matter action takes the form

$$\mathcal{S} = \frac{1}{2}\int d^4x\,\sqrt{-g}\,R + \int d^4x\sqrt{-g}\mathcal{L}_{\rm m}, \tag{2.15}$$

where g represents the metric determinant. As the metric tensor encodes all the information regarding gravitation, it represents the dynamical variable of the system [57]. Taking variations with respect to the fundamental field yields the Einstein field equations [1, 54, 57]

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\Theta_{\mu\nu}, \tag{2.16}$$

where $G_{\mu\nu}$ is the Einstein tensor and $\Theta_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm m})}{\delta g^{\mu\nu}}$ is the stress-energy tensor. For the purpose of this work, and as means to account for the vast collection of particles present throughout the universe, a perfect fluid is considered to describe their corresponding overall macroscopic features. The stress-energy tensor for a perfect fluid takes the form [54, 56, 57, 63]

$$\Theta_{\mu\nu} = (\rho + p)\,u_{\mu}u_{\nu} + pg_{\mu\nu}, \tag{2.17}$$

where ρ represents the energy density, p represents the isotropic pressure (meaning that the pressure in each spatial direction is equal) and u_{μ} represents the rest frame fluid velocity, which is taken to be $u^{\mu} = (1, 0, 0, 0)$. This sets the metric tensor component to be $g_{00} = -1$ [56, 57]. Thus, the perfect fluid does not exhibit any shear forces, anisotropies or viscosity effects.

Following the argumentations of GR, a similar thought process is repeated from a teleparallel approach. In this case, the fundamental gravitational variable is the tetrad replacing the role of the metric. Next, instead of the Ricci scalar, a new torsional scalar quantity is constructed, namely the torsion scalar [52]

$$T \equiv S_{\rho}{}^{\mu\nu} T^{\rho}{}_{\mu\nu} = \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}{}^{\rho} T^{\nu\mu}{}_{\nu}, \tag{2.18}$$

where $S_{\rho}{}^{\mu\nu}$ represents the superpotential

$$S_{\rho}{}^{\mu\nu} = \frac{1}{2} \left(K^{\mu\nu}{}_{\rho} + \delta^{\mu}_{\rho} T^{\alpha\nu}{}_{\alpha} - \delta^{\nu}_{\rho} T^{\alpha\mu}{}_{\alpha} \right). \tag{2.19}$$

This replaces gravity to be expressed through torsion rather than curvature. This way, the TEGR gravitational action is given to be

$$\mathcal{S} = \frac{1}{2} \int d^4x \, e \, T + \mathcal{S}_{\mathrm{m}}, \tag{2.20}$$

which yields the field equations [52]

$$e^{-1} \partial_{\nu} \left(e S_{a}{}^{\mu\nu} \right) + \frac{1}{4} e_{a}{}^{\mu} T - T^{b}{}_{\nu a} S_{b}{}^{\nu\mu} + \omega^{b}{}_{a\nu} S_{b}{}^{\nu\mu} = \frac{1}{2} \Theta_{a}{}^{\mu}. \tag{2.21}$$

Here, $\mathcal{S}_{\mathrm{m}} = \int d^4x \sqrt{-g} \, \mathcal{L}_{\mathrm{m}}$ represents the matter action. Observe that the equivalence between TEGR and GR arises from the relationship between the curvature based quantities (Riemann tensor) and the torsional ones (torsion tensor), obtained by combining the definition of the Riemann tensor Eq.(2.9) with that being given by the contorsion tensor Eq.(2.13). As shown in Ref. [52], this reduces the teleparallel field equations to those of GR.

This link between curvature and torsion leads to an intrinsic relationship between the Ricci and torsion scalars,

$$R = -T - \frac{2}{e} \partial_{\mu} \left(e T^{\nu\mu}{}_{\nu} \right) = -T + B, \tag{2.22}$$

where B is called the boundary term. This contribution is absent between the GR and TEGR Lagrangians, which is not surprising as this yields a total derivative.

In other words, at the level of the action, B can be set to vanish at the boundary leading to the equivalence even at the level of the Lagrangian [52, 64]. It is this contribution which distinguishes between the non-equivalence from the curvature and teleparallel extensions, $f(R)$ and $f(T)$ gravity respectively.

2.2.1 $f(T)$ Gravity

The first trivial generalisation of the TEGR Lagrangian involves taking a general function of the torsion scalar, $f(T)$, yielding the torsional analogue of the GR curvature extension $f(R)$ gravity. By considering the gravitational action

$$\mathcal{S} = \frac{1}{2}\int d^4x\, e\, f(T) + \mathcal{S}_{\mathrm{m}}, \tag{2.23}$$

the $f(T)$ gravity field equations turn out to be [59]

$$\frac{1}{4}e_a{}^{\mu} f + f_T\left[e^{-1}\partial_\nu\left(eS_a{}^{\mu\nu}\right) - T^b{}_{\nu a}S_b{}^{\nu\mu} + \omega^b{}_{a\nu}S_b{}^{\nu\mu}\right] + f_{TT}S_a{}^{\mu\nu}\partial_\nu T = \frac{1}{2}\Theta_a{}^{\mu}. \tag{2.24}$$

These equations can be expressed in a more familiar form by expressing them in terms of global indices, which results in

$$f_T G_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(f - Tf_T\right) - 2S_\nu{}^\alpha{}_\mu\partial_\alpha f_T = \Theta_{\mu\nu}. \tag{2.25}$$

Evidently, the field equations are distinct from those encountered in $f(R)$ gravity as $f(R) = f(-T+B) \neq f(T)$. In fact, $f(R)$ gravity is a subclass of $f(T,B)$ gravity as shall be shown shortly. Moreover, the equations are second order in the tetrad field, in contrast to the fourth order equations obtained in $f(R)$ gravity. This difference allows for simpler computations of physical systems [21, 59, 64].

An important point is the issue of local Lorentz invariance of this theory, a topic which has been investigated in numerous works (see Ref. [21] and references therein). Formally, $f(T)$ gravity was stated to be one which breaks such invariance, sourced from the fact that the torsion scalar is not locally Lorentz invariant. At the level of TEGR, this is not an issue as the field equations still obey local Lorentz invariance. In the case of $f(T)$ gravity however, the equations would no longer be locally Lorentz invariant [59, 64]. This caused the theory to be frame dependent, leading to the so called good and bad tetrads, with the former yielding the correct equations of motion, and the others yielding inconsistency issues [65].

This issue arises depending on how the theory is formulated. If the absolute parallelism condition is imposed, meaning setting the spin connection to be strictly zero *a priori*, this forces a specific vierbein frame choice, leading to the apparent breakdown of local Lorentz invariance. However, by relaxing this constraint and introducing the spin connection in the field equations as given in Eq.(2.24), this issue is resolved as the field equations now become local Lorentz invariant [59, 66, 67].

Following the covariant approach in Ref. [59], for a given tetrad, the spin connection which gives rise to the correct field equations can be obtained as follows. Generally, the torsion tensor is a function of both the tetrad and the spin connection, and hence represents a source of gravitation and inertia. However, here torsion is constructed to act as a source of only gravitation, meaning there exists a special form of the spin connection where the effects of inertia are eliminated. In other words, in the absence of gravitation i.e. $G \to 0$, the torsion tensor must vanish. From Eqs.(2.10) and (2.12), the spin connection turns out to be given in the form

$$\omega^{a}{}_{b\mu} = \Gamma^{a}_{b\mu} - e_{b}{}^{\nu}\partial_{\mu}e^{a}{}_{\nu}\big|_{G\to 0} \,. \tag{2.26}$$

2.2.2 $f(T,B)$ Gravity

As mentioned in the previous section, $f(T)$ gravity is not equivalent to $f(R)$ gravity due to the boundary term difference between the two quantities. It is then natural to include a more general gravitational Lagrangian to be dependent on both the torsion and boundary scalars, namely $f(T,B)$ gravity, first considered in Ref. [68]. In other words, the theory is constructed from the gravitational action

$$\mathcal{S} = \frac{1}{2}\int d^4x\, e\, f(T,B) + \mathcal{S}_{\rm m}, \tag{2.27}$$

which yields the field equations [68]

$$\begin{aligned} & e_a{}^{\mu}\Box f_B - e_a{}^{\nu}\nabla^{\mu}\nabla_{\nu} f_B + \frac{1}{2}e_a{}^{\mu}\left(Bf_B - f\right) + 2S_a{}^{\nu\mu}\partial_{\nu}\left(f_B + f_T\right) \\ & + 2e^{-1}\partial_{\nu}\left(eS_a{}^{\nu\mu}\right) f_T - 2T^{\alpha}{}_{\nu a}S_{\alpha}{}^{\mu\nu} f_T = -\Theta_a{}^{\mu}, \end{aligned} \tag{2.28}$$

where $\Box \equiv \nabla^{\mu}\nabla_{\mu}$ is d'Alembert's operator. These can also be expressed in a purely spacetime indexed form to be

$$-f_T G_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu}\right) f_B + \frac{1}{2}g_{\mu\nu}\left(Bf_B + Tf_T - f\right) + 2S_{\nu}{}^{\alpha}{}_{\mu}\partial_{\alpha}\left(f_T + f_B\right) = -\Theta_{\mu\nu}, \tag{2.29}$$

Evidently, one can recover the $f(R)$ field equations in the limit $f(T,B) = f(-T + B) = f(R)$ by identifying $f_B \to f_R$ and $f_T \to -f_R$. Naturally, all analyses carried out in the context of $f(R)$ gravity are therefore recovered. For more general Lagrangians, $f(T,B)$ gravity theory has been investigated in various works, for instance in reconstruction of different cosmological behaviours [69–71], stability of homogeneous and isotropic cosmological solutions [69, 70] and gravitational waves [72]. The latter is explored in further detail in Chapter 6.

It is also remarked that the field equations Eq.(2.28) were derived under the absolute teleparallelism condition and hence are not expressed in a covariant form as those found in $f(T)$ gravity. For this reason, the issue of local Lorentz invariance remarked in Ref. [68] may be resolved once the spin connection is included. Nonetheless, the issue of local Lorentz invariance can be circumvented as long as an appropriate tetrad frame satisfying ${\omega^a}_{b\mu} = 0$ is chosen. As mentioned in Ref. [68], local Lorentz invariance is recovered in the special case when $f(T, B) = f(R)$, as expected.

2.2.3 $f(T, T_G)$ Gravity

Up to this point, the source of curvature which describes the effects of gravity (and hence yields the field equations) is described by the Ricci scalar. However, this is not the only scalar quantity capable of quantifying curvature. In fact, there exists two other alternatives which stem from the Riemann tensor, the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ and another from the Ricci tensor $R_{\mu\nu}R^{\mu\nu}$. However, these two quantities on their own introduce terms which are fourth order in the metric in the field equations, compared to the second order behaviour found in GR. In other words, the system requires more degrees of freedom than GR, as it requires more information regarding the behaviour of the higher order derivatives of the metric [73].

In 1971, Lovelock [74] formulated a particular combination of quadratic curvature terms, which eliminate this undesired fourth order effect, leaving the field equations to be second order. This combination is known as the Gauss-Bonnet invariant, defined to be

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}. \tag{2.30}$$

Despite this attractive feature, this term does not contribute to the field equations in 4-dimensions, as it becomes a total derivative unless higher order dimensions are considered [73]. For this reason, two possible alternatives to maintain the Gauss-Bonnet term contributions in the field equations have been considered: (i) either it is coupled to some scalar field, or (ii) a general function $f(\mathcal{G})$ is considered [73, 75]. In either case, the equations are no longer second order and hence the theory still contains more degrees of freedom than GR.

Focusing on the latter case, this model has been extensively investigated under both local and cosmological regimes. In the former, the Newtonian limit is effectively unchanged, leaving the theory to be compatible with Solar System constraints [31]. From a cosmological viewpoint, these models are capable of describing an accelerating cosmology, as well as generate transitions from deceleration to acceleration [31, 73, 76, 77].

Motivated by these features, the construction of a teleparallel equivalent of this scalar has been investigated. Indeed, this is achieved in Ref. [78]. In a similar notion that the relation between the Ricci scalar and the torsion scalar differ by a boundary contribution, the curvature Gauss-Bonnet scalar and its teleparallel equivalent, T_G, also differ by a boundary term B_G, i.e.

$$\mathcal{G} = T_G + B_G. \tag{2.31}$$

The formal expression of T_G is given by [78]

$$\begin{aligned} T_G = \Big[& K^{\alpha}{}_{\gamma\beta} K^{\gamma\lambda}{}_{\rho} K^{\mu}{}_{\epsilon\sigma} K^{\epsilon\nu}{}_{\varphi} - 2K^{\alpha\lambda}{}_{\beta} K^{\mu}{}_{\gamma\rho} K^{\gamma}{}_{\epsilon\sigma} K^{\epsilon\nu}{}_{\varphi} + 2K^{\alpha\lambda}{}_{\beta} K^{\mu}{}_{\gamma\rho} K^{\gamma\nu}{}_{\epsilon} K^{\epsilon}{}_{\sigma\varphi} \\ & + 2K^{\alpha\lambda}{}_{\beta} K^{\mu}{}_{\gamma\rho} \left(K^{\gamma\nu}{}_{\sigma,\varphi} + \omega^{\gamma}{}_{\theta\varphi} K^{\theta\nu}{}_{\sigma} + \omega^{\nu}{}_{\theta\varphi} K^{\gamma\theta}{}_{\sigma} - \omega^{\theta}{}_{\sigma\varphi} K^{\gamma\nu}{}_{\theta} \right) \Big] \delta^{\beta\rho\sigma\varphi}_{\alpha\lambda\mu\nu}. \end{aligned} \tag{2.32}$$

Here, commas denote partial differentiation with respect to the index, $K^{\gamma\nu}{}_{\sigma,\varphi} := \partial_{\varphi} K^{\gamma\nu}{}_{\sigma}$. Therefore, one can then consider an arbitrary functional dependence on the torsion and the TEGB scalars, say $f(T, T_G)$, to act as the gravitational source. Equivalently, this infers a gravitational (and matter) action of the form

$$\mathcal{S} = \frac{1}{2} \int d^4x \, e \, f(T, T_G) + \mathcal{S}_{\mathrm{m}}. \tag{2.33}$$

which yields the $f(T, T_G)$ field equations [78]

$$
\begin{aligned}
&2\left(H^{[ac]b}+H^{[ba]c}-H^{[cb]a}\right)_{,c}+2\left(H^{[ac]b}+H^{[ba]c}-H^{[cb]a}\right)C^{d}{}_{dc}\\
&+\left(2H^{[ac]d}+H^{dca}\right)C^{b}{}_{cd}+4H^{[db]c}C_{(dc)}{}^{a}+\left(T^{a}{}_{cd}+2\omega^{a}{}_{[cd]}\right)H^{cdb}\\
&-h^{ab}+\left(f-Tf_T-T_Gf_{T_G}\right)\eta^{ab}=\Theta^{ab},
\end{aligned}
\tag{2.34}
$$

where

$$
\begin{aligned}
H^{abc} &= f_T\left(\eta^{ac}K^{bd}{}_{d}-K^{bca}\right)+f_{T_G}\Big[\epsilon^{cprt}K^{qf}{}_{t}\left(2\epsilon^{a}{}_{dkf}K^{bk}{}_{p}K^{d}{}_{qr}+\epsilon_{qdkf}K^{ak}{}_{p}K^{bd}{}_{r}\right.\\
&\left.+\epsilon^{ab}{}_{kf}K^{k}{}_{dp}K^{d}{}_{qr}\right)+\epsilon^{cprt}\epsilon^{ab}{}_{kd}K^{fd}{}_{p}\left(K^{k}{}_{fr,t}-\frac{1}{2}K^{k}{}_{fq}C^{q}{}_{tr}+\omega^{k}{}_{qt}K^{q}{}_{fr}+\omega^{q}{}_{fr}K^{k}{}_{qt}\right)\\
&+\epsilon^{cprt}\epsilon^{ak}{}_{df}K^{df}{}_{p}\left(K^{b}{}_{kr,t}-\frac{1}{2}K^{b}{}_{kq}C^{q}{}_{tr}+\omega^{b}{}_{qt}K^{q}{}_{kr}+\omega^{q}{}_{kr}K^{b}{}_{qt}\right)\Big]\\
&+\epsilon^{cprt}\epsilon^{a}{}_{kdf}\Big[\left(f_{T_G}K^{bk}{}_{p}K^{df}{}_{r}\right)_{,t}+\frac{1}{2}f_{T_G}C^{q}{}_{pt}K^{bk}{}_{q}K^{df}{}_{r}-\frac{1}{2}f_{T_G}C^{q}{}_{pt}K^{bk}{}_{r}K^{df}{}_{q}\\
&+f_{T_G}\left(\omega^{b}{}_{qp}K^{qk}{}_{r}+\omega^{k}{}_{qp}K^{bq}{}_{r}\right)K^{df}{}_{t}+f_{T_G}\left(\omega^{d}{}_{qp}K^{qf}{}_{t}+\omega^{f}{}_{qp}K^{dq}{}_{t}\right)K^{bk}{}_{r}\Big],
\end{aligned}
\tag{2.35}
$$

$$
h^{ab}=f_T\epsilon^{a}{}_{kcd}\epsilon^{bpqd}K^{k}{}_{fp}K^{fc}{}_{q},
\tag{2.36}
$$

$$
C^{c}{}_{ab}=e_{a}{}^{\mu}e_{b}{}^{\nu}\left(\partial_\nu e^{c}{}_{\mu}-\partial_\mu e^{c}{}_{\nu}\right).
\tag{2.37}
$$

Similar to its curvature counterpart, this theory of gravity has proven to be successful in describing an early (inflation) and late time acceleration without invoking dark energy [79,80]. Moreover, the specific model $f(T, T_G) = -T + \alpha\sqrt{T^2 + \beta T_G}$, where α and β are constants, has been investigated in further detail using dynamical systems where a viable cosmology can be realised [81].

2.2.4 $f(T, \mathcal{T})$ Gravity

The next teleparallel extension considers a matter-gravitational coupling through an arbitrary function comprising of the torsion scalar and the trace of the stress-energy tensor $\mathcal{T}$, a model first considered in Ref. [82]. Matter coupling theories (see Ref. [83] for a review on the topic) have been extensively studied in literature, most notably in their curvature analogue, where these exhibit features different from the behaviours discussed so far. Introducing such coupling generalises the possible interaction between gravitation and matter, possibly providing further insight about dark matter and dark energy without introducing new exotic forms of matter.

In the curvature formulation, these theories cause the motion of particles to be non-geodesic and hence lead to the presence of an extra force [84–87]. Furthermore, the energy-momentum tensor is no longer covariantly conserved [85–88]. However, these two issues can be resolved by appropriate choices of the Lagrangian even for non-trivial choices which do not reduce to GR [85, 89–91].

In order to obtain the corresponding field equations, in contrast to the previous teleparallel models, the form of the matter Lagrangian is required. As discussed in detail in [84], the form of the matter Lagrangian for a perfect fluid is not unique. For instance, both $\mathcal{L}_{\rm m} = p$ and $\mathcal{L}_{\rm m} = -\rho$ give rise to the same stress-energy tensor. However, it has been argued that the most natural choice is $\mathcal{L}_{\rm m} = p$ as this preserves the covariance of the field equations. This choice also appears in Refs. [82, 86, 88, 92] amongst others. For this reason, this form shall be considered.

Going back to the teleparallel gravity extension of interest, $f(T,\mathcal{T})$ gravity, the field equations are

$$\begin{aligned}
&\frac{1}{4}e_a{}^{\rho}f + f_T\left[e^{-1}\partial_\sigma\left(eS_a{}^{\rho\sigma}\right) - T^b{}_{\nu a}S_b{}^{\nu\rho} + \omega^b{}_{a\nu}S_b{}^{\nu\rho}\right] \\
&+ S_a{}^{\rho\sigma}\partial_\sigma f_T + \frac{f_{\mathcal{T}}}{2}\left(\Theta_a{}^{\rho} + pe_a{}^{\rho}\right) = \frac{1}{2}\Theta_a{}^{\rho},
\end{aligned} \tag{2.38}$$

which in spacetime indices take the form of

$$f_T G_{\mu\nu} + \frac{g_{\mu\nu}}{2}\left(f - Tf_T\right) + 2S_{\nu\mu}{}^{\sigma}\partial_\sigma f_T + f_{\mathcal{T}}\left(\Theta_{\mu\nu} + pg_{\mu\nu}\right) = \Theta_{\mu\nu}. \tag{2.39}$$

From this expression, the non-conservation of the stress-energy tensor becomes evident as, instead, one obtains

$$\nabla_\mu \Theta^{\mu\nu} = \frac{f_{\mathcal{T}}}{1 - f_{\mathcal{T}}} \left[\left(\Theta^{\mu\nu} + p g^{\mu\nu} \right) \partial_\mu \ln f_{\mathcal{T}} + g^{\mu\nu} \partial_\mu \left(p + \frac{\mathcal{T}}{2} \right) \right] \tag{2.40}$$

unless specific model ansatzes are chosen [93–95]. Similar to the curvature analogue, $f(T, \mathcal{T})$ gravity has been investigated under different regimes, including as a possible viable alternative in explaining the early and late time cosmology [82] along with their stability [93,94,96], to match with the observational constraints given by SNe Ia data on the expansion rate [95] and their ability to reconstruct various cosmological histories [97, 98].

2.3 The FLRW Geometry

Having all necessary ingredients to generate the equations of motion defined, the final step is to construct a geometry (hence a tetrad and a metric) which describes the basic key features of the universe. Two assumptions are sufficient, it being homogeneous and isotropic on sufficiently large scales. This is known as the cosmological principle, a property which is in agreement with observations [58,99,100]. For small scales, presence of inhomogeneity and anisotropy is indeed observed (for example large scale structure and CMB) [56, 99, 101] and hence the following is not directly applicable. These features are investigated in Chapter 7.

A cosmological metric describing these two properties is achieved through the FLRW metric

$$ds^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2 \theta \mathrm{d}\phi^2 \right), \tag{2.41}$$

where $a(t)$ is the scale factor describing the expansion of space, normalised to be $a(t_0) = 1$ at present times t_0, and k is the spatial curvature of the geometry which can be flat ($k = 0$), open ($k = -1$) or closed ($k = 1$). According to Planck 2018 data [35], the universe is observed to be flat, and hence only the case $k = 0$ shall be considered throughout this work. Here, the FLRW metric can be expressed in a simpler Cartesian form,

$$ds^2 = -\mathrm{d}t^2 + a(t)^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{2.42}$$

Next, a tetrad which yields this metric is considered. As the tetrad is not unique, the choice of tetrad must be accompanied by an appropriate spin connection such that the field equations are consistent. For instance, a tetrad which realises a vanishing spin connection is the diagonal tetrad $e^a{}_\mu = \mathrm{Diag}(1, a, a, a)$. For other possible choices, the appropriate spin connection is then computed as shown in Refs. [59,67].

Equipped with the FLRW metric and the perfect fluid description (which is conformal with the cosmological principle assumptions) allows for the generation of the resulting field equations. Starting with GR, the equations (called Friedmann equations) are given to be

$$3H^2 = \rho, \tag{2.43}$$

$$2\dot{H} + 3H^2 = -p, \tag{2.44}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, which describes the expansion rate of the universe. Here, overdots represent derivatives with respect to time. Using the Friedmann equations, or through the conservation of the stress-energy tensor $\nabla_\mu \Theta^{\mu\nu} = 0$, the continuity equation for the perfect fluid is obtained, [58, 100, 101]

$$\dot{\rho} + 3H\left(\rho + p\right) = 0. \tag{2.45}$$

The perfect fluid can be expressed in terms of various matter constituents present in the universe (such as baryons, photons and neutrinos) each satisfying their own continuity equation. A simple parametrisation for the different models is to assume a special equation of state relation of the form $p = \omega\rho$, where ω is called the equation of state (EoS) parameter whose value determines the nature of the fluid (e.g. dust

$\omega = 0$, radiation $\omega = \frac{1}{3}$ and a cosmological constant $\omega = -1$). From the continuity equation, assuming the EoS is a constant reveals the general relation

$$\rho \propto a^{-3(1+\omega)}. \tag{2.46}$$

In this way, the Friedmann equation takes a more simple form

$$H^2 = {H_0}^2 \sum_i \Omega_{i,0} a^{-3(1+\omega_i)}, \tag{2.47}$$

where $\Omega_i := \frac{\rho_i}{3H^2}$ represent the fluid density parameters for each fluid i. Zero subscripts denote that the parameters are evaluated using present time values, a convention which shall be used throughout this work. Finally, the deceleration parameter [56, 99, 100]

$$q := -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} \tag{2.48}$$

determines whether the universe is in a decelerating phase ($q > 0$), transitioning phase ($q = 0$), or accelerating phase ($q < 0$). This shall prove useful to constrain models which are capable of explaining the present time observed acceleration $q_0 < 0$.

Chapter 3

Stability of the FLRW Metric in Torsional Gravity

Study of the background FLRW universe cosmology has been pivotal to the investigation of various phenomena observed in nature, ranging from the earliest of times to present times. Although there have been major developments in constructing the evolution of this background cosmology, it could still be insufficient to describe all observables. While a gravitational model might be able to account for the late time cosmic acceleration, it might not agree with Solar System tests or CMB spectrum data.

Cosmological stability can serve as another useful tool to analyse the behaviour of the gravitational theory under study. However, the term 'stability' can take various meanings, forms and approaches. In this chapter, one particular type of stability shall be investigated in detail. A brief overview of the various forms of stability is provided.

The first type of stability arises from the study of dynamical systems, where the evolution of a set of variables is studied to determine the paths which the system has undertaken. Encountered in many forms in classical mechanics (for instance, the simple pendulum [102]), dynamic systems allow for the investigation of a system of differential equations without the requirement for the system to be solved exactly. Through what are known as critical points, the behaviour of the system is investigated through a phase-space. Stability here refers to the type of behaviours these critical points exhibit, which are classified into three categories: attractors, repellers and saddle points. As the name implies, attractors are points which attract the system towards the critical point, while repellers are points which repel the system away from the critical point. Saddle points are neither attractors nor repellers. For certain trajectories, the system approaches the saddle point but for others, the system moves away from the saddle critical point state.

In the context of cosmology, the dynamical systems approach offers a strong and yet simple tool in examining the cosmological dynamics of a specific gravitational model independent of initial conditions (see Ref. [103] for a detailed review on the topic). As the critical points represent cosmological behaviours while the stability analysis determines the nature of the critical point, a cosmological history can be examined without the requirement of any initial conditions. This turns out to be very useful, especially for theories which invoke higher order terms such as $f(R)$ and $f(R, G)$ theories as the complexity of the field equations make them difficult to extract any analytical (or even numerical) cosmological behaviours [90, 104–109]. Furthermore, this approach can be useful in the study of anisotropic metrics as means to investigate which models lead to an isotropic universe at late times. In other words, isotropisation of the anisotropy cosmology behaves as a late-time attractor (see, for instance, Refs. [110–112]).

Based on the observed cosmological history, it is postulated that the universe passes through a sequence of phases in the order of inflation, radiation and matter domination epochs onward to an accelerating universe (which is potentially asymptotically de Sitter). As discussed in Ref. [113], the inflationary epoch is expected to be unstable, and matter and radiation epochs to be saddle points. As it is unknown what the final stage of the universe is going to be, the stability of the acceleration phase is left open. However, if the universe is indeed approaching a final accelerated stage, then this should be an attractor [105]. Models can then be constrained by demanding that these phases are realised.

The second type of stability, which will be investigated in greater detail, deals with small perturbations of the background cosmology. This can be approached in two distinct ways. The first is through perturbations of the background cosmology parameters, primarily the scale factor and matter energy density components, while the second through a more general perturbation of the metric tensor. In the former, perturbations are taken to be homogeneous while in the latter they are inhomogeneous and anisotropic.

The case of homogeneous perturbations, which will be the main topic of this chapter, deals with small perturbations which are only time dependent. Their role is to investigate whether the initial background cosmology retains its behaviour with time, basically questioning the validity of the background cosmological solution at various times. If these perturbations do grow large, this means that the initial assumption of the background cosmology is insufficient to correctly describe the universe during times where these perturbations become significant.

Inhomogeneous and anisotropic perturbations provide a richer picture of the universe as will be discussed in further detail in Chapter 7. These include the CMB spectrum, growth structure and gravitational waves amongst others. In the case of this type of perturbation, stability determines whether these perturbations grow or decay with time, and are thus generally different from the homogeneous ones. For instance, consider an Einstein static universe in GR. Under homogeneous perturbations, the model is never stable, which infers either a gravitational collapse or an expansion (towards an attractor solution of a de Sitter form). However, inhomogeneous perturbations can be stable, provided $\frac{\partial \rho}{\partial p} > \frac{1}{\sqrt{5}}$. In other words, while the homogeneous perturbation deviates from the Einstein static behaviour, the inhomogeneous perturbations do not grow as they are damped by the background evolution [114]. A similar behaviour is observed in the $f(R)$ case as illustrated in Ref. [115]. Inhomogeneous instability has also been investigated in the case of $f(T)$ gravity [116] and $f(T,\mathcal{T})$ gravity [82] where classes of functions have been shown to be viable.

Although these two types of perturbations are distinctively different, there are instances in which the stability conditions can indeed coincide. For instance, this is observed for a de Sitter phase in Refs. [117, 118] in the case of $f(R)$ gravity and Refs. [75, 119] in the case of $f(G)$ gravity. Thus, homogeneous stability can prove as a simple way to examine the stability of the theory without giving a detailed account of the resulting phenomenology from an inhomogeneous point of view.

3.1 Homogeneous Perturbations

The study of homogeneous perturbations will be presented in the context of $f(T)$ gravity in a flat FLRW metric background. Although these types of perturbations have been previously considered in Refs. [120, 121] (and in the extension to $f(T,B)$ gravity in Ref. [69]), only specific cosmological behaviours were investigated, while here, a broader class of gravitational models are investigated. Furthermore, it will be shown that the perturbation parameters take a simple analytical form in the case of $f(T)$ gravity. For this chapter, the functional form of $f(T)$ is assumed to take $f(T) = T + F(T)$ for some function $F(T)$. In the absence of the latter function, TEGR is obtained. With this in mind, the $F(T)$ contribution represents the deviation from TEGR and hence acts as a gravitational source for dark energy.

For a spatially flat FLRW metric Eq.(2.42), the torsion scalar takes the simple form $T = -6H^2$ with modified Friedmann equations [21]

$$F - T - 2TF_T = 2\rho, \tag{3.1}$$
$$\dot{H} = -\frac{\rho + p}{2(1 + F_T + 2TF_{TT})}. \tag{3.2}$$

It can be easily observed that Eq.(3.2) is valid provided that the denominator is non-zero. This yields the simple constraint $F(T) \neq -T + c_1\sqrt{-T} + c_2$, where $c_{1,2}$ are integration constants.

The c_1 term takes the role of a boundary contribution as this does not give rise to any cosmological dynamics [122–126]. For this reason, this term will be ignored and will not be included in the work that follows. On the other hand, c_2 plays a role of a cosmological constant. Overall, the gravitational action will only consist of this contribution, meaning that the resulting cosmology would be de Sitter throughout the whole universe's history. Given the clear evidence of other epochs, this gravitational action is clearly not suitable to describe the overall expansion history [127]. Thus, this condition will not be satisfied for any physically motivated cosmological behaviour.

As for the matter component, this obeys the usual continuity equation Eq.(2.45), sourced by ordinary matter and radiation, with the $F(T)$ contribution taking the role of dark energy. Furthermore, the fluid components are assumed to be non-interacting, and hence individually satisfy their associated continuity equations

$$\dot{\rho}_\mathrm{M} + 3H\rho_\mathrm{M} = 0 \quad \Longrightarrow \quad \rho_\mathrm{M} = \rho_{\mathrm{M},0}a^{-3}, \tag{3.3}$$
$$\dot{\rho}_\mathrm{R} + 4H\rho_\mathrm{R} = 0 \quad \Longrightarrow \quad \rho_\mathrm{R} = \rho_{\mathrm{R},0}a^{-4}, \tag{3.4}$$

where the subscripts 'M' and 'R' represent the matter and radiation components respectively. As stated previously, the $F(T)$ contribution acts as a source for dark energy as some sort of exotic fluid, which has energy density and pressure [21]

$$\rho_T \equiv TF_T - \frac{F}{2}, \tag{3.5}$$

$$p_T \equiv -\rho_T + 2\dot{H}(F_T + 2TF_{TT}). \tag{3.6}$$

Consequently, an associated equation of state can be defined for this exotic fluid

$$\omega_T \equiv \frac{p_T}{\rho_T} = -1 - 4\dot{H}\frac{F_T + 2TF_{TT}}{F - 2TF_T}. \tag{3.7}$$

For instance, if a standard cosmological constant-like behaviour is desired, one needs to set $\omega_T = -1$, which is satisfied when $f_T + 2Tf_{TT} = 0$, leading to $F(T) = c_3$, a cosmological constant. Therefore, the model simply reduces to standard TEGR with a cosmological constant as expected.

With this reformulation, the modified Friedmann equations Eqs.(3.1) and (3.2) can be recast in a more familiar form associated to the Friedmann equations obtained in GR

$$-T = 2(\rho + \rho_T), \tag{3.8}$$

$$2\dot{H} = -(\rho + p + \rho_T + p_T). \tag{3.9}$$

Trivially, it can easily be realised that this gravitational exotic fluid also obeys the standard continuity equation

$$\dot{\rho}_T + 3H(1 + \omega_T)\rho_T = 0. \tag{3.10}$$

As the equation of state parameter is not necessarily constant, a relation for its density evolution is not recovered in general, unless specific configurations are considered (see, for instance, Ref. [128]).

In the case of $f(T)$ gravity, the deceleration parameter Eq.(2.48) takes the form

$$q = -1 - \frac{3}{2T}\frac{F - T - 2TF_T}{1 + F_T + 2TF_{TT}} + \frac{1}{2}\frac{\Omega_{\mathrm{R}}}{1 + F_T + 2TF_{TT}}, \tag{3.11}$$

where the Friedmann equations Eqs.(3.1) and (3.2) have been used. Observe that in the limit when the radiation density is negligible, the deceleration parameter can be fully expressed in terms of the torsion scalar.

The next step is to derive the evolution equations for homogeneous perturbations and hence determine the stability of the cosmology for a given $F(T)$ function. Such perturbations are obtained by taking the background solutions and perturbing them about some small parameter. For instance, the perturbation about the scale factor is [120, 129–134]

$$a(t) \to a(t)\left[1 + \delta_a(t)\right], \tag{3.12}$$

where $a(t)$ now represents the background evolution which obeys Eqs.(3.1) and (3.2), while $\delta_a(t)$ represents the homogeneous perturbation. In order for said perturbation to be small, it is required that $|\delta_a| \ll 1$. Hence, the Friedmann equations will be evaluated by order of δ_a. If this perturbation grows with time, then the cosmology will be considered to be unstable.

Observe that since the torsion scalar is expressed purely in terms of the Hubble parameter, it is more convenient to consider its perturbation and obtain its evolution instead. Based on the perturbation regime for the scale factor Eq.(3.12), one can assume that the Hubble parameter will take a similar form [69, 75, 117, 118, 121]

$$H(t) \to H(t)\left[1 + \delta(t)\right], \tag{3.13}$$

with an associated perturbation $\delta(t)$ satisfying $|\delta| \ll 1$. Indeed, it is trivial to show that these parametrisations are directly related. Starting from the definition of the

Hubble parameter, based on Eq.(3.12) and the fact that $|\delta_a| \ll 1$, one finds

$$\bar{H}(t) = \frac{\dot{a}(t)\left[1+\delta_a(t)\right] + a(t)\dot{\delta}_a(t)}{a(t)\left[1+\delta_a(t)\right]} = H(t) + \dot{\delta}_a(t), \tag{3.14}$$

which leads to the simple relation $\dot{\delta}_a = H\delta$. Here, $\bar{H}$ represents the Hubble parameter describing the background and the first order homogeneous and isotropic correction evolution.

Lastly, the matter and radiation density perturbations are also defined

$$\rho_{\rm M}(t) \rightarrow \rho_{\rm M}(t)\left[1+\delta_{\rm M}(t)\right], \qquad \rho_{\rm R}(t) \rightarrow \rho_{\rm R}(t)\left[1+\delta_{\rm R}(t)\right], \tag{3.15}$$

where $\delta_{\rm M}$ and $\delta_{\rm R}$ represent the matter and radiation perturbations respectively, which satisfy the condition $|\delta_{\rm M,R}| \ll 1$. Naturally, the background evolution does not determine the evolution of the perturbation of the matter components. As such, these are instead sourced by the background perturbations, meaning that $\delta \sim \delta_{\rm M,R}$.

With these assumptions, from the Friedmann and continuity equations Eqs.(3.1), (3.3) and (3.4), the perturbation relations are given to be

$$-T(1+F_T+2TF_{TT})\delta = \left(\rho_{\rm M}\delta_{\rm M} + \rho_{\rm R}\delta_{\rm R}\right), \tag{3.16}$$

$$\dot{\delta}_{\rm M} + 3H\delta = 0, \tag{3.17}$$

$$\dot{\delta}_{\rm R} + 4H\delta = 0. \tag{3.18}$$

From the matter and radiation perturbation relations, it easily results that the latter perturbations are simply related by

$$\delta_{\rm M} = \frac{3}{4}\delta_{\rm R}. \tag{3.19}$$

This relationship is expected as in this formulation no matter interaction is assumed.

Taking Eq.(3.19) in mind, Eq.(3.16) can be expressed as

$$-T(1+F_T+2TF_{TT})\delta = \left(\rho_{\mathrm{M}} + \frac{4}{3}\rho_{\mathrm{R}}\right)\delta_{\mathrm{M}} = -2(1+F_T+2TF_{TT})\dot{H}\delta_{\mathrm{M}},$$

$$\implies T\delta = 2\dot{H}\delta_{\mathrm{M}}. \tag{3.20}$$

Here, Eq.(3.2) has been used. Therefore, a direct relation between the Hubble and matter perturbation has been obtained, which can be used to solve for the evolution. For instance, from Eq.(3.17), one finds

$$0 = \dot{\delta}_{\mathrm{M}} - \frac{\dot{H}}{H}\delta_{\mathrm{M}} \implies \delta_{\mathrm{M}} = kH, \tag{3.21}$$

for some constant k, which implies the Hubble perturbation is simply given to be

$$\delta = -\frac{\dot{H}}{3H}k. \tag{3.22}$$

The constant k can be obtained by evaluating the expressions at some time, which is taken to be the present time to normalise it to observed present values. As the value of k appears in both the matter and Hubble perturbations, this sets a relationship between the presents values of these quantities. Starting from the matter perturbation relation Eq.(3.21), k is given to be

$$k = \frac{\delta_{\mathrm{M},0}}{H_0}, \tag{3.23}$$

while from the Hubble perturbation, Eq.(3.22) is

$$k = -3\delta_0\frac{H_0}{\dot{H}(t_0)}. \tag{3.24}$$

Together, these equations relate the present values through

$$\delta_{\mathrm{M},0} = -3\delta_0\frac{{H_0}^2}{\dot{H}(t_0)}, \tag{3.25}$$

which is expected as seen from Eq.(3.20).

Having obtained the evolution of the perturbation parameters, the next step would be to determine the stability of the parameters based on the particular $F(T)$ model. As seen from the resulting expressions, the perturbations are entirely dependent on the Hubble parameter and hence must decrease with time in order to retain stability. Modifications of the gravitational Lagrangian through $F(T)$ affects the growth of this parameter as seen from the background Friedmann equations and hence the choice of $F(T)$ is crucial to determine which models are stable. The choice of this $F(T)$ function is tackled under two distinct ways. Furthermore, for the purpose of this work, the models considered are those that are able to realise the late time acceleration.

Firstly, following Ref. [135], a set of possible viable $F(T)$ functions have been considered to be alternative choices to ΛCDM through analysis of various cosmological parameters. In particular, two viable models will be investigated in detail, the power law model $F(T) = \alpha(-T)^n$ [136] and the exponential model $F(T) = \alpha T_0 \left(1 - \exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right)$ [122], for which technical details are explored in their relevant sections. Here, α, $p \neq 0$ and n are constants.

Another second viable approach is to consider specific cosmological evolutions *a priori*, those which are not determined according to an $F(T)$ model choice. As will be discussed in the following section, this choice restricts uniquely the $F(T)$ Lagrangian which does not necessarily correspond to the latter two models.

Before examining said stability, observe that the exotic fluid can also be studied to see how its perturbation evolves with time. In spirit of the construction of the homogeneous perturbation for the matter components Eqs.(3.15), the exotic fluid is parametrised as

$$\rho_T \to \rho_T \left[1 + \delta_T(t)\right], \tag{3.26}$$

with $\delta_T(t)$ representing its perturbation. From the definition of its energy density

Eq.(3.5), the latter is related to the Hubble perturbation as

$$\delta_T = \frac{2T\left(F_T + 2TF_{TT}\right)}{2TF_T - F}\delta = \frac{T(1+\omega_T)}{2\dot{H}}\delta = (1+\omega_T)\delta_{\mathrm{M}}, \tag{3.27}$$

where Eqs.(3.2), (3.7) and (3.20) have been used to obtain the final expression in terms of δ_{M}. Furthermore, using Eq.(3.21), the perturbation can be renormalised to its present value to be

$$\frac{\delta_T(t)}{\delta_{T,0}} = \frac{1+\omega_T(t)}{1+\omega_{T,0}}\frac{H(t)}{H_0}. \tag{3.28}$$

Therefore, the choice of $F(T)$ determines ω_T and the evolution for the Hubble parameter, which consequently yields the growth history of the perturbation. Observe that for a cosmological constant, i.e. in the case when $\omega_T = -1$, the perturbation is zero, which is expected as due to its repulsive nature, the fluid can never overdense [137–139].

3.2 Specific Choices of Scale Factor

Choosing a scale factor *a priori* allows an easy determination of the Hubble parameter, which facilitates the study of stability. However, this approach comes at a price. The choice of scale factor determines the Hubble parameter, and hence the torsion scalar's evolution, uniquely. However, the Friedmann equations in Eqs.(3.1) and (3.2) imply that only a specific choice of the $F(T)$ Lagrangian satisfies the equations, essentially setting a constraint on the model. This approach is the basis of reconstruction, which will be explored in further detail in Chapter 4. For the purpose of this section, the resulting $F(T)$ function and resulting stability will be considered.

The choice of scale factor can be arbitrarily set, even though it should be chosen with physical motivation. Two models are thus explored, (*i*) power-law scale factors $a(t) \propto t^n$ for some constant n, and (*ii*) de Sitter behaviour $a(t) \propto e^{Ht}$ where the

Hubble parameter becomes a (positive) constant.

The power-law behaviour is a simple yet rich model with various applications in different aspects of cosmology. When a certain fluid with EoS ω starts to dominate the cosmic expansion, the scale factor takes the approximate form $a(t) \propto t^{\frac{2}{3(1+\omega)}}$ provided $\omega \neq 1$. This implies that for fluids with EoS $\omega < -\frac{1}{3}$, an accelerating universe is achieved which corresponds to $n < 0$ or $n > 1$. Use of such models to describe the accelerating universe appear in Refs. [140–142] and references therein.

The use of power-law scale factors also appears in the context of inflation (known as power-law inflation) which has been considered as a simple model that resolves the so called horizon and flatness problems.[4] Inflation is realised only for $n > 1$, for which the corresponding inflaton potential takes an exponential form [146, 147]. However, the model suffers from issues. The resulting scalar and tensor spectral indices become constant, with values that do not match with observable data [148–150] (this was ruled out by 2013 Planck data [151] as well as recent Planck data [152]), unless deviations from a standard inflaton potential are considered (for instance, k-essence [153]). Furthermore, standard power-law inflation lacks an exit mechanism as it becomes an everlasting accelerated expansion, requiring the need of other factors to exit the inflationary phase. Such considerations appear in, for instance, Ref. [153].

Towards the beginning of the universe, a power-law scale factor can also be used to construct a bounce known as a superbounce, where the scale factor is modified to be $a(t) \propto (-t + t_c)^{\frac{2}{c^2}}$, where t_c represents the crunch time (the time when the universe collapses) and $c > \sqrt{6}$ [154, 155]. Originally considered in [156], superbounce models are used to construct a universe which collapses and rebirths through a Big Bang without a singularity (a scale factor for which the minimal value is non-zero – see

[4] The horizon problem discusses how photons which could not have been in casual contact still have the same uniform temperature at the time of decoupling. The flatness problem requires the universe to be extremely fine tuned at early times to obtain the observed flat universe today [143–145].

also [157] for further details).

Lastly, in the special case when $n = 1$, a linear scale factor is obtained which represents a coasting cosmology. This special case has appeared in various sources as it does not suffer from the horizon problem, does not constrain the matter parameter and hence cures the flatness problem, gives an age estimate (in the absence of a cosmological constant) which agrees with the estimated age of old clusters, and can also tackle the cosmological constant problem [158–160]. Furthermore, this simple model agrees with SNe Ia data [140, 161].

On the other hand, de Sitter cosmology has been a recurring feature of multiple studies appearing in both the early and the late-time universe. Characterised by an exponential scale factor, this corresponds to a period dominated by a cosmological constant-like fluid. In the case of inflation, for instance, a de Sitter expansion is common based on the assumption that inflation is driven by a scalar field, causing the exponential expansion [143, 162, 163]. This leads to the observed scalar-dominated power spectrum supported by Planck 2018 data [152], while also resolving the horizon problem. At late times, a dark energy fluid with an EoS close to that of a cosmological constant, leads to another accelerating domination period in de Sitter phase. Indeed, in the context of GR with a cosmological constant, this corresponds to the final asymptotic state of the universe. Current observational data also seems to infer a present dark energy component having $\omega \simeq -1$ [35], leaving the de Sitter cosmology to be a possible asymptotic future universe state.

3.2.1 Power-law Model

Starting with the power-law scale factor $a(t) \propto t^n$, the Hubble parameter becomes $H = \frac{n}{t}$. The perturbation parameters are thus simply given to be

$$\delta \propto t^{-1}, \qquad \delta_{\mathrm{M}} = \frac{4}{3}\delta_{\mathrm{R}} \propto t^{-1}, \tag{3.29}$$

which indicate stability at late times. For this behaviour, the torsion scalar turns out to be $T = -\frac{6n^2}{t^2}$. This yields a relation between time and the torsion scalar, which infers a direct relationship between the torsion scalar and the scale factor. Indeed, this is achieved through

$$\frac{t}{t_0} = \left(\frac{T}{T_0}\right)^{-\frac{1}{2}} \implies a(t) = \left(\frac{T}{T_0}\right)^{-\frac{n}{2}}. \tag{3.30}$$

Using the Friedmann equation Eq.(3.1), an ordinary second order differential equation for $F(T)$ is obtained,

$$F - T - 2TF_T = -T_0 \left[\Omega_{\mathrm{M},0}\left(\frac{T}{T_0}\right)^{\frac{3n}{2}} + \Omega_{\mathrm{R},0}\left(\frac{T}{T_0}\right)^{2n}\right], \tag{3.31}$$

the solution for which is dependent on the choice of n. For $n \neq \frac{1}{3}, \frac{1}{4}$, the solution is

$$F(T) = -T + \frac{\Omega_{\mathrm{M},0}T_0}{3n-1}\left(\frac{T}{T_0}\right)^{\frac{3n}{2}} + \frac{\Omega_{\mathrm{R},0}T_0}{4n-1}\left(\frac{T}{T_0}\right)^{2n}, \tag{3.32}$$

which leads for the exotic fluid perturbation to evolve as

$$\delta_T = \frac{3n\Omega_{\mathrm{M},0}\left(\frac{t_0}{t}\right)^{3n} + 4n\Omega_{\mathrm{R},0}\left(\frac{t_0}{t}\right)^{4n} - 2\left(\frac{t_0}{t}\right)^2}{\Omega_{\mathrm{M},0}\left(\frac{t_0}{t}\right)^{3n} + \Omega_{\mathrm{R},0}\left(\frac{t_0}{t}\right)^{4n} - \left(\frac{t_0}{t}\right)^2}\frac{k}{3t}. \tag{3.33}$$

Irrespective of value of n, the first term approaches a constant value with increasing time and hence the perturbation decays with time in the order of t^{-1}, leading towards stability. When $n = \frac{1}{3}$, the $F(T)$ function takes the form

$$F(T) = -T + \frac{1}{2}\Omega_{\mathrm{M},0}T_0\sqrt{\frac{T}{T_0}}\ln\left(\frac{T}{T_0}\right) + 3\Omega_{\mathrm{R},0}T_0\left(\frac{T}{T_0}\right)^{2/3}, \tag{3.34}$$

with associated exotic fluid perturbation evolution

$$\delta_T = \frac{3\Omega_{\mathrm{M},0} + 4\Omega_{\mathrm{R},0}\left(\frac{t_0}{t}\right)^{\frac{1}{3}} - 6\frac{t_0}{t}}{\Omega_{\mathrm{M},0} + \Omega_{\mathrm{R},0}\left(\frac{t_0}{t}\right)^{\frac{1}{3}} - \frac{t_0}{t}}\frac{k}{9t}. \tag{3.35}$$

At late times, $\delta_T \approx \frac{k}{3t}$ and hence is also stable. Lastly, when $n = \frac{1}{4}$, one finds

$$F(T) = -T - 4\Omega_{\mathrm{M},0}T_0 \left(\frac{T}{T_0}\right)^{3/8} + \frac{1}{2}\Omega_{\mathrm{R},0}T_0\sqrt{\frac{T}{T_0}}\ln\left(\frac{T}{T_0}\right), \tag{3.36}$$

which yields an exotic fluid perturbation evolution in the form

$$\delta_T = \frac{3\Omega_{\mathrm{M},0} + 4\Omega_{\mathrm{R},0}\left(\frac{t_0}{t}\right)^{\frac{1}{4}} - 8\left(\frac{t_0}{t}\right)^{\frac{5}{4}}}{\Omega_{\mathrm{M},0} + \Omega_{\mathrm{R},0}\left(\frac{t_0}{t}\right)^{\frac{1}{4}} - \left(\frac{t_0}{t}\right)^{\frac{5}{4}}}\frac{k}{12t}. \tag{3.37}$$

Similar to the previous scenario, at late times $\delta_T \approx \frac{k}{4t}$, which is once again stable. Therefore, this shows that power-law evolution is stable at late times for any choice of n. As a closing note, the reconstructed $F(T)$ solution for power-law behaviours also appear in Refs. [69, 128, 154, 164, 165]. Furthermore, a power-law behaviour results from a Noether symmetry consideration with the previously obtained reconstructed $F(T)$ solutions as shown in Refs. [166–172].

3.2.2 de Sitter Behaviour

For the case of de Sitter cosmology, the Hubble parameter achieves a constant (positive) value, leading to the behaviour

$$\delta = 0, \qquad \delta_{\mathrm{M}} = \frac{4}{3}\delta_{\mathrm{R}} = \text{constant}. \tag{3.38}$$

An interesting behaviour is observed, as the Hubble perturbation becomes identically zero (the scale factor perturbation δ_a in this case becomes constant). This suggests that once the universe enters a de Sitter phase, the background cosmology remains intact without any perturbation corrections. This feature is expected as dynamical system analysis shows that the de Sitter cosmology, in the case of $f(T)$ gravity, is a future attractor, meaning that the universe approaches a de Sitter phase and remains in that phase indefinitely.

To obtain the resulting function, note that the torsion scalar is a constant which, contrary to the previous case, does not exhibit a functional relationship between cosmic time t and T. From the Friedmann equations Eqs.(3.1) and (3.2), one obtains

$$F - T - 2TF_T = 2\left[\rho_{\mathrm{M},0}e^{-3Ht} + \rho_{\mathrm{R},0}e^{-4Ht}\right], \tag{3.39}$$

$$0 = \frac{\rho_{\mathrm{M},0}e^{-3Ht} + \frac{4}{3}\rho_{\mathrm{R},0}e^{-4Ht}}{1 + F_T + 2TF_{TT}}, \tag{3.40}$$

which is only satisfied under the condition that $\rho_{\mathrm{M},0} = \rho_{\mathrm{R},0} = 0$. This is reasonable as during those times, the contribution from both matter and radiation is negligible compared to the dark energy component, as these decay exponentially with time. Therefore, this leaves us with $F - T - 2TF_T = 0$. This can be investigated in two ways, either as a differential equation or as an algebriac equation for the torsion scalar (and hence the Hubble parameter). Naturally, standard ΛCDM is recovered if $F(T) = 2\Lambda$ but other functional forms can also be considered [69, 173, 174].

As anticipated, for this type of behaviour, the exotic fluid behaviour becomes that of a cosmological constant $\omega_T = -1$ (which agrees with the result obtained in [128]) which sets $\delta_T = 0$ confirming its stability. However, this behaviour is now achieved for models which are not necessarily standard ΛCDM.

3.3 Specific Choices of the $F(T)$ Function

In this section, the power-law and exponential $F(T)$ functions will be examined in detail, as they act as a viable alternative to ΛCDM. Furthermore, since the major interest is at late times, the effect of radiation is practically negligible (supported by Planck data, which indicates a value of $\Omega_{\mathrm{R},0} \sim 10^{-5}$ [35]). Thus, the radiation density will not be considered in the background and perturbation equations.

To make realistic choices of the model parameters of each model, Planck data shall be used to serve as restrictions to the parameters. Firstly, acceleration is observed at present times which implies $q_0 < 0$. Actually, according to recent Planck data, $q_0 \sim -0.5$ [35].

A secondary constraint can be achieved by demanding the exotic fluid EoS be close to that observed. Although the physical nature of the EoS is unknown, as discussed in Chapter 1, various forms describing its EoS nature have been considered; whether it originates from ΛCDM, has some other constant value or whether it is dynamical (for instance, linear evolution EoS dark energy parameter models like Chevallier-Polarski-Linder (CPL) [175, 176]). For a constant value, one finds the constraint $\omega_T = -1.03 \pm 0.03$ suggesting a phantom nature. Restricting the fluid to have a non-phantom nature, the constraint then becomes $\omega_{T,0} < -0.95$ [35]. Both values will be used to constrain the respective model.

To obtain numerical simulations of the models, the present value of the matter density $\Omega_{\mathrm{M},0} = 0.315 \pm 0.007$ and the Hubble parameter $H_0 = (67.4 \pm 0.5)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [35] shall be used. Furthermore, these will also serve to help constrain the parameters together with the previous conditions.

3.3.1 Power-law Model Stability

Starting with the power-law ansatz model $F(T) = \alpha(-T)^n$, the Friedmann equation (Eq. 3.1) becomes

$$\alpha(2n-1)\frac{(-T)^n}{T_0} + \frac{T}{T_0} = \frac{\Omega_{\mathrm{M},0}}{a(t)^3}, \tag{3.41}$$

where the value of α can be found by evaluating the expression at current times to give

$$\alpha = \frac{(\Omega_{\mathrm{M},0} - 1)\,(-T_0)^{1-n}}{1-2n}. \tag{3.42}$$

Trivially, this relation does not hold for $n = \frac{1}{2}$ as this leads to the boundary contribution. Using this value of α, the equation can be recast into a simpler form

$$\frac{T}{T_0} + (\Omega_{\mathrm{M},0} - 1)\left(\frac{T}{T_0}\right)^n = \frac{\Omega_{\mathrm{M},0}}{a(t)^3}. \tag{3.43}$$

For this model, the exotic fluid EoS Eq.(3.7) takes the form

$$w_T = \frac{(n-1)}{1 + n(\Omega_{\mathrm{M},0} - 1)\left(\frac{T}{T_0}\right)^{n-1}}. \tag{3.44}$$

At present times, the EoS parameter has a magnitude of

$$w_{T,0} = -\left[1 + \frac{n}{1-n}\Omega_{\mathrm{M},0}\right]^{-1}. \tag{3.45}$$

From this expression, standard behaviours can be extracted. For $n = 1$, $\omega_{T,0} = 0$, meaning it behaves as a dust fluid which is expected as this model reduces to a rescaling of TEGR, which is not of importance. A cosmological constant is otherwise achieved for $n = 0$ as the $F(T)$ function reduces to a cosmological constant. A singularity appears for $n^\star \equiv (1 - \Omega_{\mathrm{M}})^{-1}$, but its physical significance will not be explored. However, it is observed that for $n > n^\star$, $\omega_{T,0} < -n^\star$ while for $n < n^\star$, $\omega_{T,0} > -n^\star$. From present values, this means that $\omega_{T,0} \lesseqgtr -\frac{10}{7} \approx -1.43$. Thus, models which realise the observational constraint are for values of $n < n^\star \approx 1.43$.

On the other hand, the deceleration parameter Eq.(2.48) for this model takes the form of

$$q(t) = \frac{1 - (2n-3)(\Omega_{\mathrm{M},0} - 1)\left(\dfrac{T}{T_0}\right)^{n-1}}{2 + 2n(\Omega_{\mathrm{M},0} - 1)\left(\dfrac{T}{T_0}\right)^{n-1}}, \tag{3.46}$$

which when evaluated at present times yields

$$q_0 = \frac{1 - (2n-3)(\Omega_{\mathrm{M},0} - 1)}{2 + 2n(\Omega_{\mathrm{M},0} - 1)}. \tag{3.47}$$

Acceleration is obtained when

$$n < \frac{3\Omega_{M,0} - 2}{2\Omega_{M,0} - 2} \quad \text{or} \quad n > \frac{1}{1 - \Omega_{M,0}}, \tag{3.48}$$

which, using observed values, yields $n \lessapprox 0.79$ or $n \gtrapprox 1.43$. As the value of $n = 0$ yields standard ΛCDM behaviour, other values will be considered. Following the EoS discussion, values $n \lessapprox 0.79$ have been chosen for consideration. To simplify the necessary numerical computations to obtain the desired evolutions, the power-law index values considered are taken to be negative integers, namely $n = -1$ ($w_{T,0} \approx -1.18$) and $n = -2$ ($w_{T,0} = -1.25$).

The models are first analysed for their background behaviour features to verify that the above claims are satisfied. Furthermore, this will allow for an insight of what happens to the cosmological parameters and the EoS of the exotic fluid throughout time. In what follows, the initial condition $a\,(10^{-5}\,\text{Gyr}) = 10^{-4}$ was used, as it describes a time at the matter-radiation equality [35]

$$a_{\text{eq.}} = \frac{\Omega_{r,0}}{\Omega_{M,0}} \sim 10^{-4}. \tag{3.49}$$

After this time, the universe starts to become matter dominated, satisfying the claim that the radiation component becomes negligible in the equations. In the plots that follow, ΛCDM is represented by the solid curve, $n = -1$ is dotted and $n = -2$ is dashed.

Fig. 3.1 illustrates the expansion history of the power-law model in comparison with ΛCDM. The behaviour is practically identical at early times but deviates at late times due to differences in the exotic fluid behaviour with $n = -2$ yielding a greater accelerated expansion. The reason for this becomes apparent through the EoS behaviour shown in Fig. 3.2.

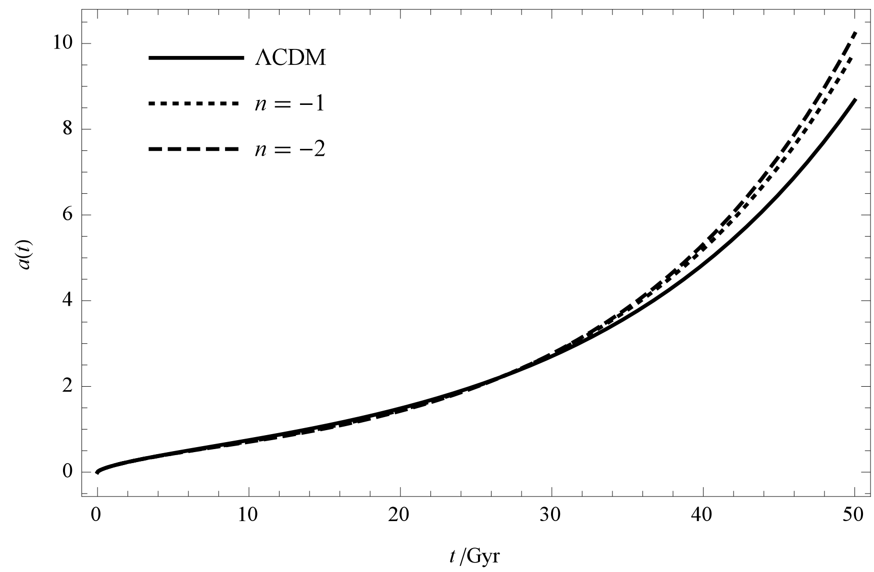


Figure 3.1: The evolution of the scale factor $a(t)$ with cosmic time for the model $F(T) = \alpha\,(-T)^n$ for $n = 0$, -1 and -2. For early and current times, the scale factor for each model is almost identical. However, during later times, the $n = -1$ and $n = -2$ models start to deviate from the ΛCDM model yielding a faster accelerated expansion.

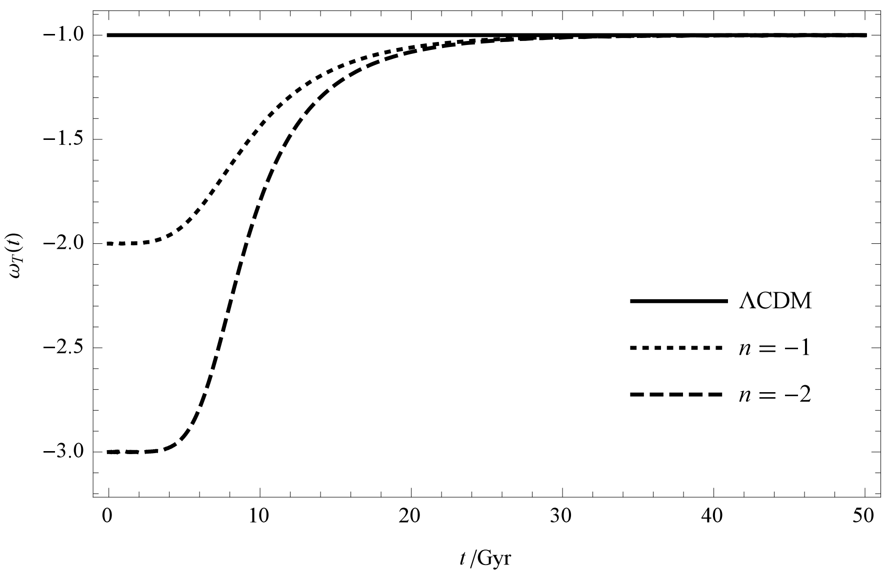


Figure 3.2: The evolutionary behaviour of the exotic fluid's EoS parameter $w_T(t)$ with cosmic time for the model $F(T) = \alpha\,(-T)^n$ for $n = 0$, -1 and -2. As expected, $n = 0$ leads the ΛCDM behaviour with $\omega_T = -1$ at all times. On the other hand, the $n = -1$ and $n = -2$ models describe a varying EoS parameter, which is phantom in nature, starting at $n - 1$ at $t = 0$ and approaches -1 during late times, meaning it approaches the behaviour of a cosmological constant.

The evolutionary behaviour of the exotic fluid shows that its EoS is not constant, leading to a dynamical energy model. Overall, the general behaviour of the model depends on the choice of n which starts at a value of $\omega_T(0) = n - 1$ initially and approaches $\omega_T \to -1$ at late times, meaning that it behaves as a cosmological constant at late times. However, the phantom-divide line (the $\omega = -1$ barrier) is never crossed, and hence does not behave as a quintom (a fluid which behaves both as a phantom and as a non-phantom fluid throughout the cosmological history). In fact, the fluid is phantom in nature. As suggested from the scale factor evolution, $n = -2$ leads to a faster accelerated expansion as the EoS is smaller than $n = -1$ and ΛCDM.

Discussions about the accelerated expansion are further supported by the deceleration parameter illustrated in Fig. 3.3, which clearly indicates that $n = -1$ and $n = -2$ lead to a faster accelerated expansion compared to ΛCDM. Furthermore, since in each case the exotic fluid approaches a cosmological constant behaviour, the deceleration parameter approaches $q \to -1$ at late times. As expected, the absence of the exotic fluid at early times also confirms the initial value of $q(0) = 0.5$. Each model transitions from deceleration to acceleration at practically the same time, at a value of around $t \approx 8\,\text{Gyr}$.

Fig. 3.4 illustrates the behaviour of the matter and exotic fluid density parameters with time. At early times, the matter density approaches a value of 1 while the exotic fluid approaches zero. The opposite is observed at late times. This reflects the domination epochs of each fluid, corresponding to the observed behaviour in Fig. 3.1. One notes that the matter-exotic fluid equality occurs at different times, with the $n = -1$ and -2 models at approximately the same time of $t \approx 12\,\text{Gyr}$ (for ΛCDM, this is $t \approx 10\,\text{Gyr}$). This shift occurs due to the different dynamical nature of the exotic fluid.

With the background evolution analysed, the perturbation parameters are now analysed. Starting with the Hubble perturbation δ illustrated in Fig. 3.5, it is evident

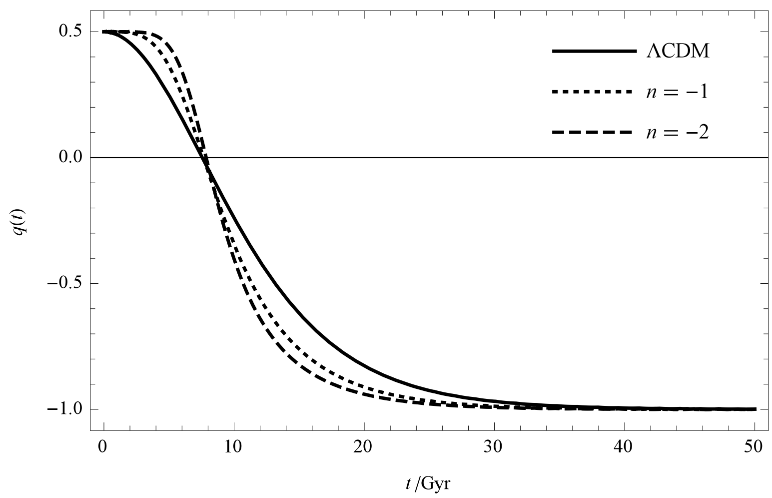


Figure 3.3: The deceleration parameter $q(t)$ for the model $F(T) = \alpha\,(-T)^n$ for $n = 0, -1, -2$. All models mimic the behaviour of the ΛCDM model, with each model transitioning from a decelerated to an accelerated expansion at approximately the same time ($t \approx 8\,\text{Gyr}$), however with different acceleration rates with $n = -2$ being the fastest accelerated expansion. Due to the nature of the exotic fluid, each model starts with $q(0) = 0.5$ and ends with $q \to -1$ as the fluid approaches a cosmological constant behaviour as shown in Fig. 3.2.

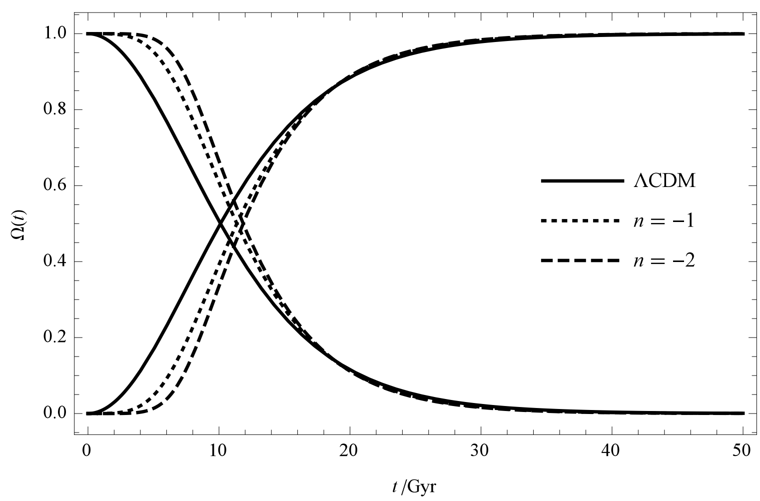


Figure 3.4: Density parameter evolution for matter and the exotic fluid with cosmic time for the model $F(T) = \alpha\,(-T)^n$ for $n = 0, -1, -2$. The models exhibit similar behaviour, however with differences. The most notable difference is how the matter-exotic fluid equality occurs at different stages, with the power-law model occurring at a later time. Overall, the early stages of the universe is dominated by matter while the exotic fluid dominates at late times, conforming with the scale factor behaviour observed in Fig. 3.1.

that the model is stable for all choices of n as the perturbation parameters decays with time. The fact that the parameter decays to zero means that the late time universe (which is now asymptotically approaching a de Sitter phase) is asymptotically approaching a perfectly homogeneous and isotropic universe. This also agrees with the attractor behaviour observed from a dynamical system approach analysis [177–179].

Figs. 3.6 and 3.7 indicate the behaviour of the matter and exotic fluid perturbations respectively. For the former, the perturbation decays with time but, in contrary to the Hubble perturbation, this approaches a non-zero constant value. Such behaviour is expected as this illustrates structure formation throughout the universe timeline. The fact that it approaches a constant value shows that structure stops to form at late times, which is expected as the accelerated expansion does not allow for more structure to form. The behaviour for the exotic fluid perturbation, on the other hand, is slightly different. As the ΛCDM result shows, the perturbation is strictly

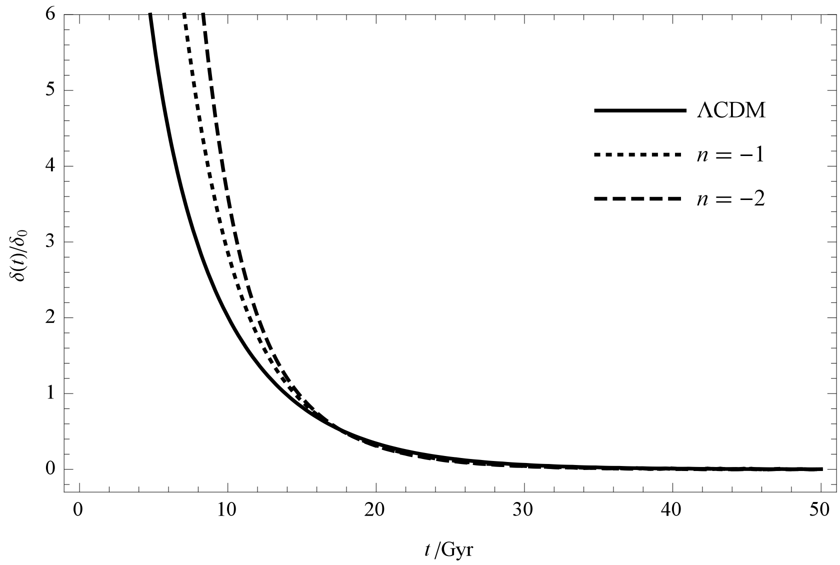


Figure 3.5: The Hubble perturbation parameter $\delta(t)$ evolution with cosmic time for the model $F(T) = \alpha\,(-T)^n$ for $n = 0$, -1 and -2. Here, the ratio δ/δ_0 is plotted to examine the behaviour. As the ratio is shown to decay with time, it means that the model is stable. The $n = -1$ and $n = -2$ models mimic the ΛCDM behaviour, with the only difference being an earlier decay rate.

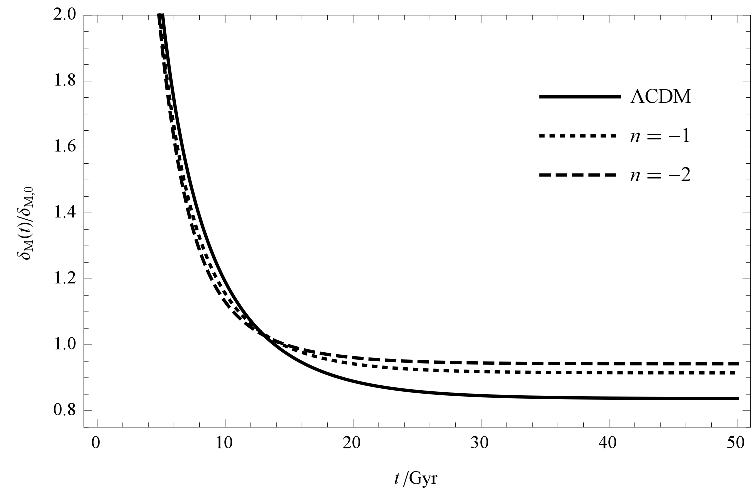


Figure 3.6: Matter perturbation $\delta_{\rm M}(t)$ evolution with cosmic time for the model $F(T) = \alpha\,(-T)^n$ for $n = 0$, -1 and -2. The models all exhibit a similar behaviour in which the parameters decay with time leading to stability, while approaching a constant value at later times, indicating the structure formation throughout the universe's history. The only difference is the limiting value of $\delta_{\rm M}/\delta_{\rm M,0}$ for each model. For ΛCDM, the limiting value is 0.837, for $n = -1$ it is 0.915 and for $n = -2$ it is 0.942.

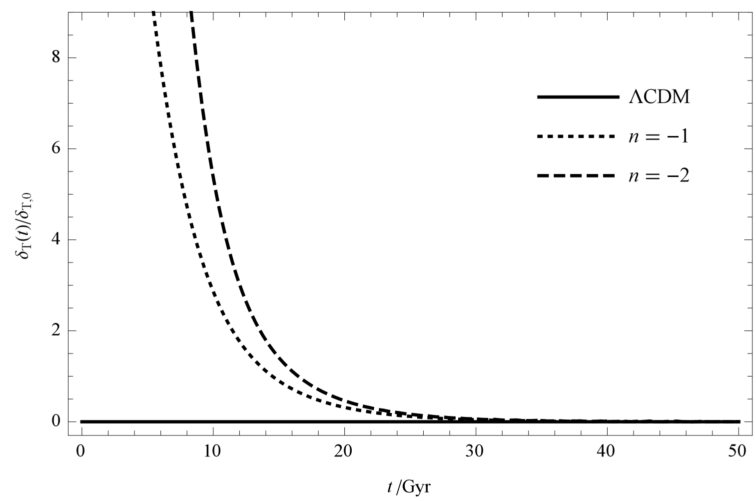


Figure 3.7: The exotic fluid perturbation parameter $\delta_T(t)$ evolution with cosmic time for the model $F(T) = \alpha\,(-T)^n$ for $n = 0$, -1 and -2. Similarly to the matter perturbation Fig. 3.6, the $n = -1$ (dotted) and $n = -2$ (dashed) model perturbations decay with time, however it approaches a zero value at late times. This is expected as the fluid approaches a cosmological constant, one which by definition cannot overdense, and hence has a constant perturbation value of zero (solid). Due to their phantom nature, however, a non-zero perturbation value is observed at earlier times.

zero, illustrating the repulsive nature of the cosmological constant. For $n = -1$ and $n = -2$, however, its phantom property allows for an initial non-zero value clustering, which decays as it approaches a cosmological constant behaviour at late times, as expected. Nonetheless, the solution is stable.

3.3.2 Exponential Model Stability

For the exponential model, a procedure and analysis similar to that carried out for the power-law $F(T)$ ansatz shall be explored. The Friedmann equation Eq.(3.1) for this model takes the form of

$$\frac{T}{T_0} - \alpha\left\{1 - \left(1 + p\sqrt{\frac{T}{T_0}}\right)\exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right\} = \frac{\Omega_{\mathrm{M},0}}{a(t)^3}, \tag{3.50}$$

where α can once more be obtained by evaluating the expression at current times

$$\alpha = \frac{1 - \Omega_{\mathrm{M},0}}{1 - (1+p)\, e^{-p}}. \tag{3.51}$$

Substituting the value of α back into the Friedmann equation yields

$$\frac{T}{T_0} + \frac{(\Omega_{\mathrm{M},0} - 1)}{1 - (1+p)\, e^{-p}}\left\{1 - \left(1 + p\sqrt{\frac{T}{T_0}}\right)\exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right\} = \frac{\Omega_{\mathrm{M},0}}{a(t)^3}. \tag{3.52}$$

For the exponential model, the exotic fluid EoS parameter takes the form of

$$w_T = -\left[2 - \frac{p^2 x^2 e^{-px}}{1 - (1+px)\, e^{-px}}\right]\left[2 + \frac{(\Omega_{\mathrm{M},0} - 1)}{1 - (1+p)\, e^{-p}} p^2 e^{-px}\right]^{-1}, \tag{3.53}$$

where $x = \sqrt{\frac{T}{T_0}}$, which when evaluated at current times ($x = 1$) yields

$$w_{T,0} = -\left\{1 + \frac{\Omega_{\mathrm{M},0} p^2 e^{-p}}{2\left[1 - (1+p)\, e^{-p}\right] - p^2 e^{-p}}\right\}^{-1}. \tag{3.54}$$

To determine which values of p are to be considered to realise the desired cosmology, the behaviour of $\omega_{T,0}$ with p is analysed. A singularity arises when

$$2\left[1-(1+p)\,e^{-p}\right]+\left(\Omega_{\mathrm{M},0}-1\right)p^2e^{-p}=0, \tag{3.55}$$

which cannot be solved analytically as it is a transcendental equation. Nonetheless, a numerical value is obtained using observational data, yielding $p^\star \approx -1.18$. Then, it is observed that for $p > p^\star$, $\omega_T > -1$ while $p < p^\star$, $\omega_T < \frac{1}{\Omega_{\mathrm{M},0}-1} \approx -1.43$. Based on these considerations, it is more suited to choose values for $p > p^\star$. The constraint can be restricted further to $p > 0$ since for values between $p^\star < p \leq 0$, the EoS is non-negative and hence does not compare with observations.

Next, the deceleration parameter is found to be

$$q=-1+\frac{3}{x^2}\frac{x^2\left[1-(1+p)e^{-p}\right]+\left(\Omega_{\mathrm{M},0}-1\right)\left[1-(1+px)\,e^{-px}\right]}{2\left[1-(1+p)e^{-p}\right]+\left(\Omega_{\mathrm{M},0}-1\right)p^2e^{-px}}, \tag{3.56}$$

which when evaluated at present times is reduced to

$$q_0=-1+\frac{3\Omega_{\mathrm{M},0}\left[1-(1+p)e^{-p}\right]}{2\left[1-(1+p)e^{-p}\right]+\left(\Omega_{\mathrm{M},0}-1\right)p^2e^{-p}}. \tag{3.57}$$

Similar to the previous instance, determining a general relation between p and $\Omega_{\mathrm{M},0}$ which gives rise to an accelerating behaviour is not analytically possible due to the presence of the exponential functions. Nonetheless, a numerical bound is obtained, yielding $p < -1.18$ or $p > 0.68$. From the EoS discussion, since $p > 0$ is favoured, values of $p > 0.68$ shall be considered. Two values are taken, being $p = 2$ ($\omega_T \approx -0.80$) and $p = 5$ ($\omega_T \approx -0.97$).

With the models defined, the background behaviour shall be the first to be analysed, followed by the model stability. Once again, the initial condition $a\,(10^{-5}\,\mathrm{Gyr}) = 10^{-4}$ was used while in the plots, ΛCDM is represented by the solid curve, $p = 2$ as dotted and $p = 5$ as dashed.

Fig. 3.8 represents the expansion history with cosmic time for the models compared with ΛCDM. As the EoS of the exotic fluid lies in a quintessence state[5] as shown in Fig. 3.9, this causes a slower accelerated expansion than ΛCDM at late times. For the $p = 5$ model however, this is practically indistinguishable in this regard. At early times, the evolutionary behaviours for both p model values are practically identical.

The EoS of the exotic fluid is explored in further detail. This is illustrated in Fig. 3.9. It is evident that the $p = 5$ closely mimics the ΛCDM behaviour with an almost constant value $\omega_T \sim -1$. On the other hand, the $p = 2$ model deviates from such behaviour at present times. Both models exhibit an early and late time cosmological constant behaviour.

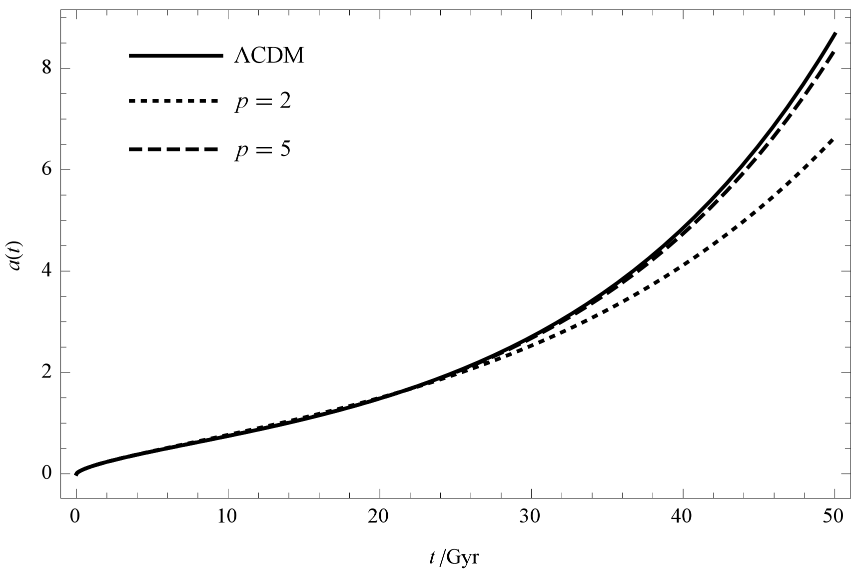


Figure 3.8: The evolution of the scale factor $a(t)$ with cosmic time for the model $F(T) = \alpha T_0 \left(1 - \exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right)$ for $p = 2$ and 5, compared to the ΛCDM model. The overall behaviour of the models mimics that of ΛCDM, albeit at slower rates, most notably that for $p = 2$. $p = 5$ yields a behaviour which is practically indistinguishable from ΛCDM.

[5]Quintessence here is used in the context of having an equation of state $\omega > -1$ and not the scalar field model presented in Chapter 1. See, for instance, Ref. [81] where this definition has been used.

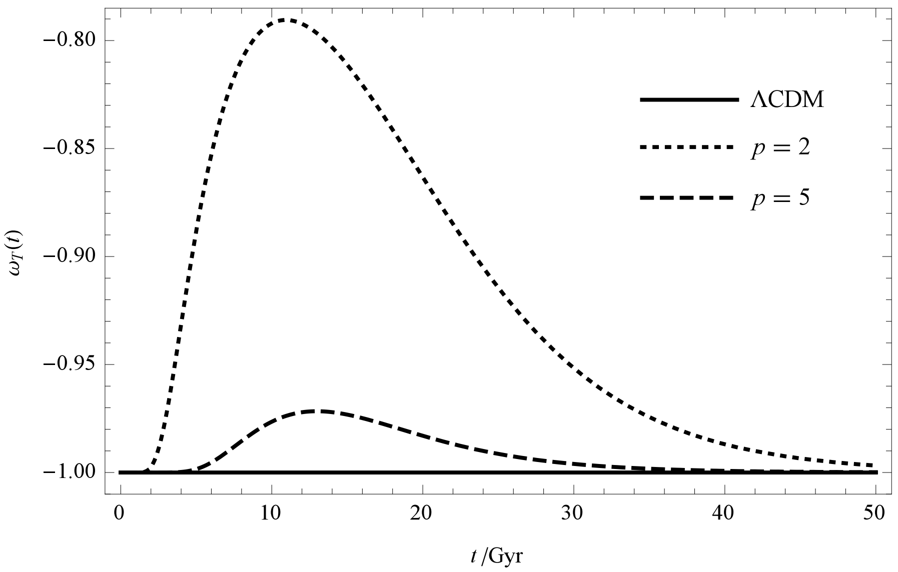


Figure 3.9: EoS behaviour of the exotic fluid for the model $F(T) = \alpha T_0 \left(1 - \exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right)$ for $p = 2$ and $p = 5$ in comparison with the ΛCDM model. Here, the models exhibit a quintessence behaviour, at which they peak at some maximum value at a certain stage during the evolution. At early and late times, both models approach a cosmological constant behaviour. Observe that the $p = 5$ model closely mimics that of ΛCDM as the value is almost constant ($\omega_T \sim -1$).

The previous claims are further supported by the evolution of the deceleration parameter illustrated in Fig. 3.10. Clearly, the $p = 2$ model exhibits a slower accelerated expansion at late times, while the $p = 5$ closely mimics the ΛCDM behaviour. Nonetheless, the standard features where the models start at $q(0) = 0.5$ and approach $q \to -1$ at late times are retained. Furthermore, the transition from deceleration to acceleration all happen at approximately the same time ($t \approx 8\,\text{Gyr}$).

The density parameters for matter and the exotic fluid are shown in Fig. 3.11. In this case, the quintessence behaviour of the exotic fluid causes the matter-exotic fluid equality to occur at an earlier time compared to ΛCDM (although for the $p = 5$ model this shift is negligible). The standard behaviour of matter being dominant at early times and the exotic fluid to dominate at late times is retained.

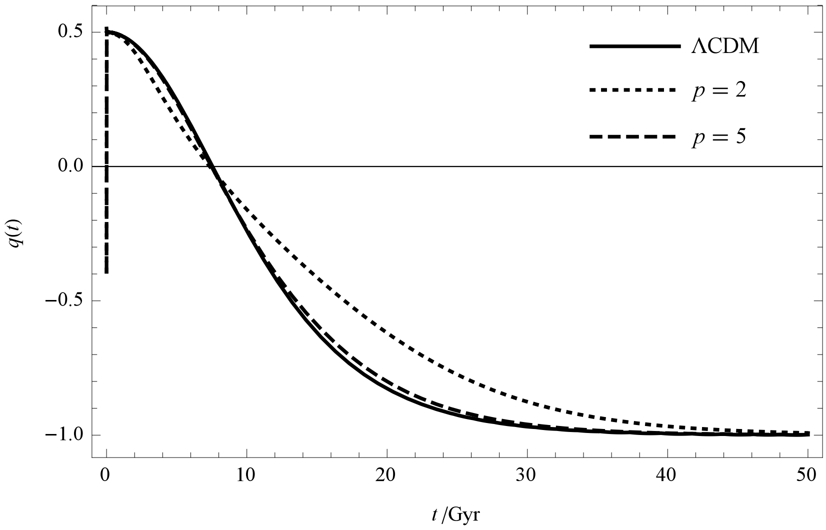


Figure 3.10: The evolution of the deceleration parameter $q(t)$ with cosmic time for the model $F(T) = \alpha T_0 \left(1 - \exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right)$ for $p = 2$ and 5, compared to the ΛCDM model. The overall behaviour of the models mimics that of ΛCDM, albeit at slower rates, most notably that for $p = 2$. $p = 5$ yields a behaviour which is practically indistinguishable from ΛCDM. The initial drop results from computational limitations towards the initial condition.

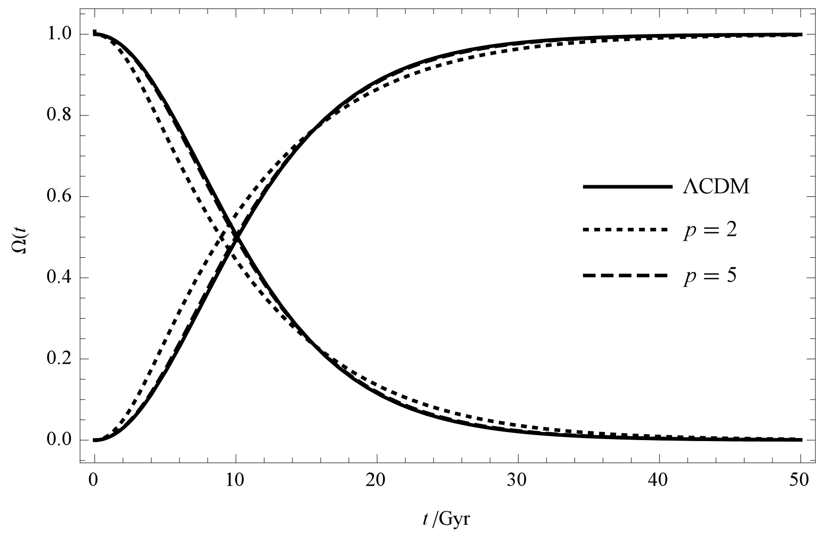


Figure 3.11: Density parameter evolution for the matter and exotic fluids with cosmic time for the model $F(T) = \alpha T_0 \left(1 - \exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right)$ for $p = 2$ and $p = 5$ compared to the ΛCDM model is shown. Overall, the models exhibit similar behaviour with $p = 5$ closely mimicking ΛCDM. However, due to the quintessence nature of the exotic fluid, the matter-exotic fluid equality occurs at an earlier stage. Nonetheless, the early domination by matter and exotic fluid domination at late times is retained.

Moving towards the stability behaviour of the model, the Hubble perturbation is first presented, as shown in Fig. 3.12. Clearly, the perturbations decay with time and hence the solution is stable. Similar to the power-law model, the Hubble perturbation decays to a zero value, indicating the attractor behaviour associated with de Sitter cosmologies found in $f(T)$ models [178].

Once again, the matter perturbation Fig. 3.13 yields stability as the parameter decays with time while approaching a constant value indicating structure formation. From the models considered, $p = 2$ takes the longest time to approach this constant value, which is an indicator that structure continued to form for a longer time compared to $p = 5$ and ΛCDM. This is expected as the accelerated expansion is slower, allowing for more structure to form during the universe's lifetime.

Contrary to the power-law model, however, lies in the exotic fluid behaviour shown in Fig. 3.14. The exponential model starts and approaches a value of zero at late

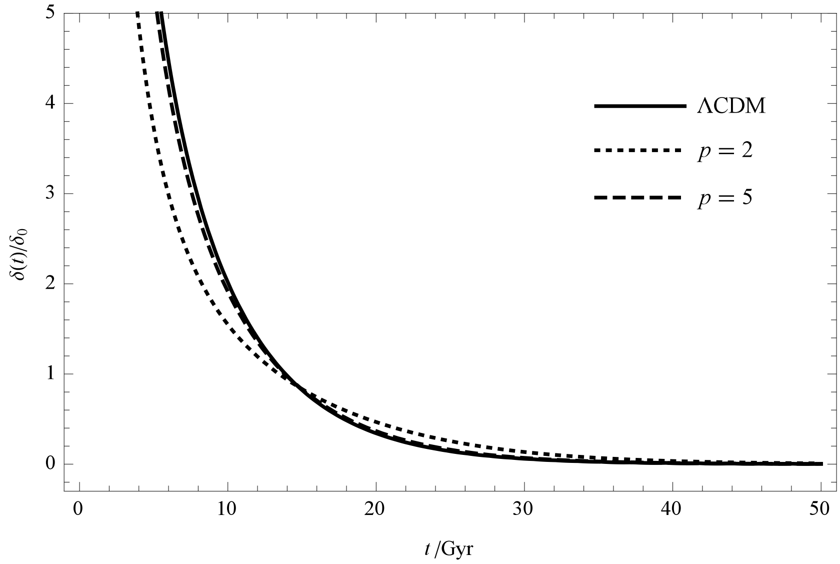


Figure 3.12: Hubble parameter perturbation evolution for the exponential model $F(T) = \alpha T_0\left(1 - \exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right)$ for $p = 2$ and 5 compared to ΛCDM. Each model mimics the behaviour obtained in ΛCDM, albeit with slight value differences at early times and present times. Regardless, the perturbations decay to zero indicating stability.

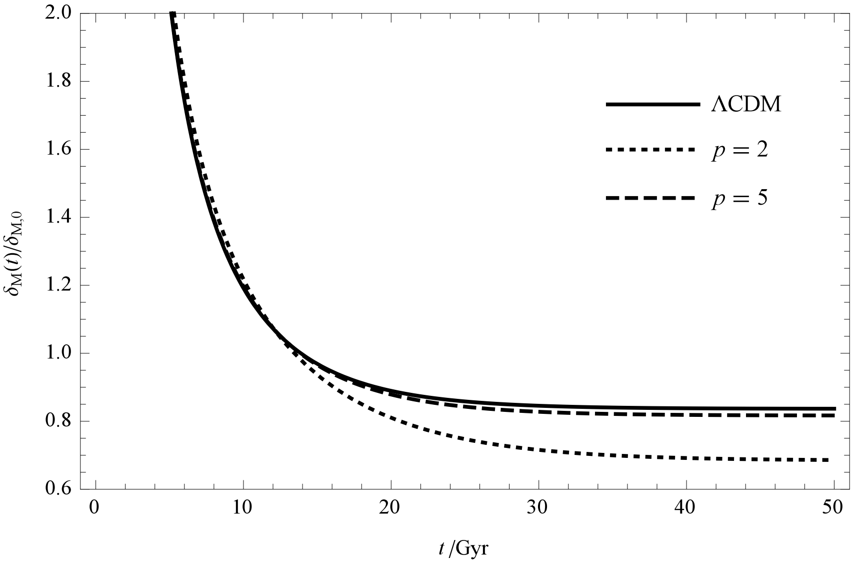


Figure 3.13: Evolution of the matter perturbation with cosmic time for the model $F(T) = \alpha T_0 \left(1 - \exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right)$ for $p = 2$ and 5 compared to the ΛCDM model. Stability for each model is retained, as the evolution decays with time until it approaches a constant value. This asympototic limiting value varies from model to model, in which it is 0.837 for ΛCDM, 0.686 for $p = 2$, and 0.817 for $p = 5$.

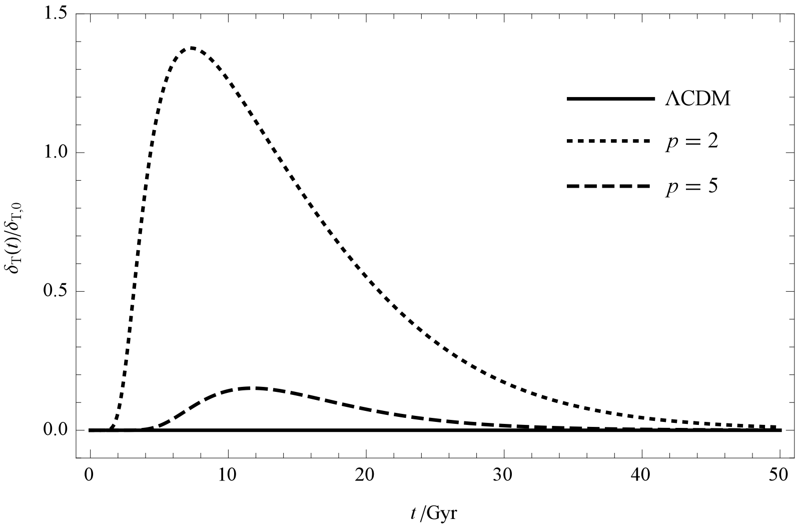


Figure 3.14: Exotic fluid perturbation evolution with cosmic time for the model $F(T) = \alpha T_0 \left(1 - \exp\left[-p\sqrt{\frac{T}{T_0}}\right]\right)$ for $p = 2$ and $p = 5$, compared to the ΛCDM model. Here, the exponential model deviates from the ΛCDM behaviour during intermediate times due to the quintessence nature of the fluid. However, at early and late times, the perturbation approaches a zero value which is expected since during those times, the fluid closely mimics a cosmological constant.

times, as expected by the behaviour of the EoS. However, the quintessence behaviour during intermediate times indicate a non-zero value in the perturbation parameter, leading to possible formations of overdense regions. Stability for both models is, nonetheless, retained.

3.4 Discussion

The FLRW background in an $f(T)$ gravity theory was examined and studied under homogeneous perturbations to determine whether the cosmological behaviour for the given Lagrangian is deemed stable. Study of stability determines whether the solution retains that behaviour throughout its time, and hence serves as means to investigate whether the description is valid throughout the universe history. For $f(T)$ gravity, the matter homogeneous perturbation and Hubble perturbation parameters, which are the main results of this chapter, evolve as given in Eqs.(3.21) and (3.22). As both quantities are solely dependent on the Hubble parameter, stability is determined depending on its evolution. Furthermore, the exotic fluid component which is associated from the TEGR Lagrangian deviation was also obtained as listed in Eq.(3.27).

Initially, stability was investigated for specific ansatz choices of scale factor, the power-law and de Sitter models, which have a diverse number of applications in cosmology. In both instances, the resulting evolution is stable but with minor differences. In the power-law model, each perturbation parameter decays inversely with time while in the de Sitter case, the Hubble and exotic fluid perturbations are identically zero, while the matter and radiation perturbations result in a constant. For the former, this implies that the solution remains stable and 'attracted' towards the solution with time while in the latter, as the Hubble perturbation is identically zero, means that the solution remains indefinitely at that point. The matter and radiation perturbation differences refer to the growth of such perturbations, in which power-law models realise a growth, whereas during the de Sitter phases growth stops as the perturbation remains constant.

Next, specific $f(T)$ ansatzes are chosen to examine their resulting cosmological histories and features together with their associated stability analysis. Two viable ansatz models were chosen, the power-law model $F(T) = \alpha(-T)^n$ and the exponential model $F(T) = \alpha T_0 \left(1 - \exp\left[1 - p\sqrt{\frac{T}{T_0}}\right]\right)$. Since both models have a free parameter n and p respectively, these parameters have been restricted through the use of Planck data by demanding an exotic fluid EoS close to that observed $\omega_T \sim 1$ which also yields an accelerated expansion today.

Starting with the power-law model, these restrictions have set the value of n to be $n = -1$ and $n = -2$, which yield a behaviour close to ΛCDM especially towards late times as the exotic fluid approaches a cosmological constant behaviour during these periods. For this model, the exotic fluid is strictly phantom in nature. In each case, the resulting solution is stable as each parameter decays with increasing time and also retains the feature that matter perturbations do grow throughout the universe's timeline. However, as the exotic fluid is not a cosmological constant, an initial non-zero value of the density parameter δ_T is observed. Overall, the $n = -1$ model mimics the ΛCDM behaviour better.

Next, for the exponential model, the parameters $p = 2$ and $p = 5$ were chosen to satisfy the observational constraints. Contrary to the power-law model, the exotic fluid now behaves as a quintessence fluid, attributing to a slower acceleration as evident by the comparisons of Figs.3.3 and 3.10. At late times, however, the fluid approaches a cosmological constant linking a close behaviour to ΛCDM. Due to this behaviour, the exotic fluid perturbation is non-zero at early and present times, and approaches a zero value at late times. Both models exhibit stability against Hubble and matter perturbations, with the latter persisting at late times.

All in all, the $p = 5$ exponential model exhibits the closest behaviour to ΛCDM in terms of evolution and nature, making it a viable candidate to study for cosmology. Nonetheless, it is worth noting that the model still has to obey other observational criteria as well (for example, Big Bang Nucleosynthesis bounds are satisfied for this model [180]). Despite the power-law model not being as favourable as the exponential model (at least for the parameters considered), it can still be a viable model as discussed in Ref. [181], where it has been shown to be in agreement with CMB observations. The requirement of stability can therefore help restrict the possible Lagrangian models in hope of realising a more complete picture for the description of the universe.

Chapter 4

Reconstruction in $f(T, T_G)$ Theories of Gravity

The ongoing research to find a gravitational model which is able to completely describe the universe is still an open problem. From a phenomenological point of view, it is questionable whether the basis of the theory, the form and behaviour of matter, or the gravitational Lagrangian sector are of the correct form to match with observations. Up to this point, this question is tackled under the viewpoint of teleparallel gravity with an associated gravitational Lagrangian, coupled with (perfect) matter components. As seen in the previous chapter, the choice of the Lagrangian is important to check whether the model is cosmologically stable.

In particular, two distinct approaches were investigated. One approach, which has been carried out in Section 3.3, assumes a given Lagrangian ansatz which, according to its form, determines the resulting cosmology. Although viable models can be constructed, as seen by the power-law and exponential $f(T)$ models, there is no method which hints a specific form of this Lagrangian. At best, these models are constrained from theoretical considerations and observational data, leaving the possible $f(T)$ Lagrangian form clouded by numerous models with no clear sign of its true form [77, 182].

This shortcoming is avoided in the second approach. Given a specific evolutionary ansatz (being power-law cosmology and de Sitter evolution in Section 3.2), these evolutions are only valid in the $f(T)$ theory, provided that the field equations give rise to the solution. This is only possible through a restriction in the $f(T)$ Lagrangian, a method known as reconstruction.

Reconstruction serves as a way to conform with observations by constructing a Lagrangian (or dark energy model) which directly yields the desired behaviour provided by observations. Using SNe Ia data as an example, a relation between redshift z and the Hubble parameter can be constructed. From this correspondence, the quantities which give rise to the considered Lagrangian could then be expressed purely in terms of either variable, reducing the field equations into a set of differential equations which, in principle, can then be solved to yield the required Lagrangian [183–187]. Otherwise, as seen in the previous chapter, it can be used to examine the Lagrangian behaviour during specific periods, determining whether the model under study is capable of generating the cosmological behaviours [188–192]. This can be investigated throughout various epochs including early times (for example, inflation through reconstruction of the inflaton potential [193,194]), transition periods [195], domination periods, or late time acceleration periods (for instance, de Sitter phases).

Solving these differential equations, however, is not always possible. Taking for instance $f(R)$ gravity, the resulting equations are fourth order, leading to a complex system of equations which can be difficult to solve analytically. This becomes even more exhaustive if higher order corrections are included. In the case of $f(T)$ gravity, however, this becomes relatively simpler as it is a second order theory [128]. For cases where analytical expressions are not possible, numerical computations can be considered, with boundary conditions set either from local solar system tests or from cosmological sources [182].

In cases where a solution is indeed found, this does not make the model necessarily viable. Although it will generate the desired behaviour, it must still pass through other observational tests, both local and cosmological [196]. Furthermore, the reconstructed solution can result in complicated expressions which would not be suitable for use [188] and can also suffer from instabilities. Nonetheless, it may well still be a step closer towards a more formal expression of the gravitational Lagrangian.

For the work presented in this chapter, the reconstruction technique shall be applied to the teleparallel Gauss-Bonnet extension, $f(T, T_G)$ gravity in the context of a spatially flat FLRW metric. Contrary to what is found in $f(T)$ gravity, the introduction of the Gauss-Bonnet term will introduce an increased difficulty to solve the Friedmann equations, as these become partial differential equations (PDEs) instead of ordinary differential equations (ODEs). Therefore, particular ansatz choices shall be considered as means to simplify the problem. Furthermore, in this work, the metric signature is reversed, leading to the expressions of the torsion scalar and TEGB term to be

$$T = 6H^2, \qquad T_G = 24H^2(H^2 + \dot{H}). \tag{4.1}$$

It is observed that for this particular spacetime, the TEGB term matches identically with the curvature based Gauss-Bonnet term, G. Therefore, any uncoupled $f(G)$ Lagrangian is equivalent to an uncoupled $f(T_G)$ Lagrangian leaving a sense of equivalence between the two theories, and allowing for a comparison between the reconstructed solutions obtained from either formulation. However, this is not true for more general spacetimes due to the boundary term contribution [197].

The resulting modified Friedmann equations for the spatially flat FLRW spacetime in $f(T, T_G)$ gravity are [79]

$$f - 2Tf_T - T_G f_{T_G} + 24H^3 \dot{f}_{T_G} = 2\rho, \tag{4.2}$$

$$-\dot{H}f_T - H\dot{f}_T + 2H\left(2\dot{H} - H^2\right)\dot{f}_{T_G} + 4H^2\ddot{f}_{T_G} = -\frac{1}{2}\left(\rho + p\right). \tag{4.3}$$

For the reconstruction technique, it is sufficient to consider the first equation Eq.(4.2).

The work concerning reconstruction shall be applied as follows. Firstly, the model ansatz and cosmological models considered shall be presented and discussed, together with a constraint to reduce the number of viable Lagrangians. Afterwards, the resulting Lagrangian models are reconstructed together with their viability constraints.

4.1 $f(T, T_G)$ Ansatz and Cosmological Models

As previously discussed, the introduction of T_G in the Lagrangian results into a more complicated system to solve. Ansatz choices for the Lagrangian are therefore considered to simplify the problem. Motivated by the ansatz models which have been considered in literature, such as those in $f(R, G)$ gravity in Refs. [198,199] and those which appear in $f(T, T_G)$ gravity in Refs. [80,200], the following are considered

$$(i) \quad f(T, T_G) = g(T) + h(T_G), \qquad (iv) \quad f(T, T_G) = -T + T_G g(T),$$
$$(ii) \quad f(T, T_G) = T g(T_G), \qquad (v) \quad f(T, T_G) = -T + \mu T^{\beta} {T_G}^{\gamma}.$$
$$(iii) \quad f(T, T_G) = T_G g(T),$$

The first model is a separable additive type, one of the simplest forms that can be considered for reconstruction which can realise various well known Lagrangians. For instance, if $g(T) = -T$, the $h(T_G)$ contribution takes the role of the deviation from TEGR (and hence of dark energy). On the other hand, an arbitrary choice of g and h may lead to a different formulation from TEGR. In the case of reconstruction, a further remark about this model is observed. From the Friedmann equation Eq.(4.2), the additive model takes the form of

$$h + g - 2Tg_T - T_G h_{T_G} + 24H^3 \dot{T}_G h_{T_G T_G} = T_0 \Omega_{\omega,0} a^{-3(1+\omega)}. \tag{4.4}$$

Given a specific cosmology $a = a(t)$, the T and T_G functions can be expressed in terms of t. Suppose that these relations are invertible, i.e. $t = t(T)$ and $t = t(T_G)$. The left-hand side (LHS) of this equation infers that it could be separated as an expression solely in T and another in T_G. However, the right-hand side (RHS) can now be expressed in either variable, a choice between whether the matter is sourced by the torsion scalar or the TEGB scalar. Since in the TEGR limit matter is sourced by the torsion scalar, without loss of generality, this will be assumed to be the case in the following work. Nonetheless, the choice for the TEGB scalar can be considered instead.[6] Now that the equation can be fully separated, each T and T_G contribution becomes equal to a constant, say λ, yielding the relations

$$g - 2Tg_T - T_0\Omega_{\omega,0}a(T)^{-3(1+\omega)} = -h + T_G h_{T_G} - 24H(T_G)^3\dot{T}_G(T_G)h_{T_GT_G} = \lambda. \quad (4.5)$$

Evidently, $g(T) = \lambda = -h(T_G)$ is a solution. However, this implies that λ does not appear in the overall gravitational $f(T, T_G)$ Lagrangian and hence the source of this λ can be neglected. Thus, the resulting system of ODEs for an additive model results in

$$g - 2Tg_T = T_0\Omega_{\omega,0}a(T)^{-3(1+\omega)}, \quad (4.6)$$

$$h - T_G h_{T_G} + 24H(T_G)^3\dot{T}_G(T_G)h_{T_GT_G} = 0. \quad (4.7)$$

Models (*ii*) and (*iii*) represent rescaling solutions for T and T_G respectively. In the former model, for $g(T_G) = \alpha + h(T_G)$ where α is some constant, yields a TEGR rescaling with $h(T_G)$ acting as the source of deviation from TEGR. However, for non-trivial forms of $g(T_G)$, this could realise non-trivial functions which in turn realise the observed gravitational behaviour. In the case of model (*iii*), this will naturally realise a non-trivial behaviour, as TEGR cannot be constructed. However, model (*iv*) accounts for this difference by the introduction of a TEGR contribution in the

[6] This consideration has also been applied in $f(R, G)$ gravity [198, 199] and in $f(T, B)$ gravity [69].

gravitational Lagrangian, leaving the $T_G g(T)$ contribution to act as the deviation from TEGR. Nonetheless, the similarity between the last two models yields an interesting result. Taking both ansatz models in the Friedmann equation Eq.(4.2), the resulting Lagrangians are related through

$$\text{Model } (iv)\ f(T, T_G) = -T - \frac{3T_G}{4T} + \text{Solution of Model } (iii). \tag{4.8}$$

Thus, although the models are fundamentally different, obtaining the solution for model (iii) automatically yields the resulting behaviour of model (iv). Thus, only the solutions for model (iii) will be listed.

Lastly, a power-law model with a TEGR contribution is considered. In this case, the ΛCDM model can be achieved in the event that $\beta = \gamma = 0$. Furthermore, the $f(T)$ power-law model considered in Chapter 3 can also be recovered in the case when $\gamma = 0$. More generally, this functional form also appears in the study of Noether symmetry in the absence of the TEGR contribution with the constraint that $\beta = 1 - \gamma$ which yields scale factor power-law solutions $a(t) \propto t^{2\gamma+1}$, as well as oscillatory $a(t) \propto \tan t$ and hyperbolic forms $a(t) \propto \tanh t$ for $\gamma = 1$ [80, 200].

With the ansatz models defined, the cosmological behaviours considered for reconstruction shall be the following (a) power-law cosmology, (b) expanding de Sitter phase and (c) ΛCDM behaviour. A glimpse about the resulting reconstructed Lagrangian in the case of $f(T)$ gravity for the first two models has been derived in Chapter 3. Here, the results for the first two models will now be extended with the T_G contribution for the various model ansatz considered. Nonetheless, the previously obtained results will be recovered in the additive case model when $h(T_G) \to 0$.

The interesting application is the reconstruction for ΛCDM, to determine whether $f(T, T_G)$ gravity can produce the ΛCDM cosmology. As shown in Chapter 3, $f(T)$ gravity yields ΛCDM if it is precisely ΛCDM (at least for the case of dust, generalisations will be discussed further on). Thus, it proves to be an interesting alternative if $f(T, T_G)$ is able to generate alternative models which yield the ΛCDM cosmology.

Given the diverse number of gravitational Lagrangian which could be constructed, it is as important to distinguish between those which could be physically viable for use in other areas. A simple consideration is to recover the vacuum solutions (for instance, Minkowski and Schwarzchild), meaning that in the absence of matter, both T and T_G would be null. Examination of the Friedmann equations implies a simple Lagrangian constraint, being $f(0,0) = 0$.[7] Therefore, those which do not obey this constraint will be deemed to be non-viable.

In what follows, for simplicity, the boundary $\sqrt{T}$ term and Gauss-Bonnet contributions which appear in the reconstructed solutions will not be listed.

4.2 Power-law Reconstruction

The reconstruction of the Lagrangian is first applied to power-law models, i.e. those of the form $a(t) \propto t^\alpha$ where $\alpha \neq 0$ is some constant. The case for $\alpha = 0$ corresponds to a static universe which is not considered here. For this type of cosmology, T and T_G take the following forms,

$$T = 6\frac{\alpha^2}{t^2}, \qquad T_G = 24\frac{\alpha^3}{t^4}(\alpha - 1) = \frac{2}{3}\left(1 - \frac{1}{\alpha}\right)T^2, \tag{4.9}$$

which implies that each component (including the scale factor) can be expressed in any of the scalars. An interesting behaviour is observed for $\alpha = 1$ as this sets the TEGB term to be identically zero. For these cases, any function of T_G would become constant and hence the reconstruction technique for this type of cosmology shall be treated separately. This particular value of α is of major interest as it corresponds

[7]The $f(T, T_G)$ vacuum constraint considered here appears in constructing a viable Minkowski model as discussed in Ref. [201]. For $f(T)$ gravity, a similar condition appears in the context of Schwarzchild solutions being $f(T = 0) = 0$ [202] while a detailed account regarding vacuum solutions for $f(R)$ theories of gravity is given in Ref. [203] where a similar result arises, being $f(R = 0) = 0$.

to a coasting cosmology discussed in Chapter 3.

4.2.1 The Case $\alpha \neq 1$

Starting with $\alpha \neq 1$, the results are shown in Table 4.1. The solutions clearly indicate that the choice of the EoS parameter of the perfect fluid greatly influences the form of the solution as seen in the additive model and the T_G rescaling model. Furthermore, the relationship between α, γ and β in the power-law ansatz model determines whether the power-law contribution in the Lagrangian is present (i.e. whether μ is zero or non-zero). Note that the Gauss-Bonnet solution in the additive case matches that obtained from other sources [199, 204–208].

In these cases, the vacuum condition is satisfied, depending on parameter restrictions of the given model as illustrated in Table 4.2. Starting with the additive case, an interesting behaviour is observed if both the $g(T)$ and $h(T_G)$ solutions are required to exist. For the $\alpha(1+\omega) > 0$ and $\alpha < 1$ case, this sets $\alpha < 0$ and $\omega < -1$ or $0 < \alpha < 1$ and $\omega > -1$. This means that for phantom fluids, $\alpha < 0$, resulting in an accelerating cosmology, a property which is normally attributed to such fluids. On the other hand, non-phantom fluids lead to $0 < \alpha < 1$, which corresponds to a decelerating universe. This is interesting as quintessence with EoS $-1 < \omega < -\frac{1}{3}$ is normally associated with an accelerating behaviour but here, it still generates a decelerating phase.

On the other hand, when $1 - 3\alpha(1+\omega) = 0$ and $\alpha < 1$, one finds the following. For $0 < \alpha < 1$, the EoS $\omega > -\frac{2}{3}$. Similar to the previous scenario, it means that a decelerating universe can still be generated for fluids normally associated with an accelerating behaviour $\left(-\frac{2}{3} < \omega < -\frac{1}{3}\right)$. On the other hand, for $\alpha < 0$, $\omega < -1$. This is again the expected behaviour, where phantom fluids generate an accelerating cosmology.

$f(T,T_G) = g(T) + h(T_G)$	
$1-3\alpha(1+\omega) \neq 0$	$g(T) = \frac{\Omega_{w,0} T_0}{1-3\alpha(1+\omega)} \left(\frac{T_0}{T}\right)^{-\frac{3\alpha(1+\omega)}{2}}$
$1-3\alpha(1+\omega) = 0$	$g(T) = -\frac{1}{2}\Omega_{w,0} T \sqrt{\frac{T_0}{T}} \ln\left(\frac{T_0}{T}\right)$
In either case, $h(T_G) = c_1 {T_G}^{\frac{1-\alpha}{4}}$.	

$f(T,T_G) = Tg(T_G)$
$g(T_G) = c_2 T_G^{m_+} + c_3 T_G^{m_-} - \frac{2(\alpha-1)\Omega_{w,0}}{3[6\alpha^2 w^2 + (13\alpha^2 - 11\alpha)w + 7\alpha^2 - 11\alpha + 4]} \left(\frac{T_G}{T_{G,0}}\right)^{\frac{3\alpha(1+\omega)-2}{2}}$ provided the denominator of the final term is non-zero. No solution has been obtained when the latter is zero except for the homogeneous solution.

$f(T,T_G) = T_G g(T)$	
$4-3\alpha(1+\omega) \neq 0$	$g(T) = \frac{3\Omega_{w,0}}{2[4-3\alpha(1+w)]} \left(\frac{T}{T_0}\right)^{\frac{3\alpha(1+\omega)}{2}}$
$4-3\alpha(1+\omega) = 0$	$g(T) = \frac{3\Omega_{w,0}}{4T_0} \ln\left(\frac{T_0}{T}\right)$

$f(T,T_G) = -T + \mu T^\beta {T_G}^\gamma$
For the μ term to contribute, the conditions $\beta + 2\gamma = 1$, $2 = 3(1+\omega)\alpha$ and $\gamma \neq \frac{\alpha-1}{3\alpha-1}$ must be satisfied. If the μ term does not contribute to the field equations, then $2 = 3(1+\omega)\alpha$ with $(\alpha+\gamma-1)(2\beta+\gamma-1) + 3(\gamma-1)\gamma = 0$.

Table 4.1: Summary of the model $f(T,T_G)$ Lagrangians which reproduce a power-law cosmological solution $a(t) \propto t^\alpha$ with $\alpha \neq 0$ or 1. The solutions are dependent on the type of fluid considered. In the special case of a power-law ansatz model, it is also dependent on the relationship of the model parameters. Here, the variable $m_\pm \coloneqq \frac{1}{8}\left(3 - \alpha \pm \sqrt{\alpha^2 - 22\alpha + 25}\right)$ was defined, and $c_{1,2,3}$ represent integration constants.

In the $Tg(T_G)$ rescaling, all contributions are allowed to coexist while obeying the vacuum condition when $-3 < \alpha < 7$ and $3\alpha(1+\omega) > 1$. Here, the observed behaviour is the following. For $-3 < \alpha < 0$, the EoS is always phantom, leading to the expected accelerating behaviour of phantom fluids. For $0 < \alpha < 1$, the EoS will always obey the constraint $\omega > -\frac{2}{3}$. Thus, depending on the choice of α, fluids with EoS $-\frac{2}{3} < \omega < -\frac{1}{3}$ can still generate a deceleration. Lastly, for $\alpha > 1$, another interesting situation arises. Any $\omega > -\frac{2}{3}$ satisfies the vacuum constraint, meaning that

$f(T, T_G) = g(T) + h(T_G)$	
$1 - 3\alpha(1+\omega) \neq 0$	$\alpha(1+\omega) > 0$
$1 - 3\alpha(1+\omega) = 0$	Always
$h(T_G)$	$\alpha < 1$

$f(T, T_G) = Tg(T_G)$	
m_+ term	$\alpha < 7$
m_- term	$-3 < \alpha < 7$
Fluid term	$3\alpha(1+\omega) > 1$

$f(T, T_G) = T_G g(T)$	
$4 - 3\alpha(1+\omega) \neq 0$	$4 + 3\alpha(1+\omega) > 0$
$4 - 3\alpha(1+\omega) = 0$	Always

$f(T, T_G) = -T + \mu T^\beta {T_G}^\gamma$
Always

Table 4.2: Summary of the $f(T, T_G)$ models which satisfy the vacuum constraint for the power-law cosmological solution $a(t) \propto t^\alpha$ with $\alpha \neq 0$ or 1.

fluids which are not normally associated with an accelerating cosmology $\left(\omega > -\frac{1}{3}\right)$ can be used to realise this behaviour.

For the $T_G g(T)$ rescaling scenario, this satisfies the vacuum constraint only in the presence of other matter fluid components. In the first instance, this is only possible for $4 + 3\alpha(1+\omega) > 0$, which brings the following implications. When $\alpha < 0$, depending on the choice of α, it is possible to consider EoSs normally associated with deceleration while still yielding an accelerating universe. In this case, phantom fluids trivially satisfy the constraint. For cases when $0 < \alpha < 1$, every $\omega > -\frac{7}{3}$ satisfies the constraint meaning the existence of both phantom and non-phantom fluids associated with an accelerating cosmology, which yield a decelerating behaviour in this model. Lastly, in the case when $\alpha > 1$, every EoS $\omega \geq -1$ satisfies the constraint, inferring the possibility of fluids associated with non-accelerating behaviour

$\left(\omega \geq -\frac{1}{3}\right)$ to generate acceleration.

In the second instance when $3\alpha(1+\omega) = 4$, the following is obtained. When $\alpha < 0$, $\omega < -1$ while for $0 < \alpha < 1$, $\omega > \frac{1}{3}$. These are expected behaviours in terms of cosmological acceleration/deceleration. For the case when $\alpha > 1$, $-1 < \omega < \frac{1}{3}$. This leads to another interesting behaviour as for $-\frac{1}{3} \leq \omega < \frac{1}{3}$, an accelerating cosmology results despite these EoSs resulting into a constant velocity or decelerating behaviours.

Lastly, the power-law model always satisfies the constraint. Here, for $\alpha < 0$, $\omega < -1$, when $0 < \alpha < 1$, $\omega > -\frac{1}{3}$, and for $\alpha > 1$, $-1 < \alpha < -\frac{1}{3}$. All cases result into expected behaviours for the resulting ranges of the EoS.

4.2.2 The Case $\alpha = 1$

Moving towards the coasting cosmology case, the results are presented in Table 4.3, which clearly exhibits a distinction from the $\alpha \neq 1$ case. Here, the first model behaves as a $g(T)$ function with a cosmological constant (as $h(T_G)$ becomes constant) while the second model reduces to a TEGR rescaling (provided $g(0) \neq 0$). Interestingly, the third model reduces to a vanishing Lagrangian, and hence cannot yield any cosmological dynamics while the fourth model reduces to TEGR. Lastly, the power-law model reduces to either TEGR or its corresponding rescaled version.

With these simplifications in mind, the possible reconstructed solutions are the following. In the additive case, new solutions appear in the forms of η_1 and η_2 while the solution which describes the matter content remains unchanged. In the $Tg(T_G)$ rescaling case, solutions only appear for a stiff fluid ($\omega = 1$) and for $\omega = -\frac{1}{3}$. For the $T_G g(T)$ rescaling, a solution is obtained provided $\omega = -\frac{1}{3}$ with no apparent constraint on $g(T)$. Lastly, in the power-law ansatz model case, a solution exists only for $\omega = -\frac{1}{3}$ with the β parameter taking different restrictions depending on the magnitude of γ.

For the given existence conditions, the models satisfy the vacuum criterion in the following instances: model (i) for $T_G g_{T_G}|_{T_G\to 0} = 0$ and in the presence of fluids $\omega > -1$, (ii) $g(0)$ must be finite, (iv) for $T_G g(T)|_{T,T_G\to 0} = 0$ and (v) is always satisfied. Note that here, the additive and $Tg(T_G)$ solutions infer the existence of other fluids, which can generate a constant velocity expansion besides $\omega = -\frac{1}{3}$, a non-phantom fluid usually associated with such a coasting cosmology.

$f(T,T_G) = g(T) + h(T_G)$
$g(T) = -\eta_1 - \frac{8\eta_2 T^2}{9} + \begin{cases} \frac{\Omega_{\omega,0}T_0\left(\frac{T}{T_0}\right)^{\frac{3}{2}(\omega+1)}}{1-3(\omega+1)}, & \text{for } \omega \neq -\frac{2}{3} \\ -\frac{1}{2}\Omega_{\omega,0}T_0\sqrt{\frac{T}{T_0}}\ln\left(\frac{T}{T_0}\right), & \text{for } \omega = -\frac{2}{3} \end{cases}$

$f(T,T_G) = Tg(T_G)$	
$\omega = 1$	$g + T_G g_{T_G}\|_{T_G\to 0} = 0$, $g_{T_G} + 2T_G g_{T_G T_G}\|_{T_G\to 0} = -\frac{\Omega_{1,0}{t_0}^4}{48}$.
$\omega = -\frac{1}{3}$	$g + T_G g_{T_G}\|_{T_G\to 0} = -\Omega_{-\frac{1}{3},0}$, $g_{T_G} + 2T_G g_{T_G T_G}\|_{T_G\to 0} = 0$.

$f(T,T_G) = T_G g(T)$
No solution.

$f(T,T_G) = -T + T_G g(T)$
$g(T)$ is unconstrained and $\omega = -\frac{1}{3}$.

$f(T,T_G) = -T + \mu T^\beta {T_G}^\gamma$	
$\omega = -\frac{1}{3}$	For $\gamma > 0$, β is unconstrained while for $\gamma = 0$, $\beta = 1$.

Table 4.3: Summary of the model $f(T,T_G)$ Lagrangians which reproduce a linear coasting cosmology $a(t) \propto t$. Contrary to the general power law case, models (iii) and (iv) are distinct as the T_G term does not contribute in this limit. Furthermore, a fluid with EoS $\omega = -\frac{1}{3}$ is mostly required to have any physical solutions which is expected for such coasting cosmologies. Here, the variables $\eta_1 := h - T_G h_{T_G}|_{T_G\to 0}$ and $\eta_2 := T_G h_{T_G T_G}|_{T_G\to 0}$ were defined.

4.3 de Sitter Reconstruction

During a de Sitter phase, the Hubble parameter approaches a constant value, say $H = H_{\rm dS}$, which results in $T = 6{H_{\rm dS}}^2$ and $T_G = 24{H_{\rm dS}}^4$ to become constant. In this case, the Friedmann equation Eq.(4.2) takes the form

$$f - 2T f_T - T_G f_{T_G} = T_0 \Omega_{\omega,0} e^{-3(1+\omega)H_{\rm dS}t}. \tag{4.10}$$

As the LHS is time-independent, the only source of matter which can satisfy the equation is one which behaves as a cosmological constant $\omega = -1$. In other words, the resulting equation to solve is

$$f - 2T f_T - T_G f_{T_G} = T_0 \Omega_{\Lambda,0}. \tag{4.11}$$

This equation can be treated in two ways. The first is as a PDE, which can be solved analytically to give

$$f(T, T_G) = T_0 \Omega_{\Lambda,0} + \sqrt{T}\, g\left(\frac{T_G}{\sqrt{T}}\right), \tag{4.12}$$

where g is some arbitrary function, leading to an infinite class of solutions which describe the de Sitter cosmology. However, if the vacuum condition must be satisfied, only a subset of these solutions will result to be viable, as the function g must satisfy the constraint

$$T_0 \Omega_{\Lambda,0} = -\sqrt{T}\, g\left(\frac{T_G}{\sqrt{T}}\right) |_{T,\, T_G \to 0}\ . \tag{4.13}$$

For example, the function $g = -T_0 \Omega_{\Lambda,0} \left(\frac{3T_G}{2\sqrt{T}}\right)^{-\frac{1}{3}}$ satisfies this constraint. However, here, a cosmological constant fluid has been assumed to be present in the universe. If the gravitational Lagrangian is to replace this source, then the system must be solved in the absence of this fluid. This alters the vacuum condition to be $\sqrt{T} g\left(\frac{T_G}{\sqrt{T}}\right) |_{T,\, T_G \to 0} = 0$. For instance, the model $g \propto \frac{{T_G}^2}{T}$ satisfies this constraint but not the previous one, meaning that de Sitter cosmologies can be generated without the requirement of a cosmological constant source. Observe that the Lagrangians $f \propto e^{\frac{T}{12H_{\rm dS}^2}}$ and $f \propto e^{\frac{T_G}{24H_{\rm dS}^4}}$ which appear in Refs. [69,204] also satisfy the PDE but not the vacuum constraint as both models reduce to a non-zero constant.

A secondary approach is to treat Eq.(4.11) as an algebraic expression for the Hubble parameter $H_{\rm dS}$. Given a function $f(T,T_G)$, the LHS yields an expression in terms of the Hubble parameter which, in principle, can be solved to yield a value during de Sitter times. Taking TEGR as a first example, this results in $H_{\rm dS} = H_0\sqrt{\Omega_{\Lambda,0}} \equiv \sqrt{\frac{\Lambda}{3}}$ with Λ representing the standard cosmological constant which appears in ΛCDM. As previously discussed, the gravitational Lagrangian is constructed to act as a source for the de Sitter phase without introducing a cosmological constant fluid, meaning the RHS will be set to zero. This shall be presented through some model examples.

In the case of $f(T)$ gravity, taking the power-law model $f(T) = -T + \alpha T^\beta$, for parameters α and $\beta \neq \left\{\frac{1}{2}, 1\right\}$, the Hubble parameter adopts the value of

$$H_{\rm dS} = \left[6^{\beta-1}\alpha(2\beta-1)\right]^{\frac{1}{2(1-\beta)}} . \tag{4.14}$$

The vacuum condition is satisfied provided that $\beta > 0$ and since an expanding universe is observed, this constrains $\alpha < 0$ for $0 < \beta < \frac{1}{2}$ and $\alpha > 0$ for $\beta > \frac{1}{2}$. If the parameter choices considered in Ref. [178] are chosen, this model yields a de Sitter attractor with the same resulting Hubble value, as shown in Eq.(4.14). A similar analysis can be carried out for other $f(T)$ Lagrangian forms.

However, this approach does not always yield a Hubble value. For instance, the model $f(T,T_G) = -T + \mu\sqrt{T^2 + \eta T_G}$ for constants μ and η yields the constraint

$$\mu = -\frac{\sqrt{3}\sqrt{2\eta+3}}{\eta-3}. \tag{4.15}$$

Despite this, it does impose a constraint when the model generates a de Sitter behaviour. Furthermore for this model ansatz, the vacuum constraint is trivially satisfied. Note that the parameter constraints also arise from a dynamical stability approach as shown in Ref. [81].

4.4 Reconstruction of ΛCDM Cosmology

In this section, the reconstruction of ΛCDM cosmology is considered, in the presence of matter and a cosmological constant-like fluid. Thus, the effects of radiation is neglected for simplicity. Such a cosmology is realised for

$$\frac{T}{T_0} = \Omega_{\mathrm{M},0} a^{-3} + \left(1 - \Omega_{\mathrm{M},0}\right), \tag{4.16}$$

which generates the analytical scale factor solution

$$a(t) = \left(\frac{\Omega_{\mathrm{M},0}}{1 - \Omega_{\mathrm{M},0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3\sqrt{1 - \Omega_{\mathrm{M},0}}}{2} H_0 t\right). \tag{4.17}$$

From Eq.(4.16), the scale factor can be analytically expressed in terms of the torsion scalar

$$a^3 = \frac{T_0 \Omega_{\mathrm{M},0}}{T - T_0\left(1 - \Omega_{\mathrm{M},0}\right)}. \tag{4.18}$$

On the other hand, the TEGB term is related to the torsion scalar through

$$T_G = \left(1 - \Omega_{\mathrm{M},0}\right) T_0 T - \frac{T^2}{3}, \tag{4.19}$$

which can be inverted to yield an expression for $T = T(T_G)$. The aim here is to determine whether an $f(T, T_G)$ Lagrangian is capable of reproducing the ΛCDM expansion history (not the ΛCDM model Lagrangian) without invoking any dark fluid or the addition of any cosmological constant, while also satisfying the vacuum condition. For the model ansatz considered, the resulting functional forms are summarised in Table 4.4.

Overall, the derived functions are fairly complicated due to the introduction of the hypergeometric functions except for the power-law model, which is able to generate simple solutions. Despite these apparent complicated results, the solutions do simplify to simple forms in the special case of dust. For instance, the additive $g(T)$ function reduces to the standard ΛCDM result $g(T) = -T - \Omega_{\Lambda,0}T_0$, as shown in Refs. [121, 128]. In the T_G rescaling model, the Lagrangian takes the simple form

$$f(T, T_G) = (2T - \Omega_{\Lambda,0}T_0)\,\frac{3T_G}{8T^2}. \tag{4.20}$$

However, the $Tg(T_G)$ model case does not yield any analytical solutions that describe the matter sector. For the power-law model, no solutions are derived for an equation of state different from ordinary (dust) matter. In this case, the resulting solutions are the standard ΛCDM Lagrangian and the non-trivial solution having $\beta = -2$ and $\gamma = 1$. For a general EoS, the $g(T)$ function derived in the additive case matches with those found in Refs. [69, 128].

However, none of the reconstructed solutions satisfy the vacuum condition in the presence of matter fluids. For the additive model, $g(0) \neq 0$ and $h(0) = 0$ while $g(0) = 0$ in vacuum satisfying the condition. In the second model, the homogeneous solution satisfies the constraint. The third model $T_G g(T)$ always diverges, and in the absence of fluids, the Lagrangian becomes null, making it impossible to satisfy the constraint. The extended model with a TEGR contribution $-T + T_G g(T)$ either suffers from divergences (in the presence of matter fluids), or reduces to a non-zero constant (in vacuum), and hence cannot host vacuum solutions either. A similar result emerges in the power-law case and hence, the condition is never satisfied. Therefore, only the first two models can satisfy the constraint while describing a ΛCDM cosmology, at the expense that the universe is an empty state solely dominated by some effective fluid arising from the TEGB contribution. Given the presence of matter in the universe, questions arise regarding the validity of such solutions.

$f(T,T_G)=g(T)+h(T_G)$
$g(x)=T_0\Omega_{\omega,0}\left(\frac{\Omega_{\text{M},0}}{\Omega_\Lambda(x-1)}\right)^{-(1+\omega)}\left[{}_2F_1\left(1,\omega+\frac{1}{2};\frac{1}{2};x\right)+x\,{}_2F_1\left(1,\omega+\frac{3}{2};\frac{3}{2};x\right)\right]$
$h(y)=c_1(1-3y)^{2/3}\left[(40y+24)\sqrt{1-y}+5\sqrt{6}\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};\frac{5}{3};\frac{3y-1}{2}\right)(y^2-1)\right]$

$f(T,T_G)=Tg(T_G)$
Only the homogeneous solution is found, having a power series solution $g(y)=\sum_{n=0}^{\infty}a_ny^n$ where the coefficients a_n obey the recurrence relation $(2+7n-6n^2)\,a_n+(20n^2+13n-9)\,a_{n+1}-(n+2)(24n+11)a_{n+2}$ $+3(n+3)(4n+5)a_{n+3}-2(n+2)(n+4)a_{n+4}=0,$ having initial conditions $a_0=3(a_2-a_3)$, $a_1=0$ with $a_{2,3}$ acting as the constants of integration.

	$f(T,T_G)=T_Gg(T)$
$\omega\neq n\in\mathbb{Z}^+$	$g(T)=\frac{3\Omega_{w,0}T_0}{8T^2}\left(\frac{\Omega_{\text{M},0}T_0}{T-\Omega_\Lambda T_0}\right)^{-(1+\omega)}\left[1+\frac{1+\omega}{1-\omega}\,{}_2F_1\left(1,2;2-\omega;\frac{\Omega_\Lambda T_0}{T}\right)\right]$
$\omega=n\in\mathbb{Z}^+$	$g(T)=-\frac{3}{4}T_0\Omega_{\omega,0}\left(T_0\Omega_{\text{M},0}\right)^{-(1+\omega)}\left\{\binom{1+\omega}{\omega-1}\ln T\left[-T_0\Omega_\Lambda\right]^{\omega-1}+\sum_{\substack{k=0\\k\neq\omega-1}}^{1+\omega}\binom{1+\omega}{k}\frac{T^{\omega-k-1}}{\omega-k-1}\left[-T_0\Omega_\Lambda\right]^k\right\}$

	$f(T,T_G)=-T+\mu T^\beta {T_G}^\gamma$
$\omega=0$	$\{\beta,\gamma\}=\{0,0\},\ \{-2,1\}$
$\omega\neq0$	No solutions

Table 4.4: Summary of the reconstructed $f(T,T_G)$ Lagrangians, which reproduce an exact ΛCDM model. In the second model, an analytical solution describing the matter content could not be derived. In this table, $x=\frac{T}{\Omega_\Lambda T_0}$, $y=\sqrt{1-\frac{4T_G}{3{\Omega_\Lambda}^2{T_0}^2}}$, c_1 is an integration constant and ${}_2F_1(a,b;c;x)$ is Gauss's hypergeometric function.

4.5 Discussion

In this chapter, the resourceful tool of reconstruction has been considered in the context of a general gravitational Lagrangian that is dependent on the torsion scalar and the TEGB contribution to yield desired cosmological histories in terms of analytical functions. Due to the introduction of a higher-order contribution, it is not always possible to produce a general analytical solution for the Lagrangian, and hence specific ansatz are chosen. In particular, the power-law, the de Sitter and ΛCDM behaviours have been considered for reconstruction. As means to restrict the possible physical Lagrangian, a further constraint was set, being that the model is capable of exhibiting vacuum solutions (for instance, Minkowski) for which the sufficient condition turns out to be $f(0,0) = 0$.

Starting with power-law solutions, the special case of a coasting cosmology had to be investigated separately since in these cases, T_G is exactly zero. Overall, as the summarised results in Table 4.3 show, one finds that models can be constructed to satisfy the vacuum constraint even for fluids not normally associated with a coasting cosmology (those being $\omega = -\frac{1}{3}$). For the remaining instances, as shown in Table 4.1, a reconstructed Lagrangian can also be constructed. In this case, vacuum conditions depend on the model considered, where in certain instances, further restrictions on the parameters α and ω are required, as summarised in Table 4.2. These restrictions also pose an interesting behaviour in terms of the resulting cosmological dynamics. Fluids which are not normally associated with, say, a decelerating universe can still possibly result in such a cosmology.

De Sitter cosmology has also been investigated. In this case, a fluid can only contribute if it behaves as a cosmological constant. Since the aim of this modified theory of gravity is to sway away from the introduction of such a constant, de Sitter solutions have been constructed in its absence. The resulting equation Eq.(4.11) with $\Omega_{\Lambda,0} = 0$ can be viewed either as a PDE or as an algebraic equation. In the former, the resulting models exhibit de Sitter behaviour naturally, albeit at the expense of not inferring a de Sitter Hubble value. As for the latter, depending on the form of the chosen Lagrangian, it infers either a constraint on the parameters which generate a de Sitter cosmology or infers a specific Hubble parameter value. In both cases, the vacuum constraint can be satisfied.

Lastly, the $f(T, T_G)$ Lagrangian was investigated to see whether it is capable of reproducing an exact ΛCDM cosmology without invoking a cosmological constant. As shown in Table 4.4, although the models do not contain an explicit cosmological constant dependence, none of the solutions satisfy the vacuum constraint except for the additive and the $Tg(T_G)$ rescaling models in an empty universe. However, this consideration is unrealistic when based on observational criteria, leaving the reconstructed $f(T, T_G)$ models incapable of hosting vacuum solutions. Despite this issue, the models also suffer from complicated Lagrangian forms arising from the resulting hypergeometric functions (with the exception of the power-law model). However, this only appears when expressed for an arbitrary choice of EoS as taking specific fluid choices (for example, ordinary pressureless matter) can simplify the Lagrangian significantly, making the models viable for use in other fields.

Therefore, $f(T, T_G)$ gravity has been shown to be able to realise various cosmological solutions while also being capable of recovering vacuum solutions. Nonetheless, the reconstructed solutions are by no means necessarily viable models as they are still required to be tested against observations.

Chapter 5

Reconstructing $f(T, T_G)$ Lagrangian for Bounce Cosmologies

A physical description of the very early time cosmology is still not well understood. Although a major leap towards a better understanding has been achieved through the formulation of inflation, this is by no means necessarily correct. Amongst the various issues encountered in inflation (see Refs. [145, 209–213] and references therein), one finds two major problems: the singularity problem, and the so-called trans-Planckian problem.

Starting with the former, this is better understood from the Penrose-Hawking singularity theorems [214, 215]. Roughly speaking, these state that, given the GR field equations and certain constraints in the stress-energy momentum sector, there exists a time before inflation where a singularity arises, namely the Big Bang singularity. This singularity represents a point at which the equations break down, and nothing can be inferred of what happens past the singularity [209, 213, 216]. Although, in general, the singularity theorems do not exactly describe the nature of the singularity, here the Big Bang singularity corresponds to a diverging Ricci scalar R due to the vanishing scale factor (and hence divergent H). This breakdown is expected from a GR viewpoint as it is a classical theory and hence not UV complete, meaning that GR is not applicable at very high energy scales. Thus, the application of the Einstein field equations throughout such high curvature regimes is no longer applicable [216].

The second issue refers to the origin of the relevant scales which generate the inhomogeneous and anisotropic structures observed today. Given that inflation lasted long enough to resolve the horizon and flatness issues, the relevant modes which generate the observed structures must originate from a time in which the physical scale is smaller than a Planck length, a distance in which GR breaks down due to its classical nature. Therefore, inflation becomes dependent on the unknown behaviour of modes originating from such scales [210, 213].

Because of these two issues, amongst others, have therefore put into question the validity of the standard inflationary model. In fact, the observed large scale structure and CMB spectra were not originally considered from an inflationary viewpoint [217–220]. Here, inflation formulated a decade later, only provided a physical mechanism which generates these spectra. Therefore, other alternatives to inflation are to be considered, as long as they are still capable of reproducing these observations while also resolving the problems encountered in inflation. This is the role of what are called bouncing cosmologies.

A bouncing cosmology is one which primarily tackles the two previously mentioned issues. This is made possible through replacing the Big Bang singularity by some preceding contracting phase, followed by an expanding (possibly later inflationary) phase. To avoid the singularity, the scale factor decreases to some non-zero value, with the resulting universe volume being sufficiently large where the ensuing physics does not break down. This therefore resolves the singularity problem. Note that if the bounce occurs at some time $t = t_{\mathrm{B}}$, it is characterised by $H(t < t_{\mathrm{B}}) < 0$, $H(t_{\mathrm{B}}) = 0$ and $H(t > t_{\mathrm{B}}) > 0$ [212, 221, 222]. Introducing a contraction phase allows for all relevant scales to not originate from sub-Planckian scales as the Hubble horizon ends up being sufficiently large for all modes of interest, hence resolving the trans-Planckian problem [145, 210, 213].

Despite this attractive feature of a bouncing cosmology, they are by no means perfect. Here, some of these issues are highlighted whereas further details are given in Refs. [211–213]. If the universe is fairly anisotropic during the contracting phase, these can still remain fairly dominant as they grow with scale factor as a^{-6}. In turn, this leaves the early universe to be anisotropic instead of homogeneous, which is not observed. To account for this, the presence of fluids with EoS $\omega \gg 1$ are required to exist in order to maintain the homogeneity condition. Bouncing cosmologies may also suffer from instabilities with perturbations growing sufficiently large, thus affecting the different spectra by not being compatible with observations.

In what follows, different types of bouncing behaviours are discussed, namely (a) symmetric bounce, (b) oscillatory bounce (cyclic/ekpyrotic), (c) superbounce, (d) critical bounce, and (e) future (past) singularities, primarily those of Type I–IV. The physical interpretation and application for each model shall be explored in further detail in their corresponding sections.

To this end, the method of reconstruction presented in the previous chapter has proven to be a powerful tool to help construct gravitational Lagrangians based on a given cosmological expansion history. It is therefore natural to consider this approach for the bouncing cosmologies in question, as it will prove useful to be able to determine whether such models can be constructed in the realm of modified gravity. In what follows, $f(T, T_G)$ gravity shall be considered. As discussed in the previous chapter, solving for a general $f(T, T_G)$ Lagrangian proves to be intractable, and hence specific ansatz are chosen, namely those considered previously, being

$$(i) \quad f(T, T_G) = g(T) + h(T_G), \qquad (iv) \quad f(T, T_G) = -T + T_G g(T),$$

$$(ii) \quad f(T, T_G) = T g(T_G), \qquad (v) \quad f(T, T_G) = -T + \mu T^{\beta} {T_G}^{\gamma}.$$

(iii) $f(T, T_G) = T_G g(T)$,

For simplicity, the boundary $\sqrt{T}$ and Gauss-Bonnet contributions are again not listed, and models deemed as viable are those which are capable of generating vacuum solutions, i.e. satisfy the constraint $f(0, 0) = 0$.

5.1 Symmetric Bounce

Originally considered in Ref. [223], a symmetric bounce model describes an exponential scale factor of the form

$$a(t) = A \exp\left(\alpha \frac{t^2}{{t_*}^2}\right), \tag{5.1}$$

where t_* is some arbitrary time, $A > 0$ and $\alpha > 0$ are constants. This model yields a Hubble parameter of the form

$$H = \frac{2\alpha t}{{t_*}^2}, \tag{5.2}$$

which implies a bounce located at $t = 0$ as $H < 0$ for $t < 0$, $H = 0$ at $t = 0$ and $H > 0$ for $t > 0$. As discussed in Ref. [155], this type of bounce cannot fully describe the universe behaviour near the bounce. Any primordial modes cannot occur before the bounce point, since the Hubble parameter decreases as it approaches the bouncing time. On the other hand, if any primordial modes did occur, the Hubble horizon decreases after the bouncing point, meaning that no modes re-enter the horizon. Therefore, this model needs to be used in conjunction with other models or transition periods to give a more complete picture. For instance, an ekpyrotic behaviour was considered in Ref. [211], while the bounce together with a late-time acceleration phase was considered in Ref. [224].

In this case, T and T_G are given by

$$T = 6H^2 = \frac{24\alpha^2 t^2}{{t_*}^4}, \qquad\qquad T_G = \frac{8\alpha}{{t_*}^2}T + \frac{2T^2}{3}, \tag{5.3}$$

where the scale factor can be solely expressed in terms of the torsion scalar T

$$a(T) = A\exp\left(\frac{T{t_*}^2}{24\alpha}\right) = A\exp\left(\alpha\frac{T}{T_*}\right), \tag{5.4}$$

where $T_* \equiv T(t = t_*) = 24\alpha^2/{t_*}^2$. To simplify the reconstruction analysis for the sake of simplicity, the scale factor is set to unity at some arbitrary time $t_0 > 0$, i.e. $a(t_0) = 1$. This yields the relation

$${t_0}^2 = -\frac{{t_*}^2}{\alpha}\ln A. \tag{5.5}$$

Since α is positive, the equation yields real values for the time t_0 if and only if $0 < A < 1$, which will be assumed in the following expressions.

Table 5.1 illustrates the reconstructed solutions for the considered $f(T, T_G)$ ansatz. It is observed that the $Tg(T_G)$ model only yields analytical solutions for the homogeneous solution, as it could not be derived for the particular solution. From the Green's function and Wronskian method [225], the resulting particular solution is derived from the integral

$$g_{\text{part.}}(y) = -\frac{\Omega_{\omega,0}T_0{t_*}^2A^{-3(1+\omega)}}{3456\alpha}\int^y \frac{e^{-(4+3\omega)s}}{\sqrt{s}(1+4s)}\left[g_2(y)g_1(s) - g_1(y)g_2(s)\right]\, ds, \tag{5.6}$$

which could not be analytically computed. Of interest is the power-law ansatz model case, where the choice and presence of fluids greatly influences the existence of solutions. In the case of vacuum, the power-law modification allows for the symmetric bounce to occur. If the fluids exist, then they behave as a cosmological constant, which not only restricts the choice of the power-law indices β and γ, but also the Hubble rate factor α and the value of the cosmological constant $\Omega_{\Lambda,0}$.

For the reconstructed solutions obtained, the vacuum condition is only satisfied for the additive and $Tg(T_G)$ models in the absence of fluids. In the remaining models, the Lagrangian either diverges or reduces to a non-zero constant, and hence does not satisfy the viability condition. Since the condition only becomes satisfied in the absence of fluids, viable symmetric bounce models are only realised due to the presence of modified gravity.

$f(T,T_G) = g(T) + h(T_G)$
$g(x) = T_0\Omega_{\omega,0}A^{-3(1+\omega)}\left[\sqrt{\pi}\, x\, \mathrm{erf}\,(x) + \mathrm{e}^{-x^2}\right]$, $h(y) = c_1 e^y\left[-\sqrt{y}(y+1) + y\,(2y+1)\,F\left(\sqrt{y}\right)\right]$

$f(T,T_G) = Tg(T_G)$
Only the homogeneous solutions are found, being: $g_1(y) = c_2\left[2\left(8y^2+14y-3\right) + \sqrt{\frac{\pi}{y}}\left(16y^3+36y^2+3\right)\mathrm{erf}\left(\sqrt{y}\right)e^y\right]$, and $g_2(y) = c_3\frac{e^y}{\sqrt{y}}\left(16y^3+36y^2+3\right)$.   An analytical form of the particular solution is not obtained.

$f(T,T_G) = T_G g(T)$
$g(x) = \dfrac{3\Omega_{\omega,0}A^{-3(1+\omega)}{x_0}^4}{8T_0}\left[\dfrac{1-x^2}{x^4}e^{-x^2} - \mathrm{Ei}\left(-x^2\right)\right]$

	$f(T,T_G) = -T + \mu T^\beta {T_G}^\gamma$
In vacuum	$\beta=-1$, $\gamma=1$ and $\mu=-\frac{3}{4}$
$\omega=-1$	$\beta=-3$, $\gamma=2$ provided that $\alpha=\frac{2{t_*}^2}{7}$, $\mu=-\frac{9}{20}$ and $\Omega_{\Lambda,0}T_0=\frac{128\mu}{7}$.

Table 5.1: The $f(T,T_G)$ reconstructed Lagrangian that reproduces the symmetric bounce model is obtained. The following variables and functions have been defined and used: $x := \sqrt{\frac{T(1+\omega){t_*}^2}{8\alpha}}$, $y := \frac{1}{4}\left(-1+\sqrt{1+\frac{T_G{t_*}^4}{24\alpha^2}}\right)$, $F(z)$ is the Dawson integral, which is defined as $F(z) = e^{-z^2}\int\limits_0^z e^{p^2}\,dp$, $\mathrm{Ei}(z)$ is the exponential integral function $\mathrm{Ei}(z) = -\int\limits_{-z}^{\infty}\frac{e^{-r}}{r}\,dr$, and $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}\,ds$ is the error function. Furthermore, $c_{1,2,3}$ are constants of integration.

5.2 Oscillatory Bouncing Model

The second bouncing model considered is one with an oscillatory behaviour having the form

$$a(t) = A \sin^2 \left(B \frac{t}{t_*} \right) , \tag{5.7}$$

where t_* is some reference time which, for simplicity, is taken to be positive, and A and B are real positive dimensionless constants. Such a model represents the behaviour of what is commonly referred to as a cyclic universe, originally proposed by Tolman in Ref. [226] (further details on cyclic universes is available in Ref. [209]). This proposal introduces the concept where the universe exhibits a sequence of continuous contractions and expansions, thus leading to a new insight towards the beginning of the universe [227]. Oscillatory scale factors allow for a simple construction of said behaviour, as discussed, for instance, in Refs. [227–230]. Furthermore, there exists a special class of cyclic cosmologies, known as conformally cyclic universes (see for example Ref. [228]) where whilst the universe has a cyclic behaviour, the overall expansion is monotonically increasing.

However, the choice of the scale factor may suffer from singularities. As the universe is continuously expanding and contracting, implying a continuous crunch and rebirth, these points might correspond to a Big Crunch and a Big Bang respectively. These two behaviours represent a singularity since, when the scale factor approaches a zero value, the Hubble parameter diverges. This problem could be circumvented through various mechanisms such as Loop Quantum Cosmology (LQC) [229], matter couplings [228], or by demanding the scale factor to not approach a zero value.

The features of the considered scale factor are examined further. It is useful to determine its associated Hubble parameter, which takes the form of

$$H = \frac{2B}{t_*} \cot \left(B \frac{t}{t_*} \right) . \tag{5.8}$$

The scale factor is zero for times $t_{\text{min}} = \frac{n\pi t_*}{B}$ with $|H(t_{\text{min}})| \to \infty$ and reaches a maximum size of A at times $t_{\text{max}} = \frac{(2n+1)\pi t_*}{2B}$ where $H(t_{\text{max}}) = 0$. Here, n is an integer. For times $t_{\text{min}} < t < t_{\text{max}}$, the Hubble parameter is positive and approaching zero as $t \to t_{\text{max}}$ whilst for $t_{\text{max}} < t < t_{\text{min}}$, the Hubble parameter decreases and becomes singular while approaching negative infinity as $t \to t_{\text{min}}$. Therefore, when the scale factor approaches its minimal size as $t \to t_{\text{min}}$, the universe experiences a Big Crunch, which then rebirths into a Big Bang, both being a singularity (as $a(t_{\text{min}}) = 0$ and $|H(t_{\text{min}})| \to \infty$). This behaviour can be associated with that of a superbounce. On the other hand, at maximal size, the universe experiences a bounce called a cosmological turnaround, as the Hubble parameter transitions from positive to a negative value as $t_1 \to t_{\text{max}} \to t_2$ $(t_1 < t_{\text{max}} < t_2)$. This represents a period where the universe stops expanding post-Big Bang, and starts to contract towards a Big Crunch [221].

For this cyclic model, the torsion and TEGB scalars take the form of

$$T = \frac{24B^2}{{t_*}^2} \cot^2\left(B\frac{t}{t_*}\right), \qquad T_G = 4T\left(\frac{T}{12} - \frac{2B^2}{{t_*}^2}\right), \tag{5.9}$$

where they are directly related through

$$T = \frac{12B^2}{{t_*}^2}\left(1 - \sqrt{1 + \frac{T_G {t_*}^4}{48B^4}}\right). \tag{5.10}$$

In this way, the scale factor can be expressed purely in terms of the torsion scalar

$$a(T) = A\left(1 + \frac{T{t_*}^2}{24B^2}\right)^{-1}. \tag{5.11}$$

For simplicity, assume that the Big Bang singularity occurs at a time $t_{\text{min}} = 0$. Then, at some time $t_{\text{min}} < t_0 < t_{\text{max}}$, the scale factor is unity, which leads to the condition

$$\frac{1}{A} = \sin^2\left(B\frac{t_0}{t_*}\right). \tag{5.12}$$

This solution yields real values only when $A > 1$ (the case $A = 1$ would imply that $t_0 = t_{\max}$ hence not satisfying the assumption). Therefore, the present time would lie at $t_0 = \frac{t_*}{B} \sin^{-1}\left(\frac{1}{A}\right)$. In this way, the present time parameters $\Omega_{\omega,0}$, T_0 and $T_{G,0}$ can be defined and may represent current values.

Table 5.2 illustrates the reconstructed solutions for the various $f(T, T_G)$ ansatz considered for the oscillatory model. Of note is the $Tg(T_G)$ rescaling model, where a lack of closed forms for the homogeneous solution result. Furthermore, only one of the homogeneous solutions is obtained, and the particular solution could not be derived. On the other hand, the $T_G g(T)$ model solution depends on the EoS parameter value. Similar to the symmetric bounce case, the power-law model yields a solution in vacuum with the same $\{\beta, \gamma, \mu\}$ parameter choices. Contrary to the symmetric bounce model, however, is the absence of solutions for non-vacuum considerations. This means that an oscillatory behaviour is only generated in the absence of matter fluids.

The viability condition is thus only satisfied for the additive and the $Tg(T_G)$ models, only in the absence of matter fluids similar to the symmetric bounce case. This is due to divergences or non-zero finite limits of the resulting Lagrangian in the remaining models and cases. Thus, the oscillatory model given in the form of Eq.(5.7) is only generated in the first two ansatz models, provided that matter fluids are absent.

5.3 Superbounce

As mentioned in Chapter 3, a superbounce model takes a power-law solution of the form $a(t) \propto (-t + t_c)^{2/c^2}$, where t_c represents the crunch time and $c > \sqrt{6}$. However, a simple transformation of the coordinate system allows for a shifting of the crunch time to any arbitrary time, which for simplicity is taken to occur at zero. Thus, the scale factor reduces to the standard simple power-law model $a(t) \propto t^{\alpha}$ with

$\alpha \coloneqq \frac{2}{c^2}$. Therefore, the reconstructed solutions obtained in Table 4.1 will apply, but with the constraint that $0 < \alpha < \frac{1}{3}$. Although this constraint does not affect the reconstructed solutions themselves, the vacuum condition constraints can be further simplified.

$f(T,T_G) = g(T) + h(T_G)$
$g(x) = T_0 \Omega_{\omega,0} A^{-3(1+\omega)} \left\{ \frac{1}{5} x^3 \, {}_2F_1\left(\frac{5}{2}, -3\omega; \frac{7}{2}; x\right) - x^2 \, {}_2F_1\left(\frac{3}{2}, -3\omega; \frac{5}{2}; x\right) + 3x \, {}_2F_1\left(\frac{1}{2}, -3\omega; \frac{3}{2}; x\right) + {}_2F_1\left(-\frac{1}{2}, -3\omega; \frac{1}{2}; x\right) \right\}$
$h(y) = \left[(8-3y)\sqrt{y} - 6y(y-1)\tan^{-1}\left(\sqrt{y}\right)\right] c_1$

$f(T,T_G) = Tg(T_G)$
Only one homogeneous solution is obtained, which has a power-series solution of the form $g(x) = \sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^n$, where the coefficients a_n obey the recurrence relation $2(n-1)^2 a_n + n a_{n-2} + (n-5)(2n-1) a_{n-1} = 4n(n+1) a_{n+1}$, with $a_{-2,-1} = 0$. Particular solutions describing the matter source are not analytically solvable.

$f(T,T_G) = T_G g(T)$	
For $n \in \mathbb{Z}$, $n \geq -1$:	
$w \neq \frac{n}{3}$	$g(x) = -\frac{\xi_\omega}{6\omega x^3}(1+x)^{1+3\omega} \left\{ x\left[1 + 3\omega + x(3\omega - 1)\right] {}_2F_1\left(1, 1; 1-3\omega; -\frac{1}{x}\right) - (1+x)\left[1 + x(3\omega+5)\right] {}_2F_1\left(1, 2; 1-3\omega; -\frac{1}{x}\right) - \frac{6\omega x^3}{1+3\omega} \right\}$
$w = \frac{n}{3}$	$g(x) = \xi_{\frac{n}{3}} \left[\binom{n+3}{2} \ln x + \sum_{\substack{k=0 \\ k\neq 2}}^{n+3} \binom{n+3}{k} \frac{x^{k-2}}{k-2} \right]$

$f(T,T_G) = -T + \mu T^\beta {T_G}^\gamma$	
In vacuum	$\beta = -1$, $\gamma = 1$ and $\mu = -\frac{3}{4}$

Table 5.2: Summary of the reconstructed $f(T,T_G)$ Lagrangians which reproduce the oscillatory bounce model. Here, the following variables and functions have been defined and used: $x \coloneqq -\frac{T{t_*}^2}{24B^2}$, $y \coloneqq 2\left(1 - \sqrt{1 + \frac{T_G {t_*}^4}{48B^4}}\right)$, $\xi_\omega \coloneqq -\frac{3}{4}\left[\frac{{t_*}^2}{24B^2}\right]^2 T_0 \Omega_{\omega,0} A^{-3(1+\omega)}$ and c_1 is an integration constant.

In the additive case, the condition simply reduces to $\omega > -1$ for $1 - 3\alpha(1+\omega) \neq 0$ whilst the second case $1 - 3\alpha(1+\omega) = 0$ always satisfies the vacuum condition. For the $Tg(T_G)$ rescaling model, the homogeneous solutions always satisfy the constraint, while the fluid component that satisfies the condition is $3\alpha(1+\omega) > 1$ (in these cases, $\omega > 0$). On the other hand, the TEGB rescaling model requires the same constraint $4 + 3\alpha(1+\omega) > 0$, as the other case remains trivially satisfied. Here, every value of $\omega > -5$ satisfies the constraint. Lastly, the power-law model always satisfies the viability condition.

5.4 Matter Bounce

The bouncing model referred to as a matter bounce originates from LQC theory, where the scale factor evolves in a form

$$a(t) = A\left(\frac{3}{2}\rho_{\mathrm{cr}}t^2 + 1\right)^{1/3}, \tag{5.13}$$

where ρ_{cr} is a critical density, a value which originates from LQC theory, and $A > 0$ is a dimensionless constant. This scale factor solution stems from the resulting Friedmann equation constructed from LQC theory

$$H^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_{\mathrm{cr}}}\right), \tag{5.14}$$

where ρ is some fluid energy density. If this is taken to be pressureless dust, the above scale factor solution Eq.(5.13) is obtained. In this case, the Hubble parameter takes the simple form of

$$H = \frac{2t\rho_{\mathrm{cr}}}{2 + 3t^2\rho_{\mathrm{cr}}}, \tag{5.15}$$

from which it can easily be observed that a non-singular bounce is generated at a time $t = 0$, the time at which the universe reaches its minimal size as $H(t < 0) < 0$, $H(t = 0) = 0$ and $H(t > 0) > 0$ [231,232]. This type of bouncing cosmology has been studied as a means to investigate the early stages of the universe. In particular, dust has been considered due to its ability to generate a scale-invariant power spectrum arising from the matter contraction and expansion phases whilst not suffering from any singularities. However, for a perfect dust fluid, the scale-invariant spectrum is actually a problem, as observations indicate a slight red tilt[8]. Furthermore, for dust, the tensor-scalar ratio is much larger than what is observed [234,235]. However, small deviations from dust [211, 232] or introduction of other scalar fields [223] can address this problem.

For the given matter bounce scale factor Eq.(5.13), the teleparallel scalars are given to be

$$T = 6\left(\frac{2t\rho_{\rm cr}}{2+3t^2\rho_{\rm cr}}\right)^2, \qquad T_G = -\frac{4T^2}{3} + 2T\rho_{\rm cr}\left(1+\sqrt{1-\frac{T}{\rho_{\rm cr}}}\right), \tag{5.16}$$

which allows the scale factor to be expressed in terms of T, as

$$a(T) = A\left[\frac{2\rho_{\rm cr}}{T}\left(1-\sqrt{1-\frac{T}{\rho_{\rm cr}}}\right)\right]^{1/3}. \tag{5.17}$$

Assuming that there exists a time $t_0 > 0$ when $a(t_0) = 1$, yields the relation

$${t_0}^2 = \frac{2}{3\rho_{\rm cr}}\left(\frac{1}{A^3} - 1\right). \tag{5.18}$$

Since $\rho_{\rm cr} > 0$, this equation holds provided that $A < 1$, which will be assumed from here thereon. This allows for the definition of the present quantities of the density parameters and the teleparallel scalars at this time. Furthermore, note that the T_G and T relation can be inverted to yield

[8] For a given power spectrum $P(k) \propto k^n$, where n is some real number, the spectrum is said to be red tilted if $n < 0$ and blue tilted if $n > 0$, otherwise it is invariant for $n = 0$. As observations indicate a scalar spectral index of $n_s = 0.9649 \pm 0.0042$, the matter power spectrum $P(k) \propto k^{n_s-1}$ is red tilted [152, 233].

$$\begin{aligned}
T &= \frac{3\rho_{\rm cr}}{16} + \frac{\sqrt{\xi}}{2} - \frac{1}{32}\sqrt{96T_G\left(\frac{9\rho_{\rm cr}}{\sqrt{\xi}} - 8\right) + 27{\rho_{\rm cr}}^2\left(\frac{\rho_{\rm cr}}{\sqrt{\xi}} + 4\right) - 256\xi},\\
\xi &:= \frac{9{\rho_{\rm cr}}^2}{64} + \frac{z}{8} + \frac{16{T_G}^2 - 9{\rho_{\rm cr}}^2 T_G}{2z} - T_G,\\
z^3 &:= -512{T_G}^3 + 1161{\rho_{\rm cr}}^2{T_G}^2 + 9\sqrt{3}\sqrt{-3072{\rho_{\rm cr}}^2{T_G}^5 + 4523{\rho_{\rm cr}}^4{T_G}^4 + 192{\rho_{\rm cr}}^6{T_G}^3},
\end{aligned} \tag{5.19}$$

which will prove to be useful to describe the $Tg(T_G)$ solution.

For the $f(T, T_G)$ ansatz being considered, a matter bounce cosmology can be realised as shown in Table 5.3. In the additive and $T_G g(T)$ model, the solution takes different forms depending on the EoS of the fluid. For the additive solution, the dust case agrees with the $f(T)$ reconstructed solutions obtained in Refs. [234,236,237]. For the $Tg(T_G)$ model, only the homogeneous solutions are obtained, which are nonetheless inexpressible in terms of known functions. In this case, the coefficients a_n satisfy the recurrence relation

$$\begin{aligned}
&4a_{n-10}(147n^2 - 3170n + 16608) + 2a_{n-9}(14407 - 2783n + 126n^2)\\
&- a_{n-8}(1317n^2 - 23757n + 104888) - a_{n-7}(792n^2 - 13671n + 55822)\\
&+ a_{n-6}(828n^2 - 11964n + 42043) + a_{n-5}(864n^2 - 10905n + 32555)\\
&- 2a_{n-4}\left(15n^2 - 235n + 763\right) - a_{n-3}(360n^2 - 2917n + 5484)\\
&- 3a_{n-2}\left(32n^2 - 148n + 159\right) + 9(n-1)\left[a_{n-1}(4n-9) + 3a_n n\right] = 0,
\end{aligned} \tag{5.20}$$

with $a_{-10} = \ldots = a_{-1} = 0$. Although a general analytical solution was not found, two solutions are still generated, as shown by the solution of the first few terms

$$a_2 = \frac{a_1 - 3a_0}{6},\ a_3 = 0,\ a_4 = \frac{15a_0 - 769a_1}{648},\ a_5 = \frac{1851a_0 + 1706a_1}{2430},\ \ldots, \tag{5.21}$$

where $a_{0,1}$ act as the arbitrary constants. Similar to the previous bouncing models, the power-law solution only exists in vacuum for specific parameter choices.

The reconstructed solutions which can be deemed as viable are only the additive and $Tg(T_G)$ Lagrangians only in the absence of matter fluids, since the remaining models do not satisfy the vacuum constraint. This leaves these two models to be the only two ways to generate a matter bounce.

$f(T, T_G) = g(T) + h(T_G)$	
$\omega \neq 0$	$g(x) = \frac{\Omega_{\omega,0} T_0}{2\omega} A^{-3(1+\omega)} x \left[(1+\omega)\, {}_2F_1\left(-\frac{1}{2}, \omega; \frac{1}{2}; \frac{x-2}{x}\right) - \left(\frac{x}{2}\right)^{\omega}\right]$
$\omega = 0$	$g(x) = \frac{\Omega_{\mathrm{M},0} T_0}{2A^3} x \left[1 + \sqrt{\frac{2-x}{x}} \arcsin \sqrt{x(2-x)}\right]$
No analytical solution found for $h(T_G)$.	

$f(T, T_G) = Tg(T_G)$
Only the homogeneous solutions can be derived, which take a power-series solution of the form $g(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$, with the coefficients a_n satisfying the recurring relation given in Eq.(5.20). As the the g function is to be expressed in terms of T_G, the $x = x(T)$ variable can be expressed in terms of T_G through the use of Eq.(5.19). No analytical results have been found for the particular solution.

$f(T, T_G) = T_G g(T)$	
$\omega \neq 0$	$g(x) = \frac{1}{16}\xi_\omega x^\omega \left[\frac{2}{(\omega-1)x} + \frac{1}{\omega} + 4\left(\frac{x}{x-2}\right)^{-\omega} \frac{{}_2F_1\left(2-\omega, -\omega; 3-\omega; \frac{2}{2-x}\right)}{(\omega-2)(x-2)^2} - \frac{1}{\omega}\left(\frac{x}{x-2}\right)^{-\omega} {}_2F_1\left(-\omega, -\omega; 1-\omega; \frac{2}{2-x}\right)\right]$
$\omega = 0$	$g(x) = -\frac{\xi_0}{8}\left[\frac{1}{(x-2)^2} + \frac{1}{x} + \ln\sqrt{\frac{2-x}{x}}\right]$

$f(T, T_G) = -T + \mu T^\beta T_G{}^\gamma$	
In vacuum	$\beta = -1$, $\gamma = 1$ and $\mu = -\frac{3}{4}$

Table 5.3: Reconstruction of the $f(T, T_G)$ Lagrangian for the matter bounce cosmological model. For the above solutions, the following variables and functions have been defined: $x \coloneqq 1 + \sqrt{1 - \frac{T}{\rho_{\mathrm{cr}}}}$ and $\xi_\omega \coloneqq -\frac{3T_0}{\rho_{\mathrm{cr}}{}^2}\Omega_{\omega,0} 2^{-(2+\omega)} A^{-3(1+\omega)}$.

5.5 Type I–IV Past (Future) Singularities

The final bouncing model involves taking a power-law Hubble parameter ansatz in the form of

$$H = f_0(t - t_s)^\alpha, \tag{5.22}$$

where $f_0 > 0$, α is a real number and t_s is some reference time, which as will be seen, will correspond to the singularity (bouncing) time. For cases when $t < t_s$, this corresponds to a future type singularity, while for $t_s < t$, it yields a past singularity. In the former case, the Hubble parameter could instead be expressed as $H = f_0(t_s - t)^\alpha$. The parameter α is then chosen to be $\alpha \neq 0$, 1 as these correspond to the de Sitter and superbounce scenarios respectively. Integrating for the scale factor yields the solution

$$a(t) = A \exp\left[\frac{f_0}{\alpha+1}(t - t_s)^{\alpha+1}\right], \tag{5.23}$$

where $A > 0$ is a dimensionless constant which corresponds to the scale factor at the bouncing time i.e. $A = a(t_s)$. In this case, the torsion and TEGB scalars are given by

$$T = 6{f_0}^2(t - t_s)^{2\alpha}, \qquad T_G = 4T\left[\frac{T}{6} + f_0\alpha\left(\frac{T}{6{f_0}^2}\right)^{\frac{\alpha-1}{2\alpha}}\right], \tag{5.24}$$

which allows the scale factor to be solely expressed in terms of the torsion scalar to be

$$a(T) = A \exp\left[\frac{f_0}{\alpha+1}\left(\frac{T}{6{f_0}^2}\right)^{\frac{\alpha+1}{2\alpha}}\right]. \tag{5.25}$$

In order to keep the cosmological parameters real, the parameter α can be conveniently chosen as[9]

$$\alpha = \frac{2n+1}{2m+1}, \tag{5.26}$$

[9] A similar consideration appears, for instance, in Refs. [238–240].

where n, $m \in \mathbb{Z}$. In this way, for times $t < t_s$, both the scale factor and Hubble parameter (and their corresponding derivatives) yield real values. Naturally, for times $t > t_s$, all quantities are always real and well defined.

Depending on the choice of the parameter α, various types of bouncing cosmologies can be constructed, which are primarily classified as follows:[10]

1. **Type I singularity:** The first type corresponds to models where at a finite time, the scale factor $a(t_s) \to \infty$ with ρ, $|p| \to \infty$ as $t \to t_s$ causing the so called Big Rip singularity (alternatively, this can be defined as $a(t_s)$, $H(t_s) \to \infty$). In this model, the universe experiences an accelerated expansion which causes a dissociation of gravitationally bound structures. This kind of model is usually associated with a phantom type cosmology, as such fluids are able to generate such a singularity [247]. However, this type of singularity can be avoided through the use of fluids having a dynamical behaviour [248, 249]. Furthermore, close to the singularity, quantum effects might become more dominant and an account for these effects might prevent this singularity [242, 243, 250]. An extension of the Big Rip model is the so called Little Rip, where the scale factor and density are not singular but grow with time (equivalently, $H(t) \to \infty$ as $t \to \infty$), while still causing a dissociation of structure [251]. In the instance that the density increases up to a bounded limit (equivalently, H approaches a constant value as $t \to \infty$), the singularity is called a Pseudo Rip [252]. This still causes a dissolution of structures whilst asymptotically approaching a de Sitter behaviour provided that $\dot{H} \to 0$ [33, 252, 253]. Observe that the Little Rip and Pseudo Rip models are not Type I singularities.

2. **Type II singularity:** Known also as sudden singularities, these occur when scale factor and density are finite at $t \to t_s$ but the pressure diverges (or equivalently $\ddot{a}(t_s) \to -\infty$) [254]. Here, the universe experiences a strong acceleration

[10]For a general overview of the discussed singularities, see Refs. [207, 221, 241–246].

as it approaches the singularity which is stable against inhomogeneous perturbations, irrespective of the universe's spatial geometry [255]. Furthermore, this type of singularity can prevent a closed universe from re-collapsing [256], and may conform with cosmological observations [257, 258]. Despite the existence of the singularity however, this may not present a final state of the universe [257, 259]. A special class of Type II singularity is referred to as the Big Brake singularity where the universe exhibits a deceleration phase (instead of acceleration) which causes the scale factor evolution to stop at the singularity. This is characterised by a finite scale factor $a(t_s) < \infty$ with $\dot{a}(t_s) = 0$ and $\ddot{a}(t_s) \to -\infty$ [260]. These models can arise from tachyon models where after a quasi-de Sitter phase, the universe enters a decelerating Big Brake phase [261, 262]. As discussed in [263], the Big Brake is not the final fate of the universe as after this contraction phase, a Big Crunch ensues.

3. **Type III singularity:** This singularity exhibits a similar behaviour to a Big Rip singularity while having the scale factor become finite at the singularity (an alternative definition requires the first and higher derivatives of the scale factor to diverge at this singularity [245, 264]). This type of singularity (also referred to as Big Freeze singularity, first formulated in Ref. [265]) has been studied as a model for inflation as it provides a decreasing comoving horizon close to the singularity [264], and can conform with observations, provided the existence of a non-zero dark matter contribution [266]. However, quantum effects for times close to the singularity may prevent its formation [243, 266].

4. **Type IV singularity:** This type of singularity (first introduced in Ref. [243]) yields a finite value for the scale factor, density and pressure at the singularity but higher order derivatives of the Hubble parameter diverge i.e. $H^{(n)}(t_s) \to \infty$ for some $n \geq 2$. Contrary to previous models, the universe continues to evolve smoothly past the singularity, avoiding the need of quantum corrections [238, 240]. When applied during inflationary epochs, the Type IV singularity

can end that epoch, leading to a graceful exit mechanism [239]. However, this model suffers from issues including: (*i*) primordial modes which grow after horizon crossing and (*ii*) a scalar power spectrum which is not invariant and may be very red tilted, deviating from observed values (this may be addressed through quantum considerations) [240, 246].

Based on these classifications, the α domain values which generate the described singularity models are as follows: (*i*) Type I for $\alpha < -1$, with a Little Rip for $\alpha > 0$. However, a Pseudo Rip is not possible; (*ii*) Type II for the range of values $0 < \alpha < 1$. Furthermore, it is observed that a Big Brake sudden singularity cannot be constructed; (*iii*) Type III for $-1 < \alpha < 0$, and (*iv*) Type IV for the remaining values $\alpha > 1$.

Before reconstructing for the Lagrangian, an instant of time $0 < t_s < t_0$ at which $a(t_0) = 1$ is first chosen to exist with the aim of simplifying the calculations. Here, this means that t_s acts as a past singularity. The time occurs when

$$t_0 = t_s + \left[-\frac{\alpha + 1}{f_0} \ln A \right]^{\frac{1}{\alpha+1}}. \tag{5.27}$$

A simple constraint to satisfy the assumption that $0 < t_s < t_0$ is to consider $(\alpha + 1) \ln A < 0$, which imposes the constraints $0 < A < 1$ for $\alpha > -1$ (meaning Types II–IV) and $A > 1$ for $\alpha < -1$ (meaning for Type I), for which this shall be assumed in what follows.

The resulting reconstructed solutions for arbitrary α are summarised in Table 5.4. As shown, the $h(T_G)$ function in the additive model and the rescaling $Tg(T_G)$ model were not obtained, which arises due to the invertibility issues arising from the torsion and TEGB scalars relation Eq.(5.24). With the exception of the power-law model, the solutions are expressed in terms of the confluent hypergeometric function of the first kind ${}_1F_1(a; b; x)$, which is always defined provided $b \leq 0$ is not an integer. For the considered form for α, the additive solution is always defined while the rescaled

$f(T,T_G) = g(T) + h(T_G)$
$g(T) = \Omega_{\omega,0} T_0 A^{-3(1+\omega)} \, {}_1F_1\left(-\frac{\alpha}{\alpha+1}; \frac{1}{\alpha+1}; -\frac{3f_0(1+\omega)}{\alpha+1}\left(\frac{T}{6{f_0}^2}\right)^{\frac{\alpha+1}{2\alpha}}\right)$ No analytical solution found for $h(T_G)$.

$f(T,T_G) = Tg(T_G)$
No analytical solution obtained.

	$f(T,T_G) = T_G g(T)$
$\frac{1-3\alpha}{\alpha+1} \neq N$	$g(T) = \frac{3\Omega_{\omega,0} T_0 A^{-3(1+\omega)}}{8T^2} \, {}_1F_1\left(-\frac{4\alpha}{\alpha+1}; \frac{1-3\alpha}{\alpha+1}; -\frac{3f_0(1+\omega)}{\alpha+1}\left(\frac{T}{6{f_0}^2}\right)^{\frac{\alpha+1}{2\alpha}}\right)$
$\alpha = \frac{1}{3}$	$g(x) = \frac{3\Omega_{\omega,0} T_0 A^{-3(1+\omega)}(1+\omega)}{128{f_0}^3 x}\left[x\mathrm{Ei}\left(-x\right) + e^{-x}\right]$
$\alpha = 3$	$g(y) = \frac{9\Omega_{\omega,0} T_0 A^{-3(1+\omega)}(1+\omega)^3}{2048 f_0 y^3}\left[\frac{y^3}{2}\mathrm{Ei}\left(-y\right) + e^{-y}\left(1 - \frac{y}{2} + \frac{y^2}{16}\right)\right]$
$\frac{1-3\alpha}{\alpha+1} = N \leq -4$	No general solution is derived.

	$f(T,T_G) = -T + \mu T^\beta {T_G}^\gamma$
In vacuum	$\beta = -1$, $\gamma = 1$ and $\mu = -\frac{3}{4}$

Table 5.4: The reconstructed $f(T, T_G)$ Lagrangian which is able to describe a Type I–IV (past) singularity. Due to the lack of invertibility between T and T_G, analytical solutions for $h(T_G)$ and $Tg(T_G)$ could not be obtained. $x \equiv \frac{T^2(1+\omega)}{16{f_0}^3}$, $y \equiv \left(\frac{3T^2(1+\omega)^3}{256 f_0}\right)^{\frac{1}{3}}$ and $N \leq 0$ being a negative integer have been defined for simplicity. Here, ${}_1F_1(a;b;x)$ is the confluent hypergeometric function of the first kind and $\mathrm{Ei}(z) \equiv -\int_{-z}^{\infty} \frac{e^{-r}}{r}\, dr$ is the exponential integral function.

solution $T_G g(T)$ is not defined for specific values of α when $\frac{1-3\alpha}{\alpha+1} = N$ for some negative integer $N \leq 0$. This yields a specific value of $\alpha = \frac{1}{3}$ for Type II, $\alpha = 3$ for Type IV and the remaining values in Type I. This case results in solutions expressed in terms of the exponential integral function. Despite a general solution for the Type I cases is not derived, an analytical solution can be derived for specific values. Lastly, for the power law case, a simple solution arises only in the absence of matter fluids.

Based on the obtained solutions, the vacuum condition is satisfied in the following instances. For the additive case, it appears that this is satisfied only for $-1 < \alpha < 0$ with $\omega > -1$ (Type III with non-phantom fluid). For the $T_G g(T)$ rescaling model, this appears to hold for the same latter constraint[11] while for the $-T + T_G g(T)$ case, this depends on the existence of fluids. If these exist, then the condition is satisfied for Type III with non-phantom fluids.Otherwise, in the absence of matter fluids, this is satisfied for $\alpha < 0$ or $\alpha > 1$ (Types I, III and IV). This constraint also appears for the power law model. Therefore, overall it is observed that for the model ansatz considered

1. Type II singularities do not satisfy the vacuum constraint;
2. Type III singularities equipped with a non-phantom fluid satisfy the constraint;
3. Type I and IV singularities can only appear in an empty universe.

5.6 Discussion

Bouncing cosmological models prove to be a useful mechanism to tackle the initial singularity problem, as well as provide an insight towards the possible final fate of the universe. Various bouncing cosmologies have been discussed throughout this chapter, each having their own applicability to various scenarios, with some being more applicable than others in certain regimes. Here, $f(T, T_G)$ gravity has been investigated in context of whether it is capable of reproducing such cosmological scenarios through the use of reconstruction. Given the large number of possible reconstructed Lagrangian solutions, similar to Chapter 4, the models are constrained by investigating whether the reconstructed solution is capable of hosting vacuum solutions by obeying the constraint $f(0, 0) = 0$.

[11] Although a general solution is not derived for $\alpha < -1$, the solutions appear to diverge.

The first bounce model considered is the symmetric bounce, which can be used to replace the initial Big Bang singularity albeit together with the introduction of other mechanisms (to avoid issues with primordial perturbation modes). For the model ansatzes considered, the reconstructed solutions obtained are summarised in Table 5.1 where only the additive and $Tg(T_G)$ models are capable of satisfying the vacuum constraint, although only in the absence of matter fluids.

Next, an oscillatory cyclic bouncing model has been investigated, which experiences a series of expansions and contractions which realise two distinct bounces, one at minimal size, which corresponds to a Big Crunch/Big Bang singularity, and another at maximal size, which describes a cosmological turnaround. The resulting reconstructed solutions are listed in Table 5.2. Similar to the symmetric bounce model, the vacuum condition is only satisfied for the first two reconstructed models in the absence of matter.

The superbounce model was then investigated; a model which describes a universe rebirth without reducing the universe to a zero size. Given the simple nature of the scale factor to be that of a power-law model with an exponent constrained to be $0 < \frac{2}{c^2} < \frac{1}{3}$, the same results obtained in Chapter 4 are retained, with the only difference being that some of the vacuum condition constraints are simplified. Overall, a superbounce behaviour can be generated whilst obeying said condition.

An important investigated model was the matter bounce scenario, a model originating from LQC. Similar to the superbounce scenario, the universe experiences a rebirth, however the Hubble parameter does not diverge at the bounce point making it free from singularities. However, it can suffer from a large tensor-scalar ratio that does not conform with observations. From the obtained reconstructed Lagrangians summarised in Table 5.3, a similar behaviour to the first two bouncing models is obtained, where the vacuum condition is only satisfied for the additive and $Tg(T_G)$ rescaling model in the absence of matter fluids.

Lastly, the past (future) singularities of Type I–IV as well as the Little Rip singularity have been investigated. These type of singularities have been studied extensively in various areas of research, ranging from the early universe to the final fate of the universe. The reconstructed Lagrangians for each ansatz were found and listed in Table 5.4, for which it is observed that the Type II singularity never satisfied the vacuum constraint. Type I and IV only appear in an empty universe, and Type III appears in a universe filled with non-phantom matter, making it the only model capable of describing the presence of matter contributions while satisfying the vacuum constraint.

Overall, $f(T, T_G)$ gravity proposes a possible viable alternative to describe various models of a bouncing cosmology. However, it has been shown that the vacuum constraint poses to be a significant restriction, as most models are only valid in the absence of matter fluids. In fact, only the superbounce and Type III singularity are capable of hosting a non-empty universe. Thus, this serves as a significant constraint towards constructing a more concise description of the mechanics of gravity.

Chapter 6

Gravitational Waves in Modified Teleparallel Theories

Newtonian gravity, despite its first initial success, has been plagued by the concept of action at a distance. With the introduction of GR, Einstein realised a year after the publication of his theory in 1916 [6] (and later in 1918 [7]), that gravity, similar to electromagnetism, propagates as transverse waves at the speed of light while having two distinct polarisations. These waves, adequately named gravitational waves, resolved the problem which plagued Newtonian gravity.

Despite this successful resolution, it was straightforwardly concluded that if these waves did exist, their strength would be too small to detect. This was in fact true as it took over many years until an indirect measurement and existence of GWs was imposed. The discovery of the binary system PSR B1913+16 by Hulse and Taylor in 1974 changed history [267]. Since the prediction of GWs, it was theorised that a pair of massive rotating bodies would emit gravitational radiation causing an energy loss in the system and hence causing an orbital decay [268]. Observation of the Hulse-Taylor binary indeed showed the presence of such decay at a rate which matches the one predicted by GR [269], a statement which holds true until today [270, 271].

Nonetheless, there had been continuous research in hopes of a direct detection of a GW (see, for instance, Ref. [272] for a detailed account). It took exactly 100 years from the formulation of GR for the first ever detection of a GW. The detection by LIGO in 2015 [8] opened the realm of GW astronomy. Further observations joined by other collaborations such as Virgo, verified the existence on the two polarisation states, as well as imposed constraints on the propagating speed. Starting with the former, results show that the GWs exhibit a polarisation behaviour as predicted by GR disfavouring the existence of other polarisation states [273]. In the case of propagation speed, observations infer that the waves travel at light speed [274, 275].

However, these only represent the first steps toward a full thorough understanding of the properties of GWs. In fact, a more thorough analysis is necessary to fully determine whether there exist more GW polarisations. In fact, the LIGO detectors have been designed to be co-aligned, meaning that they would not be sensitive to more polarisations besides the predicted modes [276, 277]. As the main goal was to detect any GWs, this alignment permitted to maximise the signal to noise ratio. Even with the introduction of Virgo, this is insufficient to detect all possible polarisation states. In fact, as there is a possible number of six distinct polarisations, in principle, at least six detectors are necessary to uniquely distinguish between the polarisations. Construction of other detectors such as Kamioka Gravitational Wave Detector (KAGRA) and LIGO-India will help constrain such polarisations [276].

Existence of GWs have stronger implications and importance in the realm of cosmology. Their existence may imply the possible existence of the so called stochastic GW background, which is a background of gravitational waves sourced by countless independent systems and processes in a random (stochastic) manner. The source for this background could be either astrophysical or cosmological [278–280]. From the former, this could infer further information regarding the astrophysical processes beyond the CMB whereas cosmological sources could originate from the Big Bang or inflation.[12]

In this chapter, the existence and polarisations of GWs in teleparallel theories of gravity are investigated. The GWs in TEGR had been well established since 1983 by Müller-Hoissen and Nitschin [281] (and more recently by Obukhov and Pereira [282]) confirming their existence, as well as exhibiting the same polarisations as those in GR, as expected. Later, modifications to TEGR have been investigated. Most notably, in the first trivial modification of $f(T)$ gravity, it has been shown that the same polarisations in GR result in a linearised, weak-field, gravity limit [283]. For a more general cosmological FLRW background, the speed of the waves has been confirmed to remain at light speed, unaffected by the $f(T)$ Lagrangian, although the amplitude of the waves is indeed altered [284, 285]. Other teleparallel extensions have also been investigated, notably $f(T^{\mu\nu\rho}T_{\mu\nu\rho}, T^{\mu\nu\rho}T_{\rho\nu\mu}, T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho})$ and its sub-case New General Relativity

$$f(T^{\mu\nu\rho}T_{\mu\nu\rho}, T^{\mu\nu\rho}T_{\rho\nu\mu}, T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}) = a_1 T^{\mu\nu\rho}T_{\mu\nu\rho} + a_2 T^{\mu\nu\rho}T_{\rho\nu\mu} + a_3 T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}, \quad (6.1)$$

with $a_{1,2,3}$ being coupling constants, where it has been shown that despite the fact that these waves travel at light speed, other polarisations besides the tensor ones might arise for certain configurations [286, 287]. Lastly, $f(T, B)$ gravity has been analysed in [72] where it has been shown that an extra polarisation mode propagating at a speed different than that of light arises with properties similar to those encountered in $f(R)$ gravity.

In light of the given observational results, the speed of the waves can therefore impose strong constraints on the models, ultimately confirming (or debunking) their viability with future detectors, hopefully imposing stronger constraints on the polarisation states. This chapter shall be organised as follows. An introduction regarding

[12] Regarding further implications and possible future detections of this background see Refs. [278–280] and references therein. The details concerning these features will not be explored as they lie beyond the scope of this work.

the mathematical tools necessary to extract the existence and behaviour of the GWs is first presented, followed by the GW analysis in $f(T)$ gravity. This shall be derived under two major fronts, the metric and tetrad approaches. A higher order perturbation analysis is then explored. Next, the GWs in $f(T,B)$ gravity are then explored under a Minkowski and a cosmological constant background, wherein the latter's effect shall be explored. Finally, the GWs in $f(T,T_G)$ gravity are analysed, followed by a discussion of the results.

6.1 Linearised Gravity Approach

Linearised gravity offers a relatively simple way to examine the GW features which arise depending on the gravitational model. Otherwise referred to as the weak-field approximation, the gravitational strength is assumed to be weak, meaning that the metric is expressed as Minkowski space at a background level plus a small correction, $h_{\mu\nu}$. In other words, the metric tensor can be expanded as a series of n corrections

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \ldots + \mathcal{O}\left(h^{(n)}_{\mu\nu}\right), \tag{6.2}$$

where each perturbation satisfies $|h^{(n)}_{\mu\nu}| \ll |h^{(n-1)}_{\mu\nu}| \ll \ldots \ll |h^{(1)}_{\mu\nu}| \ll 1$ and $h^{(i)}h^{(j)} \sim h^{(i+j)}$.[13] This weak-field limit is an understandable approximation in the given context. As the GW carries both energy and momentum, it is difficult to solve the non-linear equations exactly, and hence this limit proposes a simpler alternative. Furthermore, the gravitational radiation is expected to be small and the weak-field limit provides a more direct understanding of the GW behaviour in the absence of other sources [1].

[13] An alternative description is to consider the perturbations to be expressed as $h^{(n)}_{\mu\nu} = \epsilon^n H^{(n)}_{\mu\nu}$ where $|\epsilon| \ll 1$ and $H^{(n)}_{\mu\nu}$ is some function. Then the n^{th} order is defined by comparing the coefficients of ϵ^n. This trivially gives rise to the order relation property $h^{(i)}h^{(j)} \sim h^{(i+j)}$. See, for instance Ref. [55] for further details.

A similar application of the weak-field approximation can be considered for the tetrad fields, which in this case is crucial as it serves as the basis fundamental variable of the theory. In a similar notion to the metric, the tetrad is expanded in terms of a background value $\gamma_\mu^{(0)a}$ plus some small corrections, namely

$$e^a{}_\mu = \gamma_\mu^{(0)a} + \gamma_\mu^{(1)a} + \gamma_\mu^{(2)a} + \ldots + \mathcal{O}\left(\gamma_\mu^{(n)a}\right), \tag{6.3}$$

with $|\gamma_\mu^{(n)a}| \ll |\gamma_\mu^{(n-1)a}| \ll \ldots \ll |\gamma_\mu^{(1)a}| \ll 1$ and $\gamma_\mu^{(i)a}\gamma_\mu^{(j)a} \sim \gamma_\mu^{(i+j)a}$. As the tetrads construct the metric through Eq.(2.3), the perturbed quantities are interlinked. As an illustration, up to second order, the relations are

$$\eta_{\mu\nu} = \eta_{ab}\gamma_\mu^{(0)a}\gamma_\nu^{(0)b}, \tag{6.4}$$

$$h_{\mu\nu}^{(1)} = \eta_{ab}\left(\gamma_\mu^{(0)a}\gamma_\nu^{(1)b} + \gamma_\mu^{(1)a}\gamma_\nu^{(0)b}\right), \tag{6.5}$$

$$h_{\mu\nu}^{(2)} = \eta_{ab}\left(\gamma_\mu^{(0)a}\gamma_\nu^{(2)b} + \gamma_\mu^{(1)a}\gamma_\nu^{(1)b} + \gamma_\mu^{(2)a}\gamma_\nu^{(0)b}\right). \tag{6.6}$$

To analyse the perturbed field equations, it is also necessary to investigate the order behaviour of the tetrad determinant. Only the zeroth order of the tetrad determinant is presented as it suffices for the following analysis. As the tetrad and metric determinants are related through the relation $g = -e^2$, it is trivial to show that $e^{(0)} = \pm 1$.

The next step is to extract the order behaviour of the gravitational components which gives rise to the gravitational action. As all relevant quantities in teleparallelism are constructed from the torsion tensor, this is considered first. For the work which follows, the first order tetrad expansion suffices for the relevant analysis. Nonetheless, this could easily be generalised for higher order behaviours. From Eq.(2.10), up to first order, this becomes

$$\begin{aligned} T^a{}_{\mu\nu} = {} & \partial_\mu\gamma_\nu^{(0)a} - \partial_\nu\gamma_\mu^{(0)a} + \omega^a{}_{b\mu}\gamma_\nu^{(0)b} - \omega^a{}_{b\nu}\gamma_\mu^{(0)b} \\ & + \partial_\mu\gamma_\nu^{(1)a} - \partial_\nu\gamma_\mu^{(1)a} + \omega^a{}_{b\mu}\gamma_\nu^{(1)b} - \omega^a{}_{b\nu}\gamma_\mu^{(1)b}. \end{aligned} \tag{6.7}$$

As the purely inertial spin connection does not represent any source of gravitation, ${\omega^a}_{b\mu}$ represents a zeroth order quantity. As discussed in Chapter 2, the spin connection can be obtained by demanding the torsion tensor to be zero in the absence of a gravitational field, namely in the limit $G \to 0$. In this regime, only the zeroth order contributions remain, as higher order terms are gravitationally sourced, meaning ${e^a}_\mu\big|_{G\to 0} = \gamma_\mu^{(0)a}$. Thus, in this purely inertial frame, Eq.(6.7) reveals the condition

$$ {T^a}_{\mu\nu}\big|_{G\to 0} = \partial_\mu \gamma_\nu^{(0)a} - \partial_\nu \gamma_\mu^{(0)a} + {\omega^a}_{b\mu}\gamma_\nu^{(0)b} - {\omega^a}_{b\nu}\gamma_\mu^{(0)b} = 0, \tag{6.8} $$

which shows that the torsion tensor is a first order quantity given by

$$ {T^a}_{\mu\nu} = \partial_\mu \gamma_\nu^{(1)a} - \partial_\nu \gamma_\mu^{(1)a} + {\omega^a}_{b\mu}\gamma_\nu^{(1)b} - {\omega^a}_{b\nu}\gamma_\mu^{(1)b}. \tag{6.9} $$

This derivation is also consistent with the derived expression for the spin connection Eq.(2.26), namely

$$ {\omega^a}_{b\mu} = \Gamma^a_{b\mu} - {e_b}^\nu \partial_\mu {e^a}_\nu\big|_{G\to 0}\,. \tag{6.10} $$

In the inertial limit, the metric tensor reduces to the Minkowski background metric, leading to the Levi-Civita connection to be zero. Therefore, only the last term remains, leaving the spin connection to be expressed as

$$ {\omega^a}_{b\mu} = -\gamma_b^{(0)\nu}\partial_\mu \gamma_\nu^{(0)a}. \tag{6.11} $$

Substituting in the derived condition Eq.(6.8) reveals the consistency of the result. The form of the spin connection is expected. Recall that the purely inertial spin connection is defined as Eq.(2.7)

$$ {\omega^a}_{b\mu} = -{\Lambda_b}^c \partial_\mu {\Lambda^a}_c, \tag{6.12} $$

with ${\Lambda_d}^c$ representing a Lorentz matrix. This is of precisely the same form obtained in Eq.(6.11), which leads to the direct association of the tetrads to be exactly the

Lorentz matrices. In fact, the determinant of Lorentz matrices are indeed ± 1, which agrees with the result obtained previously, and the expression for $\eta_{\mu\nu}$ is precisely the Lorentz transformation property of the Minkowski metric. Such tetrads are called trivial tetrads (or trivial frames) [52].[14]

As the torsion tensor is first order, since both the contorsion and superpotential tensors are linearly dependent on the latter, these consequently contribute at least at first order. Ultimately, the torsion scalar is therefore at least second order in perturbations. On the other hand, the boundary term is first order. These results combined show that the Ricci scalar is at least of first order, a result which can also be obtained from direct computation of perturbations of the metric tensor. Indeed, the Ricci scalar at first order is given to be

$$R^{(1)} = \partial_\rho \partial_\nu h^{(1)\rho\nu} - \Box h^{(1)}, \tag{6.13}$$

where $h^{(1)} = h^{(1)\mu}{}_{\mu}$ represents the trace. It is remarked that from here onwards, indices are raised and lowered with respect to the Minkowski (background) metric. At this level, the d'Alembert operator reduces to $\Box = \partial_\mu \partial^\mu$. Expanding in terms of tetrads yields

$$R^{(1)} = 2\eta_{ab} \left[\partial^\mu \partial^\nu \left(\gamma_\mu^{(0)a} \gamma_\nu^{(1)b}\right) - \eta^{\mu\nu} \Box \left(\gamma_\mu^{(0)a} \gamma_\nu^{(1)b}\right)\right]. \tag{6.14}$$

On the other hand, expanding the boundary term at first order yields

$$B^{(1)} = -2 \left(\nabla^\mu T^\nu{}_{\mu\nu}\right)^{(1)} = -2\eta^{\mu\rho} \partial_\rho \left(\gamma_a^{(0)\nu} T^{(1)a}{}_{\mu\nu}\right), \tag{6.15}$$

Expanding using the torsion tensor definition Eq.(6.9) and the spin connection

[14] The derived expressions for the zeroth order condition Eq. (6.8) and the properties of the background tetrad can be observed in Ref. [52] Eqs.(2.3), (2.5) and (2.12).

Eq.(6.11) together with the relation

$$\gamma_b^{(0)\gamma} = \eta_{cb}\eta^{\alpha\gamma}\gamma_\alpha^{(0)c} \tag{6.16}$$

obtained from inverting Eq.(6.4) yields the Ricci scalar result. This affirms the consistency of the linearisation claims. Note that the Ricci and Riemann tensors are also of at least first order as both are zero at a background level.

Having the orders of the relevant gravitational-based quantities defined, the next step would be to extract the perturbed field equations. In order to do this, the behaviour of the Lagrangian function is first investigated. As both T and B are null at a background level and the weak-field limit is being considered, it is reasonable to assume that the function is Taylor expandable about those background values, i.e.

$$\begin{aligned} f(T,B) = f(0,0) + f_T(0,0)T + f_B(0,0)B + \frac{1}{2}f_{TT}(0,0)T^2 \\ + \frac{1}{2}f_{BB}(0,0)B^2 + f_{TB}(0,0)TB + \ldots . \end{aligned} \tag{6.17}$$

With this assumption in mind, the Minkowski spacetime is expected to be recovered in the absence of a gravitational field. In this way, similar to the vacuum constraint considered in the $f(T, T_G)$ reconstruction (see Chapter 4), a necessary requirement is $f(0,0) = 0$ as otherwise it would represent a cosmological constant. As will be shown, this is also recovered as a constraint from the perturbed equations. Next, the coefficient of $f_T(0,0) \neq 0$ as this represents the TEGR contribution (or the effective Newtonian gravitational constant, see for instance [127, 288] – this shall be investigated in further detail in Chapter 7).

With this consideration in mind, while also having all the relevant quantities defined in terms of expansion order, the GWs can now be investigated. Before doing so, the role of the stress-energy tensor is first addressed. The main aim of this chapter is to investigate the existence of any wave solutions together with their associated physical behaviour. To do so, it is sufficient to investigate such properties far away from the source of such waves. In other words, one can set the stress-energy to zero (i.e. in vacuum). However, any physical observation will not be able to detect any GWs of this sort as their wave amplitude would be extremely weak and hence would need to be detected from stronger astrophysical sources [55]. In such cases, the contribution of the stress-energy has to be included (see, for instance, Refs. [1, 55–57]).

6.2 Gravitational Waves in $f(T)$ gravity

The first investigation is the resulting GWs, which arise in the simplest teleparallel extension of gravity, namely that of $f(T)$ gravity. Originally investigated in Ref. [283], the resulting GW modes were extracted using the spacetime indexed form of the equations Eq.(2.25) through a combination of linearisation of the metric tensor and the tetrads. In what follows, the same procedure shall be followed. However, in contrast to Ref. [283], the zeroth order contribution of the tetrad will not be strictly set to be the Kronecker delta (which sets the spin connection to zero), but will be left arbitrary. This way, the background tetrad is not necessarily constrained, thus allowing for the investigation even in the presence of non-zero spin connections.

Besides this 'metric' investigation, the GWs shall be also analysed from a pure tetrad perspective, an approach originally considered in an absolute parallelism setting in Refs. [281, 282]. However, only the final perturbed field equations are given, whereas here the perturbed tetrad field shall be solved explicitly based on the results obtained from the metric approach.

Lastly, higher order perturbations shall be examined in order to investigate whether deviations from GR arise. Such higher order corrections to the metric are of great importance as these reveal crucial information about the GW, for instance about gravitational radiation.

6.2.1 Metric Approach

Starting off with the metric approach introduced in Ref. [283], the spacetime indexed equations Eq.(2.25), which are listed below for the sake of convenience,

$$f_T G_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left(f - T f_T\right) - 2 S_\nu{}^\alpha{}_\mu \partial_\alpha f_T = 0, \tag{6.18}$$

shall be perturbed up to first order. The resulting system of equations order by order results in

$$\eta_{\mu\nu} f(0) = 0, \tag{6.19}$$

$$f_T(0) G^{(1)}_{\mu\nu} = 0, \tag{6.20}$$

where the zeroth order equation justifies the absence of a cosmological constant in the theory while the second shows that the same expression encountered in GR is recovered provided $f_T(0) \neq 0$. This is a clear indication that the resulting GW modes shall be identical to those obtained in GR. Therefore, from here onwards, the GW solutions shall be extracted following the techniques used in GR (see, for instance, Refs. [1,55–58]). Rewriting the equations in terms of the first order metric perturbation $h^{(1)}_{\mu\nu}$ results into

$$\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\sigma \partial_\rho h^{\sigma\rho} + \eta_{\mu\nu} \Box h = 0. \tag{6.21}$$

To simplify the expression, the trace-reversed perturbed metric $\bar{h}_{\mu\nu}$ defined as[15]

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \tag{6.22}$$

is introduced which simplifies the equations to

$$\partial_\sigma \partial_\nu \bar{h}^\sigma{}_\mu + \partial_\sigma \partial_\mu \bar{h}^\sigma{}_\nu - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu} \partial_\sigma \partial_\rho \bar{h}^{\sigma\rho} = 0. \tag{6.23}$$

[15]The term trace-reversed stems from the fact that $\bar{h} = \bar{h}^\mu{}_\mu = -h$.

Solving the equations of motion is relatively complicated in the given form. However, this can be circumvented by choosing another coordinate system which eliminates some of the terms to reduce it to a simpler equation. This can be addressed through coordinate (gauge) transformations [1,54,55,289]. For the given weak field, one can consider the coordinate transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}\left(x^{\alpha}\right), \tag{6.24}$$

where primes shall denote the new coordinate system and $\xi^{\mu}(x)$ is an infinitesimal function of position which is of the order of the perturbation (i.e. $\xi \sim h$). In this way, the trace-reversed perturbed metric in the new coordinate system is given to be

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\partial_{\rho}\xi^{\rho}. \tag{6.25}$$

As the equations are derived from components of the Riemann tensor, application of this change of coordinates shows that the field equations are gauge invariant as $R'_{\alpha\beta\gamma\rho} = R_{\alpha\beta\gamma\rho}$ leaving the system unchanged. Thus, any solution $\bar{h}_{\mu\nu}$ implies that $\bar{h}'_{\mu\nu}$ is also a solution for any arbitrary choice of the function ξ^{μ}. Thus, the latter can be chosen in a way that the equations for $\bar{h}'_{\mu\nu}$ are expressed in a simpler form. One such suitable choice is the Lorenz gauge $\partial^{\mu}\bar{h}'_{\mu\nu} = 0$ which eliminates the gauge freedom of the metric equations. This can be achieved by demanding

$$\Box\xi^{\mu} = \partial^{\mu}\bar{h}_{\mu\nu}. \tag{6.26}$$

This reduces the equations to simply that of a plane wave $\Box\bar{h}_{\mu\nu} = 0$. So far, this leaves six dynamical degrees of freedom out of a possible ten from the perturbed metric. However, there is still some residual gauge freedom left. For the Lorenz gauge condition, one can introduce another coordinate transformation ζ^{μ} satisfying $\Box\zeta^{\mu} = 0$ which leaves the Lorenz condition unchanged. This allows for another gauge fixing. For the trace-reversed perturbed metric, the trace transforms as

$$\bar{h}' = \bar{h} + 2\partial_\rho \zeta^\rho. \tag{6.27}$$

From this, one imposes the transverse-traceless (TT) gauge by demanding the coordinate transformation to also obey

$$2\partial_\rho \zeta^\rho = -\bar{h}, \tag{6.28}$$

which sets $\bar{h}' = 0$ leading to the perturbed metric to be traceless (as $\bar{h} = -h$). Transverse here is realised by $\Box\zeta^\mu = 0$ as this implies that $h_{0\mu} = 0$ and $\partial_i h^{ij} = 0$ meaning the metric is purely spatial. As shown in Refs. [1, 54, 55, 57, 289], this TT gauge can always be imposed in the case of vacuum as one can always find a function ζ^ρ satisfying those constraints. With these gauge fixing conditions, the number of degrees of freedom decrease the current six to only two, while the wave equation can be simply expressed in terms of metric perturbation $h_{\mu\nu}$ as

$$\Box h_{\mu\nu} = 0. \tag{6.29}$$

Assuming a plane wave solution, the solution in Fourier space is given to be

$$h_{\mu\nu} = A_{\mu\nu} \exp(ik_\sigma x^\sigma), \tag{6.30}$$

where k^σ is some wavevector such that $k_\sigma k^\sigma = 0$ and $A_{\mu\nu}$ are amplitude coefficients. The wavevector signifies that the wave is null, meaning the wave is propagating at the speed of light. On the other hand, the transverse-traceless gauge condition imposes the following constraints on the amplitude coefficients, $k^\nu A_{\mu\nu} = 0$, $A_{0\nu} = 0$ and $A^i{}_i = 0$. If, without loss of generality, the wave is assumed to be propagating in the z-direction, the Lorenz gauge implies the condition $A_{3i} = 0$ which simplifies the traceless condition to $A_{11} = -A_{22}$, leaving A_{21} as the other remaining undetermined coefficient. Overall, this means that the perturbed metric tensor takes the form of

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11}\exp(ik_\sigma x^\sigma) & A_{12}\exp(ik_\sigma x^\sigma) & 0 \\ 0 & A_{12}\exp(ik_\sigma x^\sigma) & -A_{11}\exp(ik_\sigma x^\sigma) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (6.31)$$

where the definitions $h_+ \coloneqq A_{11}\exp(ik_\sigma x^\sigma)$ and $h_\times \coloneqq A_{12}\exp(ik_\sigma x^\sigma)$ have been introduced. These are the two propagating degrees of freedom of GR (and equivalently, TEGR) as well as for $f(T)$ gravity.

The next step would be to investigate the physical behaviour of these waves. One may first consider the effect of the waves on free particles to see whether the passage of the wave causes a change in behaviour. Starting from a free particle initially at rest with an initial four-velocity $U^\mu = (1,0,0,0)$ (which means the particle is slowly moving), the geodesic equation for the particle

$$\frac{dU^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta = 0, \quad (6.32)$$

reduces to $\frac{dU^\mu}{d\tau} + \Gamma^\mu_{00} = 0$. In the TT gauge, the metric perturbation solution Eq.(6.31) reveals that $\Gamma^\mu_{00} = 0$ meaning that $\frac{dU^\mu}{d\tau} = 0$, or equivalently that the particle will remain stationary and hence does not feel any effects of the passing wave [56, 57]. Observe that despite the fact that the particles in teleparallel gravity do not travel along geodesics but instead according to force equations given to be

$$\frac{dU^\mu}{d\tau} + \hat{\Gamma}^\mu_{\alpha\beta}U^\alpha U^\beta = K^\mu{}_{\alpha\beta}U^\alpha U^\beta, \quad (6.33)$$

the relative motion is equivalent due to the relation Eq.(2.13) $K^\mu{}_{\alpha\beta} = \hat{\Gamma}^\mu_{\alpha\beta} - \Gamma^\mu_{\alpha\beta}$ which relates the two particle motion equations [52].

As free particles do not infer any information regarding the behaviour of the wave, another alternative would be to investigate the relative motion of freely falling neighbouring particles. This is achieved by observing the relative motion of their respective geodesics, namely through the geodesic deviation formula [57]

$$A^\mu \equiv U^\rho \nabla_\rho \left(U^\sigma \nabla_\sigma S^\mu\right) = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma \tag{6.34}$$

where A^μ represents the relative acceleration between geodesics and S^σ represents the separation vector, a measure of the relative distance between geodesics. As before, despite the fact that particles do not follow geodesics in teleparallel based theories, the resulting deviation between particles following their respective force equation turns out to be identical. This was also shown in Ref. [290]. Consequently, this approach should be in principle applicable to any teleparallel extension considered in this work. Back on the implications of the geodesic deviation formula, the Riemann tensor is a first order quantity in the linearisation procedure, meaning the particles are to be expressed in terms of their behaviour at a Minkowski level. Assuming once more the particles to be slowly moving [57, 277]

$$\ddot{S}^i = -R^i{}_{0j0} S^j, \tag{6.35}$$

where $A^i = \ddot{S}^i$ with dots representing coordinate time derivatives.[16] The remaining Riemann tensor components $R^i{}_{0j0}$ (which amount to a total of 6 independent components and is referred to as the electric part of the Riemann tensor) is the quantity which contains all the relevant information regarding the polarisations. In the TT gauge, this Riemann tensor component turns out to be

$$R_{i0j0} = -\frac{1}{2}\partial_t^2 \bar{h}_{ij}, \tag{6.36}$$

which in turn yields the following equations of motion for the separation vector

$$\ddot{S}^1 = \frac{1}{2}\ddot{h}_\times S^2 + \frac{1}{2}\ddot{h}_+ S^1, \tag{6.37}$$

[16]In the slow moving particle limit, the proper time τ is identical to the coordinate time t.

$$\ddot{S}^2 = \frac{1}{2}\ddot{h}_\times S^1 - \frac{1}{2}\ddot{h}_+ S^2, \tag{6.38}$$

$$\ddot{S}^3 = 0. \tag{6.39}$$

To determine the resulting behaviour, each mode is investigated separately.

In the absence of the $h_\times$ polarisation, the particles will oscillate in a form of a '+' while in the absence of the h_+ polarisation, this expands in and contracts in the shape of a '×'. Hence, the polarisations are referred to as the plus and cross tensor polarisations [63]. The term tensor shall be explained further shortly. For an initial stationary ring configuration, this oscillating behaviour is illustrated in Fig. 6.1. As these modes are invariant under rotations of 180°, this means the graviton, i.e. the particle responsible for such a wave, is a massless spin-2 particle.

In more general theories of gravity however, the existence of other possible polarisations might arise as will be seen in the case of $f(T,B)$ gravity. There have been various approaches to try and identify all possible polarisations which can be gen-

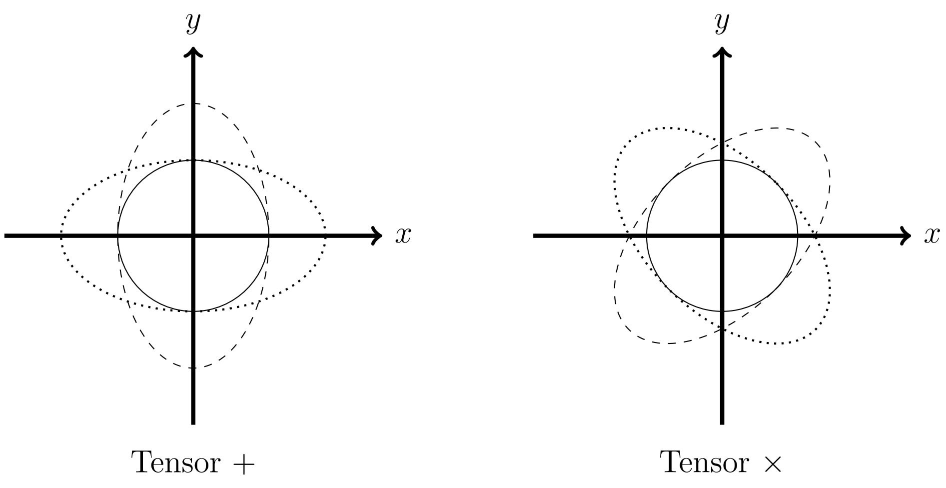


Figure 6.1: The effect of the tensor + and × polarisations for the GWs in predicted in $f(T)$ gravity on a ring of slowly moving test particles. Here, the wave is propagating along the z-direction causing distortions along the x and y directions. The latter occurs as a function of time. Taking $h_{+,\times} \sim \cos(\omega t)$, with ω representing the frequency, the ring distorts as a function of time as: solid $(\omega t = 0, \pi)$, dashed $\left(\omega t = \frac{\pi}{2}\right)$ and dotted $\left(\omega t = \frac{3\pi}{2}\right)$. Illustration based off Refs. [63, 277, 291].

erated which, as shall be seen, ultimately depend on the values of the electric part of the Riemann tensor. In what follows, two main approaches are discussed.

The first and by far most commonly used method is the so called Newman-Penrose (NP) formalism [292] with further developments on the work by Eardley et al. [293]. A brief overview of the NP formalism shall be presented. For further details on the method, one can refer to Refs. [63, 294, 295]. As seen in the GW analysis, the waves are plane waves, which are null and are functions of the retarded time $u = t - z$ and advanced time $v = t + z$ for waves propagating along the z-direction. A null basis is then constructed with basis vectors[17]

$$l^\mu = (1, 0, 0, 1), \qquad n^\mu = \frac{1}{2}(1, 0, 0, -1), \tag{6.40}$$

$$m^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0), \qquad \bar{m}^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0). \tag{6.41}$$

In this way, one can construct a null basis tetrad $e_a{}^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ which constructs the metric through the standard tetrad relations. Assuming that the waves are almost null, it can be shown that through symmetry properties of the Riemann tensor together with the Bianchi identities reveal that the only non-zero components of the Riemann tensor are of the form

$$R_{npnq} \equiv R_{rpsq} n^r n^s, \qquad (p, q \text{ are indices which run over } l, m \text{ and } \bar{m}) \tag{6.42}$$

leaving a total of six non-vanishing components of the Riemann tensor. These are denoted by

$$\Psi_2 \equiv -\frac{1}{6} R_{nlnl}, \qquad \Psi_3 \equiv -\frac{1}{2} R_{nln\bar{m}}, \tag{6.43}$$

$$\Psi_4 \equiv -R_{n\bar{m}n\bar{m}}, \qquad \Phi_{22} \equiv -R_{nmn\bar{m}}, \tag{6.44}$$

where $\Psi_{3,4}$ are complex and hence adding up to a total of six polarisations. It can

[17] The coefficients are not uniquely defined as seen between the works presented in Refs. [63, 294, 295]. This is not important as it will not alter the final results.

be shown that Ψ_4 corresponds to spin-2 behaviour (tensor), Ψ_3 to spin-1 (vector) while Ψ_2 and Φ_{22} are spin-0 (scalar). Observe that R_{npnq} corresponds to, precisely, components of the electric part of the Riemann tensor, which means that the latter can be expressed in terms of the NP quantities to be

$$R_{i0j0} = \begin{pmatrix} -\Re\Psi_4 - \Phi_{22} & \Im\Psi_4 & -2\sqrt{2}\Re\Psi_3 \\ \Im\Psi_4 & \Re\Psi_4 - \Phi_{22} & -2\sqrt{2}\Im\Psi_3 \\ -2\sqrt{2}\Re\Psi_3 & -2\sqrt{2}\Im\Psi_3 & -6\Psi_2 \end{pmatrix}, \tag{6.45}$$

where $\Re$ and $\Im$ represent the real and imaginary parts respectively. Plugging into the geodesic deviation formula Eq.(6.35) gives rise to the two tensor polarisations illustrated in Fig. 6.1 and the two vector and two scalar polarisations shown in Fig. 6.2. Here, the plus tensor mode corresponds to $\Re\Psi_4$ while the cross tensor mode corresponds to $\Im\Psi_4$. On the other hand, the real and imaginary part of Ψ_3 correspond to the vector-x and y polarisations respectively while Φ_{22} yields the breathing (b) mode and Ψ_2 the longitudinal (l) mode.

Despite the success of the NP formalism as a method to identify the various types of GW polarisations, it does suffer from issues. The approach is only valid for waves travelling at the speed of light, and hence, for theories which do predict more polarisation modes but propagate at a different speed cannot be identified using the NP approach. This is observed, for instance, in $f(R)$ gravity. It is known that besides the standard tensor polarisations, a scalar polarisation state which exhibits a mixture of both the breathing and longitudinal modes does exist(see Refs. [296–298] amongst others) but there had been debate whether this represents two distinct polarisations or one. Following the NP approach as shown in Refs. [294, 299], as $\Phi_{22} \neq 0$ and $\Psi_2 \neq 0$, this would infer that both scalar modes exist and are independent of one another. However, as discussed in Ref. [300], this is not true as the state exists only as a mixture of the two. A similar argument is given in Ref. [301].

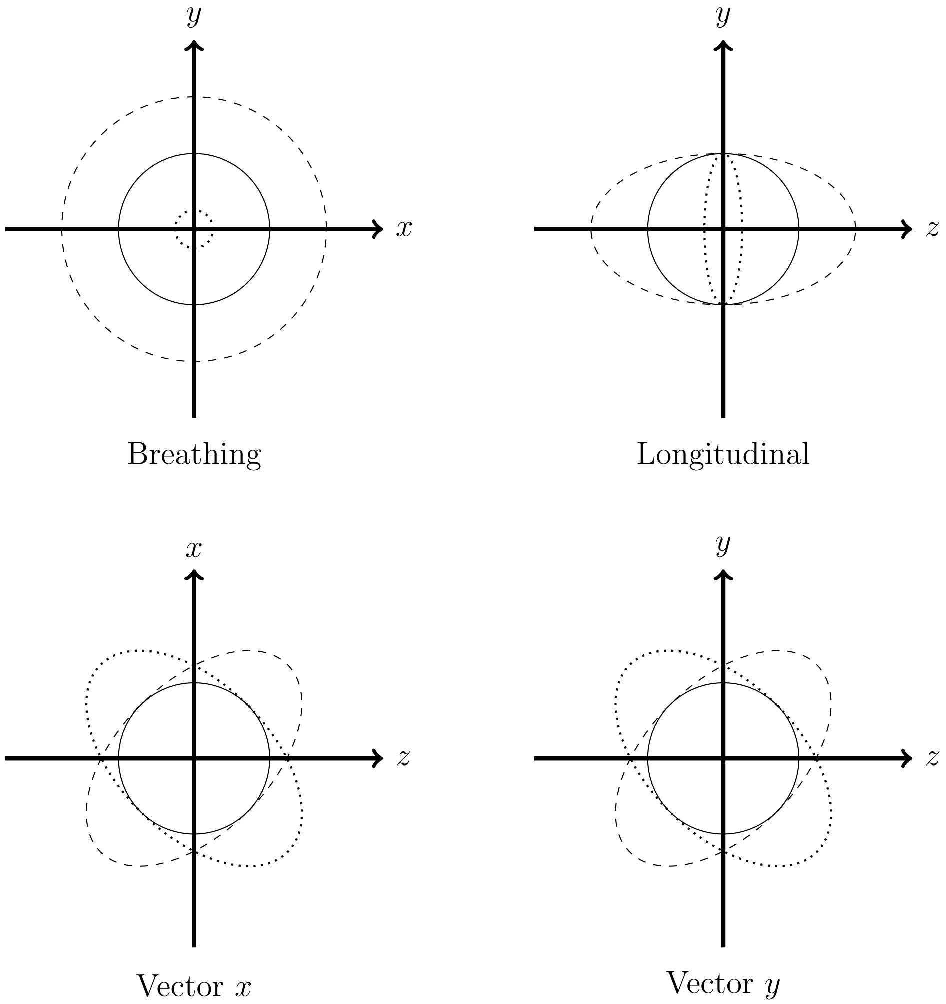


Figure 6.2: The effect of scalar (breathing and longitudinal) and vector polarisations (vector-x and y) for a GW propagating in the z-direction, which may arise in modified theories of gravity. As the distortions are a function of time due to the GW dependence $h \sim \cos(\omega t)$, with ω representing the frequency, the initial particle ring configuration distorts with time as: solid $(\omega t = 0, \pi)$, dashed $\left(\omega t = \frac{\pi}{2}\right)$ and dotted $\left(\omega t = \frac{3\pi}{2}\right)$. Illustration based off Refs. [63, 277, 291].

The second alternative would then be to take the electric component of the Riemann tensor and use the result from the NP formalism as a foundation to identify the components which give rise to all six different polarisations without invoking a null plane wave condition. This has been considered in Refs. [291, 301] where, for a wave propagating in the z-direction, the electric part of the Riemann tensor can be expressed as

$$R_{i0j0} = \begin{pmatrix} A_L + A_+ & A_\times & A_{V_x} \\ A_\times & A_L - A_+ & A_{V_y} \\ A_{V_x} & A_{V_y} & A_B \end{pmatrix}, \tag{6.46}$$

where $A_{+,\times}$ realise the plus and cross tensor polarisations, $A_{V_{x,y}}$ the vector x and y polarisations, and A_L and A_B the longitudinal and breathing scalar modes respectively. A similar identification was proposed in Ref. [277] but instead of the Riemann tensor, the decomposition is applied to the perturbed metric $h^{(1)}_{\mu\nu}$ where the spatial part is decomposed in the same fashion. In this way, the non-zero components of the Riemann tensor infer the type of mode without imposing any constraint on the wave speed, meaning that this would not impose any constraint on the number of modes as mixed states can then be constructed. Since in the work which follows non-null waves are found, the NP formalism shall not be considered to investigate the resulting polarisation states.

6.2.2 Tetrad Approach

The metric approach gives a clear indication that there are no further GW modes in $f(T)$ gravity. Motivated by the results obtained, in what follows, the same result is derived in tetrad formulation.

Starting from the $f(T)$ field equations Eq.(2.24),

$$\frac{1}{4}e_a{}^\mu f + f_T\left[e^{-1}\partial_\nu\left(eS_a{}^{\mu\nu}\right) - T^b{}_{\nu a}S_b{}^{\nu\mu} + \omega^b{}_{a\nu}S_b{}^{\nu\mu}\right] + f_{TT}S_a{}^{\mu\nu}\partial_\nu T = 0, \tag{6.47}$$

expanding order by order yields

$$\gamma_a^{(0)\rho} f(0) = 0, \tag{6.48}$$

$$f_T(0)\left[\partial_\nu S_a{}^{(1)\mu\nu} + \omega^b{}_{a\nu}S_b{}^{(1)\nu\mu}\right] = 0. \tag{6.49}$$

As before, the first condition implies that no cosmological constant is present, while the second expression yields the GW behaviour provided that $f_T(0) \neq 0$. In a purely local indexed form, the expression changes to

$$\partial_c S^{(1)abc} + S^{(1)cbd}\omega^a{}_{cd} + S^{(1)acd}\omega^b{}_{cd} + S^{(1)abc}\omega^{dc}{}_d = 0, \tag{6.50}$$

which shall prove useful in discussing the behaviour of $f(T, T_G)$ gravity later on. Eq.(6.49) is an extension of the result which appears in Ref. [282] with the spin dependent term appearing in the expression. The next step would be to solve for the perturbed tetrad. However, it turns out that this is not possible for an arbitrary spin connection due to the introduction of the background tetrad, the form of which is unknown even if relevant gauge constraints are imposed. For this reason, this shall not be investigated in further detail.

If one instead chooses the proper frame of reference (i.e. $\omega^a{}_{b\mu} = 0$), then Eq.(6.49) simplifies significantly. Note that in this proper frame, a zero spin connection implies that the Lorentz matrices (or alternatively the background tetrad) are constant [52]. Nonetheless, this still turns out to be insufficient as the resulting equations of motion are generally difficult to solve. This is where the metric approach result comes into play. Based on the previous analysis, it is expected that the gauge conditions used to eliminate the extra degrees of freedom in the metric are still applicable. In other words, the traceless condition

$$h^{(1)\mu}{}_\mu = 2\eta^{\mu\nu}\eta_{ab}\gamma_\mu^{(0)a}\gamma_\nu^{(1)b} = 0 \tag{6.51}$$

and the Lorenz gauge condition

$$0 = \partial^\mu h^{(1)}_{\mu\nu} = \partial_b \gamma_\nu^{(1)b} + \eta_{ab}\left[\gamma_\mu^{(1)a}\partial^\mu\gamma_\nu^{(0)b} + \gamma_\nu^{(1)b}\partial^\mu\gamma_\mu^{(0)a} + \gamma_\nu^{(1)b}\partial^\mu\gamma_\mu^{(0)a}\right] \tag{6.52}$$

can be used to simplify the field equations. In fact, these conditions allow for the field equations to be purely expressed in terms of the perturbed tetrad variable as

$$\eta^{\mu\alpha}\eta_{df}\Box\gamma_{\alpha}^{(1)f} - \Box\gamma_{d}^{(1)\mu} = 0, \tag{6.53}$$

where the relation $\gamma_{b}^{(1)\nu} = -\gamma_{a}^{(0)\nu}\gamma_{\mu}^{(1)a}\gamma_{b}^{(0)\mu}$ which arises from the identity property of tetrads has been used.

For the sake of simplicity, the case when $\gamma_{\mu}^{(0)a} = \delta_{\mu}^{a}$ is considered, one which trivially satisfies the absolute teleparallelism constraint. With this formulation, the field equations can be solely expressed in terms of $\gamma_{\mu}^{(1)a}$ to give rise to the following system of equations

$$A_{0}^{0}: \qquad \Box\gamma_{0}^{(1)0} = 0, \tag{6.54}$$

$$A_{i}^{0} = -A_{0}^{i}: \qquad \Box\left(\gamma_{i}^{(1)0} - \gamma_{0}^{(1)i}\right) = 0, \tag{6.55}$$

$$A_{j}^{i}\,(i \neq j): \qquad \Box\left(\gamma_{i}^{(1)j} + \gamma_{j}^{(1)i}\right) = 0, \tag{6.56}$$

$$A_{m}^{i}\,(i = m): \qquad \Box\gamma_{i}^{(1)i} = 0, \tag{6.57}$$

where the indices $i, j = \{1, 2, 3\}$. Next, the transverse gauge condition is introduced which imposes the constraints $\gamma_{0}^{(1)0} = 0$ and $\gamma_{0}^{(1)i} = \gamma_{i}^{(1)0}$, essentially eliminating the first two equations. Together with the traceless and Lorenz gauge conditions, the final set of equations left to solve are

$$\text{Traceless condition:} \qquad \gamma_{i}^{(1)i} = 0, \tag{6.58}$$

$$\text{Lorenz gauge condition:} \qquad \partial_{j}\left(\gamma_{i}^{(1)j} + \gamma_{j}^{(1)i}\right) = 0, \tag{6.59}$$

$$A_{j}^{i}\,(i \neq j): \qquad \Box\left(\gamma_{i}^{(1)j} + \gamma_{j}^{(1)i}\right) = 0, \tag{6.60}$$

$$A_{m}^{i}\,(i = m): \qquad \Box\gamma_{i}^{(1)i} = 0. \tag{6.61}$$

Without loss of generality, the GW shall be assumed to propagate in the z-direction.

Solving in Fourier space, the solutions are

$$\gamma_i^{(1)i} = A_i{}^i \exp(ik_\mu x^\mu),\ (i \text{ fixed index}), \tag{6.62}$$

$$\gamma_i^{(1)j} + \gamma_j^{(1)i} = B_i{}^j \exp(ik_\mu x^\mu),\ i \neq j, \tag{6.63}$$

where k_μ is a wavevector such that $k_\mu k^\mu = 0$ (which again confirms that the plane wave is propagating at the speed of light), and $A_i{}^i$ and $B_i{}^j$ are amplitude coefficients such that $A_1{}^1 = -A_2{}^2$, $A_3{}^3 = 0$ and $B_1{}^3 = B_2{}^3 = 0$, which arise from the traceless and Lorenz gauge conditions. This leaves $A_1{}^1, B_1{}^3, B_2{}^3$ and $B_1{}^2$ as the undetermined coefficients, resulting in the perturbed tetrad adopting the final form of

$$\gamma_\mu^{(1)a} = \begin{pmatrix} 0 & \gamma_0^{(1)1} & \gamma_0^{(1)2} & \gamma_0^{(1)3} \\ \gamma_0^{(1)1} & A_1{}^1 \exp(ik_\mu x^\mu) & \gamma_1^{(1)2} & \gamma_1^{(1)3} \\ \gamma_0^{(1)2} & B_1{}^2 \exp(ip_\mu x^\mu) - \gamma_1^{(1)2} & -A_1{}^1 \exp(ik_\mu x^\mu) & \gamma_2^{(1)3} \\ \gamma_0^{(1)3} & -\gamma_1^{(1)3} & -\gamma_2^{(1)3} & 0 \end{pmatrix}. \tag{6.64}$$

Note that there are six undetermined tetrad components as they are not constrained by the equations. However, these are not physical degrees of freedom, as they do not appear in the perturbed metric Eq.(6.5), which in this case takes the form of

$$h_{\mu\nu}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2A_1{}^1 \exp(ik_\mu x^\mu) & B_1{}^2 \exp(ik_\mu x^\mu) & 0 \\ 0 & B_1{}^2 \exp(ik_\mu x^\mu) & -2A_1{}^1 \exp(ik_\mu x^\mu) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.65}$$

Thus, these correspond to the Lorentz freedom of the tetrad.

Comparing with the solution obtained from the metric approach Eq.(6.31), the $+$ and $\times$ polarizations are clearly obtained after defining $h_+ := 2A_1{}^1 \exp(ik_\mu x^\mu)$ and $h_\times := B_1{}^2 \exp(ik_\mu x^\mu)$. This confirms the result obtained from the metric approach. In other words, it is entirely possible to solve the teleparallel form of the field equations through tetrads and obtain the GWs by appropriate choice of gauge conditions while not losing any information about the tetrad. This tetrad approach could be especially useful in theories where it is not possible apply a metric-like approach.

6.2.3 Higher Order Perturbations

Taking the linearised equations up to first order revealed no deviations from GR on a Minkowski background. Here, higher order perturbations are investigated to determine whether deviations from GR appear. Despite the small magnitudes of these higher order perturbations, these contain important information regarding the GWs. Following the metric approach Sec. 6.2.1, the second and third order perturbation equations take the simple form of

$$G^{(2)}_{\mu\nu} = 0, \tag{6.66}$$

$$G^{(3)}_{\mu\nu} = \frac{2f_{TT}(0)}{f_T(0)} S_{\nu}^{(1)\alpha}{}_{\mu} \partial_\alpha T^{(2)}. \tag{6.67}$$

Clearly, the second order equation coincides with the one which arises from GR. This equation is important as this accounts for the gravitational radiation of a GW. To illustrate this, one can separate the $G^{(2)}_{\mu\nu}$ perturbation in two parts, terms which are linearly proportional to $h^{(2)}_{\mu\nu}$ and those quadratic in $h^{(1)}_{\mu\nu}$. This leads to

$$G^{(2)}_{\mu\nu}\left[h^{(2)}\right] = -G^{(2)}_{\mu\nu}\left[h^{(1)}h^{(1)}\right] \equiv t_{\mu\nu}, \tag{6.68}$$

where $t_{\mu\nu} \equiv -G^{(2)}_{\mu\nu}\left(h^{(1)}h^{(1)}\right)$ has been defined. This term represents the pseudo energy-momentum tensor for the GWs. In fact, this term satisfies a local conservation law $\partial_\mu t^{\mu\nu} = 0$. However, this is not an actual stress-energy tensor for various reasons. 'Pseudo' here refers to the fact that it is not an actual tensor as it is not invariant under gauge transformations [57]. Furthermore, there is no way to define the local energy-momentum of gravity by virtue of the equivalence principle [54–56].

However, one can instead define a macroscopic value for the energy of the field in a specified coordinate system by averaging over a finite volume that is much larger than the wavelength of the GW. In the TT gauge, this results in the expression

$$t_{\mu\nu} \sim \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle \,. \tag{6.69}$$

This formula has been used as means to quantify the energy loss of a system such as the Hulse-Taylor binary system, which predicted the orbital decay rate (originally derived in Ref. [268]) which has remained to be in continuous agreement with observations [270,271]. As discussed in Ref. [302], this effect implies that the energy-momentum of a GW is not a first order effect but a non-linear one. This follows from the fact that there is no linear stress-energy pseudo-tensor (in fact, the least order term appearing in the pseudo-tensor is of second order [1]). In Ref. [303], an exact form of $h^{(2)}_{\mu\nu}$ is derived.

Moving on to the third order perturbation equation, it is evident that a clear deviation from GR is present due to a new contribution arising from the f_{TT} term coefficient, unless the latter is zero, in which case the equation gets reduced to that constructed from GR. However, no physical conclusion could be drawn, as the system has not been solved (and this would be relatively difficult to do due to non-linearities in the equations of motion). These higher order contributions represent corrections to the gravitational radiation obtained at second order. In the case of GR, these have been analysed up to any order in Ref. [304] and references therein.

6.3 Gravitational Waves in $f(T, B)$ gravity

Following the GW analysis carried out in the case of $f(T)$ gravity, the extension which includes a functional dependence on the boundary term scalar is investigated, namely $f(T, B)$ gravity. The inclusion of this scalar shall introduce the presence of a new mode, which is expected, since $f(R)$ gravity, which exhibits three gravitational modes, is a sub-class of the theory. Gravitational waves in $f(T, B)$ gravity have been investigated in Ref. [72] where the former statement turned out to be correct. However, in what follows, a minor modification in the coefficients is found, while the full polarisation of the waves shall be carried out in detail. Furthermore, the GWs shall be investigated under two distinct regimes; first in a Minkowskian background and second in the presence of a cosmological constant. The latter regime has not yet been investigated in literature.

6.3.1 GWs in the Absence of a Cosmological Constant

Metric Approach

Following the approach applied to the $f(T)$ case Sec. 6.2.1, the zeroth and first order equations for the spacetime indexed form of the $f(T, B)$ gravity field equations Eq.(2.29) given to be

$$-f_T G_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right) f_B + \frac{1}{2} g_{\mu\nu}\left(B f_B + T f_T - f\right) + 2 S_\nu{}^\alpha{}_\mu \partial_\alpha \left(f_T + f_B\right) = 0, \tag{6.70}$$

are

$$\eta_{\mu\nu} f(0,0) = 0, \tag{6.71}$$

$$-f_T(0,0) G^{(1)}_{\mu\nu} + f_{BB}(0,0)\left(\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu\right) R^{(1)} = 0, \tag{6.72}$$

where the result $R^{(1)} = B^{(1)}$ has been used, which shall be useful in simplifying the forthcoming equations. Once more, the zeroth order equation confirms the absence of a cosmological constant. As mentioned previously, as $f(R)$ gravity is a sub-case of $f(T, B)$ gravity, new modes are expected to appear. In fact, the resulting first order equation Eq.(6.72) is strikingly similar to that found in $f(R)$ gravity, with the only discrepancies being the different coefficients [20, 296–301, 305–307].[18] Motivated by this, the same procedure shall be followed, a method which has also been used in deriving the result obtained in Ref. [72].

First, the trace of Eq.(6.72) is considered which yields the relation

$$f_T(0,0)R^{(1)} + 3f_{BB}(0,0)\Box R^{(1)} = 0. \tag{6.73}$$

This is of the same form as the Klein-Gordon equation of a scalar field (in this case $R^{(1)}$) of effective mass m, $(\Box - m^2)\, R^{(1)} = 0$, which allows for an identification of the effective mass m to be

$$m^2 \equiv -\frac{f_T(0,0)}{3f_{BB}(0,0)}. \tag{6.74}$$

In the $|m^2| \to \infty$ limit (for instance $f(T,B) = f(T)$ is one example), the equation reduces to the constraint $R^{(1)} = 0$, the same as found in standard GR. Thus, Eq. (6.72) reduces to that encountered in $f(T)$ gravity yielding the same GW mode behaviour. In cases when the mass is finite, this yields a plane-wave solution for the Ricci scalar, which in Fourier space can be expressed as

$$R^{(1)} = B \exp\left(ip_\mu x^\mu\right), \tag{6.75}$$

for some amplitude B and wavevector p^μ which satisfies the condition $p_\mu p^\mu = -m^2$. This implies a wave which, as dictated by the wavevector constraint, does not travel at the speed of light. Instead, it travels at some group velocity v [20,296–298,306,308]

$$v = \frac{\sqrt{\omega^2 - m^2}}{\omega}, \tag{6.76}$$

with frequency ω. Therefore, there exists a class of $f(T,B)$ functions which imply the existence of a new scalar mode.

[18]Indeed, the $f(R)$ limit is recovered when $f(T,B) = f(-T+B)$ which sets $f_T \to -f_R$ and $f_{BB} \to f_{RR}$, leading to the same expression encountered in $f(R)$ gravity.

Next, the perturbed metric tensor is found. Similar to the $f(T)$ case, a new perturbed tensor quantity shall be introduced to simplify the equations. In this case, one convenient choice is

$$h^{(1)}_{\mu\nu} = \bar{h}^{(1)}_{\mu\nu} - \frac{1}{2}\bar{h}^{(1)}\eta_{\mu\nu} - \frac{1}{3m^2}\eta_{\mu\nu}R^{(1)}, \tag{6.77}$$

where once again $\bar{h}^{(1)}$ represents the trace of $\bar{h}^{(1)}_{\mu\nu}$. Note that contrary to the $f(T)$ case, this tensor is not trace-reversed unless the scalar mode vanishes. Thanks to this newly defined perturbed tensor, Eq.(6.72) takes a much simpler form

$$\partial^\rho\partial_\nu\bar{h}^{(1)}_{\rho\mu} + \partial^\rho\partial_\mu\bar{h}^{(1)}_{\nu\rho} - \eta_{\mu\nu}\partial^\rho\partial^\alpha\bar{h}^{(1)}_{\rho\alpha} - \Box\bar{h}^{(1)}_{\mu\nu} = 0. \tag{6.78}$$

To simplify the equations further, the transverse-traceless gauge ($\partial^\mu\bar{h}^{(1)}_{\mu\nu} = 0$, $\bar{h}^{(1)} = 0$ and $\bar{h}^{(1)}_{0\mu} = 0$) is considered. The applicability of whether such gauge freedom is permitted has been of discussion in the context of $f(R)$ gravity. According to Ref. [299], one can set either transversality or tracelessness but not both at the same time. However, it was shown later in Ref. [308] that both conditions can indeed be set as also remarked in Ref. [300]. Thus, in a similar analogy, such gauge conditions are then allowed to be imposed, which reduce the field equations to $\Box\bar{h}^{(1)}_{\mu\nu} = 0$, which reveals a null plane-wave solution

$$\bar{h}^{(1)}_{\mu\nu} = A_{\mu\nu}\exp\left(ik_\rho x^\rho\right), \tag{6.79}$$

with k^ρ representing the null four-wavevector $k_\rho k^\rho = 0$ and $A_{\mu\nu}$ are the amplitude coefficients. Due to the gauge fixing, the latter satisfy $k^\mu A_{\mu\nu} = 0$, $A^\mu{}_\mu = 0$ and $A_{0\nu} = 0$. Assuming the wave to be propagating in the z-direction sets $A_{11} = -A_{22}$ and $A_{12} = A_{21}$ as the undetermined coefficients whereas the remaining unlisted coefficients are zero. Combining the results obtained from the scalar mode Eq.(6.75), the full solution for the perturbed metric $h_{\mu\nu}^{(1)}$ is found to be

$$h_{\mu\nu}^{(1)} = A_{\mu\nu} \exp\left(ik_\rho x^\rho\right) - \frac{1}{3m^2}\eta_{\mu\nu} B \exp\left(ip_\rho x^\rho\right). \tag{6.80}$$

To finally determine the polarisations of these GWs, one should again resort to the use of the geodesic deviation formula, where as discussed in in Sec. 6.2.1, it is sufficient to determine and compare the resulting non-zero components of the electric part of the Riemann tensor. In Fourier space, this turns out to be

$$\begin{aligned} R_{i0j0} &= \frac{1}{2}{k_0}^2 \bar{h}_{ij} - \frac{1}{6m^2}\left[\delta_{ij}{p_0}^2 - p_i p_j\right] R^{(1)} \\ &= \begin{pmatrix} \frac{1}{2}{k_0}^2\bar{h}_{11} - \frac{1}{6m^2}{p_0}^2 R^{(1)} & \frac{1}{2}{k_0}^2\bar{h}_{12} & 0 \\ \frac{1}{2}{k_0}^2\bar{h}_{12} & -\frac{1}{2}{k_0}^2\bar{h}_{11} - \frac{1}{6m^2}{p_0}^2 R^{(1)} & 0 \\ 0 & 0 & -R^{(1)} \end{pmatrix}, \end{aligned} \tag{6.81}$$

where the condition $p_\mu p^\mu = -m^2$ has been applied. Comparing with Eq.(6.46) reveals $\bar{h}_{11}$ and $\bar{h}_{12}$ correspond to the massless $+$ and $\times$ polarisations respectively, while the scalar mode appears in the breathing and longitudinal components and hence exists as a mixture of both modes. Note that similar to the $f(R)$ case, application of the NP formulation would infer that these two scalar modes are distinct from one another ($\Phi_{22} \neq 0$ and $\Psi_2 \neq 0$), leaving a total of four polarisations instead of three. However, as the wave is propagating at a different speed from that of light, this formulation does not apply and the scalar mode exists only as a mixture of both scalar polarisations [300, 301, 307]. Nonetheless, in the absence of this mode ($|m^2| \to \infty$), only the tensor modes remain, which is in agreement with the NP formulation as expected since these waves are null.

Tetrad Solutions for GWs in $f(T,B)$ gravity

So far, $f(T,B)$ gravity revealed the existence of an extra massive scalar GW mode besides the two massless tensor $+$ and $\times$ polarisations, obtained through a metric approach. In the following, the latter results shall be obtained through a tetrad formulation. Based on the tetrad approach used in $f(T)$ gravity, although introducing the spin connection keeps a sense of generality, it does not allow for finding general solutions. Despite this difficulty, in the proper frame, the system simplifies considerably, especially when the choice of having the background tetrad as the Kronecker delta is considered. This shall be assumed in the following. The resulting perturbed field equations arising from Eq.(2.28), which are again listed for simplicity

$$\begin{aligned} & e_a{}^{\mu}\Box f_B - e_a{}^{\nu}\nabla^{\mu}\nabla_{\nu} f_B + \frac{1}{2} e_a{}^{\mu}\left(B f_B - f\right) + 2 S_a{}^{\nu\mu}\partial_{\nu}\left(f_B + f_T\right) \\ & + 2e^{-1}\partial_{\nu}\left(e S_a{}^{\nu\mu}\right) f_T - 2 T^{\alpha}{}_{\nu a} S_{\alpha}{}^{\mu\nu} f_T = 0, \end{aligned} \tag{6.82}$$

yield

$$\delta^{\mu}_{a} f(0,0) = 0, \tag{6.83}$$

$$\partial_{\nu} S_a{}^{(1)\nu\mu} + \frac{f_{BB}(0,0)}{2 f_T(0,0)}\left[\delta^{\mu}_{a}\Box B^{(1)} - \delta^{\nu}_{a}\partial^{\mu}\partial_{\nu} B^{(1)}\right] = 0. \tag{6.84}$$

Here, the consistency of an absence of a cosmological constant in the gravitational Lagrangian is retained from the zeroth order equation Eq.(6.83), while the first order equation Eq.(6.84) clearly shows the effect of the boundary term on the equations. This shows a clear deviation from TEGR and $f(T)$ gravity unless $f_{BB}(0,0) = 0$. In the latter, this results in the two massless $+$ and $\times$ polarisations, agreeing with the metric and tetrad approach results.

As shown in the $f(T)$ case (Sec. 6.2.2), solving the first order equation is not, in general, possible unless the use of gauge conditions obtained from the metric approach are used. In the avenue that $f_{BB}(0,0) \neq 0$, the metric approach showed the existence of a massive scalar mode arising from $B^{(1)}$, which led to the transverse-traceless gauge conditions $\partial^\mu \bar{h}^{(1)}_{\mu\nu} = 0$ and $\bar{h}^{(1)} = 0$ with $\bar{h}^{(1)}_{\mu\nu}$ as defined in Eq.(6.77). Expressing in terms of tetrads, the traceless condition $\bar{h}^{(1)} = 0$ becomes

$$\delta^\nu_b \gamma^{(1)b}_\nu = \frac{2 f_{BB}(0,0) B^{(1)}}{f_T(0,0)}, \tag{6.85}$$

and the Lorenz gauge condition $\partial^\mu \bar{h}^{(1)}_{\mu\nu} = 0$ takes the form of

$$\eta_{ab}\left(\partial^a \gamma^{(1)b}_\nu + \delta^b_\nu \partial^\mu \gamma^{(1)a}_\mu\right) = \frac{f_{BB}(0,0)\partial_\nu B^{(1)}}{f_T(0,0)}. \tag{6.86}$$

These conditions allow for the field equations to simplify to

$$A^\mu_a \equiv \eta^{\mu\alpha}\eta_{ab}\Box\gamma^{(1)b}_\alpha + \delta^\mu_b \delta^\rho_a \Box\gamma^{(1)b}_\rho - \delta^\mu_a \frac{f_{BB}(0,0)\Box B^{(1)}}{f_T(0,0)} = 0, \tag{6.87}$$

which yields the following system of plane-wave equations:

$$A^0_0: \qquad \Box\left(\gamma^{(1)0}_0 - \frac{f_{BB}(0,0)B^{(1)}}{2f_T(0,0)}\right) = 0, \tag{6.88}$$

$$A^0_i = -A^i_0: \qquad \Box\left(\gamma^{(1)0}_i - \gamma^{(1)i}_0\right) = 0, \tag{6.89}$$

$$A^i_j\,(i \neq j): \qquad \Box\left(\gamma^{(1)j}_i + \gamma^{(1)i}_j\right) = 0, \tag{6.90}$$

$$A^i_m\,(i = m): \qquad \Box\left(\gamma^{(1)i}_i - \frac{f_{BB}(0,0)B^{(1)}}{2f_T(0,0)}\right) = 0. \tag{6.91}$$

From the transverse gauge constraint, one obtains that $\gamma^{(1)0}_0 = \frac{f_{BB}(0,0)B^{(1)}}{2f_T(0,0)}$ and $\gamma^{(1)0}_i = \gamma^{(1)i}_0$. Therefore, only the final two equations remain which, in Fourier space, yield

$$\gamma^{(1)i}_i = C_i{}^i \exp(ik_\mu x^\mu) + \frac{f_{BB}(0,0)B^{(1)}}{2f_T(0,0)}, \qquad i \text{ fixed index}, \tag{6.92}$$

$$\gamma^{(1)j}_i + \gamma^{(1)i}_j = D_i{}^j \exp(ik_\mu x^\mu), \qquad i \neq j, \tag{6.93}$$

for amplitude coefficients $C_i{}^i$ and $D_i{}^j$ and wavevector such that $k_\mu k^\mu = 0$, i.e. they are propagating at the speed of light. Assuming that the waves propagate along the z-direction, the amplitude coefficients satisfy the constraints ${C_3}^3 = {D_1}^3 = {D_2}^3 = 0$ and ${C_1}^1 = -{C_2}^2$, which leads to the perturbed first order tetrad to take the form of

$$\gamma_\mu^{(1)a} = \begin{pmatrix} \frac{f_{BB}(0,0)B^{(1)}}{2f_T(0,0)} & \gamma_0^{(1)1} & \gamma_0^{(1)2} & \gamma_0^{(1)3} \\ \gamma_0^{(1)1} & {C_1}^1 \exp(ik_\mu x^\mu) + \frac{f_{BB}(0,0)B^{(1)}}{2f_T(0,0)} & \gamma_1^{(1)2} & \gamma_1^{(1)3} \\ \gamma_0^{(1)2} & {D_1}^2 \exp(ik_\mu x^\mu) - \gamma_1^{(1)2} & -{C_1}^1 \exp(ik_\mu x^\mu) + \frac{f_{BB}(0,0)B^{(1)}}{2f_T(0,0)} & \gamma_2^{(1)3} \\ \gamma_0^{(1)3} & -\gamma_1^{(1)3} & -\gamma_2^{(1)3} & \frac{f_{BB}(0,0)B^{(1)}}{2f_T(0,0)} \end{pmatrix}. \tag{6.94}$$

Once more, there are six undetermined components not constrained by the equations which correspond to the Lorentz symmetry degrees of freedom leaving three undetermined coefficients, being ${C_1}^1$, ${D_1}^2$ and $B^{(1)}$, acting as the GWs. In fact, computing the perturbed metric tensor reveals

$$h_{\mu\nu}^{(1)} = \begin{pmatrix} -\frac{f_{BB}(0,0)B^{(1)}}{f_T(0,0)} & 0 & 0 & 0 \\ 0 & h_+ + \frac{f_{BB}(0,0)B^{(1)}}{f_T(0,0)} & h_\times & 0 \\ 0 & h_\times & -h_+ + \frac{f_{BB}(0,0)B^{(1)}}{f_T(0,0)} & 0 \\ 0 & 0 & 0 & \frac{f_{BB}(0,0)B^{(1)}}{f_T(0,0)} \end{pmatrix}, \tag{6.95}$$

which is precisely the solution found in Eq.(6.80) after exchanging $B^{(1)} = R^{(1)}$ with the identification $h_+ \coloneqq 2{C_1}^1 \exp(ik_\mu x^\mu)$ and $h_\times \coloneqq {D_1}^2 \exp(ik_\mu x^\mu)$.

6.3.2 Effect of a Cosmological Constant in $f(T, B)$ gravity

On a Minkowski background, the GWs arising from an $f(T, B)$ gravity model have indicated the presence of three modes with the tensor modes travelling at the speed of light and the scalar mode travelling at a different speed. Still within the linearised gravity regime, the effect of a cosmological constant to the background Minkowski geometry shall now be investigated. The introduction of this constant does not alter the GW polarisation behaviour but its presence causes an effect in the amplitude and phase of these waves, which occurs as the background geometry is no longer

Minkowski spacetime. Due to the change in the background geometry, the linearised approach must be slightly revised to include the effects of the cosmological constant. For the purpose of this work, the linearisation procedure presented in Refs. [309–311] shall be followed.

Denoting Λ as the cosmological constant, effects of this quantity in both the background and perturbed level require an order expansion at each level of perturbation about Λ. This is necessary in order to introduce Λ into the background geometry as it is no longer Minkowski, and also to introduce the effect of the cosmological constant onto the GWs. As the cosmological constant is constrained observationally to be a very small quantity, one can follow this linearisation procedure where

$$g_{\mu\nu} = h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu} + \mathcal{O}\left(h^{(2)}_{\mu\nu}\right) , \tag{6.96}$$

where $\left|h^{(2)}_{\mu\nu}\right| \ll \left|h^{(1)}_{\mu\nu}\right| \ll 1$ represents the overall metric perturbation, in which each perturbation is expanded in orders of Λ to be

$$h^{(0)}_{\mu\nu} = \eta_{\mu\nu} + \Lambda h^{(0,\Lambda)}_{\mu\nu} + \mathcal{O}\left(\Lambda^2\right) , \tag{6.97}$$

$$h^{(1)}_{\mu\nu} = h^{(1,\mathrm{GW})}_{\mu\nu} + \Lambda h^{(1,\Lambda)}_{\mu\nu} + \mathcal{O}\left(\Lambda^2\right) . \tag{6.98}$$

Here, the superscripts define the type of order, $(0, \Lambda)$ represents the first Λ dependence on the background metric, $(1, \mathrm{GW})$ represents the GW contribution independent of the cosmological constant while $(1, \Lambda)$ represents the latter correction to the GWs. This notation shall be assumed in what follows.

In a similar fashion, a comparable expansion is considered for the tetrads, namely

$$e^a{}_\mu = \bar{e}^{(0)a}_\mu + \bar{e}^{(1)a}_\mu + \mathcal{O}\left(\bar{e}^{(2)a}_\mu\right) , \tag{6.99}$$

where $\left|\bar{e}_{\mu}^{(2)a}\right| \ll \left|\bar{e}_{\mu}^{(1)a}\right| \ll 1$, and

$$\bar{e}_{\mu}^{(0)a} = \gamma_{\mu}^{(0)a} + \Lambda\gamma_{\mu}^{(0,\Lambda)a} + \mathcal{O}(\Lambda^2), \tag{6.100}$$
$$\bar{e}_{\mu}^{(1)a} = \gamma_{\mu}^{(1,\mathrm{GW})a} + \Lambda\gamma_{\mu}^{(1,\Lambda)a} + \mathcal{O}(\Lambda^2). \tag{6.101}$$

Overall, the metric perturbations can therefore be expressed in terms of tetrad perturbations through the relations

$$\eta_{\mu\nu} = \eta_{ab}\gamma_{\mu}^{(0)a}\gamma_{\nu}^{(0)b}, \tag{6.102}$$
$$h_{\mu\nu}^{(0,\Lambda)} = \eta_{ab}\left(\gamma_{\mu}^{(0)a}\gamma_{\nu}^{(0,\Lambda)b} + \gamma_{\mu}^{(0,\Lambda)a}\gamma_{\nu}^{(0)b}\right), \tag{6.103}$$
$$h_{\mu\nu}^{(1,\mathrm{GW})} = \eta_{ab}\left(\gamma_{\mu}^{(0)a}\gamma_{\nu}^{(1,\mathrm{GW})b} + \gamma_{\mu}^{(1,\mathrm{GW})a}\gamma_{\nu}^{(0)b}\right), \tag{6.104}$$
$$h_{\mu\nu}^{(1)\mathrm{GW}\Lambda} = \eta_{ab}\left(\gamma_{\mu}^{(0)a}\gamma_{\nu}^{(1,\Lambda)b} + \gamma_{\mu}^{(0,\Lambda)a}\gamma_{\nu}^{(1,\mathrm{GW})b} + \gamma_{\mu}^{(1,\mathrm{GW})a}\gamma_{\nu}^{(0,\Lambda)b} + \gamma_{\mu}^{(1,\Lambda)a}\gamma_{\nu}^{(0)b}\right). \tag{6.105}$$

The next step is to revisit and examine the order behaviour of the relevant gravitational quantities. Starting with the definition of the torsion tensor Eq.(2.10) up to first order in both perturbation and Λ yields

$$\begin{aligned} T^{a}{}_{\mu\nu} = {} & \partial_{\mu}\gamma_{\nu}^{(0)a} - \partial_{\nu}\gamma_{\mu}^{(0)a} + \omega^{a}{}_{b\mu}\gamma_{\nu}^{(0)b} - \omega^{a}{}_{b\mu}\gamma_{\mu}^{(0)b} \\ & + \partial_{\mu}\gamma_{\nu}^{(0,\Lambda)a} - \partial_{\nu}\gamma_{\mu}^{(0,\Lambda)a} + \omega^{a}{}_{b\mu}\gamma_{\nu}^{(0,\Lambda)b} - \omega^{a}{}_{b\mu}\gamma_{\mu}^{(0,\Lambda)b} \\ & + \partial_{\mu}\gamma_{\nu}^{(1,\mathrm{GW})a} - \partial_{\nu}\gamma_{\mu}^{(1,\mathrm{GW})a} + \omega^{a}{}_{b\mu}\gamma_{\nu}^{(1,\mathrm{GW})b} - \omega^{a}{}_{b\mu}\gamma_{\mu}^{(1,\mathrm{GW})b} \\ & + \partial_{\mu}\gamma_{\nu}^{(1,\Lambda)a} - \partial_{\nu}\gamma_{\mu}^{(1,\Lambda)a} + \omega^{a}{}_{b\mu}\gamma_{\nu}^{(1,\Lambda)b} - \omega^{a}{}_{b\mu}\gamma_{\mu}^{(1,\Lambda)b}. \end{aligned} \tag{6.106}$$

Using the fact that in the $G \to 0$ limit the torsion tensor should be zero (as this reduces to Minkowski space), the previously obtained condition Eq.(6.8) is once more obtained. In other words, this shows that at a background level, the torsion tensor is of Λ order while it exhibits GW contributions at both zeroth and first order Λ contributions. Consequently, the same applies for the boundary term. For the torsion scalar, it is second order in Λ at a background level and has a GW correction at first Λ order. Therefore, the torsion scalar is expressed as

$$T = T^{(0,\Lambda^2)} + T^{(1,\Lambda)} + \mathcal{O}\left(T^{(2)}\right), \tag{6.107}$$

with only $T^{(1,\Lambda)}$ being relevant to the forthcoming analysis.

With all relevant quantities having their order defined, the final step requires a method to deal with orders of the gravitational Lagrangian $f(T,B)$. For simplicity, assume that the function can be expanded about $T = B = 0$, namely

$$\begin{aligned} f(T,B) = f(0,0) + f_T(0,0)T + f_B(0,0)B + \frac{1}{2}f_{TT}(0,0)T^2 \\ + \frac{1}{2}f_{BB}(0,0)B^2 + f_{TB}(0,0)TB + \dots . \end{aligned} \tag{6.108}$$

Due to the cosmological constant background contribution, $f(0,0)$ represents the latter, and is hence treated to be at least of first Λ order. The remaining Taylor coefficients, however, will depend on the functional limit, as they may contain terms proportional to Λ. To illustrate this, consider the following examples where $f_1(T,B) = T + \Lambda T e^B$ and $f_2(T,B) = T + \alpha T e^B + 2\Lambda$ for some constant $|\alpha| \gg |\Lambda|$. Then

$$f_{1T}^{(0,\Lambda)} = f_{1T}(0,0) + f_{1TB}(0,0)B^{(0,\Lambda)} = 1 + \Lambda + \Lambda B^{(0,\Lambda)} = \Lambda + \mathcal{O}\left(\Lambda^2\right), \tag{6.109}$$

$$f_{2T}^{(0,\Lambda)} = f_{2T}(0,0) + f_{2TB}(0,0)B^{(0,\Lambda)} = 1 + \alpha + \alpha B^{(0,\Lambda)} = \alpha B^{(0,\Lambda)} + \mathcal{O}\left(\Lambda^2\right). \tag{6.110}$$

As the examples illustrate, f_{1T} contributes at first Λ order while f_{2T} does not. On the other hand, the f_{1TB} term becomes second order while f_{2TB} yields a first order contribution. In order to distinguish between these two cases, Taylor coefficients denoted by a superscript Λ, i.e. $f^{\Lambda}_{(n)}(0,0)$ where n represents the various possible combination of derivatives, will represent the explicit Λ dependency of the Taylor coefficient. On the other hand, those without the superscript shall represent the Λ

independent contribution.

The perturbed ordered equations can now be examined in detail. Starting from the spacetime indexed form of the equations Eq.(2.29), at zeroth order, the Minkowski background is identically zero while at first Λ order yields

$$f_T(0,0)G^{(0,\Lambda)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}f(0,0) = 0. \tag{6.111}$$

This describes the cosmological constant dependent background metric solution, which evidently depends on the coefficients $f(0,0)$ and $f_T(0,0)$. Essentially, this modifies the de Sitter metric solution, which can be easily derived following [310]. To illustrate this, an effective cosmological constant

$$\tilde{\Lambda} \equiv \frac{f(0,0)}{2f_T(0,0)} \tag{6.112}$$

is defined. Then, Eq.(6.111) is reduced to a GR-like form $G^{(0,\Lambda)}_{\mu\nu} + \tilde{\Lambda}\eta_{\mu\nu} = 0$ which is the one given in Refs. [309–311]. Indeed, the background value of the Ricci scalar at first order in Λ, which is obtained after taking the trace, reveals $R^{(0,\Lambda)} = 4\tilde{\Lambda}$, which is precisely the value in GR in the presence of a cosmological constant.

Moving towards the first order perturbation, expanding in orders of Λ, the following equations are derived:

$$- f_T(0,0)G^{(1,\mathrm{GW})} + f_{BB}(0,0)\left(\eta_{\mu\nu}\Box^{(0)} - \partial_\mu\partial_\nu\right)B^{(1,\mathrm{GW})} = 0, \tag{6.113}$$

$$\begin{aligned}
&- f_{TB}(0,0)B^{(1,\mathrm{GW})}G^{(0,\Lambda)}_{\mu\nu} - \left[f_{TB}(0,0)B^{(0,\Lambda)} + f^{\Lambda}_T(0,0)\right]G^{(1,\mathrm{GW})}_{\mu\nu} - f_T(0,0)G^{(1,\Lambda)}_{\mu\nu}\\
&+ \frac{1}{2}\eta_{\mu\nu}\left[f_{BB}(0,0)B^{(0,\Lambda)} + f^{\Lambda}_B(0,0)\right]B^{(1,\mathrm{GW})} - \frac{1}{2}h^{(1,\mathrm{GW})}_{\mu\nu}f(0,0)\\
&+ 2\left[f_{TB}(0,0) + f_{BB}(0,0)\right]S_\nu^{(0,\Lambda)\alpha}{}_\mu\partial_\alpha B^{(1,\mathrm{GW})}\\
&+ \left(\eta_{\mu\nu}\Box^{(0)} - \partial_\mu\partial_\nu\right)\left(f_{TB}(0,0)T^{(1,\Lambda)} + f_{BB}(0,0)B^{(1,\Lambda)} + f_{BBB}(0,0)B^{(0,\Lambda)}B^{(1,\mathrm{GW})}\right)\\
&+ f_{BB}(0,0)\left(h^{(0,\Lambda)}_{\mu\nu}\Box^{(0)} + \eta_{\mu\nu}\Box^{(0,\Lambda)} - \nabla^{(0,\Lambda)}_\mu\partial_\nu\right)B^{(1,\mathrm{GW})} = 0,
\end{aligned} \tag{6.114}$$

where $\square^{(0)} \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. The $f(T,B)$ GW equation Eq.(6.72) is recovered as seen from Eq.(6.113), confirming the consistency of the result. Eq.(6.114) then describes the effect of the cosmological constant to the GWs. However, this turns out to be intractable to solve for the following reasons. Here, the goal is to solve the unknown components which appear in $G^{(1,\Lambda)}_{\mu\nu}$, $T^{(1,\Lambda)}$ and $B^{(1,\Lambda)}$. However, as $T^{(1,\Lambda)}$ and $B^{(1,\Lambda)}$ are not expressed solely in terms of metric perturbations, this results into a complicated system of equations in terms of tetrads. A viable alternative would be to express the combination of these two quantities in terms of $R^{(1,\Lambda)}$. Alas, the presence of the f_{TB} and f_{BB} coefficients make this non-viable, therefore making the system generally very difficult to solve.

Nonetheless, in the simple $f(T,B) = f(T)$ limit, the Λ dependent order solution Eq.(6.114) reduces massively to

$$f^{\Lambda}_T(0)G^{(1,\mathrm{GW})}_{\mu\nu} + f_T(0)G^{(1,\Lambda)}_{\mu\nu} + \frac{1}{2}h^{(1,\mathrm{GW})}_{\mu\nu}f(0) = 0. \tag{6.115}$$

In this limit, Eq.(6.72) reduces to the constraint $G^{(1,\mathrm{GW})}_{\mu\nu} = 0$ (in the same way as shown in the $f(T)$ analysis (Sec. 6.2.1) and hence leads to the massless $+$ and $\times$ polarisations), leading to a further simplification to

$$G^{(1,\Lambda)}_{\mu\nu} + \tilde{\Lambda}h^{(1,\mathrm{GW})}_{\mu\nu} = 0, \tag{6.116}$$

where the effective cosmological constant definition Eq.(6.112) has been used. This equation is precisely the one found in GR in Refs. [309–311][19] albeit having a different cosmological constant, meaning that the same procedure can be followed to obtain the resulting effect of Λ onto the GWs.

Without going into the mathematical details, this equation implies that no extra polarisation modes besides the massless tensor modes exist. However, the latter

[19]This is apparent after applying the corresponding gauge choices to yield the simplified expression $\square h^{(1,\Lambda)}_{\mu\nu} + 2\Lambda h^{(1,\mathrm{GW})}_{\mu\nu} = 0$.

modes are affected, causing a time varying amplitude and phase dependent on the cosmological constant. Despite this effect, this contribution is very small due to the small nature of Λ and hence such an observation would be very difficult to detect [309–313].

In conclusion, contrary to the background GW solutions which turn out to be identical to those of GR, a change in the background causes deviations from GR. Including the cosmological constant causes the choice of the $f(T)$ function to infer an effective cosmological constant, which could be different from that defined in standard GR with a cosmological constant. This choice is important as it affects the GW behaviour. Such behaviour is also observed in an FLRW setting, where it has been shown that despite the fact that tensor modes travel at the speed of light, their amplitude is still dependent on the $f(T)$ function [284, 285].

6.4 Gravitational Waves in $f(T, T_G)$ gravity

The final gravitational model investigated for GWs is the one which includes the TEGB extension, T_G. To investigate the existence and behaviour of such waves, first order perturbations in the tetrad are sufficient for the analysis. Recall that the definition of the TEGB scalar Eq.(2.32) is given to be

$$T_G = \Big[K^{\alpha}{}_{\gamma\beta}K^{\gamma\lambda}{}_{\rho}K^{\mu}{}_{\epsilon\sigma}K^{\epsilon\nu}{}_{\varphi} - 2K^{\alpha\lambda}{}_{\beta}K^{\mu}{}_{\gamma\rho}K^{\gamma}{}_{\epsilon\sigma}K^{\epsilon\nu}{}_{\varphi} + 2K^{\alpha\lambda}{}_{\beta}K^{\mu}{}_{\gamma\rho}K^{\gamma\nu}{}_{\epsilon}K^{\epsilon}{}_{\sigma\varphi} + 2K^{\alpha\lambda}{}_{\beta}K^{\mu}{}_{\gamma\rho}\left(K^{\gamma\nu}{}_{\sigma,\varphi} + \omega^{\gamma}{}_{\theta\varphi}K^{\theta\nu}{}_{\sigma} + \omega^{\nu}{}_{\theta\varphi}K^{\gamma\theta}{}_{\sigma} - \omega^{\theta}{}_{\sigma\varphi}K^{\gamma\nu}{}_{\theta}\right)\Big]\delta^{\beta\rho\sigma\varphi}_{\alpha\lambda\mu\nu}. \quad (6.117)$$

As shown previously, since the contorsion tensor is, at least, a first order contribution in the linearised gravity regime, this implies that the T_G scalar is of at least third order. Next, the field equations for $f(T, T_G)$ gravity Eq.(2.34),

$$2\left(H^{[ac]b} + H^{[ba]c} - H^{[cb]a}\right)_{,c} + 2\left(H^{[ac]b} + H^{[ba]c} - H^{[cb]a}\right)C^{d}{}_{dc}$$

$$+\left(2H^{[ac]d}+H^{dca}\right)C^{b}{}_{cd}+4H^{[db]c}C_{(dc)}{}^{a}+\left(T^{a}{}_{cd}+2\omega^{a}{}_{[cd]}\right)H^{cdb}$$
$$-h^{ab}+\left(f-Tf_{T}-T_{G}f_{T_G}\right)\eta^{ab}=0, \tag{6.118}$$

are expanded and solved order by order. Similar to the previous considerations, the $f(T,T_G)$ Lagrangian can be Taylor expanded about $T = T_G = 0$ since the background values of each scalar is null. With this in mind, one finds that the field equations up to first order can be simply reduced to

$$\eta^{ab}f(0,0)=0, \tag{6.119}$$
$$2\left(H^{(1)[ac]b}+H^{(1)[ba]c}-H^{(1)[cb]a}\right)_{,c}+2\left(H^{(1)[ac]b}+H^{(1)[ba]c}-H^{(1)[cb]a}\right)C^{(0)d}{}_{dc}$$
$$+\left(2H^{(1)[ac]d}+H^{(1)dca}\right)C^{(0)b}{}_{cd}+4H^{(1)[db]c}C^{(0)}_{(dc)}{}^{a}+2\omega^{a}{}_{[cd]}H^{(1)cdb}=0, \tag{6.120}$$

where

$$H^{(1)abc}=f_{T}(0,0)\left(\eta^{ac}K^{(1)bd}{}_{d}-K^{(1)bca}\right). \tag{6.121}$$

The zeroth order equation clearly indicates that no cosmological constant is present in the theory, maintaining consistency with the linearised gravity approach. From the definition of the coefficients $C^{c}{}_{ab}$ Eq.(2.37) and the resulting form of the spin connection Eq.(6.11), it is straightforward to show that

$$C^{(0)c}{}_{ab}=2\omega^{c}{}_{[ba]}. \tag{6.122}$$

Combining this result with Eq.(6.121) into the first order field equation Eq.(6.120), the equations simplify considerably to

$$f_{T}(0,0)\left[\partial_{c}S^{(1)abc}+S^{(1)cbd}\omega^{a}{}_{cd}+S^{(1)acd}\omega^{b}{}_{cd}+S^{(1)abc}\omega^{dc}{}_{d}\right]=0, \tag{6.123}$$

where the definition of the superpotential Eq.(2.19) has been used.

As $f_T(0,0) \neq 0$ (otherwise one does not recover TEGR as a limit and no GWs result), the first order equation reduces to the same equation obtained in the context of $f(T)$ gravity Eq.(6.50). This is expected since the TEGB term is a higher order contribution, and thus it has no effect at lower perturbation orders. This leaves the behaviour of the GWs, at least at a Minkowski background level, identical to that obtained in $f(T)$ gravity. However, the given result does not completely define the GW speed of propagation.

In the case of curvature, the introduction of higher order invariants together with (non)-minimal couplings of a scalar field has been extensively investigated in literature. As shown in Ref. [314], for a Minkowski background, the speed of the waves in a general Horndeski theory (which encompasses a large class of gravitational models) indicate the existence of extra non-tensor polarisations which do not necessarily propagate at the speed of light, whereas the tensor modes propagate at said speed. However, in a presence of a FLRW background, the waves do not necessarily propagate at the speed of light. Due to recent GW observations, this imposes strong constraints on the model, thus reducing it to that of a conformal coupling $\mathcal{L}_{\text{grav}} = \sqrt{-g}f(\phi)R$, where ϕ is some scalar field [315–319]. It would therefore be of interest to investigate whether a similar behaviour arises in the teleparallel approach, as means to constrain the viability of $f(T, T_G)$ gravity theory.

6.5 Discussion

This chapter has seen the in-depth investigation of the existence and nature of the GWs in modified teleparallel theories of gravity, namely $f(T)$, $f(T, B)$ and $f(T, T_G)$ gravity, using a linearised gravity approach. The results are summarised in Table 6.1. This linearisation procedure has been performed under two viewpoints. The first is the metric approach, where the spacetime indexed forms of the field equations have been used. In this way, the resulting quantity to solve is the perturbed metric. The second approach deals with the equations expressed purely in terms of tetrads, where the resulting GWs are generated from solving the tetrad rather than the

Teleparallel Model	Polarisations	Speed of Wave
$f(T)$	Tensor $+$, $\times$	c
$f(T,B)$	Tensor $+$, $\times$	c
	Scalar mixed b and l state	$< c$
$f(T,T_G)$	Tensor $+$, $\times$	c

Table 6.1: A summary of the different possible polarisations of the GWs arising in $f(T)$, $f(T,B)$ and $f(T,T_G)$ gravity derived using a linearised gravity procedure within a Minkowski background. Only $f(T,B)$ gravity exhibits an extra massive scalar mode, which only exists if the effective mass defined in Eq.(6.74) satisfies $|m^2| < \infty$.

metric. Despite the fact that the metric approach appears to be overall simpler to solve, especially since it does not invoke any constraint on the spin connection, it does not reveal any possible properties of the tetrad. Nonetheless, solving with the tetrad still revealed to retain the same physical results obtained from the metric formulation albeit only in a preferred frame setting as the equations are not solved for an arbitrary background tetrad and spin connection.

Starting with $f(T)$ theory, the theory does not predict any further modes besides those of GR gravitational as shown in Ref. [283]. However, when a cosmological constant is introduced, although the resulting behaviour is effectively equivalent to that of GR with a cosmological constant, the choice of the $f(T)$ function affects the effective cosmological constant $\tilde{\Lambda}$ strength defined in Eq.(6.112), thus deviating from standard behaviour. Nonetheless, as the deviation from GR is expected to be small, $f_T(0) \simeq 1$ meaning $\tilde{\Lambda}$ would not differ by much from Λ, resulting in an effect which is practically negligible for the GWs. Interestingly, a clear deviation from GR is observed for higher-order perturbations. While second order perturbations are identical to GR, meaning that the gravitational energy radiation behaves in the same way, third order perturbations differ from GR due to the presence of the $f_{TT}(0)$ contribution. It is unknown what effects this will induce unless the relevant equations are solved.

Next, in the case of $f(T,B)$ gravity, an extra polarisation mode arises, which is scalar in nature and exhibits a mix between the longitudinal and breathing polarisation states. The existence of this mode is entirely dependent on the effective mass term, meaning one can find sub-cases where this mode does not appear. Furthermore, the results from $f(R)$ gravity are recovered as expected, since this is a sub-case of $f(T, B)$ gravity. The effect of a cosmological constant has also been considered for this model, in which the complicated expression Eq.(6.114) results. This expression could not be solved in general, but it is expected that the cosmological constant affects the modes as indicative from the $f(T)$ gravity sub-case.

In the final teleparallel extension which includes the TEGB scalar, $f(T, T_G)$ gravity, it is observed that the first order perturbation is identical to that obtained in $f(T)$ gravity, or equivalently, that in GR. In other words, no extra polarisations other than the tensor polarisations exist. Such behaviour is expected as the Gauss-Bonnet teleparallel invariant is a higher order quantity and hence should contribute at higher order corrections.

Overall, $f(T, B)$ is found to be the only model which may exhibit extra polarisations. With the current observations, it appears that the GWs travel at the speed of light and exhibit two polarisations, which would put tight constraints on the form of the $f(T, B)$ Lagrangian. Furthermore, these observations should also be tested in a more general background, such as an FLRW one, as the GW speed might be different from that obtained using a linearised approach, as observed in curvature based theories. This would then impose tighter constrains especially in the $f(T, T_G)$ model if it exhibits similar behaviour to its curvature counterpart.

Chapter 7

Growth of Structure in $f(T, \mathcal{T})$ Gravity

Based on the considerations discussed in Chapter 2, the background geometry of the universe so far has been taken to be described by the spatially flat FLRW cosmology. This geometry is based on the assumption that the universe obeys the cosmological principle, meaning that it is homogeneous and isotropic. Supported by observations from the distribution of structure at large scales, and due to the very isotropic nature of the CMB temperature, the basis for considering such a cosmology is well founded.

However, small temperature fluctuations in the CMB and formation of overdense regions are clear indicators of deviations of these two assumptions, meaning that the FLRW geometry is insufficient to fully describe the universe's features. Furthermore, from a more basic conceptual perspective, even if the overall universe appears to obey the cosmological principle, it is not necessarily implied that this is the case [320]. For instance, the very early and the late future states of the universe may not be strictly isotropic and homogeneous [321, 322]. To further expand the concept of introducing anisotropy and inhomogeneity in the cosmological metric, two viewpoints are discussed.

The first is to consider a completely alternative metric to the FLRW one, such as the Lemaître-Tolman-Bondi (isotropic and inhomogeneous) and Bianchi type (anisotropic and homogeneous) metrics[20]. In doing so, this allows a more detailed investigation of the background geometry dynamics (for instance through dynamical systems), and of the various possible states the universe may experience, some of which may not be clearly evident from a strict FLRW geometry.

In order for these non-FLRW metrics to be viable, they must be able to recover the observed homogeneity and isotropy at specific epochs. This is possible through isotropisation of the parameters. Taking Bianchi metrics as an example, these can become isotropised at late times although not all of them exhibit stable behaviour. Another application arises during inflation, where a pre-inflationary universe could be anisotropic but becomes isotropised through inflation (or through some other mechanism). These alternative metrics also affect the CMB spectrum [321, 322].

The second consideration, which will be the main focus of this chapter, involves extending the weak-field approximation used in Chapter 6 by replacing the background Minkowski metric by the FLRW metric. This approach, more commonly known as cosmological perturbation theory, was first introduced by Lifshitz in 1946 [323]. Here, the inhomogeneous and anisotropic structure is assumed to be generated by small perturbations of the FLRW metric. It is these perturbations which yield the observed CMB anisotropy and large scale structure. Despite being a very successful technique, the approach does suffer from limitations, as it is not applicable when the perturbations become more pronounced (for instance, the matter overdensities at later times become significantly more dominant) while leaving the linearised regime [324].

Throughout this chapter, the large scale structure formation of CDM overdense regions shall be explored in the context of $f(T, \mathcal{T})$ gravity. This serves as an ex-

[20] See, for instance, Refs. [321, 322] for an extended review on the applications of the mentioned (and other) metrics.

tension of the investigations carried out in Refs. [127, 325–327] in the case of $f(T)$ gravity. For this model, it has been shown that structure forms similarly to that described by GR, except for a redefined Newtonian gravitational constant. Furthermore, in its curvature analogue $f(R, \mathcal{T})$ gravity, it was found that the growth becomes strongly scale dependent, contrary to what is encountered in GR and $f(T)$ gravity [328]. Thus, this chapter shall investigate whether $f(T, \mathcal{T})$ gravity exhibits the same features.

The chapter is subdivided as follows. A brief introduction on cosmological perturbation theory is first presented, where gauge invariance and the SVT decomposition are explored. Next, through a focus on the scalar perturbations, the CDM growth equation for sub-horizon scales is derived and examined under distinct regimes. The resulting growth behaviour is then explored for two $f(T, \mathcal{T})$ models derived on the assumption that the stress-energy tensor is covariantly conserved. A final discussion of the results is then given.

7.1 Cosmological Perturbations

Throughout this section, a brief overview of the necessary key ingredients to perform cosmological perturbations is presented. Effectively, this approach is an extension of the weak field approximation used in the context of GWs given in Chapter 6. Here, the main difference lies in the treatment of the background metric to represent the background cosmology, namely the FLRW metric, and the perturbed metric which will contain all the relevant information regarding the inhomogeneous and anisotropic aspect of the universe. In other words, the metric tensor is expressed simply as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{7.1}$$

with $\bar{g}_{\mu\nu}$ representing the FLRW metric and $h_{\mu\nu}$ the small inhomogeneous and anisotropic correction to the background spacetime, which satisfies $|h_{\mu\nu}| \ll |g_{\mu\nu}$. In this way, any higher order corrections shall be ignored in what follows.

As the background evolution is relatively straightforward to compute, the focus here lies in the form of the perturbation. The most general cosmologically perturbed FLRW metric, firstly introduced by Lifshitz, takes the form of [57, 61, 100, 144, 324, 329, 330]

$$ds^2 = (1 + 2\phi)dt^2 - 2a(t)B_i dx^i dt - a^2(t)[(1 - 2\psi)\delta_{ij} + 2E_{ij}]dx^i dx^j, \tag{7.2}$$

where ϕ and ψ are scalar functions (named lapse and spatial curvature respectively), B_i is a vector function also known as the shift vector, and E_{ij} is a symmetric and traceless tensor function referred to as the shear tensor [331]. Contrary to the scale factor, these functions (which correspond to 10 unknowns as expected from the metric) are dependent on both space and time, confirming their inhomogeneous and anisotropic nature. Furthermore, their magnitudes are small to conform with the linear perturbation scheme.

However, the metric tensor has not yet been expressed in a convenient form. Particularly, the vector and tensor functions can be expressed in terms of more basic components. Thanks to Helmholtz's theorem [332], this states that any vector field can be decomposed in terms of a gradient of a scalar field and a divergenceless vector field uniquely (provided certain conditions are satisfied), meaning that the vector function B_i can be expressed as

$$B_i = \partial_i B + S_i, \tag{7.3}$$

with B being the scalar component and S_i being the vector component satisfying $\partial^i S_i = 0$. In a similar analogy, the tensor function E_{ij} can be decomposed in terms of scalar, vector and tensor components according to

$$E_{ij} = \partial_{\langle i}\partial_{j\rangle}E + 2\partial_{(i}F_{j)} + \chi_{ij}, \tag{7.4}$$

with E being the scalar component, F_i representing a divergenceless vector field and χ_{ij} being a symmetric traceless divergenceless tensor. Here, the angled brackets are defined as,

$$\partial_{\langle i}\partial_{j\rangle}E \coloneqq \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2\right)E. \tag{7.5}$$

Thanks to Helmholtz's theorem, the metric can now be purely expressed in terms of SVT components. Observe that the number of degrees of freedom still remains 10, amounting to four scalars, four vectors and two tensors. The choice to opt for this form of decomposition is simple; when finding the resulting field equations, each type of perturbation decouples from one another, i.e. the evolution of each perturbation is independent from each other. This allows for the investigation of the perturbed cosmology to be solved in a more straightforward approach.

This decomposition, however, does not apply for all different cosmological spacetimes. For instance, in the case of the Lemaître-Tolman-Bondi metric, an alternative decomposition is considered, with this being composed of polar and axial perturbations, which "replace" the scalar and vector perturbations discussed previously [333]. In the case of an anisotropic Bianchi Type I spacetime, for instance, a similar SVT decomposition to that of the FLRW one applies, but requires some minor alterations in order to obtain the correct perturbed spacetime [334].

The decoupling of the perturbations leads to different physical implications for each SVT component. Briefly, these infer the following:

1. Scalar modes are capable of describing the evolution of large scale structure of the universe, which is the main purpose of this chapter. They are also important to describe the anisotropies in the CMB [144, 331, 335, 336];

2. Vector perturbations had seen less physical impact compared to the other two perturbations since, in the general context of GR and standard inflation, these perturbations decay with the universe's expansion, leaving no physical imprint [337]. However, this does not mean they are not important. If such perturbations are more dominant, these would leave an imprint in the CMB B-polarisation spectrum [331]. This measurement could be explained if inflation is sourced by more than one scalar field [338]. Furthermore, in ekpyrotic and bouncing cosmologies, the presence of a time of collapse before the "Big Bang" bounce would cause a growth in the vector perturbation, rather than a decay, which would break the homogeneity assumption in such small volume regimes [337, 339].

3. Tensor perturbations describe the primordial gravitational mode spectrum, particularly the tensor modes (plus and cross polarisations). As mentioned in Chapter 6, tensor perturbations describe various properties of the primordial waves, namely their amplitude, speed and effective mass [277]. Furthermore, this allows for the computation of the tensor power spectrum, which quantifies the strength of the stochastic background (see for instance Ref. [340]).

Clearly, the main focus will lie on the evolution of the scalar perturbations, leaving vector and tensor perturbations to not be explored further.

Before advancing, a small comment regarding higher-order SVT perturbations is necessary. The SVT decomposition procedure applies very well at first order due to its decoupling nature. However, this is not true once higher order perturbations are introduced. Indeed, the higher-order variables become dependent on cross contributions from lower order perturbations [341]. Despite this difficulty, these higher-order corrections have physical implications most notably for inflation. Contributions from second order perturbations could infer a deviation from the Gaussian nature of the CMB leading to so called non-Gaussianities. These are beyond the scope of this work, see for instance Refs. [342–344] for further details on the topic.

Focusing solely on scalar perturbations simplifies the metric to

$$ds^2 = (1+2\phi)dt^2 - 2a(t)B_{,i}dx^i dt - a^2(t)[(1-2\psi)\delta_{ij} + 2E_{,ij}]dx^i dx^j. \tag{7.6}$$

Now, one has to clearly distinguish between the background and perturbed quantities from the aspect of coordinate transformations. Focusing on the background metric, one can always define some coordinate transformation which makes the FLRW metric to appear as if it was inhomogeneous and anisotropic. This is the aspect of covariance of the metric [100, 329]. However, it does not truly represent the inhomogeneous and anisotropic structure of the universe, as it is only a mere coordinate transformation. To truly distinguish between the background and perturbed metrics, the coordinate transformations have to also represent the smallness of the perturbed metric. Similar to the procedure carried out in gravitational waves, a gauge transformation of the form[21]

$$x^\mu \to x^\mu + \xi^\mu(x^\nu) \tag{7.7}$$

for some vector ξ^μ such that $|\xi^\mu| \ll 1$ is applied to investigate how the scalar functions transform under such infinitesimal coordinate gauge transformations. Clearly, this keeps the background metric gauge-invariant and hence only the scalar perturbations change under coordinate transformations. Since only scalar perturbations are considered, ξ^μ is decomposed in terms of SVT components while keeping the scalar perturbations. This leads to the coordinate transformations

$$t \to t + \alpha, \tag{7.8}$$

$$x^i \to x^i + \partial^i \beta, \tag{7.9}$$

for some scalar functions $\alpha(t, x^i)$ and $\beta(t, x^i)$. In this way, the scalar perturbations

[21] A more detailed mathematical interpretation of the gauge coordinate transformations is given in Refs. [57, 329, 330].

transform according to [144, 331]

$$\phi \to \phi - \dot{\alpha}, \tag{7.10}$$

$$B \to B + a^{-1}\alpha - a\dot{\beta}, \tag{7.11}$$

$$E \to E - \beta, \tag{7.12}$$

$$\psi \to \psi + H\alpha. \tag{7.13}$$

As there are two coordinate degrees of freedom and four metric scalar perturbations, it is therefore always possible to eliminate two scalar perturbations by appropriate fixation of α and β. This leads to gauge fixing i.e. a preferred coordinate frame of reference which shall prove to be useful in solving the field equations. There are various gauge choices; here, only the conformal Newtonian (or also known as longitudinal) gauge is addressed. For further details regarding other gauges, see Refs. [100, 144, 329–331] and references therein.

The conformal Newtonian gauge fixes $E = B = 0$, reducing the metric to be only dependent on the gravitational potentials ϕ and ψ, which has been a very popular gauge in the study of large scale structures. Here, the term Newtonian stems from the fact that in the weak-field limit, these potentials correspond to the standard Newtonian gravitational potential $\phi = \psi = -GM/r$ [57, 345].

Although one can gauge fix the coordinates, the physical interpretation of these scalar perturbation quantities is still left unanswered as these will change between different coordinate systems (i.e. different coordinate gauge transformations). Physical quantities are independent of coordinates and hence, one should look for perturbation variables which are gauge-invariant, i.e. quantities which do not change under gauge transformations [329].

Bardeen constructed two simple gauge invariant perturbations, rightfully named Bardeen potentials, being [346]

$$\phi_{\rm B} \equiv \phi - \frac{d}{dt}\left[a^2\left(\dot{E} - a^{-1}B\right)\right], \tag{7.14}$$

$$\psi_{\rm B} \equiv \psi + a^2 H\left(\dot{E} - a^{-1}B\right). \tag{7.15}$$

From the given forms, in the conformal Newtonian gauge, the Bardeen potentials would exactly correspond to the gravitational potentials ϕ and ψ respectively. Hence, the resulting evolution equation obtained in terms of these variables would remain unchanged.

Up to this point, the SVT decomposition and the issue of gauge invariance has only been investigated in the context of the metric. However, the field equations consist of contributions from the matter sector, namely the stress-energy tensor. Hence, these properties have to be investigated there as well.

Starting with the SVT decomposition, for which only the scalar perturbations are presented, the main interest lies in the scalar perturbations of a perfect fluid. Trivially, two natural scalar perturbations arise from the energy density $\delta\rho$ and pressure δp respectively. However, other components also arise.

Any perfect fluid within a homogeneous and isotropic background has a temperature which depends solely on time and not on space. In this way, only density and pressure components arise. However, in an inhomogeneous and anisotropic background, the temperature now also depends on position, which gives rise to, momenta and anisotropic stresses, beside perturbations in density and pressure. From a Boltzmann viewpoint, once spatial effects are introduced, the distribution function is perturbed to account for the spatial dependence on temperature. For relativistic fluids, for instance, this yields monopole, dipole and quadrupole moments which correspond to energy perturbation, velocity perturbation and anisotropic stress respectively. Non-relativistic fluids (e.g. CDM) on the other hand do not give rise to anisotropic stress [61, 99, 100, 144].

An alternative simple way to characterise this description, is to include an anisotropic stress tensor $\Pi^{\mu\nu}$ and to consider velocity perturbations on u^μ. This tensor satisfies the properties $\Pi^0{}_0 = \Pi^0{}_i = u^\mu \Pi_{\mu\nu} = 0$. Furthermore, without loss of generality, this can be defined to be traceless $\Pi^i{}_i = 0$ as the anisotropic component can be redefined in the pressure component. However, this is not a necessary condition, and will thus not be considered further.

The velocity and the anisotropic stress being vectors and tensors respectively can therefore be decomposed according to the SVT decomposition. The corresponding scalar perturbation for the velocity vector is the velocity potential v [329, 330] while the scalar component of the anisotropic stress tensor is denoted by π^{S}. Overall, the perturbed stress-energy tensor components take the final form[22]

$$\begin{aligned} \delta T_0{}^0 &= \delta\rho, & \delta T_0{}^i &= (\rho + p)\,\partial^i v, \\ \delta T_i{}^0 &= -a^2 (\rho + p)\,\partial_i v, & \delta T_i{}^j &= -\delta p \delta_i^j - \partial_i \partial^j \pi^{\mathrm{S}}. \end{aligned} \tag{7.16}$$

Hence, the trace of the stress-energy tensor $\mathcal{T}$ is

$$\mathcal{T} = \rho + \delta\rho - 3(p + \delta p) - \partial^2 \pi^{\mathrm{S}}. \tag{7.17}$$

Similar to the Bardeen potentials, the scalar components of the stress-energy are not gauge invariant, and hence their behaviour changes from one coordinate system to another. In order to account for this, one should find a gauge invariant quantity which describes the inhomogeneous overdense regions in space, which is the main aim of this part of the work. This is given by the gauge-invariant comoving density,

[22] If the anisotropic stress is chosen to be traceless, then the scalar component would take the form of $\Pi_i{}^j = \partial_{\langle i}\partial_{j\rangle}\pi^{\mathrm{S}}$ [99, 101, 329, 331].

which, in conformal Newtonian gauge takes the form of [100, 101, 144, 329]

$$\delta\rho_{\mathrm{m}} \equiv \delta\rho + a^2\dot{\rho}v. \tag{7.18}$$

As a measure of the fractional evolution of structure, it is also useful to define the fractional comoving density, namely

$$\delta \equiv \frac{\delta\rho_{\mathrm{m}}}{\rho}. \tag{7.19}$$

7.2 Scalar Perturbations in $f(T, \mathcal{T})$ Gravity

Following the discussion and motivation of the previous section, all necessary tools to investigate the growth of large scale structure have been defined and can now be applied to $f(T, \mathcal{T})$ gravity. In particular, the model ansatz $f(T, \mathcal{T}) = T + F(T, \mathcal{T})$ for some arbitrary function $F(T, \mathcal{T})$ is considered, which corresponds to TEGR plus a modification. Before progressing forward, it is important to first list the background evolution equations in a flat FLRW background. From the field equations Eq.(2.38), these result in

$$(1 + F_T)\, 3H^2 + \frac{F + T}{4} + \frac{F_{\mathcal{T}}}{2}(\rho + p) = \frac{\rho}{2}, \tag{7.20}$$

$$(1 + F_T)\, \dot{H} - 12H^2\dot{H}F_{TT} + H(\dot{\rho} - 3\dot{p})\, F_{T\mathcal{T}} = -\frac{1 - F_{\mathcal{T}}}{2}(p + \rho). \tag{7.21}$$

Together they yield the modified continuity equation (or directly from Eq.(2.40))

$$(1 - F_{\mathcal{T}})\left[\dot{\rho} + 3H(\rho + p)\right] = \frac{F_{\mathcal{T}}}{2}\left(\dot{\rho} - \dot{p}\right) + (\rho + p)\left[-12H\dot{H}F_{T\mathcal{T}} + (\dot{\rho} - 3\dot{p})\, F_{\mathcal{T}\mathcal{T}}\right]. \tag{7.22}$$

As discussed in Chapter 2, the introduction of the trace of the stress-energy tensor modifies the continuity equation, which leads to the non-conservation of the stress-energy tensor, as is evident by the non-zero value of the RHS. However, it is possible to impose its conservation as a constraint as this would limit the possible gravitational Lagrangian form. For the FLRW cosmology, this imposes the constraint

$$0 = F_{\mathcal{T}}\left(\dot{p}-\dot{\rho}\right) + 6(\rho+p)\left[4H\dot{H}F_{T\mathcal{T}} + \left(\dot{p}-\frac{\dot{\rho}}{3}\right)F_{\mathcal{T}\mathcal{T}}\right]. \tag{7.23}$$

This shall be explored in further detail in Section 7.3. It is also useful to define the effective dark energy density and pressure components, namely

$$\rho_{\mathrm{DE}} := TF_T - \frac{f}{2} - F_{\mathcal{T}}(\rho+p), \tag{7.24}$$

$$p_{\mathrm{DE}} := -\rho_{\mathrm{DE}} + 4T\dot{H}F_{TT} + 2H(\dot{\rho}-3\dot{p})F_{T\mathcal{T}} + 2\dot{H}F_T - F_{\mathcal{T}}(\rho+p), \tag{7.25}$$

which together yield the dark energy EoS

$$\omega_{\mathrm{DE}} = -1 + \frac{4T\dot{H}F_{TT} + 2H(\dot{\rho}-3\dot{p})F_{T\mathcal{T}} + 2\dot{H}F_T - f_{\mathcal{T}}(\rho+p)}{TF_T - \frac{f}{2} - F_{\mathcal{T}}(\rho+p)}. \tag{7.26}$$

With all the relevant background cosmological equations derived, the focus now falls on deriving the scalar perturbed Friedmann equations in a Newtonian gauge. However, the scalar perturbed metric cannot be applied directly to derive the perturbed equations in the context of teleparallel based theories. This is based on the founding principle of the theory, since the fundamental variable is the tetrad and not the metric. Hence, one must first consider the SVT perturbations of the tetrad which would then generate the metric.

Similar to the metric tensor, the tetrad consists of a background tetrad $\bar{e}^a{}_\mu$ which describes the FLRW geometry and a perturbed one $\gamma^a{}_\mu$ which describes the inhomogeneous and anisotropic geometry, i.e. it takes the form of

$$e^a{}_\mu = \bar{e}^a{}_\mu + \gamma^a{}_\mu, \tag{7.27}$$

with $|\gamma^a{}_\mu| \ll |\bar{e}^a{}_\mu|$. For simplicity, the background tetrad is taken to be the diagonal tetrad $\bar{e}^a{}_\mu = \text{Diag}(1,a,a,a)$, thus setting the spin connection to zero. On the other hand, the most general SVT perturbed tetrad takes the form of [325, 327, 347]

$$\gamma^a{}_\mu = \begin{pmatrix} \phi & -\partial_i w - U_i \\ a(\partial_i \tilde{w} + V_i) & a\left(-\psi\delta_{ij} + \partial_{\langle i}\partial_{j\rangle}h + \epsilon_{ijk}\partial^k \tilde{h} + \partial_j c_i + \epsilon_{ijk}w^k + h_{ij}\right) \end{pmatrix}, \quad (7.28)$$

which consists of six scalars ϕ, w, $\tilde{w}$, ψ, h and $\tilde{h}$, four vectors U_i, V_i, c_i and w_k, and a tensor h_{ij}. The vectors are divergenceless, while the tensor is traceless and transverse. This amounts to a total of 16 degrees of freedom, as expected from the tetrad.

The corresponding perturbed metric tensor then takes the general form

$$g_{\mu\nu} = \begin{pmatrix} 1+2\phi & a\left[\partial_i(w+\tilde{w}) + U_i + V_i\right] \\ a\left[\partial_i(w+\tilde{w}) + U_i + V_i\right] & -a^2\left[(1-2\psi)\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}h + \partial_i c_j + \partial_j c_i + 2h_{ij}\right] \end{pmatrix}. \quad (7.29)$$

One finds some interesting properties. Firstly, the scalar $\tilde{h}$ and vector w^k do not appear in the metric tensor. However, this does not infer anything about their behaviour and are to be constrained through the field equations. Secondly, when comparing with the initial metric perturbation Eq.(7.2) in a pure SVT decomposed form, one finds the correspondence that $-2B \equiv w + \tilde{w}$, $-2S_i \equiv U_i + V_i$, $E \equiv h$, $F_i \equiv c_i$ and $\chi_{ij} \equiv h_{ij}$. This shall prove to be a crucial point when considering the longitudinal gauge. As the focus lies on scalar perturbations, the effects of the vector and tensor perturbations shall not be developed further. For more details regarding their application in the aspect of teleparallel theory, see Refs. [116,284,325,327,347].

Considering only the scalar perturbations, the metric tensor reduces to

$$g_{\mu\nu} = \begin{pmatrix} 1+2\phi & a\partial_i(w+\tilde{w}) \\ a\partial_i(w+\tilde{w}) & -a^2\left[(1-2\psi)\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}h\right] \end{pmatrix}. \quad (7.30)$$

To reduce the metric to be in the longitudinal gauge, one has to set $h = 0$ and $w + \tilde{w} = 0$. The latter constraint poses two possibilities, either $w = \tilde{w} = 0$ or $w = -\tilde{w}$ which value may or may not be zero depending on the equations of motion. This choice is of great importance in the theory and will be discussed shortly. In this gauge, the torsion scalar takes the form of

$$T = -6H^2 + 12H\left(\dot{\psi} + H\phi\right) - 4a^{-1}H\partial^2 w. \tag{7.31}$$

Clearly, the choice on w (or equivalently $\tilde{w}$) affects the value of the torsion scalar. On the other hand, the scalar perturbed field equations obtained from Eq.(2.38) result in the following

$$\begin{aligned}
E^0{}_0 : \ & (1+F_T)\left[a^{-2}\partial^2\psi - 3H\dot{\psi} - 3H^2\phi\right] + 3H^2\left[F_{TT}\delta T + F_{T\mathcal{T}}\delta\mathcal{T}\right] \\
& + \frac{F_{\mathcal{T}}}{4}\left(3\delta\rho - \delta p - \partial^2\pi^{\mathrm{S}}\right) + \frac{\rho+p}{2}\left[F_{T\mathcal{T}}\delta T + F_{\mathcal{T}\mathcal{T}}\delta\mathcal{T}\right] = \frac{1}{2}\delta\rho \qquad (7.32) \\
E^i{}_0 : \ & -(1+F_T)\partial^i\left(\dot{\psi} + H\phi\right) - \partial^i\psi\left[-12H\dot{H}F_{TT} + (\dot{\rho} - 3\dot{p})F_{T\mathcal{T}}\right] \\
& = \frac{a^2}{2}(\rho+p)(1-F_{\mathcal{T}})\partial^i v, \qquad (7.33) \\
E^0{}_i : \ & (1+F_T)\partial_i\left(\dot{\psi} + H\phi\right) - H\left(F_{TT}\partial_i\delta T + F_{T\mathcal{T}}\partial_i\delta\mathcal{T}\right) \\
& = -\frac{a^2}{2}(\rho+p)(1-F_{\mathcal{T}})\partial_i v, \qquad (7.34) \\
\mathrm{Tr}\left(E^i{}_j\right) : \ & (1+F_T)\left[H\dot{\phi} + 3H^2\phi + 3H\dot{\psi} + 2\dot{H}\phi + \ddot{\psi} - \frac{1}{3}a^{-2}\partial^2(\psi-\phi)\right] \\
& - F_{TT}\Big(3H^2\delta T + 2\dot{H}\delta T + H\delta\dot{T} + 12H^2\dot{H}\phi\Big) - \frac{F_{\mathcal{T}}}{4}\left(\delta\mathcal{T} - \frac{2}{3}\partial^2\pi^{\mathrm{S}}\right) \\
& + F_{T\mathcal{T}}\left[-\left(3H^2 + \dot{H}\right)\delta\mathcal{T} - H\delta\dot{\mathcal{T}} + (\dot{\rho} - 3\dot{p})\left(\frac{\delta T}{12H} + H\phi\right)\right] \\
& + 12H^2\dot{H}\left(F_{TTT}\delta T + F_{TT\mathcal{T}}\delta\mathcal{T}\right) - H(\dot{\rho} - 3\dot{p})\left[F_{TT\mathcal{T}}\delta T + F_{T\mathcal{T}\mathcal{T}}\delta\mathcal{T}\right] \\
& = \frac{1}{2}\left(\delta p + \frac{\partial^2\pi^{\mathrm{S}}}{3}\right), \qquad (7.35) \\
E^i{}_j,\ i\neq j : \ & a^{-2}(1+f_T)\partial_j\partial^i(\psi-\phi) + a^{-1}\partial_j\partial^i w\left[-12H\dot{H}F_{TT} + (\dot{\rho} - 3\dot{p})F_{T\mathcal{T}}\right] \\
& = (1-F_{\mathcal{T}})\partial_j\partial^i\pi^{\mathrm{S}}. \qquad (7.36)
\end{aligned}$$

where $E^{\rho}{}_{A}$ corresponds to the free indices in the field equations, $\delta T = 12H\left(\dot{\psi} + H\phi\right)$ $-4a^{-1}H\partial^2 w$ and $\delta\mathcal{T} = \delta\rho - 3(p+\delta p) - \partial^2\pi^{\mathrm{S}}$. Here, one finds a matching result between the works in Refs. [82, 127], with contributions from w and the anisotropic stress respectively. Furthermore, the scalar perturbation $\tilde{h}$ does not appear in the field equations, leaving its effect to be negligible [127, 327].

To better understand the role of w, it is useful to investigate the cases between TEGR and $f(T)$ gravity in the absence of anisotropic stresses. The TEGR perturbed field equations lead to

$$E^{0}{}_{0} : a^{-2}\partial^2\psi - 3H\dot{\psi} - 3H^2\phi = \frac{1}{2}\delta\rho, \tag{7.37}$$

$$E^{0}{}_{i} : \partial_i\left(\dot{\psi} + H\phi\right) = -\frac{a^2}{2}(\rho+p)\,\partial_i v, \tag{7.38}$$

$$\mathrm{Tr}\left(E^{i}{}_{j}\right) : H\dot{\phi} + 3H^2\phi + 3H\dot{\psi} + 2\dot{H}\phi + \ddot{\psi} - \frac{1}{3}a^{-2}\partial^2(\psi-\phi) = \frac{1}{2}\delta p, \tag{7.39}$$

$$E^{i}{}_{j},\ i\neq j : \partial_j\partial^i(\psi-\phi) = 0, \tag{7.40}$$

where the field equation $E^{i}{}_{0}$ matches with $E^{0}{}_{i}$. There are four undetermined degrees of freedom ψ, ϕ, $\delta\rho$ and v (once the fluid is specified, δp can be determined from $\delta\rho$, see for instance Ref. [99]) and four constraint equations, leading to consistency since the system is not overdetermined. In the case of $f(T)$ gravity, however, the equations become

$$E^{0}{}_{0} : (1+F_T)\left[a^{-2}\partial^2\psi - 3H\dot{\psi} - 3H^2\phi\right] + 3H^2F_{TT}\delta T = \frac{1}{2}\delta\rho, \tag{7.41}$$

$$E^{i}{}_{0} : -(1+F_T)\,\partial^i\left(\dot{\psi} + H\phi\right) + 12H\dot{H}F_{TT}\partial^i\psi = \frac{a^2}{2}(\rho+p)\,\partial^i v, \tag{7.42}$$

$$E^{0}{}_{i} : (1+F_T)\,\partial_i\left(\dot{\psi} + H\phi\right) - HF_{TT}\partial_i\delta T = -\frac{a^2}{2}(\rho+p)\,\partial_i v, \tag{7.43}$$

$$\begin{aligned}\mathrm{Tr}\left(E^{i}{}_{j}\right) : (1+F_T)&\left[H\dot{\phi} + 3H^2\phi + 3H\dot{\psi} + 2\dot{H}\phi + \ddot{\psi} - \frac{1}{3}a^{-2}\partial^2(\psi-\phi)\right]\\ &- F_{TT}\left(3H^2\delta T + 2\dot{H}\delta T + H\delta\dot{T} + 12H^2\dot{H}\phi\right)\\ &+ 12H^2\dot{H}F_{TTT}\delta T = \frac{1}{2}\delta p,\end{aligned} \tag{7.44}$$

$$E^i_{\ j},\ i\neq j:\ (1+F_T)\,\partial_j\partial^i(\psi-\phi)-12aH\dot{H}F_{T\mathcal{T}}\partial_j\partial^i w=0. \tag{7.45}$$

Thus, the undetermined degrees of freedom are now the former plus w, with five constraint equations. This would then lead to a consistent system of equations, provided that no *a priori* constraint on w is set. For instance, if $w=\tilde{w}=0$, Eq.(7.36) implies that $\psi=\phi$. Comparing Eqs.(7.34) and (7.33) would then yield the constraint

$$F_{T\mathcal{T}}\left(12H\dot{\psi}+12H^2\phi-12\dot{H}\psi\right)=0, \tag{7.46}$$

which does not appear in GR as $F_{T\mathcal{T}}\neq 0$. This means that for the remaining three undetermined degrees of freedom ($\phi=\psi$, $\delta\rho$ and v), one has four constraint equations, and this leads to an overdetermined system that can lead to inconsistencies [116, 127]. If the gauge condition $w=-\tilde{w}$ is only considered instead, without further imposing that $w=0$ *a priori* (this could still occur from an evolutionary perspective but it is not imposed), the extra constraint equation would not infer pathological issues, as an extra degree of freedom is now retained (four degrees of freedom and four constraint equations). This would then resolve this inconsistency since the equations are now able to determine all degrees of freedom [127, 325]. Motivated by this, the extra degree of freedom w shall be retained.

Besides the field equations, the existence of scalar perturbations imposes new conservation laws that arise from the perturbed stress-energy tensor. As matter couplings do not conserve the stress-energy tensor in the traditional sense, these conservation laws arise in a more general aspect. The first equation is the the perturbed continuity equation, i.e.

$$\begin{aligned}&(1-F_{\mathcal{T}})\left[\delta\dot{\rho}+3H\left(\delta\rho+\delta p+\frac{\partial^2\pi^{\rm S}}{3}\right)-3(\rho+p)\dot{\psi}+(\rho+p)\partial^2 v\right]=\\&\frac{F_{\mathcal{T}}}{2}\left(\delta\dot{\rho}-\delta\dot{p}-\partial^2\dot{\pi}^{\rm S}\right)+F_{T\mathcal{T}}\left[-2a^{-2}\partial^2\psi(\dot{\rho}-3\dot{p})+(\rho+p)\left(3H\delta T+\delta\dot{T}\right)\right.\\&\left.-12H\dot{H}(\delta\rho+\delta p)+6H\left(H\phi+\dot{\psi}\right)(\dot{\rho}-3\dot{p})+(\dot{\rho}+\dot{p})\,\delta T\right]\end{aligned}$$

$$
\begin{aligned}
&+ F_{\mathcal{T}\mathcal{T}}\left[(\rho+p)\left(3H\delta\mathcal{T}+\dot{\mathcal{T}}\right) + \frac{1}{2}\left(3\delta\rho-\delta p-\partial^2\pi^{\mathrm{S}}\right)(\dot{\rho}-3\dot{p}) + (\dot{\rho}+\dot{p})\,\delta\mathcal{T}\right] \\
&- 12H\dot{H}\,(\rho+p)\left(F_{TT\mathcal{T}}\delta T + F_{T\mathcal{T}\mathcal{T}}\delta\mathcal{T}\right) + (\rho+p)\,(\dot{\rho}-3\dot{p})\left(F_{T\mathcal{T}\mathcal{T}}\delta T + F_{\mathcal{T}\mathcal{T}\mathcal{T}}\delta\mathcal{T}\right).
\end{aligned}
\tag{7.47}
$$

Due to the velocity components of the fluid, a new second conservation law arises, yielding the so called Euler equation, which describes the conservation of momentum of the fluid

$$
\begin{aligned}
&(1-F_{\mathcal{T}})\left\{(\rho+p)\left[a^2\partial_i\dot{v}+2a^2H\partial_i v+\partial_i\phi\right]+a^2\dot{p}\partial_i v+\partial_i\delta p+\partial_i\partial^2\pi^{\mathrm{S}}\right\} \\
&= -\frac{1}{2}F_{\mathcal{T}}\left[a^2(\dot{\rho}-\dot{p})\,\partial_i v+\partial_i\left(\delta\rho-\delta p-\partial^2\pi^{\mathrm{S}}\right)\right].
\end{aligned}
\tag{7.48}
$$

7.2.1 Derivation of the Growth Structure Equation

So far, all the relevant zeroth and first order scalar perturbed equations have been derived. The next step is to make use of these equations in order to investigate the evolution of matter overdensities. As discussed in Ref. [144], a simple equation that can characterise the growth of structure δ is expected to be in the form of

$$
\ddot{\delta} + [\text{Pressure} - \text{Gravity}]\,\delta = 0. \tag{7.49}
$$

This follows from the Newtonian treatment of gravitational instability of interstellar structures, firstly formulated by Jeans in 1902 [348]. The role of gravity is to collapse the source, and hence it allows for the formation of overdense regions throughout the universe. On the other hand, the role of pressure is to counteract and therefore stabilise the possible collapse of the structure. If pressure is sufficiently dominant, then no overdense regions can form.

Besides these two key components, one also has to factor in the effect of the relative expansion of the universe. This should act as a dampening effect, since expansion is causing the particles to be further apart from one another. This is causing the effect of gravity to be suppressed. From a Newtonian description, this leads to [1, 61, 349, 350]

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\rho\delta - \frac{{c_s}^2}{a^2}k^2\delta, \tag{7.50}$$

where ${c_s}^2 := \frac{\partial\rho}{\partial p}$ is the adiabatic sound speed of the fluid and k is the wavenumber. Here, the gravitational constant is reintroduced for the sake of clarity. Trivially, one can identify some key components on the evolution by comparing it to the behaviour of a damped harmonic oscillator. The Hubble parameter acts as a damping term. On the other hand, the frequency of the system is sourced by a gravitational term $4\pi G\rho$ and a pressure term $\frac{{c_s}^2k^2}{a^2}$. Depending on the length scale k, one of the terms dominates and thus generates or suppresses the formation of structure. This is known as the Jean's criterion for gravitational instability.

Formally formulated in a static universe, for $k^2{c_s}^2 > 4\pi G\rho$ (pressure dominated), the system starts to oscillate, leading to no growth. On the other hand, for $k^2{c_s}^2 < 4\pi G\rho$ (gravity dominated), gravitational instability occurs and overdense regions form.

In a relativistic treatment, a similar equation does result, but one has to include the contributions of the different species which constitute the universe, namely baryonic matter, CDM, photons and neutrinos. Furthermore, one also has to define a sense of a length scale of interest. Starting with the latter, it is useful to define the comoving distance η, which measures the distance travelled by photons from the beginning of time $t = 0$ to some time t (or equivalently, from $a = 0$ if it starts from a Big Bang to some scale factor a). This is defined through [144]

$$\eta \equiv c\int_0^t \frac{dt'}{a(t')} = c\int_0^a \frac{da'}{a'}\frac{1}{a'H(a')}. \tag{7.51}$$

where the speed of light is reintroduced for simplicity. Here, t' and a' are integration variables. In this way, any two particles could have been in casual contact if their distance is within a comoving distance. However, they could not have been in casual contact if their distance is larger than the comoving distance as these must exceed the speed of light. Therefore, any relevant structure formation should form within scales falling inside a comoving distance.

In terms of the wavenumber k, which is roughly related to the inverse wavelength λ of the mode $k \sim \lambda^{-1}$, the modes of interest must have wavelengths smaller than the comoving distance to ensure the particles have entered in casual contact leading to the condition $k\eta \gg 1$ (or as an order approximation $k \gg aH$). Such modes are called sub-horizon modes (corresponding to large scales). Conversely, those which lie outside this regime are called superhorizon $k\eta \ll 1$, but they are not of interest here.

Next, one should classify which overdense region is to be investigated. As large scales are of interest, the main focus lies in the structure of CDM regions. However, one should also account for the effect of other species. Starting with radiation-dominated epochs, most particles are relativistic and hence exhibit a large radiation pressure (and non-zero anisotropic stress). For the scales of interest, this leads to an oscillatory behaviour as indicated by Eq.(7.50) [350]. The baryonic matter exhibits a similar behaviour since, during early times, these are tightly coupled to the photons and hence also oscillate [144, 330]. CDM on the other hand experiences little growth as it is less dominant than the background radiation. Furthermore, radiation causes a large expansion, thus a larger Hubble friction [61, 350]. In fact, according to Eq.(7.50), $\delta \propto \ln a$.

Once the universe enters a matter-dominated phase, the situation changes. First, the photons and neutrinos have long decoupled during these epochs leading to barely any contribution to the anisotropic stress [99, 100]. In other words, there is no growth evolution of any radiative sources. Second, the baryons have now been decoupled from the photons and therefore have become non-relativistic. However, by the time they do achieve this state, CDM would have already started to dominate the evolution of the potential since the period of matter-radiation equality. It is afterwards that baryons then grow similar to CDM [61, 144]. Therefore, the only major source of growth stems from CDM (and later by baryonic matter) which according to Eq.(7.50) yields a linear growth $\delta \propto a$. This is known as the Mészáros effect, first derived by P. Mészáros in 1974 [351].

Lastly, the universe enters a dark energy-dominated phase. During these epochs, the effects of CDM, radiation and baryonic matter are practically negligible compared to dark energy. This leads to barely any growth, which is understandable due to the fact that during such times, the scale factor is expanding at an exponential rate, causing the Hubble friction to be significantly dominant. In fact, from Eq.(7.50), the fractional growth does not grow since δ becomes constant [144].

With these considerations, in order to investigate the growth formation for CDM (which sets the pressure components $p = \delta p = 0$), epochs will within the matter domination periods are considered. Furthermore, the absence of anisotropic stress is assumed. Since models which obey the conservation of the stress-energy tensor are assumed, they shall be considered throughout the derivation. In general, where this condition does not necessarily hold, the result as derived in Ref. [352] is obtained. However, the conclusions regarding the resulting growth evolution remain unchanged.

As per above motivations, the growth equation for large scale structure can now be derived, for which the method described in Ref. [127] is followed. It is first remarked that in this case, the gauge invariant fractional matter perturbation density Eq.(7.19) becomes

$$\delta_{\mathrm{M}} = \frac{\delta\rho}{\rho} - 3Ha^2 v. \tag{7.52}$$

From Eqs.(7.33) and (7.34), the relation

$$F_{T\mathcal{T}}\left(12H\dot{\psi} + 12H^2\phi - 12\dot{H}\psi - 4a^{-2}k^2\xi\right) = -F_{\mathcal{T}\mathcal{T}}\left(\delta\rho - 3\rho\psi\right), \tag{7.53}$$

is obtained where the quantity $\xi := aHw$ has been defined. To eliminate ϕ from the equation, Eq.(7.36) can be used to yield

$$\phi = \psi - \frac{3\xi}{1+F_T}\left(4\dot{H}F_{TT} + \rho F_{T\mathcal{T}}\right), \tag{7.54}$$

which, when combined with the former equation, results in

$$4\xi F_{TT} = \frac{\rho F_{T\mathcal{T}}\left(\delta_{\mathrm{M}} + 3Ha^2 v - 3\psi\right) + F_{TT}\left[12H\dot{\psi} + 12\psi\left(H^2 - \dot{H}\right)\right]}{\frac{k^2}{a^2} + \frac{9H^2}{1+F_T}\left(4\dot{H}F_{TT} + \rho F_{T\mathcal{T}}\right)}. \tag{7.55}$$

In the sub-horizon scale approximation, the expression simplifies to

$$4\xi F_{TT} \approx \frac{a^2}{k^2}\left\{\rho F_{T\mathcal{T}}\left(\delta_{\mathrm{M}} + 3Ha^2 v - 3\psi\right) + F_{TT}\left[12H\dot{\psi} + 12\psi\left(H^2 - \dot{H}\right)\right]\right\}. \tag{7.56}$$

As an order approximation, the expression reduces to the more simple form

$$\xi \sim \frac{a^2H^2}{k^2}\left(\delta_{\mathrm{M}} + a^2 Hv + \psi\right) \tag{7.57}$$

where the presence of $\delta_{\mathrm{M}} + a^2Hv$ holds provided $F_{T\mathcal{T}} \neq 0$ and $F_{TT} \neq 0$. This expression can be simplified further through the use of the Euler equation Eq.(7.48), which for CDM, becomes

$$\left(\dot{v} + 2Hv + \frac{\phi}{a^2}\right)\left(1 - F_{\mathcal{T}}\right) = -\frac{F_{\mathcal{T}}}{2a^2}\delta_{\mathrm{M}}. \tag{7.58}$$

As an order approximation, one has $a^2Hv + \phi \sim \delta_{\mathrm{M}}$ where δ_{M} only appears for $F_{\mathcal{T}} \neq 0$. From Eq.(7.54), the order expression becomes $\delta_{\mathrm{M}} + a^2Hv \sim \psi + \xi$ which reduces the sub-horizon limit expression to

$$\xi \sim \frac{a^2H^2}{k^2}\left(\xi + \psi\right) \ll \psi. \tag{7.59}$$

Therefore, by definition of ϕ Eq.(7.54), the gravitational potentials are of the same order of magnitude $\phi \simeq \psi$.

Observe that throughout this approximation, certain functional constraints have been considered, namely $F_{\mathcal{T}}$, $F_{T\mathcal{T}}$, $F_{TT} \neq 0$. Irrespective of functional constraints, however, the result remains unchanged. Starting with the case when $F_{TT} = 0$, then from Eq.(7.53), either $F_{T\mathcal{T}} = 0$ or $\delta\rho = 3\rho\psi$. For the former case, by Eq.(7.54), this means $\phi = \psi$ identically as required. Note that, in this case, the Lagrangian reduces to the model $F(T,\mathcal{T}) = \alpha + \beta T + g(\mathcal{T})$, where α and β are integration constants and g is some arbitrary function. The said model shall be considered later on during the growth analysis.

In the second case, Eq.(7.54) simplifies to

$$\phi = \psi - \frac{3\xi\rho F_{T\mathcal{T}}}{1+F_T}. \tag{7.60}$$

From Eq.(7.20), in the sub-horizon limit together with the above expression yields the simple relationship

$$\psi = \frac{2\xi\rho F_{T\mathcal{T}}}{1+F_T}, \tag{7.61}$$

which therefore implies that $\psi = -2\phi$. This condition, however, does not measure up with experiments. As discussed in Refs. [353, 354], Solar system tests, particularly Shapiro's time delay effect, tightly constrains the difference in the potentials to be [355]

$$\frac{|\psi - \phi|}{\phi} < 2 \times 10^{-5}, \tag{7.62}$$

while the above constraint sets a fixed value of 3, well beyond the observed constraint. An analysis carried out in the context of large scale structure also suggests that, despite being a weaker constraint, this difference should be relatively small i.e. [356]

$$\left|\frac{\phi - \psi}{\psi}\right| \leq 0.5. \tag{7.63}$$

Therefore, this case is discarded.

If $F_{TT} \neq 0$ and $F_{T\mathcal{T}} = 0$, Eq.(7.57) leads to the immediate result $\xi \sim \frac{a^2 H^2}{k^2}\psi \ll \psi$, as obtained in Ref. [127], thus leaving the result unchanged, irrespective of the value of $F_{\mathcal{T}}$. Therefore, all the different conditions lead to the same result that $\phi \simeq \psi$.

With this sub-horizon limit approximation, one can then proceed to derive the evolution equation for the gauge-invariant fractional overdensity $\delta_{\rm M}$. This process should start by combining Eqs.(7.32) and (7.34) while introducing the definition of $\delta_{\rm M}$ results into

$$\left(\frac{1}{2} - \frac{3F_{\mathcal{T}}}{4} - \frac{\rho F_{\mathcal{T}\mathcal{T}}}{2}\right)\delta_{\rm M} = \left(1 + F_T\right)\frac{k^2\psi}{a^2\rho} + \frac{1}{2}\Bigg\{F_{T\mathcal{T}}\left[12H\left(\dot{\psi} + H\phi\right) - 4a^{-1}Hk^2 w\right] + 3Ha^2 v\rho F_{\mathcal{T}\mathcal{T}}\Bigg\} + \frac{3F_{\mathcal{T}}}{4}a^2 Hv. \tag{7.64}$$

Since $\psi \simeq \phi$ and $\frac{k^2\psi}{a^2H^2} \gg \psi$, the equation simplifies to

$$A\delta_{\rm M} = \left(1 + F_T\right)\frac{k^2\psi}{a^2\rho} + \frac{3Ha^2v}{2}\rho F_{\mathcal{T}\mathcal{T}} + \frac{3F_{\mathcal{T}}}{4}a^2Hv, \tag{7.65}$$

where the quantity $A := \frac{1}{2} - \frac{3}{4}F_{\mathcal{T}} - \frac{1}{2}F_{\mathcal{T}\mathcal{T}}\rho$ has been conveniently defined. From Eq.(7.33), in the sub-horizon limit, the order relation $H\phi \sim \rho a^2 v$ is obtained, meaning the expression simplifies further to

$$A\delta_{\rm M} = \left(1 + F_T\right)\frac{k^2\psi}{a^2\rho}. \tag{7.66}$$

Taking the time derivative and using Eq.(7.33) once more yields

$$\dot{A}\delta_{\rm M} + A\dot{\delta}_{\rm M} = \frac{k^2 v}{2}\left(F_{\mathcal{T}} - 1\right). \tag{7.67}$$

Differentiating once again with time and using Euler's equation Eq.(7.58) and the previous expression for the velocity yields the final result of

$$A\ddot{\delta}_{\rm M} + \left[2\dot{A} + 2AH - \frac{3AH}{1-F_{\mathcal{T}}}\left(4\dot{H}F_{T\mathcal{T}} + \rho F_{\mathcal{T}\mathcal{T}}\right)\right]\dot{\delta}_{\rm M}$$
$$+\left[\ddot{A} + 2\dot{A}H - \frac{3\dot{A}H}{1-F_{\mathcal{T}}}\left(4\dot{H}f_{T\mathcal{T}} + \rho F_{\mathcal{T}\mathcal{T}}\right) - \frac{1-F_{\mathcal{T}}}{2(1+F_T)}A\rho - \frac{k^2F_{\mathcal{T}}}{4a^2}\right]\delta_{\rm M} = 0. \tag{7.68}$$

Instead of investigating the evolution of the matter overdensities with time, it is more convenient to analyse its behaviour with scale factor, thus yielding the expression

$$\delta''_{\rm M}(a) + \frac{1}{aAH}\left[aAH' + 2A'aH + 3AH - \frac{3AH}{1-F_{\mathcal{T}}}\left(4H'aHF_{T\mathcal{T}} + \rho F_{\mathcal{T}\mathcal{T}}\right)\right]\delta'_{\rm M}(a)$$
$$+\frac{1}{Aa^2H^2}\left[a^2H^2A'' + 3A'aH^2 + A'a^2HH' - \frac{3A'aH^2}{1-F_{\mathcal{T}}}\left(4H'aHF_{T\mathcal{T}} + \rho F_{\mathcal{T}\mathcal{T}}\right)\right.$$
$$\left.-\frac{1-F_{\mathcal{T}}}{2(1+f_T)}A\rho - \frac{k^2F_{\mathcal{T}}}{4a^2}\right]\delta_{\rm M}(a) = 0. \tag{7.69}$$

Clearly, the TEGR limit $A = \frac{1}{2}$ and $F(T, \mathcal{T}) = 0$ is recovered as desired. In general, this equation would yield two solutions for $\delta_{\rm M}$ referred to as the growing and decaying modes, which, as their name implies, describe the growing and decaying solutions respectively. As the interest lies in the formation of structure, only the growing solution shall be investigated. This is usually referred to as the growth factor, which is denoted by $D(a)$ [144].

7.2.2 Interpreting the Growth Structure Equation

When studying the evolutionary behaviour of the growth equation, it is evident that the equation is identical to that given by a damped harmonic oscillator. Generally, such an oscillator obeys the differential equation [357–359]

$$x''(t) + 2\zeta\omega_0 x'(t) + {\omega_0}^2 x = 0, \tag{7.70}$$

where ζ is the damping ratio and ω_0 is the natural frequency of the system. The former leads to underdamping ($0 < \zeta < 1$), critical damping ($\zeta = 1$) and underdamping ($\zeta > 1$) behaviours, while the special case of negative damping ($\zeta < 0$) causes the system to oscillate with an ever increasing amplitude [359]. In the absence of damping ($\zeta = 0$), for real values of ω_0, the system oscillates with frequency ω_0, while for complex values, the system grows exponentially with time.

The natural frequency and damping ratio take the form of

$$\begin{aligned}{\omega_0}^2 \equiv & \frac{1}{Aa^2H^2}\left[a^2H^2A'' + 3A'aH^2 + A'a^2HH' - \frac{3A'aH^2}{1-F_{\mathcal{T}}}\left(4H'aHF_{T\mathcal{T}} + \rho F_{\mathcal{T}\mathcal{T}}\right)\right. \\ & \left. - \frac{1-F_{\mathcal{T}}}{2(1+f_T)}A\rho - \frac{k^2 F_{\mathcal{T}}}{4a^2}\right], \end{aligned} \tag{7.71}$$

$$2\zeta\omega_0 \equiv \frac{1}{aAH}\left[aAH' + 2A'aH + 3AH - \frac{3AH}{1-F_{\mathcal{T}}}\left(4H'aHF_{T\mathcal{T}} + \rho F_{\mathcal{T}\mathcal{T}}\right)\right]. \tag{7.72}$$

when the large scale growth equation Eq.(7.69) is compared to the behaviour of a damped harmonic oscillator. To investigate their physical implications, particular Lagrangian limits are considered.

In the sub-case of $F(T)$ gravity, the model is practically identical to TEGR, except for a minor modification in the definition of the frequency due to the introduction of the F_T coefficient

$${\omega_0}^2 = -\frac{1}{1+F_T}\frac{2\rho}{a^2H^2}. \tag{7.73}$$

If the Newtonian gravitational constant G is reintroduced throughout the derivation, one will find that an effective gravitational strength can be defined as

$$G_{\text{eff}} \equiv \frac{G}{1+F_T}. \tag{7.74}$$

Therefore, the role of F_T determines whether the effect of gravity is stronger or weaker in the formation of structure. Nonetheless, the value of F_T is not expected to deviate much from the standard value of the Newtonian gravitational constant, thus leaving the value of $|F_T| \ll 1$. In this way, the results obtained from the TEGR description remain unchanged. However, in general, this argument would not hold true for models in which the magnitude is significantly dominant. For instance, when $F_T < -1$, the system experiences an oscillatory growth behaviour. It is remarked that the above results agree with the large scale equation derived in Refs. [127, 325–327] while the effective constant is also in agreement with Refs. [181, 347]. As various $F(T)$ models have been extensively studied, only the role of $\mathcal{T}$ shall be explored.

For more general arbitrary choices for the Lagrangian $F(T, \mathcal{T})$ which does not reduce to either GR or $F(T)$ gravity, the evolutionary behaviour changes considerably. The natural frequency is generally strongly dependent on the wavenumber k, something which does not appear in the latter cases. This can cause strong constraints on the theory since it alters the prediction for the matter power spectrum, which provides a measure of the number of overdense regions with scale (see for instance Refs. [100, 144] for further details). Such behaviour is also observed in other modified theories of gravity.

Taking the $f(R)$ theory of gravity as an example, the effective Newtonian constant takes the form of[23]

$$G_{\text{eff}} = \frac{G}{f_R} \frac{1 + 4\frac{k^2 f_{RR}}{a^2 f}}{1 + 3\frac{k^2 f_{RR}}{a^2 f}}. \tag{7.75}$$

The large scale evolution then has two distinct behaviours. At earlier times, the quantity $\frac{k^2 f_{RR}}{a^2 f} \gg 1$ which leads to an effective gravitational constant to be $G_{\text{eff}} \to \frac{4G}{3f_R}$ leading to a stronger gravitational affect and hence more growth. At later times, the situation changes $\frac{k^2 f_{RR}}{a^2 f} \ll 1$ leading to $G_{\text{eff}} \to \frac{G}{f_R}$ similar to $f(T)$ gravity. During transient periods, the dependence of k becomes more prominent and this affects the shape of the power spectrum [361–363].

[23] In general, as discussed in Ref. [360], the form of the effective gravitational constant is different from the one listed here for more general $f(R)$ models which could lead to deviations, especially at late times. However, if the model obeys Solar System tests i.e. $|f_R| \ll 1$, the effective constant reduces to the one listed here.

A similar strong k^2 dependence is also obtained in the curvature analogue of $f(T, \mathcal{T})$ theory, $f(R, \mathcal{T})$ gravity, as the comoving scale becomes dominant at all scales in the sub-horizon regime leading to disagreements with observations [328]. Therefore, for viable models, this effect should be suppressed. This is theoretically possible provided $F_{\mathcal{T}} \simeq 0$ once the universe enters a matter dominated phase.

The k^2 dependence leads to the second point, where the frequency is now not necessarily complex. Similar to the Jeans' instability criterion, this situation suggests that there exists a threshold comoving scale which causes the growth of structure to become oscillatory, leading to a halt in structure formation.

Overall, the above discussions infer the foreseeable issues of the $F(T, \mathcal{T})$ gravity prediction. This shall be shown explicitly for particular Lagrangian functions obeying the stress-energy conservation criterion where the k-dependence clearly leads to deviations from observations.

7.3 Constraints on the $f(T, \mathcal{T})$ function

As mentioned in Section 7.2, the requirement for the stress-energy tensor to be conserved leads to a constraint on the Lagrangian. Since the large scale structure has been investigated for times well within matter domination epochs with the only matter source being CDM, the background stress-energy trace is reduced to $\mathcal{T} = \rho$. Thus, the continuity equation Eq.(7.22) becomes

$$F_{\mathcal{T}} + 8\dot{H}F_{T\mathcal{T}} + 2\mathcal{T}F_{\mathcal{T}\mathcal{T}} = 0. \qquad (7.76)$$

Note that the continuity equation can be expressed purely as a PDE in terms of T and $\mathcal{T}$ by replacing $\dot{H}$ using the modified Friedmann equation Eq.(7.21) which results in

$$F_{\mathcal{T}} + 2\mathcal{T}F_{\mathcal{T}\mathcal{T}} + \frac{2F_{T\mathcal{T}}}{1 + 2TF_{TT} + F_T}\left(T - F + 2TF_T - 2T\mathcal{T}F_{T\mathcal{T}}\right) = 0. \tag{7.77}$$

It is remarked that $1+2TF_{TT}+F_T \neq 0$, otherwise it leads to the solution $F(T,\mathcal{T}) = -T + \alpha\sqrt{-T} + g(\mathcal{T})$ for some constant α corresponding to the boundary term and arbitrary function $g(\mathcal{T})$. Effectively, this reduces the gravitational Lagrangian to only be sourced by the trace of the stress-energy, thus removing any torsional gravitational effects that would make the model unrealistic.

Despite the relatively simple form, this does not yield any general analytic solutions, and hence must be solved for specific ansatz choices. Trivially, any function solely expressed in terms of torsion satisfies the constraint as expected from $f(T)$ gravity since the stress-energy tensor is conserved in such models. Therefore, non-trivial contributions from the trace must be considered if its effect on the growth rate is to be explored. As shown in Refs. [94, 95], two such models can be obtained.[24]

7.3.1 Model I: $F(T,\mathcal{T}) = g(\mathcal{T})$

In the first model, the modification is assumed to be solely sourced by matter, namely $F(T,\mathcal{T}) = g(\mathcal{T})$ for some function g. Here, one finds the solution $g(\mathcal{T}) = \alpha\sqrt{\mathcal{T}}+\beta$, where α and β are integration constants with β representing a cosmological constant.[25] The latter constants can be constrained from the Friedmann equation by evaluating at present times. Indeed, substituting in Eq.(7.20) yields

$$\alpha = \sqrt{\frac{3}{{H_0}^2\Omega_{\mathrm{M},0}}}\left[{H_0}^2\left(\Omega_{\mathrm{M},0} - 1\right) - \frac{\beta}{6}\right]. \tag{7.78}$$

[24]Other ansatz models besides the ones explored in Refs. [94, 95] have been considered as summarised in Ref. [352], being $F(T,\mathcal{T}) = \mathcal{T}g(T)$ for some unknown function $g(T)$ and $F(T,\mathcal{T}) = T^n\mathcal{T}^m$ for real constants n and m. However, no new solutions have been obtained.

[25]In Refs. [94,95], the model is generalised for an arbitrary perfect fluid having a constant EoS ω.

Since β represents the cosmological constant, it is useful to parametrise this constant in terms of another constant variable ϵ, which measures the deviation from ΛCDM via

$$\beta \equiv 6{H_0}^2(\Omega_{\mathrm{M},0} - 1 + \epsilon). \tag{7.79}$$

Indeed, for $\epsilon = 0$, β reduces to the ΛCDM value and $\alpha = 0$ reduces the Lagrangian to standard TEGR with a cosmological constant as expected. This way, the Friedmann equation Eq.(7.20) takes the simple form of

$$H^2 = {H_0}^2\left(\frac{\Omega_{\mathrm{M}}}{a^3} - \Omega_{\mathrm{M}} + 1 - \epsilon + \frac{\epsilon}{a^{3/2}}\right). \tag{7.80}$$

It is remarked that despite being in a curvature based setting, an identical solution is obtained in $f(R,\mathcal{T})$ theories for an additive model ansatz $f(R,\mathcal{T}) = g(R) + h(\mathcal{T})$ [89–91,328]. Nonetheless, the cosmological implications are only identical for $g(R) = R$, otherwise, the theories are fundamentally distinct.

In order to constrain the parameter ϵ, a similar procedure to that used in Chapter 3 to constrain the $f(T)$ models is employed. This requires the present values of the EoS of the dark energy fluid and of the deceleration parameter to be $\omega_{\mathrm{DE},0} \approx -1$ and $q_0 \sim -0.5$ respectively.

In the case of the EoS for this model, its present value is found to be

$$\omega_{\mathrm{DE},0} = -\frac{2\epsilon - 4(\Omega_{\mathrm{M},0} + \epsilon - 1)}{3\epsilon - 4(\Omega_{\mathrm{M},0} + \epsilon - 1)} \tag{7.81}$$

from Eq.(7.26), from which the following observations are noted. For $\epsilon < 0$ or $\epsilon > 4 - 4\Omega_{\mathrm{M},0}$, $\omega_{\mathrm{DE},0} < -1$ meaning it behaves as a phantom fluid. The case $\epsilon = 0$ gives a cosmological constant (which is true for all times) as expected. For $0 < \epsilon < 2 - 2\Omega_{\mathrm{M},0}$, the fluid behaves as quintessence while for $\epsilon = 2 - 2\Omega_{\mathrm{M},0}$, the fluid behaves as dust. Lastly, for $2 - 2\Omega_{\mathrm{M},0} < \epsilon < 4 - 4\Omega_{\mathrm{M},0}$, the EoS is positive. As observations indicate a value close to that of a cosmological constant, any reasonable solution should be in a domain close to $\epsilon = 0$ as for $\epsilon > 4 - 4\Omega_{\mathrm{M},0}$, the EoS is reasonably smaller from that observed, unless the value of ϵ is reasonably large.

The deceleration parameter, on the other hand, evolves with scale factor as

$$q(a) = -1 + \frac{3}{4}\frac{a^{3/2}\epsilon + 2\Omega_{\mathrm{M},0}}{a^{3/2}\epsilon - a^3(\Omega_{\mathrm{M},0} + \epsilon - 1) + \Omega_{\mathrm{M},0}}. \tag{7.82}$$

In order to obtain an observed acceleration today, the condition

$$\epsilon < \frac{1}{3}(4 - 6\Omega_{\mathrm{M},0}) \tag{7.83}$$

must be satisfied which, using the observed values, requires $\epsilon < \frac{11}{15}$. Given the constraint on ϵ based on the EoS analysis, this condition is trivially satisfied. Note that the obtained constraint also falls in line with the results obtained in Ref. [95], where for $\Omega_{\mathrm{M},0} = 0.3$, using SNe Ia data, the parameter is constrained to lie in the range of $-\frac{5}{36} \leq \epsilon \leq -\frac{1}{36}$ $(-1.04726 \leq \omega_{\mathrm{DE},0} \leq -1.00982)$.

7.3.2 Model II: $F(T,\mathcal{T}) = Tg(\mathcal{T})$

The second ansatz choice assumes a TEGR rescaling, namely $F(T,\mathcal{T}) = Tg(\mathcal{T})$ which yields the solution

$$g(\mathcal{T}) = -1 - \left(\frac{\mu}{\sqrt{\mathcal{T}}} + \nu\right)^{-1}, \tag{7.84}$$

with μ and ν being integration constants. It is worth noting that when $\mu = 0$, ν serves as a constant TEGR rescaling with the special case $\nu = -1$ reducing the Lagrangian to standard TEGR. Substituting into the Friedmann equation Eq.(7.20) and evaluating at presents times yields the constraint

$$H_0{}^2 = -\frac{1}{3\nu}\left(\mu + \nu\sqrt{3H_0{}^2\Omega_{\mathrm{M},0}}\right)^2. \tag{7.85}$$

This infers two implications, the first being that $\nu < 0$ (otherwise no real solution is obtained), while the second being that this is a quadratic expression in the integration constants. The latter implies that one of the integration constants is a spurious degree of freedom. Without loss of generality, ν is found in terms of μ to be

$$\nu = -\frac{3H_0 + 2\mu\sqrt{3\Omega_{\mathrm{M},0}} \pm \sqrt{9{H_0}^2 + 12H_0\mu\sqrt{3\Omega_{\mathrm{M},0}}}}{6H_0\Omega_{\mathrm{M},0}}. \tag{7.86}$$

To guarantee that the values for ν are real, a constraint on μ is required, being

$$9{H_0}^2 + 12H_0\mu\sqrt{3\Omega_{\mathrm{M},0}} \geq 0 \implies \mu \geq -\frac{1}{4}\sqrt{\frac{3{H_0}^2}{\Omega_{\mathrm{M},0}}}. \tag{7.87}$$

For simplicity, μ is redefined to be

$$\mu \equiv -\frac{\eta}{4}\sqrt{\frac{3{H_0}^2}{\Omega_{\mathrm{M},0}}} \tag{7.88}$$

for some new constant parameter η. The above constraint therefore implies $\eta \leq 1$. In this way, the expression for ν Eq.(7.86) takes the much simpler form of

$$\nu = \frac{\eta - 2 \pm 2\sqrt{1-\eta}}{4\Omega_{\mathrm{M},0}}. \tag{7.89}$$

Observe that the constraint $\eta \leq 1$ guarantees that the requirement $\nu < 0$, is always satisfied. The only exception occurs when the positive solution is considered with $\eta = 0$ which is not permitted as $\nu = 0$ in this instance. Finally, this simplifies the Friedmann equation to be

$$\frac{H}{H_0} = a^{-3/2} + \frac{\eta(a^{-3/2} - 1)}{-2 \pm 2\sqrt{1-\eta}}. \tag{7.90}$$

Similar to the previous model, the parameter η is constrained using present values of the dark energy EoS and the deceleration parameter. In the case of ω_{DE}, its present value obtained from Eq.(7.26) is given to be

$$\omega_{\mathrm{DE},0} = \frac{\eta\left(\Omega_{\mathrm{M},0} - 1 \mp \sqrt{1-\eta}\right)}{2\left(1 \pm \sqrt{1-\eta}\right)^2 + \Omega_{\mathrm{M},0}\eta - 4\Omega_{\mathrm{M},0}\left(1 \pm \sqrt{1-\eta}\right)}. \tag{7.91}$$

Taking $\Omega_{\mathrm{M},0} = 0.3$, the following constraints are obtained.

For the positive solution, the fluid exhibits phantom behaviour for $\eta < 0$ or $0.58 < \eta < 0.84$, it behaves as a cosmological constant for $\eta = 0.84$, while quintessence behaviour is attained for $0 < \eta < 0.51$ or $0.84 < \eta \leq 1$. The fluid behaves as dust for $\eta = 0.51$ and becomes positive for $0.51 < \eta < 0.58$. Thus, the model can compare with physical data within regions close to $\eta = 0$ and $\eta = 0.84$.

In the second case, for the negative solution, the EoS can never behave as a phantom fluid. In fact, the EoS behaves as quintessence for $0 < \eta \leq 1$, dust for $\eta = 0$ (as expected since this corresponds to TEGR rescaling) and positive for $\eta < 0$. Furthermore, it is observed that $\omega_{\mathrm{DE},0} \geq -\frac{7}{11}$, which is far from any observed value. Therefore, the negative solution cannot realise the observational constraint.

Moving on to the deceleration parameter, its present value is given to be

$$q_0 = \frac{1}{2} + \frac{3\eta}{4\left(-1 \pm \sqrt{1-\eta}\right)}. \tag{7.92}$$

To recover acceleration, for the positive solution, any value of $\eta \neq 0$ causes a present time acceleration. However, in order to match with the present time value of $q_0 \sim 0.5$, the value is to be constrained in the domain close to $\eta \sim 0.84$. Thus, regions close to $\eta = 0$ obtained from the EoS analysis are to be discarded.

In the negative case, an acceleration is obtained for $\frac{8}{9} < \eta \leq 1$. However, for this range of values, $-\frac{1}{4} \leq q_0 < 0$, which is incompatible with the observed constraint. Thus, as also inferred from the EoS analysis, the negative solution does not agree with observations.

Based on the observational constraints obtained in Ref. [95] with $\Omega_{\mathrm{M},0} = 0.3$, η lies within the range of $0.87 < \eta < 0.92$, which describes the present behaviour of dark energy to be that of quintessence. This constraint is relatively close to the one obtained through the previous considerations.

7.4 Numerical Results

The next step is to analyse the growth evolution for the two Lagrangian solutions obtained. In either case, the growth cannot be solved analytically and is thus solved numerically. As times well within matter domination epochs are considered, the initial scale factor for the numerical computation is taken to be $a_i = 0.1$, which is reasonable, given the matter-radiation equality occurs at a scale factor $a_{\mathrm{eq.}} \sim 10^{-4}$.

To set the initial conditions for $D(a)$, Eq.(7.69) is investigated during matter domination epochs. During such times, $H \propto a^{-3/2}$. Furthermore, based on observational constraints, this yields $A \approx 0.5$ and the effect of the derivatives becomes negligible.[26] This reduces the equation to its GR limit, leading to the growth factor to grow linearly with scale factor i.e. $D(a) = a$. In this way, the initial conditions are set to be $D(a_i) = 0.1$ and $D'(a_i) = 1$.

7.4.1 Model I: $F(T, \mathcal{T}) = \alpha\sqrt{\mathcal{T}} + \beta$

For the first model, the parameters α and β are related via Eq.(7.78) leading the background evolution to depend solely on the dimensionless parameter ϵ defined in Eq.(7.80). Based on observations, the parameter should lie in a range close to $\epsilon = 0$. Naturally, the choice $\epsilon = 0$ is not considered as this would otherwise reduce to ΛCDM. Therefore, the behaviour of the model shall be tested for two distinct

[26] In principle, even if $f_{\mathcal{T}}$ is small, there will be a threshold scale where the k^2-term becomes dominant. However, this feature is not explored when determining the initial conditions.

values, $\epsilon = \pm 0.3 \times 10^{-3}$.

The choice of this parameter is two-fold. Firstly, this satisfies the observational requirement. Secondly, the model is identical to the curvature equivalent $f(R, \mathcal{T}) = R + \alpha\sqrt{\mathcal{T}} + \beta$, and hence, the results are expected to match with those obtained in Ref. [328]. Since it does not lead to any new results, this model shall be used as an example to illustrate its non-viability with observations. To better present the behaviour at different scales, the evolutions of the natural frequency ω_0 and damping factor ζ are presented.

Beginning with the positive case $\epsilon = 0.3 \times 10^{-3}$ where the evolution is shown in Fig. 7.1, structure forms at a slower rate than ΛCDM for all scales. The smallest scales ($k = 50H_0$) have a growth evolution similar to ΛCDM, while for larger scales, this becomes increasingly periodic ($k = 500H_0$). Such behaviours can be traced to the values of the frequency and the damping ratio presented in Figs.7.2 and 7.3 respectively.

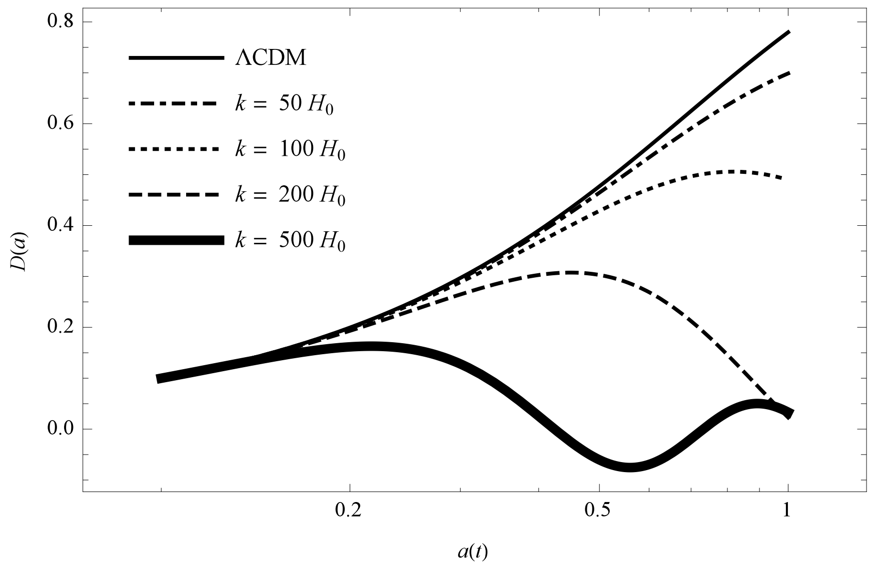


Figure 7.1: Growth factor evolution for the $F(T, \mathcal{T}) = \alpha\sqrt{\mathcal{T}} + \beta$ model with the parameters defined by Eqs.(7.78) and (7.79) for the case when $\epsilon = 0.3 \times 10^{-3}$. For sufficiently small scales, the system behaves as ΛCDM. However, when larger scales are considered, the system starts to deviate from the latter, becoming increasingly periodic, while behaving as an underdamped oscillator close to present times.

Starting with the former, the frequency is effectively complex for $k = 50H_0$ at all times except for when it is close to present times, which yields the growth observed. Once larger scales are considered, most notably for $k = 500H_0$, the frequency transitions from complex to real values towards present times. This means that initially, structures are formed but then decay in an oscillating fashion at later times. This matches the behaviour observed in the damping ratio. It is worth noting that in such cases, the damping ratio transitions from overdamping to underdamping (except for small scales), confirming the observed decaying oscillatory behaviour at late times.

For the negative ϵ case ($\epsilon = -0.3 \times 10^{-3}$), the behaviour changes completely as observed in Fig. 7.4. Growth structure forms at a faster rate than ΛCDM for each sub-horizon scale mode with faster rates achieved for larger modes, while the smallest ones yield an evolution close to ΛCDM. This agrees with the observed behaviour for the frequency in Fig. 7.5 as it is always complex.

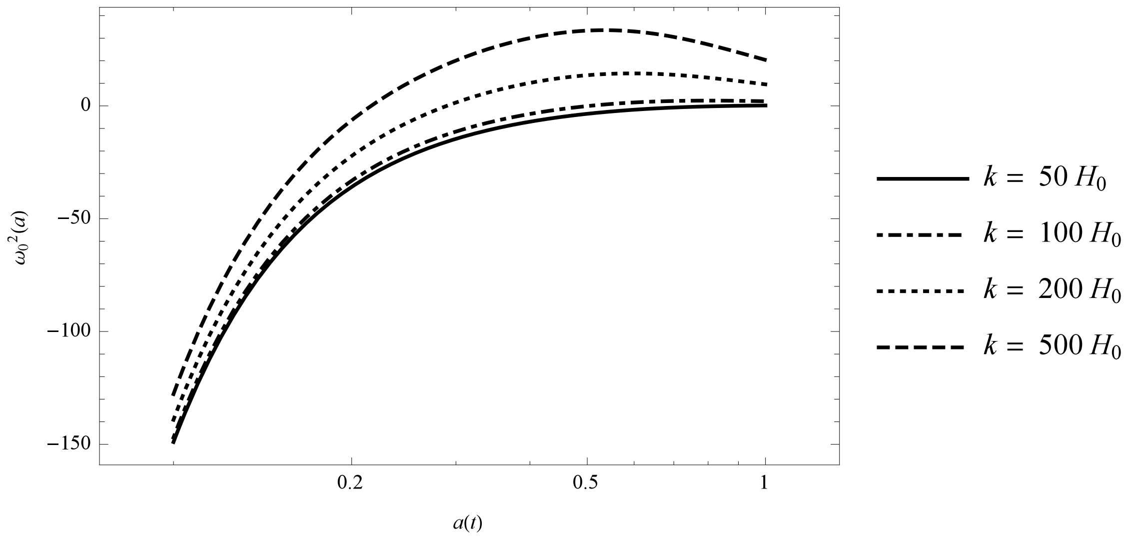


Figure 7.2: The square of the natural frequency evolution with scale factor for the model $F(T, \mathcal{T}) = \alpha\sqrt{\mathcal{T}} + \beta$ for $\epsilon = 0.3 \times 10^{-3}$. For sufficiently small scales ($k = 50H_0$), the frequency is effectively complex, leading to the growing mode similar to ΛCDM. For larger scales, ${\omega_0}^2$ transitions from negative to positive, implying a change from growth formation to oscillatory decay.

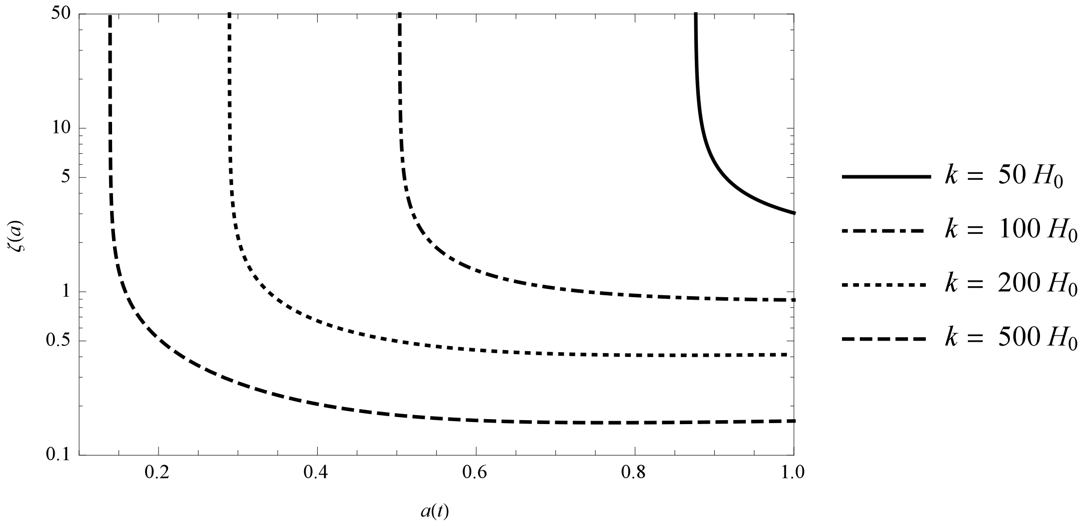


Figure 7.3: Damping ratio evolution with scale factor for the model $F(T,\mathcal{T}) = \alpha\sqrt{\mathcal{T}} + \beta$ for $\epsilon = 0.3 \times 10^{-3}$. For small sub-horizon scales ($k = 50H_0$), the damping ratio only occurs at late times, meaning its effect is mostly absent during matter domination epoch. However, for larger scales, this becomes more dominant at earlier times. Indeed, the damping ratio transitions from an overdamped state towards an underdamped state. Nonetheless, the overdamped and the critical damping states are very short, leaving little to no effect in the growth evolution (Fig. 7.1). The observed oscillations are caused by the damping ratio ending up in an underdamped state.

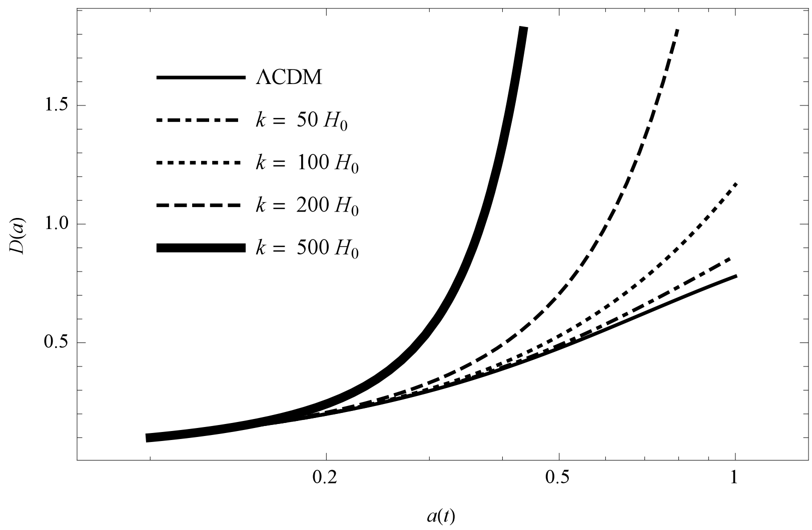


Figure 7.4: Growth factor evolution for the $F(T,\mathcal{T}) = \alpha\sqrt{\mathcal{T}} + \beta$ model with the parameters defined by Eqs.(7.78) and (7.79) for the case when $\epsilon = -0.3 \times 10^{-3}$. In this case, the growth evolution is larger than ΛCDM for all scales. For sufficiently small sub-horizon scales, this is relatively close to the latter, but departs for larger scales.

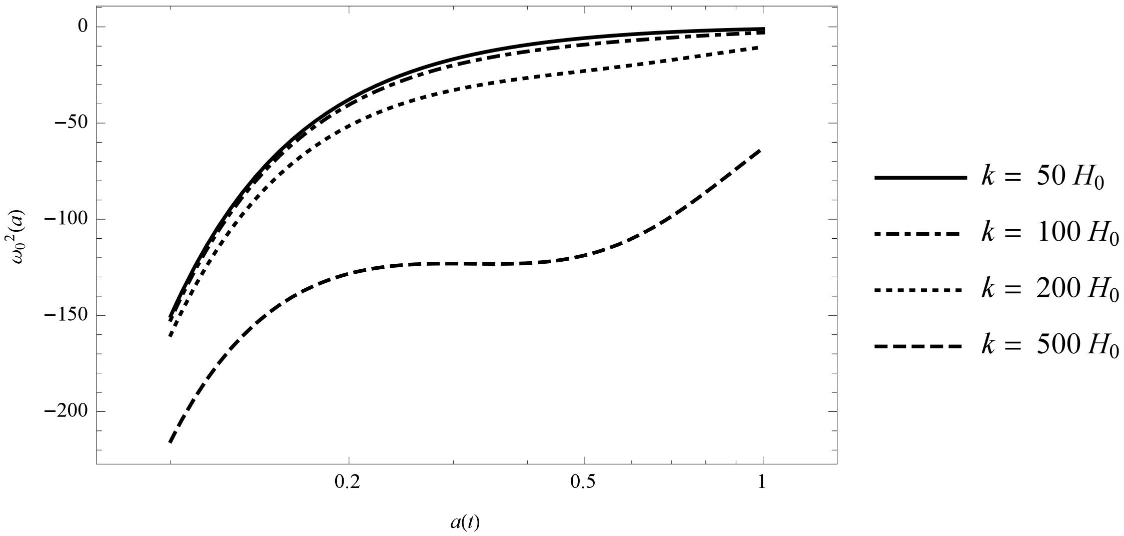


Figure 7.5: The square of the natural frequency evolution with scale factor for the model $F(T, \mathcal{T}) = \alpha\sqrt{\mathcal{T}} + \beta$ for $\epsilon = -0.3 \times 10^{-3}$. Irrespective of scale, the frequency is always complex, and this leads to the observed growing behaviour. As the scale increases, $\omega_0{}^2$ becomes smaller, causing a faster growth expansion as seen in Fig. 7.4.

7.4.2 Model II: $F(T, \mathcal{T}) = -T\left[1 + \left(\frac{\mu}{\sqrt{\mathcal{T}}} + \nu\right)^{-1}\right]$

In the case of the second model, through the relation of the constants μ and ν given in Eq.(7.86) and the introduction of the parameter η in Eq.(7.88), the latter serves as the sole degree of freedom in the system, as is evident in the solution for the Hubble parameter Eq.(7.90). From observational constraints, it was concluded that only the positive solution is viable with $\eta \sim 0.84$. However, irrespective of choice of η close to that value, it is observed that the resulting growth structure evolution remains identical.

As an example, η is chosen to be 0.9. The resulting growth structure evolution is shown in Fig. 7.6 which clearly does not describe any realistic behaviour. Barely any structure forms (small growth values), and the CDM growth is always in an underdamped state with practically no structure left at present times. The behaviour is further supported from the values of the frequency and damping ratio in Figs.7.7 and 7.8 respectively.

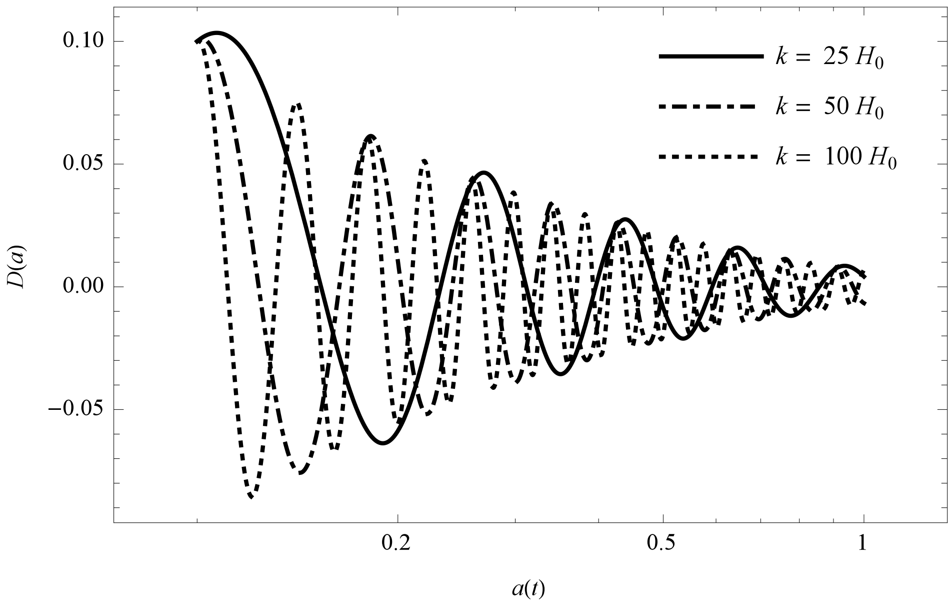


Figure 7.6: Growth evolution for the model $F(T,\mathcal{T}) = -T\left[1+\left(\frac{\mu}{\sqrt{\mathcal{T}}}+\nu\right)^{-1}\right]$ where the constants obey the relations Eqs.(7.86) and (7.88) with $\eta = 0.9$ for the positive Hubble solution Eq.(7.90). The resulting growth evolution is that of an ever under-damped oscillator, which is far from ΛCDM predictions.

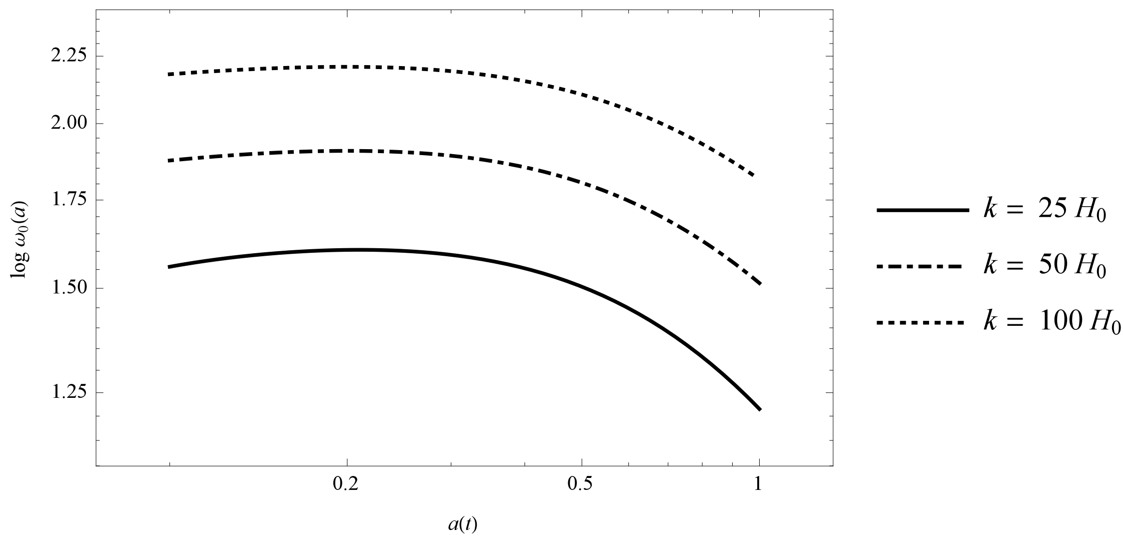


Figure 7.7: Natural frequency evolution with scale factor for the model $F(T,\mathcal{T}) = -T\left[1+\left(\frac{\mu}{\sqrt{\mathcal{T}}}+\nu\right)^{-1}\right]$ for the positive Hubble solution Eq.(7.90) with $\eta = 0.9$. As the frequency takes positive values, the resulting growth evolution becomes oscillatory. Furthermore, the magnitude of the frequency increases with scale.

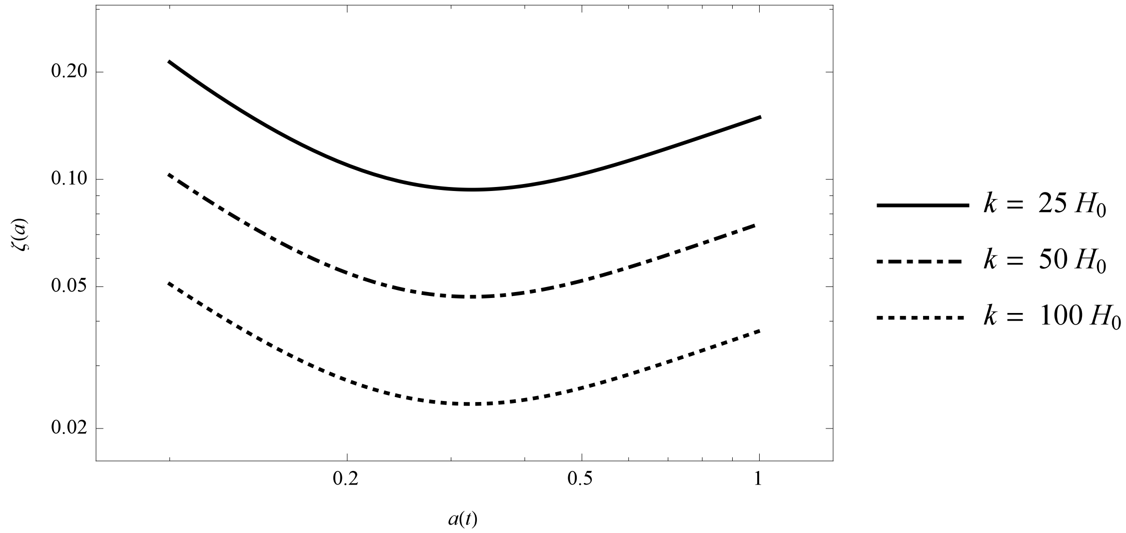


Figure 7.8: Damping ratio evolution with scale factor for the model $F(T, \mathcal{T}) = -T\left[1 + \left(\frac{\mu}{\sqrt{\mathcal{T}}} + \nu\right)^{-1}\right]$ for the positive Hubble solution Eq.(7.90) with $\eta = 0.9$. Here, since $\zeta(a) < 1$ for all the considered scales, the system behaves as an underdamped oscillator.

Starting with the frequency, this is always positive, causing the oscillatory behaviour. The frequency increases with scale, leading to the observed increased number of periods. From the damping ratio, it is observed that for all it scales lies in the underdamped state, hence causing the decay. Thus, the model is deemed to be incompatible with observations.

7.5 Discussion

Throughout this chapter, the role of scalar perturbations in the formation of large scale structure has been investigated for the class of teleparallel theories which include couplings to the trace of the stress energy tensor $\mathcal{T}$. Since the main aim was to investigate the formation of structure of CDM, sub-horizon scales were chosen. After examining the basic key features of the expected growth evolution for CDM and other primary ingredients present during various cosmological epochs, the large scale structure for CDM was then derived. This was carried out during times well within the matter epoch, and in the absence of anisotropic stress and other fluids as their contributions are small. This leads to the final growth equation Eq.(7.69) which is the main result of this chapter.

To investigate its evolutionary properties, the equation is compared to a damped harmonic oscillator. The GR and $f(T)$ gravity limits lead to a frequency which is scale-independent, a result previously obtained in literature. Furthermore, within reasonable bounds for viable $f(T)$ gravity models, the model is effectively ΛCDM. In more general $f(T,\mathcal{T})$ theories however, the frequency is observed to have a strong k^2 dependence proportional to $f_{\mathcal{T}}$. This is similar to what is obtained in $f(R,\mathcal{T})$ gravity, therefore questioning the viability of the gravitational model.

The role of the k^2 dependence is then investigated for model Lagrangians which satisfy the standard conservation of the stress-energy tensor. As derived in Refs. [94, 95], two models have been investigated, $F(T,\mathcal{T}) = \alpha\sqrt{\mathcal{T}} + \beta$ and $F(T,\mathcal{T}) = -T\left[1 + \left(\frac{\mu}{\sqrt{\mathcal{T}}} + \nu\right)^{-1}\right]$.

The first model corresponds to its $f(R,\mathcal{T})$ equivalent considered in Ref. [328] and hence the results are expected to be recovered. To match the results, ϵ is chosen to take the values $\epsilon = \pm 0.3 \times 10^{-3}$, which lie within observational constraints. For the positive ϵ case, the model exhibits less growth than ΛCDM for all relevant sub-horizon scales, which is further strongly suppressed at larger scales as it approaches an underdamped oscillating state. In the negative case, structure forms at a faster rate than ΛCDM with larger scales yielding a faster growth.

For the second model, only the positive solution for the Hubble parameter Eq.(7.90) (and, consequently, for ν Eq.(7.89)) permits an agreement with observations for the background cosmological behaviour. This is achieved for values of η which lie within domains close to $\eta = 0.84$ (if the SNe Ia data analysis carried out in Ref. [95] is included, $0.87 < \eta < 0.92$). However, for all reasonable choices of η close to the given constraint, the resulting evolution is not physical. Taking $\eta = 0.9$ as an illustrative example, the system evolves as an underdamped oscillator for all sub-horizon scales with values so small that barely any structure forms. This leads the model to be in disagreement with observations.

Overall, it is expected that most $f(T, \mathcal{T})$ models do not realise a correct growth structure formation at large scales due to the strong k^2 dependence in the theory. This effect can only be suppressed if it is possible to define a coupling where the contribution from the stress-energy trace $\mathcal{T}$ is absent after matter-radiation equality. In other words, $F_{\mathcal{T}} \simeq 0$ during these epochs, which yields a strong constraint on the Lagrangian.

Chapter 8

Conclusion

Throughout this work, an alternative description to gravity, constructed from torsion rather than curvature of spacetime, has been investigated. This has been achieved by an appropriate choice of the connection, as this determines the type of geometry. For curvature based theories of gravity, the connection is given by the Levi-Civita connection Eq.(2.11) while for teleparallel gravity, this is obtained by the Weitzenböck connection Eq.(2.12). In its basic form described as TEGR, obtained by replacing the Ricci scalar R with the torsion scalar T in the gravitational action (Eq. 2.20), the theory has been shown to be equivalent at the level of equations (although fundamentally different), and hence, most of the results predicted by GR are retained, along with some of the observational problems. In a similar way as carried out in curvature based theories, extensions to this torsional and teleparallel avenue have been constructed and studied, in order to try and explain some fundamental questions about the universe.

In particular, the extensions studied in this work are $f(T)$ gravity, the teleparallel extension of the Gauss-Bonnet term $f(T, T_G)$ gravity, effects of the boundary term B through $f(T, B)$ gravity (which includes $f(R)$ theories as a sub-case), as well as non-trivial couplings between the trace of the matter stress-energy tensor $\mathcal{T}$ with torsion in $f(T, \mathcal{T})$ gravity. These different models have been investigated under different fields, for which the results are summarised in the following sections.

8.1 Homogeneous Cosmological Stability

The stability of the FLRW metric under homogeneous and isotropic perturbations in the context of $f(T)$ gravity theory have been investigated first. Through these types of perturbations, one can determine whether the background cosmology is valid throughout the cosmic evolution, which is achieved by requiring the perturbations to remain small throughout the evolutionary history. Here, the work serves as a generalisation of previous investigations where an analytical solution for the matter and Hubble perturbations (Eqs.(3.21) and (3.22)) has now been obtained. These results show that one can investigate the stability of any $f(T)$ gravity model solely through the evolution of the Hubble parameter.

To investigate the stability behaviour, two viable $f(T)$ models have been considered: the power-law and exponential models. By constraining them using observational data, the models have been shown to be stable as they both appear to remain in a de Sitter phase at late times (hence being an attractor). Overall, the exponential model has been shown to be the closest to mimic the ΛCDM evolution.

Homogeneous perturbations have also been studied for the other gravitational models, for instance $f(T, T_G)$ [364], $f(T, B)$ [69] and $f(T, \mathcal{T})$ gravity [94,96]. In each case, however, a general analytical solution for the homogeneous perturbation variables is not recovered. This could prove to be interesting grounds for future investigations.

Although not investigated in this work, dynamical systems have proven to be a very useful tool to determine the overall behaviour of a model without having to resort to solving the system analytically. Thanks to this approach, one can easily discriminate between viable models according to the phase trajectory of the cosmology. In the case of $f(T)$ gravity, this has been thoroughly investigated with the work in Ref. [177]

describing the general behaviour for any given $f(T)$ function. For the remaining models, work has been done with the aim of determining their dynamics, but has been mostly applied for very specific models, as shown in the case of $f(T,T_G)$ gravity [81] and $f(T,\mathcal{T})$ gravity [94, 365]. Thus, it still remains to be seen what the overall picture of these theories holds in terms of general evolution viewed from the eyes of dynamical systems.

8.2 Cosmological Reconstruction

The method of cosmological reconstruction has been applied in $f(T,T_G)$ gravity to construct Lagrangians capable of describing various cosmological histories. Reconstruction therefore serves as a powerful tool to hint towards a more concrete form of the gravitational Lagrangian without resorting to testing different model ansatz. In the case of $f(T,T_G)$ gravity, the resulting PDEs are too complex to solve and hence different ansatz choices are considered. The Lagrangian is further constrained to be able to recover the vacuum solutions which require $f(0,0)=0$.

For power-law and de Sitter cosmologies, various models have been constructed and shown to be able to satisfy the vacuum constraint. However, in the case of a ΛCDM cosmology, although a Lagrangian can be reconstructed, none of the ansatz models considered are able to satisfy the vacuum constraint. The reconstructed solutions are summarised in Tables 4.1, 4.3 and 4.4.

Bouncing cosmologies have also been investigated, since they serve as an important alternative to the singularity problem encountered in a Big Bang, and as an alternative to inflation. Future singularities can also arise through bouncing cosmologies used to discuss the possible future state of the universe (e.g. Big Rip) or a mechanism for a graceful exit inflation (Type IV singularity). Overall, although different $f(T,T_G)$ Lagrangians can be reconstructed for all the bouncing models considered, only the superbounce and Type III singularity are able to satisfy the vacuum constraint within a non-empty universe. Thus, if this condition is enforced, the type of bouncing cosmology is restricted. For the bouncing models considered, the reconstructed solutions are given in Tables 5.1–5.4.

From these results, it would be an interesting viewpoint to investigate the role of $f(T, T_G)$ gravity in further detail, given its capability to describe cosmological solutions which are also able to recover vacuum solutions. This analysis can also be extended to $f(T, B)$ gravity and $f(T, \mathcal{T})$ gravity primarily in the context of bouncing cosmologies. Reconstructed solutions for power law, de Sitter and ΛCDM expansion histories have been carried out in the literature; see Refs. [69, 71, 366] in the context of $f(T, B)$ gravity and Refs. [94, 97, 98] for $f(T, \mathcal{T})$ gravity. For all the models considered, the analysis can also be extended to reconstruct the Lagrangian based on the SNe Ia data in a similar context as carried out in $f(R)$ gravity [182].

8.3 Gravitational Waves

The existence of GWs has been investigated in a weak-field linearised gravity approach, which yields linearised teleparallel field equations used to determine the basic properties of the waves predicted by said theories. In particular, the number and type of the polarisation states together with their speed has been investigated. With present and future data regarding these two properties of GWs, these will therefore serve as constraints for the theory. Furthermore, the analysis has been carried out under two formalisms, the so called metric and tetrad approaches, to illustrate that both approaches lead to the same consistent result.

Starting with the simplest case of $f(T)$ gravity, the well known result that the same polarisation states and speed as GR is recovered. However, when a cosmological constant background is introduced, the result changes slightly. Despite it exhibiting the same behaviour as encountered in GR with a cosmological constant, a new effective cosmological constant given in Eq.(6.112) is obtained. This causes an amplitude and phase shift difference, which agrees with the tensor perturbation analysis shown in Refs. [284, 327]. For higher-order perturbations, the gravitational radiation relation is the same as that found in GR but at third order, a contribution from $f_{TT}(0)$ arises deviating from GR. This was not explored further due to the complexity of the equations.

Next, in the case of $f(T, B)$ gravity, the result is effectively identical to that encountered in $f(R)$ gravity, it being a massive scalar mode with the speed of propagation being smaller than that of light for $f_{BB}(0) \neq 0$ together with the massless two tensor modes. The model has also been investigated in a cosmological constant background setting, but could not be solved in general. Nonetheless, it is expected that the cosmological constant affects the amplitude and phase of the waves. Finally, for $f(T, T_G)$ gravity theory, the standard massless tensor polarisations arise, which was expected as T_G represents a higher order term.

This work provided the basic GW features of the models. However, a more detailed account about the aspect of cosmological perturbation theory is still required. One would be able to calculate the primordial GW tensor spectrum and then compute the scalar-tensor ratio which is well constrained from Planck data [152]. For instance, in $f(T)$ gravity, models which yield a matter bounce cosmology suffer from the same issues of matter bounces themselves, namely having a scalar-tensor ratio incompatible with observations [234,367]. On the other hand, this scalar-tensor ratio could match with observations provided certain constraints on the $f(T)$ Lagrangian are satisfied [285,368]. This analysis could be extended to the other theories as well.

Another important point is that, although the tensor modes have been shown to propagate at the speed of light on a Minkowski background, this is not necessarily true on an FLRW background. As discussed in Chapter 6, $f(R, G)$ theories have been shown to yield a different wave speed, and hence are strongly constrained from observations. Thus, it would be interesting to see whether this is also true in its teleparallel analogue and in $f(T, B)$ gravity for the tensor mode propagation.

Lastly, the GW polarisations of $f(T, \mathcal{T})$ gravity could also be investigated. However, it is expected that this will reduce to the same result as $f(T)$ gravity since in the weak-field regime, the system is analysed far away from the source leading to $\mathcal{T} \to 0$. This is similar to what occurs in $f(R, \mathcal{T})$ gravity [369]. However, if this stress-energy trace is sourced by a scalar field ϕ for instance (with stress-energy trace $\mathcal{T}^{\phi}$), this could realise new modes, as shown in $f(R, \mathcal{T}^{\phi})$ gravity [369].

8.4 Large Scale Structure

Growth of large scale structure in the context of $f(T, \mathcal{T})$ theory was investigated through the use of scalar linear cosmological perturbations in an FLRW background. By using the correct form of the tetrad scalar perturbation Eq.(7.28) and focusing on sub-horizon scales, it was found that the growth factor is strongly dependent on scale due to the k^2-dependence found in Eq.(7.69). A similar feature is found in the curvature analogue of the theory, and this sets the theory to be strongly constrained against observations.

For the two $F(T, \mathcal{T})$ models considered which satisfy the stress-energy conservation constraint, the growth of the first model $F(T, \mathcal{T}) = \alpha\sqrt{\mathcal{T}} + \beta$ has been shown to match that of the curvature derived result as expected, since the Lagrangians are equivalent, which shows consistency. For the parameters considered, the effect of scale has been shown to be dominant for sufficiently large sub-horizon scales.

On the other hand, the second model $F(T, \mathcal{T}) = -T\left[1 + \left(\frac{\mu}{\sqrt{\mathcal{T}}} + \nu\right)^{-1}\right]$ does not realise any physical growth even though a satisfactory background evolution is obtained, making the model non-viable. With these results, it has been argued that a viable model is achieved provided $F_{\mathcal{T}} \simeq 0$ past matter-radiation equality, meaning the dependence on the trace should be negligible at late times.

Here, only sub-horizon scales have been investigated. However, super-horizon modes are as important especially in computing the CMB anisotropy spectrum for low multipoles. In the case of $f(T)$ gravity for instance, the super-horizon scale does not yield a constant potential as found in GR, but is scale dependent leading to a different behaviour in the low multipole regime [325, 326]. Nonetheless, one can find $f(T)$ models where this deviation is sufficiently small [181]. The anisotropy spectrum analysis can then naturally be extended to the other teleparallel models, namely $f(T, B)$, $f(T, T_G)$ and $f(T, \mathcal{T})$ gravity to be tested against observations. For instance, $f(R)$ [370–372] (amongst other works) and $f(R, G)$ [373] gravity have been shown to be able to match with CMB data, and hence it would be interesting to check whether their torsional analogues are also capable of doing so.

8.5 Final Remarks

Through the investigation of these extended teleparallel theories in various topics on cosmology, teleparallel theory has been shown to be a viable alternative to explain the mechanics of gravitation. Nonetheless, these models will always need to be continuously subjected to present and future observations in order to be able to deem whether teleparallel gravity can correctly account for all gravitational phenomena. Following the discussions in Chapter 1, a viable teleparallel theory of gravity must then be able to, amongst others:

1. Recover the Newtonian weak-field limit and pass all local Solar system tests. As shown in Refs. [374–376], despite these tests imposing strong constraints on the parameters of the theory, teleparallel gravity is still able to satisfy all local tests;

2. Explain the galactic mechanics. This can be achieved by either describing dark

matter as a manifestation of gravity, or using teleparallel theory in conjunction with the predicted behaviour and properties of dark matter [377]. The former has been shown to be possible in the case of $f(T)$ gravity [378];

3. Correctly predict the observed cosmological behaviours, both at a background level (acceleration and Hubble tension) as well as at a perturbative level (the CMB spectrum and polarisation spectra, and large scale structure formation). These features have been shown to be possible as discussed throughout this work;

4. Correctly predict the GW propagation speeds (to be the same as that of light) and through future observations, the correct polarisations. Once again, these are achieved through teleparallel gravity, as shown in Chapter 6.

Therefore, the future of teleparallel theories remains a promising alternative avenue to provide a more concrete description of gravity.

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