The Erdős-Ko-Rado properties of various graphs containing singletons

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Abstract

Let $G = (V, E)$ be a graph. For $r \geq 1$, let $\mathcal{I}_G^{(r)}$ be the family of independent vertex r-sets of G. For $v \in V(G)$, let $\mathcal{I}_G^{(r)}(v)$ denote the *star* $\{A \in \mathcal{I}_G^{(r)} \colon v \in A\}$. G is said to be *r-EKR* if there exists $v \in V(G)$ such that $|\mathcal{A}| \leq |\mathcal{I}_G^{(r)}(v)|$ for any non-star family $\mathcal{A}$ of pair-wise intersecting sets in $\mathcal{I}_G^{(r)}$. If the inequality is strict, then G is *strictly r-EKR*.

Let Γ be the family of graphs that are disjoint unions of complete graphs, paths, cycles, including at least one singleton. Holroyd, Spencer and Talbot proved that, if $G \in \Gamma$ and $2r$ is no larger than the number of connected components of G, then G is r-EKR. However, Holroyd and Talbot conjectured that, if G is any graph and $2r$ is no larger than $\mu(G)$, the size of a smallest maximal independent vertex set of G, then G is r-EKR, and strictly so if $2r < \mu(G)$. We show that in fact, if $G \in \Gamma$ and $2r$ is no larger than the independence number of G, then G is r-EKR; we do this by proving the result for all graphs that are in a suitable larger set $\Gamma' \supsetneq \Gamma$. We also confirm the conjecture for graphs in an even larger set $\Gamma'' \supsetneq \Gamma'$.

1 Introduction

Throughout this paper, we denote the set of natural numbers by $\mathbb{N}$, the set $\{x \in \mathbb{N} \colon m \leq x \leq n\}$ by $[m,n]$ and $[1,n]$ by $[n]$.

Next, we give some terminology and notation relating to graph theory.

A graph $G = (V,E) = (V(G), E(G))$ is assumed to be finite, simple and undirected unless specified otherwise. (An infinite graph is temporarily introduced in Definition 1.11, but this is the only such graph to appear.) We denote a typical edge of G by vw where $v, w \in V(G)$. For any $v \in V(G)$, the set of *neighbours* of v (that is, vertices adjacent to v) will be denoted by $N_G(v)$, and $N_G(v) \cup \{v\}$ will be denoted by $\hat{N}_G(v)$. An *independent set* of vertices of G is a set of pair-wise non-adjacent vertices.

We denote the complete graph, the path, and the cycle on n vertices by K_n, P_n and C_n, respectively. The *length* of P_n is $n-1$. A *singleton* is a vertex of G that is adjacent to no other vertex, and the *empty* graph E_n is the graph consisting of n singletons.

Let G be any graph; then the *distance* $d(v,w)$ between vertices v and w in the same connected component of G is the length of the shortest path between v and w. For $k \in \mathbb{N}$ the k^{th} *power* of G, denoted by G^k, is the graph with vertex set $V(G)$ where $vw \in E(G^k)$ iff $d(v,w) \leq k$. Note that $P_n^k = K_n$ for $k \geq n-1$, while $C_n^k = K_n$ for $k \geq n/2$.

If G is a graph and $S \subseteq V(G)$, then the subgraph H of G *induced* by S has $V(H) = S$, two vertices of H being adjacent in H iff they are adjacent in G.

Finally, the *Cartesian product* $G \times H$ of two graphs has $V(G \times H) = V(G) \times V(H)$, two vertices (v,w) and (x,y) being adjacent in $G \times H$ iff either $v = x$ and $wy \in E(H)$ or $vx \in E(G)$ and $w = y$.

Next, we introduce notation for certain families of sets of vertices of a graph.

We denote the family of all independent sets of vertices of G by $\mathcal{I}_G$. Then $\alpha(G)$ and $\mu(G)$ denote, respectively, the maximum and minimum sizes of a maximal member of $\mathcal{I}_G$ under set-inclusion.

For $r \geq 1$, let $\mathcal{I}_G^{(r)}$ be the family of independent r-sets of G, that is, $\{I \in \mathcal{I}_G \colon |I| = r\}$. For $v \in V(G)$, let $\mathcal{I}_G^{(r)}(v)$ denote the *star of* $\mathcal{I}_G^{(r)}$ *with centre* v, that is, $\{A \in \mathcal{I}_G^{(r)} \colon v \in A\}$.

More generally, for any family $\mathcal{F}$ of sets, the *stars* of $\mathcal{F}$ are the sub-families $\mathcal{F}(x) := \{F \in \mathcal{F} \colon x \in F\}$ (where we assume $x \in \bigcup_{F \in \mathcal{F}} F$). A family is said to be *intersecting* if any two sets in it intersect. Note that stars are trivially intersecting.

In [16], Holroyd and Talbot introduced the following definition that is inspired by the classical Erdős-Ko-Rado (EKR) Theorem [11]: G is said to be *r-EKR* if no intersecting family $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}$ is larger than the largest star of $\mathcal{I}_G^{(r)}$, and to be *strictly r-EKR* if no non-star intersecting family $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}$ is as large as the largest star of $\mathcal{I}_G^{(r)}$.

It is interesting that many EKR-type results can be expressed in terms of the r-EKR or strict r-EKR property of some graph G. This observation was made in [16] and inspired a number of other results about the EKR properties of certain graphs. Before coming to the crux of this paper, we give a brief review of such results, recalling certain well-known classes of graphs and also defining new ones.

The EKR Theorem [11] and the Hilton-Milner Theorem [13] may be expressed in terms of empty graphs as follows.

Theorem 1.1 (Erdős, Ko, Rado [11]; Hilton, Milner [13]) *Let $r \leq n/2$. Then E_n is r-EKR, and strictly so if $r < n/2$.*

The work of Cameron and Ku [6] (inspired by the work in [7]) on intersecting permutations and the works of Ku and Leader [17] and Li and Wang [18] on intersecting partial permutations can be summed up and phrased as follows.

Theorem 1.2 (Cameron, Ku [6]; Ku, Leader [17]; Li, Wang [18]) *$K_n \times K_n$ is strictly r-EKR for all $r \in [n]$.*

A well-known intersection theorem that was first stated by Meyer [19] and proved by Deza and Frankl [8] and Bollobás and Leader [1] can be phrased as follows.

Theorem 1.3 (Meyer [19]; Deza, Frankl [8]; Bollobás, Leader [1]) *Let $r \leq n$ and $k \geq 2$. Let G be the disjoint union of n copies of K_k. Then G is r-EKR, and strictly so unless $r = n \geq 3$ and $k = 2$.*

Other proofs are found in [9, 10] and, in a more general context, in [3]. Holroyd, Spencer and Talbot [15] extended the non-strict part of Theorem 1.3 by showing that, if G is the disjoint union of n complete graphs each of order at least 2, then G is r-EKR for all $r \leq n$.

Suppose G is a graph whose vertex set has a partition $V(G) = V_1 \cup ... \cup V_k$ into *partite sets* such that any two vertices are adjacent iff they belong to distinct partite sets. Such a graph is said to be a *complete multipartite graph*, or more particularly a *complete k-partite graph*. (Thus if $|V_1| = ... = |V_k| = 1$, then $G = K_k$.) Holroyd and Talbot [16] considered the problem for complete multipartite graphs.

Theorem 1.4 (Holroyd, Talbot [16]) *Let G be the disjoint union of two complete multipartite graphs. Let $r \leq \mu(G)/2$. Then G is r-EKR, and strictly so if $r < \mu(G)/2$.*

This result follows immediately from the case $k = 1$ of the next result (see [16]).

Theorem 1.5 (Borg, Holroyd [5]) *Let G be the disjoint union of k complete multipartite graphs and a non-empty set V_0 of singletons. Let $r \leq \mu(G)/2$. Then:*
(i) G is r-EKR;
(ii) G fails to be strictly r-EKR iff $2r = \mu(G) = \alpha(G)$, $3 \leq |V_0| \leq r$, $k = 1$.

In the recent years, a number of EKR results have been obtained for powers of paths and cycles.

Theorem 1.6 (Holroyd, Spencer, Talbot [15]) *If $d \geq 1$ and G is a d^{th} power of a path, then G is r-EKR for all $r \leq \alpha(G)$.*

A nice EKR-type result of Talbot [20] for *separated sets* can be stated as follows.

Theorem 1.7 (Talbot [20]) *Let $r \leq \alpha(C_n^k)$. Then C_n^k is r-EKR, and strictly so unless $k = 1$ and $n = 2r + 2$.*

The *clique number* $\mathrm{cl}(G)$ of graph G is the size of a largest complete sub-graph of G.

Theorem 1.8 (Hilton and Spencer [14]) *Let G be the disjoint union of graphs $G_0, G_1, ..., G_n$ such that $\mathrm{cl}(G_0) \leq \min\{\mathrm{cl}(G_i)\colon i \in [n]\}$, where G_0 is a power of a path and, for each $i \in [n]$, G_i is a power of a cycle. Then G is r-EKR for all $r \leq \alpha(G)$.*

As we explain later, the work in this paper is inspired by the following result.

Theorem 1.9 (Holroyd, Spencer, Talbot [15]) *Let G be the disjoint union of n connected components, each a complete graph, path, cycle or singleton, including at least one singleton. Then G is r-EKR for all $r \leq n/2$.*

Unlike all the preceding theorems, this result does not live up to Conjecture 1.10 (below), because for an arbitrary graph G, $\mu(G)$ is at least as large as the number of connected components of G and may be much larger.

As we hinted earlier, the idea of the graph-theoretical formulation we have been discussing emerged in [16], in which Holroyd and Talbot initiated the study of the general EKR problem for independent sets of graphs and made the following conjecture.[1]

Conjecture 1.10 (Holroyd, Talbot [16]) *Let G be any graph, and let $r \leq \mu(G)/2$. Then G is r-EKR, and strictly so if $r < \mu(G)/2$.*

By proving Theorem 1.4, they provided an example of a graph G such that G obeys the conjecture and, as we demonstrate in a stronger fashion below, G may not be r-EKR if $\mu(G)/2 < r < \alpha(G)$ (it is easy to see that for such a graph G, G is r-EKR for $r = \alpha(G)$). They gave various other examples of graphs H and values $r > \mu(H)/2$ for which H is *not* r-EKR, and one particularly interesting example of this kind has $r = \alpha(H)$. The idea behind Conjecture 1.10 is that if I is any maximal independent set of a graph G with $\mu(G) \geq 2r$, then, since $|I| \geq \mu(G)$, it holds by the EKR Theorem that $(I, \emptyset)$ (i.e. the empty graph with vertex set I) is r-EKR, and strictly so if $\mu(G) > 2r$.

We now show that there are graphs G such that $\mu(G) < \alpha(G)$ and G is not r-EKR for all $\mu(G)/2 < r < \alpha(G)$. Indeed, let G be the graph consisting of a 3-set V_0 of singletons and a complete bipartite graph with partite sets V_1 and V_2 of sizes 5 and 4 respectively. So $7 = \mu(G) < \alpha(G) = 8$. For $r \in [\alpha(G)]$, let $\mathcal{J}_r$ be a star of $\mathcal{I}_G^{(r)}$ with centre $x \in V_0$, and let $\mathcal{A}_r := \{A \in \mathcal{I}_G^{(r)}\colon |A \cap V_0| \geq 2\}$. Clearly $\mathcal{J}_r$ is a star of $\mathcal{I}_G^{(r)}$ of largest size. However, for $\mu(G)/2 < r < \alpha(G)$, we have $|\mathcal{A}_r| > |\mathcal{J}_r|$. This proves what we set out to show.

Conjecture 1.10 seems very hard to prove or disprove. However, restricting the problem to some classes of graphs with singletons makes it tractable. Theorem 1.1 and the example that we gave above demonstrate the fact that when an arbitrary number of singletons are allowed in a graph G, G may not be r-EKR for $r > \mu(G)/2$.

The following is the first of two important definitions that are needed to state the new results presented in this paper (Theorems 1.13 and 1.14).

[1] The first author [2] has recently proved this conjecture for $\mu(G) \geq \frac{1}{2}(r-1)(3r-3)(3r-4) + r$ in a more general form.

Definition 1.11 (Borg [4]) *For a monotonic non-decreasing (mnd) sequence* $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ *of non-negative integers, let* $M := M(\mathbf{d})$ *be the graph such that* $V(M) = \{x_i : i \in \mathbb{N}\}$ *and, for* $x_a, x_b \in V(M)$ *with* $a < b$, $x_a x_b \in E(M)$ *iff* $b \leq a + d_a$. *Let* $M_n := M_n(\mathbf{d})$ *be the sub-graph of* M *induced by the subset* $\{x_i : i \in [n]\}$ *of* $V(M)$. *We call* M_n *an mnd graph.*

In the case $d_i = d$ ($i \in \mathbb{N}$), the graph $M_n(\mathbf{d})$ is just the d^{th} power of P_n, i.e. P_n^d. The r-EKR problem for any mnd graph M_n and any r is addressed [2]. Here, the solution for the r-EKR problem for M_n with $d_1 = 0$ and $r \leq \alpha(M_n)/2$ is part of Theorem 1.14 below.

We now come to our second definition. We shall represent the vertices of C_n by $v_1, ..., v_n$ and take $E(C_n)$ to be in the natural way, i.e. $E(C_n) = \{v_1v_2, ..., v_{n-1}v_n, v_nv_1\}$. We shall use the term 'mod' to represent the usual modulo operation with the exception that, for any two integers a and b, ba mod a is a instead of 0.

Definition 1.12 *For* $n > 2$, $1 \leq k < n - 1$, $0 \leq q < n$, *let* ${}_qC_n^{k,k+1}$ *be the graph with vertex set* $\{v_i : i \in [n]\}$ *and edge set* $E(C_n^k) \cup \{v_i v_{i+k+1 \bmod n} : 1 \leq i \leq q\}$.

If $q > 0$, then we call ${}_qC_n^{k,k+1}$ a *modified* k^{th} *power of a cycle*; essentially it is a $(k+1)^{\text{th}}$ power for some of the cycle and a k^{th} power for the remainder of the cycle.

The objective of this paper is to provide an improvement of the techniques in [15] that enables us to confirm the conjecture for the class of graphs in Theorem 1.9 and even larger classes. The key idea that leads us to this improvement is to consider a suitable larger class of graphs, namely to allow copies of mnd graphs and modified powers of cycles in the disjoint union specified in Theorem 1.9. Since the proof goes by induction, we will need to perform certain deletions on the original graph. When a deletion is performed on a power of a cycle, which is significantly more difficult to treat than the other components, we obtain a modified power of a cycle (mpc) or a power of a path, and if a deletion is performed on an mpc then we obtain an mnd graph or an mpc. So the idea is that every time a deletion is performed, the resulting graph is in the admissible class. Although not necessary for our main aim, we show that our method allows us to include *trees* (connected cycle-free graphs) as components; the scope is to illustrate the fact that the method we employ works for many classes of graphs.

Theorem 1.13 *Conjecture 1.10 is true if* G *is a disjoint union of complete multipartite graphs, copies of mnd graphs, powers of cycles, modified powers of cycles, trees, and at least one singleton.*

Our method also allows us to improve Theorem 1.9 beyond Conjecture 1.10.

Theorem 1.14 *Let* G *be a disjoint union of complete graphs, copies of mnd graphs, powers of cycles, modified powers of cycles, and at least one singleton. Let* $r \leq \alpha(G)/2$. *Then* G *is* r*-EKR.*

Note that in this result we cannot include components like complete multipartite graphs or trees, because otherwise, as we have shown above, G may not be r-EKR for $\mu(G)/2 < r \leq \alpha(G)/2$.

2 The compression operation

In the context of set combinatorics, a *compression operation* (or simply a *compression*) is a function that maps a family of sets to another family while retaining its size and (usually) some other important properties; the survey paper [12] on the uses of this technique is recommended. Loosely speaking, a compression replaces a particular element of the ground set by another particular element whenever possible.

In the graph-theoretic context the ground set is $V(G)$ and we are interested in independent subsets of $V(G)$. The shift operation $\delta_{u,v}$ is defined on any such set as follows:

$$\delta_{u,v}(F) := \begin{cases} (F\backslash\{v\}) \cup \{u\} & \text{if } u \notin F, v \in F \text{ and } (F\backslash\{v\}) \cup \{u\} \in \mathcal{I}_G; \\ F & \text{otherwise} \end{cases}$$

Then compression $\Delta_{u,v}$ acts on sub-families of $\mathcal{I}_G$, as follows. Let $\mathcal{F}$ be a sub-family of $\mathcal{I}_G$. Then for each $A \in \mathcal{F}$, define

$$\Delta_{u,v}(\mathcal{F}) := \{\delta_{u,v}(A) \colon A \in \mathcal{F}\} \cup \{A \in \mathcal{F} \colon \delta_{u,v}(A) \in \mathcal{F}\}.$$

It should be clear that $\delta_{u,v}$ preserves the sizes of sets while $\Delta_{u,v}$ preserves the sizes of families of sets.

Let G be a graph, $v \in V(G)$. We use $G - v$ to denote the graph obtained from G by deleting $v \in V(G)$ (and hence edges incident to v), and $G \downarrow v$ to denote the graph obtained by deleting also all vertices in $N_G(v)$ (and incident edges). Next, for any $\mathcal{F} \subseteq \mathcal{I}_G$, we define the following sub-families of $\mathcal{F}$:

$$\mathcal{F}\langle v\rangle := \{A\backslash\{v\} \colon A \in \mathcal{F}(v)\} \subseteq \mathcal{I}_{G\downarrow v}, \quad \mathcal{F}\overline{(v)} := \{A \in \mathcal{F} \colon v \notin A\} \subseteq \mathcal{I}_{G-v}.$$

Lemma 2.1 *Let $uv \in E(G)$. Let $\mathcal{F} \subset \mathcal{I}_G^{(r)}$ be an intersecting family, and let $\mathcal{A}$ be the family $\Delta_{u,v}(\mathcal{F})$. Then:*
(i) $\mathcal{A}\overline{(v)}$ is intersecting;
(ii) if $|N_G(u)\backslash \hat{N}_G(v)| \leq 1$, then $\mathcal{A}\langle v\rangle$ is intersecting;
(iii) if $N_G(u)\backslash \hat{N}_G(v) = \emptyset$, then $\mathcal{A}$ and $\mathcal{A}\overline{(v)} \cup \mathcal{A}\langle v\rangle$ are intersecting.

Proof. We begin with the observation that since $uv \in E(G)$, the 2-set $\{u, v\}$ is not contained in any set of $\mathcal{I}_G$, and hence $\mathcal{F}$ may be partitioned as $\bigcup_{i=1}^{5} \mathcal{F}_i$ where
$\mathcal{F}_1 := \{F \in \mathcal{F} \colon u \in F, v \notin F\}$,
$\mathcal{F}_2 := \{F \in \mathcal{F} \colon \{u, v\} \cap F = \emptyset\}$,
$\mathcal{F}_3 := \{F \in \mathcal{F} \colon v \in F, u \notin F \text{ and } (F\backslash\{v\}) \cup \{u\} \in \mathcal{F}_1\}$,
$\mathcal{F}_4 := \{F \in \mathcal{F} \colon v \in F, u \notin F \text{ and } (F\backslash\{v\}) \cup \{u\} \notin \mathcal{I}_G\}$,
$\mathcal{F}_5 := \{F \in \mathcal{F} \colon v \in F, u \notin F \text{ and } (F\backslash\{v\}) \cup \{u\} \in \mathcal{I}_G\backslash\mathcal{F}_1\}$.
Moreover, $\mathcal{A} = \bigcup_{i=1}^{4} \mathcal{F}_i \cup \mathcal{A}_5$ where $\mathcal{A}_5 := \{(F\backslash\{v\}) \cup \{u\} \colon F \in \mathcal{F}_5\}$.

Note that $\mathcal{A}\overline{(v)} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{A}_5$. Since $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\mathcal{A}_5$ are each intersecting, to prove (i) we need merely verify that if $A \in \mathcal{F}_1 \cup \mathcal{F}_2, B \in \mathcal{A}_5$, then $A \cap B \neq \emptyset$. Now consider the set $C \in \mathcal{F}_5$ such that $(C\backslash\{v\}) \cup \{u\} = B$. Since $\mathcal{F}$ is intersecting, there exists $x \in V(G)\backslash\{v\}$ such that $x \in A \cap C$. So $x \in A \cap B$. Hence (i).

We next prove (ii). So suppose $|N_G(u)\backslash\hat{N}_G(v)| \le 1$. Clearly $\mathcal{A}\langle v\rangle = (\mathcal{F}_3 \cup \mathcal{F}_4)\langle v\rangle$. If $A \in \mathcal{F}_3$, then the set $A' := A\backslash\{v\} \cup \{u\}$ is in $\mathcal{F}_1$, and hence, for any $F \in \mathcal{F}_3 \cup \mathcal{F}_4$, $(A \cap F)\backslash\{v\} = (A' \cap F)\backslash\{v\} \ne \emptyset$ (as $u \notin F$ and $\mathcal{F}$ is intersecting). Thus we need merely show that $\mathcal{F}_4\langle v\rangle$ is intersecting. If $N_G(u)\backslash\hat{N}_G(v) = \emptyset$, then $\mathcal{F}_4 = \emptyset$, as $(A\backslash\{v\})\cup\{u\} \in \mathcal{I}_G$ whenever $A \in \mathcal{I}_G$ and $v \in A$. If $N_G(u)\backslash\hat{N}_G(v) = \{x\}$ for some $x \in V(G)$, then $x \ne v$ and every set $F \in \mathcal{F}_4$ must own x; thus $\mathcal{F}_4\langle v\rangle$ is indeed intersecting.

We finally prove (iii). So suppose $N_G(u)\backslash\hat{N}_G(v) = \emptyset$. Thus $\mathcal{F}_4 = \emptyset$. Clearly, $\bigcup_{i=1}^{3}\mathcal{F}_i$ and $\mathcal{A}_5$ are intersecting. Thus, to show that $\mathcal{A}$ is intersecting, we must show that if $A \in \bigcup_{i=1}^{3}\mathcal{F}_i, B \in \mathcal{A}_5$, then $A \cap B \ne \emptyset$. The set $C := (B\backslash\{u\}) \cup \{v\}$ is in $\mathcal{F}$ and so $A \cap C \ne \emptyset$. Suppose $A \cap C = \{v\}$. Then $A \in \mathcal{F}_3$; but then $D := (A\backslash\{v\}) \cup \{u\}$ is in $\mathcal{F}_1$ and $D \cap C = \emptyset$, a contradiction. So $(A \cap C)\backslash\{v\} \ne \emptyset$ and hence $A \cap B \ne \emptyset$. Therefore $\mathcal{A}$ is intersecting. By (ii), it follows that $\mathcal{A}\overline{(v)} \cup \mathcal{A}\langle v\rangle$ is intersecting. □

3 Vertex deletion lemmas

It frequently happens that a vertex of a graph may be deleted without decreasing μ or α. This is important to our improvement of Theorem 1.9; in this section we develop several vertex deletion lemmas that will be employed in the proofs of Theorems 1.13 and 1.14.

Lemma 3.1 *Let G be a graph, and let $v \in V(G)$. Then*

$$\min\{\mu(G \downarrow v), \mu(G - v)\} \ge \mu(G) - 1.$$

Proof: Let Z be a maximal independent set of $G \downarrow v$ of minimum size; then $Z \cup \{v\}$ is a maximal independent set of G, hence $\mu(G \downarrow v) \ge \mu(G) - 1$. Now let Z be a maximal independent set of $G - v$. If Z is not maximal in G, then $Z \cup \{v\}$ is. Thus $\mu(G - v) \ge \mu(G) - 1$. □

Corollary 3.2 *Let $r \le \frac{1}{2}\mu(G)$, and let $v, w \in V(G)$. Then:*
(i) $r - 1 < \frac{1}{2}\mu(G \downarrow v)$;
(ii) $r - 1 \le \frac{1}{2}\mu((G - v) \downarrow w)$.

Proof. Lemma 3.1 implies:
(i) $r - 1 < \frac{1}{2}(\mu(G) - 1) \le \frac{1}{2}\mu(G \downarrow v)$;
(ii) $r - 1 \le \frac{1}{2}(\mu(G) - 2) \le \frac{1}{2}(\mu(G - v) - 1) \le \frac{1}{2}\mu((G - v) \downarrow w)$. □

The next lemma relies on a well-known property of trees: any tree other than a singleton has a vertex with only one neighbour.

Lemma 3.3 *Let T be a tree with $|V(T)| \ge 2$, and let $w \in V(T)$ such that $N_T(w)$ consists only of one vertex v. Then*

$$\mu(T - v) \ge \mu(T).$$

Proof. Let Z be a maximal independent set of $T - v$. Since w is a singleton of $T - v$, we must have $w \in Z$. So Z is also a maximal independent set of T because $vw \in E(T)$. Thus $\mu(T - v) \geq \mu(T)$. $\square$

Lemma 3.4 *Let $M_n(\mathbf{d})$ be as in Definition 1.11, and let $M_n := M_n(\mathbf{d})$. Let $d_1 > 0$. Then*
(i) $\mu(M_n - x_2) \geq \mu(M_n)$;
(ii) $\alpha(M_n - x_2) \geq \alpha(M_n)$;
(iii) $\alpha(M_n \downarrow x_2) \geq \alpha(M_n) - 2$.

Proof. Let Z be a maximal independent set of $M_n - x_2$. Then $x_1 \in Z$ or $x_1x_z \in E(M_n - x_2)$ for some $x_z \in Z$. Suppose $x_1 \in Z$. Since $d_1 > 0$, we have $x_1x_2 \in E(M_n)$, and hence Z is a maximal independent set of M_n. Now suppose $x_1x_z \in E(M_n - x_2)$ for some $x_z \in Z$. Then, by definition of M_n, $z \leq 1 + d_1 < 2 + d_2$, and hence $x_2x_z \in E(M_n)$. Thus, Z is again a maximal independent set of M_n. Hence (i).

Now let I be an arbitrary independent set of M_n. If $x_2 \notin I$ then I is an independent set of $M_n - x_2$. Suppose $x_2 \in I$ instead. Since $d_1 > 0$, $x_1 \notin I$. It is therefore easy to see that $\{x_{j-1} \colon j \in [n], x_j \in I\}$ is an independent set of $M_n - x_2$ of size $|I|$. Hence (ii).

Clearly I can contain at most 2 vertices in $V(M_n)\backslash V(M_n \downarrow x_2)$. Hence (iii). $\square$

Lemma 3.5 *Let ${}_qC_n^{k,k+1}$ be as in Definition 1.12, and let $q > 0$. Then:*
(i) $\mu({}_qC_n^{k,k+1} - v_{k+2}) \geq \mu({}_qC_n^{k,k+1})$;
(ii) $\alpha({}_qC_n^{k,k+1} - v_{k+2}) \geq \alpha({}_qC_n^{k,k+1})$;
(iii) $\alpha({}_qC_n^{k,k+1} \downarrow v_{k+2}) \geq \alpha({}_qC_n^{k,k+1}) - 2$.

Proof. Let $C := {}_qC_n^{k,k+1}$ and $V := V(C)$. If $N_C(v_1) = V\backslash\{v_1\}$ then trivially $\mu(C - v_{k+2}) = \mu({}_qC_n^{k,k+1}) = 1$. So suppose $N_C(v_1) \neq V\backslash\{v_1\}$. Let Z be a maximal independent set of $C - v_{k+2}$, and let $s := \min\{i \colon v_i \in Z\}$, $t := \max\{i \colon v_i \in Z\}$. If $s \leq k+1$ then $v_sv_{k+2} \in E(C)$, and hence Z is also maximal in C. Suppose $s \geq k+3$. Suppose also that $v_{k+2}v_s \notin E(C)$. Then $v_{k+1}v_s \notin E(C - v_{k+2})$ and, since $q < n$ (by definition of C) and $s \leq t \leq n$, $v_tv_{k+1} \notin E(C - v_{k+2})$. So $Z \cup \{v_{k+1}\} \in \mathcal{I}_{C-v_{k+2}}$, but this contradicts the maximality of Z. So $v_{k+2}v_s \in E(C)$, and hence Z is also maximal in C. Hence (i).

Now let I be an arbitrary independent set of C. If $v_{k+2} \notin I$ then I is an independent set of $C - v_{k+2}$. Suppose $v_{k+2} \in I$ instead. Note that $v_1 \notin I$ as $v_1v_{k+2} \in E(C)$. By construction of C, $\{v_{j-1} \colon j \in [n], v_j \in I\}$ is an independent set of $C - v_{k+2}$ of size $|I|$. Hence (ii).

Clearly I can contain at most 2 vertices in $V(C)\backslash V(C \downarrow v_{k+2})$. Hence (iii). $\square$

Lemma 3.6 *Let $n \geq 2k+2$. Then:*
(i) $\mu(C_n^k - v_{k+1} - v_{2k+2}) \geq \mu(C_n^k)$;
(ii) $\alpha(C_n^k - v_{k+1} - v_{2k+2}) \geq \alpha(C_n^k)$;
(iii) $\alpha(C_n^k \downarrow v_{k+1}) \geq \alpha(C_n^k) - 2$.

Proof. Let Z be a maximal independent set of $C_n^k - v_{k+1} - v_{2k+2}$. If Z contains $z \in \{v_{k+2}, ..., v_{2k+1}\}$ then $zv_{k+1}, zv_{2k+2} \in E(C_n^k)$, and hence Z is also maximal in C_n^k. Now consider $Z \cap \{v_{k+2}, ..., v_{2k+1}\} = \emptyset$. Thus, if $zv_{k+1}, zv_{2k+2} \notin E(C_n^k)$ for all $z \in Z$ then $Z \cup \{v\}$ is an independent set of $C - v_{k+1} - v_{2k+2}$ for all $v \in \{v_{k+2}, ..., v_{2k+1}\}$, but this is a contradiction. We therefore have $zw \in E(C_n^k)$ for some $z \in Z$ and $w \in \{v_{k+1}, v_{2k+1}\}$. Suppose $w = v_{k+1}$ and $Z \cup \{v_{2k+2}\}$ is an independent set of C_n^k. Then $zv_{2k+1} \notin E(C_n^k - v_{k+1} - v_{2k+2})$, and hence $Z \cup \{v_{2k+1}\}$ is an independent set of $C_n^k - v_{k+1} - v_{2k+2}$, a contradiction. By symmetry, we can neither have both $w = v_{2k+2}$ and $Z \cup \{v_{k+1}\}$ an independent set of C_n^k. Therefore there exist $z_1, z_2 \in Z$ such that $z_1v_{k+1}, z_2v_{2k+2} \in E(C_n^k)$, and hence Z is maximal in C_n^k. Hence (i).

(ii) and (iii) follow by the same arguments for the corresponding parts in Lemma 3.5. □

4 Proof of Theorem 1.13

We shall now use the lower bounds obtained in Lemmas 3.4, 3.5 and 3.6 to prove Theorem 1.13. Before proceeding to the main proof, we need two straightforward lemmas concerning stars.

We remark that whenever we use a notation of the kind $\mathcal{F}(x)(y)$ we mean the family $(\mathcal{F}(x))(y)$, which, according to the notation we set up earlier, is the family $\{A \in \mathcal{F}(x) \colon y \in A\}$ $(= \{A \in \mathcal{F} \colon x, y \in A\})$. The same applies for notation like $\mathcal{F}(x)\overline{(y)}$, $\mathcal{F}(x)\langle y\rangle$, etc.

Lemma 4.1 *Let G be a graph containing an edge vw and a singleton x. Suppose $2 \leq r \leq \alpha(G)$. Then $|\mathcal{I}_G^{(r)}(v)| \leq |\mathcal{I}_G^{(r)}(x)|$, and the inequality is strict if $r \leq \mu(G)$.*

Proof. Since x is a singleton, $A\backslash\{y\} \cup \{x\} \in \mathcal{I}_G^{(r)}$ for any $A \in \mathcal{I}_G^{(r)}\overline{(x)}$ and $y \in A$. Setting $\mathcal{J} := \{A\backslash\{v\} \cup \{x\} \colon A \in \mathcal{I}_G^{(r)}(v)\overline{(x)}\}$, it follows that $\mathcal{J} \subseteq \mathcal{I}_G^{(r)}(x)\overline{(v)}$. Given that $vw \in E(G)$, we have $\mathcal{I}_G(v)(w) = \emptyset$, and hence actually $\mathcal{J} \subseteq \mathcal{I}_G^{(r)}(x)\overline{(v)}\backslash\mathcal{I}_G^{(r)}(x)(w)$; also, $\mathcal{I}_G^{(r)}(x)(w) \subseteq \mathcal{I}_G^{(r)}(x)\overline{(v)}$, and hence $|\mathcal{J}| \leq |\mathcal{I}_G^{(r)}(x)\overline{(v)}| - |\mathcal{I}_G^{(r)}(x)(w)|$. We therefore have

$$\begin{aligned}
|\mathcal{I}_G^{(r)}(v)| &= |\mathcal{I}_G^{(r)}(v)(x)| + |\mathcal{I}_G^{(r)}(v)\overline{(x)}| = |\mathcal{I}_G^{(r)}(v)(x)| + |\mathcal{J}| \\
&\leq |\mathcal{I}_G^{(r)}(x)(v)| + |\mathcal{I}_G^{(r)}(x)\overline{(v)}| - |\mathcal{I}_G^{(r)}(x)(w)| \\
&= |\mathcal{I}_G^{(r)}(x)| - |\mathcal{I}_G^{(r)}(x)(w)|.
\end{aligned}$$

Now suppose $r \leq \mu(G)$. Since $\{x, w\} \in \mathcal{I}_G^{(2)}$, there exists $I \in \mathcal{I}_G^{(r)}$ such that $\{x, w\} \subset I$, i.e. $\mathcal{I}_G^{(r)}(x)(w) \neq \emptyset$. Thus $|\mathcal{I}_G^{(r)}(v)| < |\mathcal{I}_G^{(r)}(x)|$. □

Lemma 4.2 *Let G be a graph with $\mu(G) \geq 2r$. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{I}_G^{(r)}$ such that $\mathcal{A}\langle v\rangle = \mathcal{I}_{G\downarrow v}^{(r-1)}(y) \neq \emptyset$ for some $y \in V(G \downarrow v)$. Then $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$.*

Proof. Suppose there exists $A \in \mathcal{A}\overline{(v)}$ such that $y \notin A$. We are given that $\mathcal{I}_{G\downarrow v}^{(r-1)}(y) \neq \emptyset$, and so $\mathcal{I}_G^{(r)}(v)(y) \neq \emptyset$. Therefore there exists a maximal independent set Y of G such that $v, y \in Y$. Given that $2r \leq \mu(G)$, we have $2r \leq |Y|$. Since $y, v \in Y\backslash A$, it follows that there exists an r-subset A' of $Y\backslash A$ containing $\{y, v\}$. So $A'\backslash\{v\} \in \mathcal{I}_{G\downarrow v}^{(r-1)}(y)$, and hence $A' \in \mathcal{A}(v)$. But $A \cap A' = \emptyset$, which contradicts $\mathcal{A}$ intersecting. Hence result. $\square$

Proof of Theorem 1.13. The result is trivial for $r = 1$, so we assume $r \geq 2$ and use induction on $|E(G)|$. If $|E(G)| = 0$ then the result is given by Theorem 1.1, so we assume that $|E(G)| > 0$. This means that G contains a non-singleton component. If G consists solely of complete multipartite graphs and singletons then the result is given by Theorem 1.5. We now consider the case when G contains a connected component G_1 that is neither a singleton nor a complete mulitpartite graph.

Let G_2 be the graph obtained by removing G_1 from G. Note that

$$\mu(G) = \mu(G_1) + \mu(G_2).$$

Since G_1 contains no singletons and G contains at least one singleton, G_2 contains some singleton x.

Let $r \leq \mu(G)/2$, and let $\mathcal{F}$ be an extremal intersecting sub-family of $\mathcal{I}_G^{(r)}$. Let $\mathcal{J} := \mathcal{I}_G^{(r)}(x)$. So $|\mathcal{J}| \leq |\mathcal{F}|$. Lemma 4.1 tells us that $\mathcal{J}$ is a largest star of $\mathcal{I}_G^{(r)}$ and that, for any $v \in V(G_1)$, $\mathcal{J}\langle v\rangle$ and $\mathcal{J}\overline{(v)}$ are largest stars of $\mathcal{I}_{G\downarrow v}^{(r-1)}$ and $\mathcal{I}_{G-v}^{(r)}$ respectively.

Now G_1 is one of the following: a tree, a copy of an mnd graph, a modified power of a cycle, a power of a cycle. We consider each of these four possibilities separately and in the order we have listed them. We will actually show that in each of the first three cases, G is in fact strictly r-EKR even if $r = \mu(G)/2$.

Case I: G_1 is a tree T, $|V(T)| \geq 2$. So there exists $u \in V(G_1)$ such that $N_{G_1}(u)$ consists solely of one vertex v (see the preceding section). Let $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$. Since $N_G(u) = N_{G_1}(u) = \{v\}$, it follows by Lemma 2.1(iii) that $\mathcal{A}\langle v\rangle \cup \mathcal{A}\overline{(v)}$ is intersecting.

Since G_1 contains no cycles, $G_1 - v$ and $G_1 \downarrow v$ contain no cycles, and hence $G_1 - v$ and $G_1 \downarrow v$ are disjoint unions of trees and singletons. So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

By Corollary 3.2(i), $r - 1 < \mu(G \downarrow v)/2$. By Lemma 3.3, $\mu(G_1 - v) \geq \mu(G_1)$; so $\mu(G - v) = \mu(G_1 - v) + \mu(G_2) \geq \mu(G_1) + \mu(G_2) = \mu(G) \geq 2r$.

Therefore, since $\mathcal{A}\langle v\rangle \subset \mathcal{I}_{G\downarrow v}^{(r-1)}$ and $\mathcal{A}\overline{(v)} \subset \mathcal{I}_{G-v}^{(r)}$, the inductive hypothesis gives us $|\mathcal{A}\langle v\rangle| \leq |\mathcal{J}\langle v\rangle|$ and $|\mathcal{A}\overline{(v)}| \leq |\mathcal{J}\overline{(v)}|$. So $|\mathcal{A}| \leq |\mathcal{J}|$. Since $|\mathcal{F}| = |\mathcal{A}|$ and $\mathcal{F}$ is extremal, $|\mathcal{A}\langle v\rangle| = |\mathcal{J}\langle v\rangle|$ and $|\mathcal{A}\overline{(v)}| = |\mathcal{J}\overline{(v)}|$. Since $r - 1 < \mu(G \downarrow v)/2$, it follows by the inductive hypothesis that $\mathcal{A}\langle v\rangle = \mathcal{I}_{G\downarrow v}^{(r-1)}(y)$ for some $y \in V(G \downarrow v)$. Thus, by Lemma 4.2, $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$. If y is not a singleton of G then Lemma 4.1 gives us $|\mathcal{I}_G^{(r)}(y)| < |\mathcal{J}|$, but this leads to the contradiction that $|\mathcal{F}| < |\mathcal{J}|$. So y is a singleton of G, and hence $\mathcal{F} \subseteq \mathcal{I}_G^{(r)}(y)$ (as $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$). Therefore G is strictly r-EKR.

Case II: G_1 is an mnd graph $M_n := M_n(\mathbf{d})$. Since G_1 contains no singletons, $n \geq 2$ and $d_1 \geq 1$. Let $v := x_2$ and $u := x_1$, and let $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$. By definition of M_n and $d_1 \geq 1$, $N_{G_1}(u) \subset \hat{N}_{G_1}(v)$. Since $N_G(u) = N_{G_1}(v)$, it follows by Lemma 2.1(iii) that $\mathcal{A}\langle v\rangle \cup \mathcal{A}\overline{(v)}$ is intersecting.

Clearly, $G_1 - v$ is a copy of $M_{n-1}(\{d'_i\}_{i\in\mathbb{N}})$, where $d'_1 = d_1 - 1$ and $d'_i = d_{i+1}$ for all $i \geq 2$. Also, if $n \leq 2 + d_2$ then $G_1 \downarrow v = (\emptyset, \emptyset)$, and if $n > 2 + d_2$ then $G_1 \downarrow v$ is a copy of $M_{n-2-d_2}(\{d''_i\}_{i\in\mathbb{N}})$ where $d''_i = d_{i+2+d_2}$ for all $i \geq 1$. So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

The rest follows as in the preceding case, except that we get $\mu(G_1 - v) \geq \mu(G_1)$ by Lemma 3.4(i).

Case III: G_1 is a modified kth power of a cycle, i.e. $G_1 = {}_qC_n^{k,k+1}$ for some $q > 0$. We set $u := v_{k+1}$ and $v := v_{k+2}$, and we note that the condition $q < n$ in the definition of ${}_qC_n^{k,k+1}$ implies $N_{G_1}(u) \subseteq \hat{N}_{G_1}(v)$ and hence $N_G(u) \subseteq \hat{N}_G(v)$. Thus, for $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$, we know by Lemma 2.1(iii) that $\mathcal{A}\langle v\rangle \cup \mathcal{A}\overline{(v)}$ is intersecting.

If $n = k+2$ then $G_1 = K_n$, which is a special complete multipartite graph; contradiction. So $n \geq k+3$.

Suppose $v_{k+3}v_1 \in E(G_1)$. It is easy to see that we then have $\hat{N}_{G_1}(v) = V(G_1) = \hat{N}_{G_1}(v_1)$, which gives $\mu(G_1 - v) = \mu(G_1) = 1$ and $G_1 \downarrow v = (\emptyset, \emptyset)$. Thus, by the same line of argument for the preceding cases, we conclude that G is strictly r-EKR.

So suppose $v_{k+3}v_1 \notin E(G_1)$. Then $V(G_1 \downarrow v) = \{v_m, ..., v_n\}$ where

$$m = \begin{cases} 2k+3 & \text{if } q < k+2; \\ 2k+4 & \text{if } q \geq k+2. \end{cases}$$

Let $n' := n - m + 1$. By considering the bijection $\beta\colon V(G_1 \downarrow v) \to \{x_j : j \in [n']\}$ defined by $\beta(v_l) = x_{n-l+1}$ ($l \in [m, n]$), one can see that $G_1 \downarrow v$ is a copy of $M_{n'}(\{d_i\}_{i\in\mathbb{N}})$ where

$$d_i = \begin{cases} k & \text{if } i \leq n - (q+k+1); \\ k+1 & \text{if } i > n - (q+k+1). \end{cases}$$

It is also not difficult to check that $G_1 - v$ is a path if $q = k = 1$, and that

$$G_1 - v \text{ is a copy of } \begin{cases} {}_{n+q-k-2}C_{n-1}^{k-1,k} & \text{if } q < k+1; \\ C_{n-1}^k & \text{if } k+1 \leq q \leq k+2; \\ {}_{q-k-2}C_{n-1}^{k,k+1} & \text{if } q > k+2 \end{cases}$$

otherwise. So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

The rest follows as in Case I, except that we get $\mu(G_1 - v) \geq \mu(G_1)$ by Lemma 3.5(i).

Case IV: G_1 is a k^{th} power of a cycle C_n, i.e. $G_1 = C_n^k$. Let $u := v_k$ and $v := v_{k+1}$. If $n < 2k+2$ then $G_1 = K_n$, which is a special complete multipartite graph; contradiction. So $n \geq 2k+2$. Let $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$. Since $N_G(u) \backslash \hat{N}_G(v) = \{v_n\}$, Lemma 2.1(ii) tells us that $\mathcal{A}\langle v\rangle$ and $\mathcal{A}\overline{(v)}$ are intersecting.

Clearly, $G_1 \downarrow v$ is a power of a path. As in Case I, it follows that $|\mathcal{A}\langle v\rangle| \leq |\mathcal{J}\langle v\rangle|$.

Now $G_1 - v$ is a path (if $k = 1$) or a copy of ${}_{n-k-1}C_{n-1}^{k-1,k}$ (if $k > 1$); however, we are not guaranteed that $\mu(G_1 - v) \geq \mu(G_1)$ (this is the case if, for example, $G_1 = C_4$). Let $\mathcal{G} := \mathcal{A}\overline{(v)}$. Let $u' := v_{2k+1}$ and $v' := v_{2k+2}$, and let $\mathcal{B} := \Delta_{u',v'}(\mathcal{G})$. Clearly, $N_{G-v}(u') = N_{G_1-v}(u') \subset \hat{N}_{G_1}(v')$. Thus, by Lemma 2.1(ii), $\mathcal{B}\langle v'\rangle \cup \mathcal{B}\overline{(v')}$ is intersecting.

If $k = 1$ then $G_1 - v - v'$ is a disjoint union of a path and a singleton, and if $k > 1$ then $G_1 - v - v'$ is a copy of ${}_{n-2k-2}C_{n-2}^{k-1,k}$. It is easy to see that $G_1 - v \downarrow v'$ is a power of a path. So $G - v - v'$ and $G - v \downarrow v'$ belong to the class of graphs specified in the theorem.

By Corollary 3.2(ii), $r - 1 \leq \mu(G - v \downarrow v')/2$. By Lemma 3.3, $\mu(G_1 - v - v') \geq \mu(G_1)$; so $\mu(G - v - v') = \mu(G_1 - v - v') + \mu(G_2) \geq \mu(G_1) + \mu(G_2) = \mu(G) \geq 2r$.

Therefore, since $\mathcal{B}\langle v'\rangle \subset \mathcal{I}^{(r-1)}_{G-v\downarrow v'}$ and $\mathcal{B}\overline{(v')} \subset \mathcal{I}^{(r)}_{G-v-v'}$, the inductive hypothesis gives us $|\mathcal{B}\langle v'\rangle| \leq |\mathcal{J}\overline{(v)}\langle v'\rangle|$ and $|\mathcal{B}\overline{(v')}| \leq |\mathcal{J}\overline{(v)}\overline{(v')}|$. So $|\mathcal{G}| = |\mathcal{B}| \leq |\mathcal{J}\overline{(v)}|$. Since $\mathcal{F} = |\mathcal{A}| = |\mathcal{A}\langle v\rangle| + |\mathcal{G}| \leq |\mathcal{J}\langle v\rangle| + |\mathcal{J}\overline{(v)}|$, we have $|\mathcal{F}| \leq |\mathcal{J}|$, and hence G is r-EKR.

Now suppose $r < \mu(G)/2$. Since $|\mathcal{F}| = |\mathcal{A}|$ and $\mathcal{F}$ is extremal, we must have $|\mathcal{A}\langle v\rangle| = |\mathcal{J}\langle v\rangle|$ and $|\mathcal{G}| = |\mathcal{J}\overline{(v)}|$. By Corollary 3.2(i), we have $r - 1 < \mu(G \downarrow v)/2$, and hence, by the inductive hypothesis, $\mathcal{A}\langle v\rangle = \mathcal{I}^{(r-1)}_{G\downarrow v}(y_1)$ for some $y_1 \in V(G \downarrow v) \subset V(G)\backslash\{u, v\}$. Since $|\mathcal{G}| = |\mathcal{J}\overline{(v)}|$, we have $|\mathcal{B}\langle v'\rangle| = |\mathcal{J}\overline{(v)}\langle v'\rangle|$ and $|\mathcal{B}\overline{(v')}| = |\mathcal{J}\overline{(v)}\overline{(v')}|$. Given that $r < \mu(G)/2$, we have $r - 1 < (\mu(G) - 2)/2 \leq \mu(G - v \downarrow v')/2$ by Lemma 3.1. Thus, by the inductive hypothesis, $\mathcal{B}\langle v'\rangle = \mathcal{I}^{(r-1)}_{G-v\downarrow v}(y_2)$ for some $y_2 \in V(G - v \downarrow v')$. By Lemma 4.2, $\mathcal{B} \subseteq \mathcal{I}^{(r)}_{G-v}(y_2)$. We next show that $y_1 = y_2$.

If y_2 is not a singleton of $G - v$ then Lemma 4.1 gives us $|\mathcal{I}^{(r)}_{G-v}(y_2)| < |\mathcal{J}\overline{(v)}|$, but this leads to the contradiction that $|\mathcal{G}| < |\mathcal{J}\overline{(v)}|$. So y_2 is a singleton of $G - v$, and hence, since $G_1 - v$ contains no singletons, $y_2 \in V(G)\backslash V(G_1) \subset V(G)\backslash\{u, v, u', v'\}$. Note that, by definition of $\mathcal{B}$, $\mathcal{B}(v') \subseteq \mathcal{G}$. Thus, since $\mathcal{B}\langle v'\rangle = \mathcal{I}^{(r-1)}_{G-v\downarrow v'}(y_2)$, we have $\mathcal{V} := \mathcal{I}^{(r)}_{G-v}(y_2)(v') \subseteq \mathcal{G}$. Suppose $y_1 \neq y_2$. Let $A_1 \in \{I \in \mathcal{V} \colon u, y_1 \notin I\}$ (note that A_1 exists since y_2 is a singleton of $G - v$ and, by Lemma 3.1, $\mu(G - v) \geq \mu(G) - 1 \geq 2r - 1$). So $A_1 \in \mathcal{G}$, $\{u, v\} \cap A_1 = \emptyset$, and hence $A_1 \in \mathcal{F}$. Recall that $y_1 \in V(G \downarrow v)$, which means that $y_1 v \notin E(G)$; let Y be a maximal independent set of G containing y_1 and v. Since $2r \leq \mu(G) \leq |Y|$ and $\{y_1, v\} \cap A_1 = \emptyset$, the family $\mathcal{Y} := \{A \in \binom{Y\backslash A_1}{r} \colon y_1, v \in A\}$ is non-empty. Let $A_2 \in \mathcal{Y}$; note that $A_2 \in \mathcal{I}^{(r)}_G(y_1)(v)$. Since $\mathcal{A}\langle v\rangle = \mathcal{I}^{(r-1)}_{G\downarrow v}(y_1)$, we have $\mathcal{A}(v) = \mathcal{I}^{(r)}_G(y_1)(v)$ and hence $A_2 \in \mathcal{A}(v)$. Now, by definition of $\mathcal{A}$, $\mathcal{A}(v) \subseteq \mathcal{F}$. Hence $A_2 \in \mathcal{F}$. But $A_1 \cap A_2 = \emptyset$, which contradicts $\mathcal{F}$ intersecting. So $y_1 = y_2$ indeed.

Since $y_2 \notin \{u', v'\}$ and $\mathcal{B} \subseteq \mathcal{I}^{(r)}_{G-v}(y_2)$, we clearly have $\mathcal{G} \subseteq \mathcal{I}^{(r)}_{G-v}(y_2)$. So we have $\mathcal{F} = \mathcal{A}(v) \cup \mathcal{G} \subseteq \mathcal{I}^{(r)}_G(y_2)$. This proves that G is strictly r-EKR. $\square$

5 Proof of Theorem 1.14

Theorem 1.14 is trivial for $r = 1$, so we assume $r \geq 2$ and prove the result by induction on $|E(G)|$. If $|E(G)| = 0$ then the result is given by Theorem 1.1, so we assume that $|E(G)| > 0$. This means that G contains a non-singleton component G_1. Let G_2 be the graph obtained by removing G_1 from G. Note that

$$\alpha(G) = \alpha(G_1) + \alpha(G_2).$$

Since G_1 contains no singletons and G contains at least one singleton, G_2 contains some singleton x.

Let $r \leq \alpha(G)/2$, and let $\mathcal{F}$ be an extremal intersecting sub-family of $\mathcal{I}_G^{(r)}$. Let $\mathcal{J} := \mathcal{I}_G^{(r)}(x)$. So $|\mathcal{J}| \leq |\mathcal{F}|$. By Lemma 4.1, $\mathcal{J}$ is a largest star of $\mathcal{I}_G^{(r)}$, and, for any $v \in V(G_1)$, $\mathcal{J}\langle v \rangle$ and $\mathcal{J}\overline{(v)}$ are largest stars of $\mathcal{I}_{G\downarrow v}^{(r-1)}$ and $\mathcal{I}_{G-v}^{(r)}$ respectively.

Note that a complete graph is an mnd graph, so we need to consider the following possible cases for G_1.

Case I: G_1 is an mnd graph $M_n := M_n(\mathbf{d})$. As in Case II of the Proof of Theorem 1.14, we take $v := x_2$, $u := x_1$ and $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$, and we obtain that $\mathcal{A}\langle v \rangle \cup \mathcal{A}\overline{(v)}$ is intersecting and that $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

By (ii) and (iii) of Lemma 3.4, we have $\alpha(G_1 - v) \geq \alpha(G_1)$ and $\alpha(G_1 \downarrow v) \geq \alpha(G_1) - 2$; so $\alpha(G - v) = \alpha(G_1 - v) + \alpha(G_2) \geq \alpha(G_1) + \alpha(G_2) = \alpha(G) \geq 2r$ and $\alpha(G \downarrow v) = \alpha(G_1 \downarrow v) + \alpha(G_2) \geq \alpha(G_1) - 2 + \alpha(G_2) = \alpha(G) - 2 \geq 2r - 2 = 2(r-1)$. Therefore, since $\mathcal{A}\langle v \rangle \subset \mathcal{I}_{G\downarrow v}^{(r-1)}$ and $\mathcal{A}\overline{(v)} \subset \mathcal{I}_{G-v}^{(r)}$, the inductive hypothesis gives us $|\mathcal{A}\langle v \rangle| \leq |\mathcal{J}\langle v \rangle|$ and $|\mathcal{A}\overline{(v)}| \leq |\mathcal{J}\overline{(v)}|$. So $|\mathcal{F}| = |\mathcal{A}| \leq |\mathcal{J}|$, and hence G is r-EKR.

Case II: G_1 is a modified kth power of a cycle, i.e. $G_1 = {}_qC_n^{k,k+1}$ for some $q > 0$. As in Case III of the Proof of Theorem 1.14, we take $u := v_{k+1}$, $v := v_{k+2}$ and $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$, and we obtain that $\mathcal{A}\langle v \rangle \cup \mathcal{A}\overline{(v)}$ is intersecting and that $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem. The rest follows as in Case I, except that we use Lemma 3.5 instead of Lemma 3.4.

Case III: G_1 is a k^{th} power of a cycle C_n, i.e. $G_1 = C_n^k$. As in Case IV of the Proof of Theorem 1.14, we take $u := v_k$, $v := v_{k+1}$ and $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$, and we obtain that $\mathcal{A}\langle v \rangle$ and $\mathcal{A}\overline{(v)}$ are intersecting and that $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem. As in Case I, we get $|\mathcal{A}\langle v \rangle| \leq |\mathcal{J}\langle v \rangle|$, $|\mathcal{A}\overline{(v)}| \leq |\mathcal{J}\overline{(v)}|$ and hence $|\mathcal{F}| = |\mathcal{A}| \leq |\mathcal{J}|$; the only difference is that we use Lemma 3.6 instead of Lemma 3.4. So G is r-EKR. $\square$

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References

[1] B. Bollobás and I. Leader, An Erdős-Ko-Rado theorem for signed sets, Comput. Math. Appl. 34 (1997) 9-13.

[2] P. Borg, Extremal t-intersecting sub-families of hereditary families, J. London Math. Soc., (in press).

[3] P. Borg, Intersecting systems of signed sets, Electron. J. Combin. 14 (2007) R41.

[4] P. Borg, Erdős-Ko-Rado with separation conditions, in preparation.

[5] P. Borg and F.C. Holroyd, The Erdős-Ko-Rado properties of set systems defined by double partitions, Discrete Math., in press, doi:10.1016/j.disc.2008.05.052.

[6] P.J. Cameron and C.Y. Ku, Intersecting families of permutations, European J. Combin. 24 (2003) 881-890.

[7] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Combin. Theory Ser. A 22 (1977) 352-360.

[8] M. Deza and P. Frankl, The Erdős-Ko-Rado theorem - 22 years later, SIAM J. Algebraic Discrete Methods 4 (1983) 419-431.

[9] K. Engel, An Erdős-Ko-Rado theorem for the subcubes of a cube, Combinatorica 4 (1984) 133-140.

[10] P.L. Erdős, U. Faigle and W. Kern, A group-theoretic setting for some intersecting Sperner families, Combin. Probab. Comput. 1 (1992) 323-334.

[11] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford (2) 12 (1961) 313-320.

[12] P. Frankl, The shifting technique in extremal set theory, in: C. Whitehead (Ed.), Combinatorial Surveys, Cambridge Univ. Press, London/New York, 1987, pp. 81-110.

[13] A.J.W. Hilton and E.C. Milner, Some intersection theorems for systems of finite sets, Quart. J. Math. Oxford (2) 18 (1967) 369-384.

[14] A.J.W. Hilton and C.L. Spencer, A graph-theoretical generalisation of Berge's analogue of the Erdős-Ko-Rado theorem, Trends in Graph Theory, Birkhauser Verlag, Basel, Switzerland, 2006, 225-242.

[15] F.C. Holroyd, C. Spencer and J. Talbot, Compression and Erdős-Ko-Rado graphs, Discrete Math. 293 (2005) 155-164.

[16] F.C. Holroyd and J. Talbot, Graphs with the Erdős-Ko-Rado property, Discrete Math. 293 (2005) 165-176.

[17] C.Y. Ku and I. Leader, An Erdős-Ko-Rado theorem for partial permutations, Discrete Math. 306 (2006) 74-86.

[18] Yu-Shuang Li and Jun Wang, Erdős-Ko-Rado-type theorems for colored sets, Electron. J. Combin. 14 (2007) R1.

[19] J.-C. Meyer, Quelques problèmes concernant les cliques des hypergraphes k-complets et q-parti h-complets, in: Hypergraph Seminar (Columbus, Ohio 1972), Lecture Notes in Math., Vol. 411, Springer, Berlin, 1974, 127-139.

[20] J. Talbot, Intersecting families of separated sets, J. London Math. Soc. 68 (1) (2003) 37-51.