L-Università ta' Malta

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE
EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2023 SECOND SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	I
DATE:	30th August 2023
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL** questions. There are 10 questions in all.
Each question carries 10 marks.
The total number of marks for all the questions is 100.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\text{m}\,\text{s}^{-2}$.)

1. A system of forces acts on a planar object as follows: $\boldsymbol{F}_1$ of magnitude 15 N in the direction $(-4,-3)$ acts at the point $(1,-1)$; $\boldsymbol{F}_2$ of magnitude 5 N in the direction $(3,4)$ acts at the point $(1,1)$; $\boldsymbol{F}_3$ of magnitude $3\sqrt{41}$ N in the direction $(-5,4)$ acts at the point $(-1,1)$.

   (a) Express the forces in **i**, **j** notation. (2)

   (b) Find the magnitude and direction of the resultant force. (2)

   (c) Determine the vector equation of its line of action. (4)

   (d) Hence find that single force and a point at which it acts that is required to bring the system to equilibrium. (2)

   (**Total: 10 marks**)

2. A non-uniform straight rod $ABCD$ of length 10 m rests horizontally on smooth supports at points B and C, where $AB = 4$ m and $CD = 3$ m. The rod is about to tilt either when 15 kg is placed at point A or when 5 kg is placed at D. The centre of mass of the rod is not at its centre. Find:

   (a) the mass of the rod; (5)

   (b) the position of the centre of mass. (5)

   (**Total: 10 marks**)

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3. A stone of mass 100 g is thrown vertically upwards from arm level of 1 m above ground with a speed of $15\,\mathrm{m\,s^{-1}}$. It moves freely under gravity until it hits the surface of a pond of water. As it sinks in the water, it decelerates uniformly until it hits the bottom, 50 cm below the surface, at a speed of $5\,\mathrm{m\,s^{-1}}$. The entire motion occurs in a vertical line.

   (a) Draw a velocity-time graph. (4)

   (b) Using the graph, find the velocity at impact with the water. (2)

   (c) Find the magnitude of the deceleration and the time required for the stone to sink to the bottom of the water. (4)

   (**Total: 10 marks**)

4. A pendulum consists of a particle P of mass 2 kg attached to a taut light inextensible string of length 4 m. Initially P is held at an angle of 30° to the vertical. It is released and starts to swing. Friction and air resistance can be ignored. Find:

   (a) the speed of P at the lowest point; (3)

   (b) the period of small oscillations of the pendulum. (3)

   When P is at the lowest position the string hits a nail positioned at a height of 1 m above the mass.

   (c) Find the new height reached by P, and the angle it makes with the vertical at this point. (4)

   (**Total: 10 marks**)

5. An unknown mass m is suspended from a smooth light pulley. Another mass of 8 kg is placed on a horizontal smooth surface, connected to the mass m by a light inextensible string that passes over the pulley, and the system is released from rest. Find, in terms of m:

   (a) the acceleration of the two masses; (3)

   (b) the tension in the string. (2)

   Both masses are now suspended vertically from the pulley and the mass m is observed to fall a height of 1 m in 1 s, starting from rest.

   (c) Find the value of m. (5)

   (**Total: 10 marks**)

6. Two identical smooth solid ball bearings, P and Q, having the same mass of 20 g, move directly towards each other in a straight line with a speed of $3\,\mathrm{m\,s^{-1}}$ and $6\,\mathrm{m\,s^{-1}}$ respectively. The coefficient of restitution between the ball bearings is $\frac{1}{3}$. Determine:

   (a) the change in velocities of P and Q after they collide; (7)

   (b) the change in the total kinetic energy. (3)

   (**Total: 10 marks**)

7. The pump of a fire engine can deliver water at a volume flow rate of $0.06\,\mathrm{m^3\,s^{-1}}$ through a hose of radius 3.2 cm. The water is ejected at an angle θ to the horizontal onto a target area to extinguish a fire. In the following calculations treat the jet of water as a projectile from ground level. Find:

   (a) the speed u_0 with which the water exits the hose; (2)

   (b) a simplified equation for the horizontal range, in terms of the initial speed u_0 and θ; (5)

   (c) the angle with the ground that the hose should be set so that the jet lands at a horizontal distance of 25 m. (3)

   (**Total: 10 marks**)

8. A particle of mass 2 kg is shot horizontally with a speed of $20\,\mathrm{m\,s^{-1}}$ directly towards a spring of natural length 1 m, causing it to compress. The modulus of elasticity of the spring is equal to 4×10^5 N.

   (a) Determine the maximum compression of the spring. (3)

   The particle is next dropped from a height of 20 m onto soft soil where it penetrates a distance of 4 cm before coming to a halt. Find:

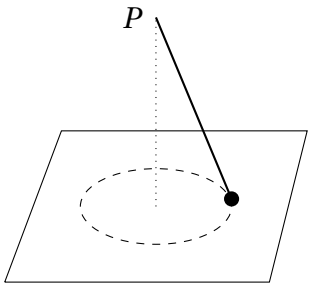
   (b) the kinetic energy of the sphere just before impact; (2)

   (c) the average retarding force acting on the sphere as it penetrates the soil; (2)

   (d) the time taken for the sphere to come to rest, assuming a uniform deceleration. (3)

   (**Total: 10 marks**)

9. A particle of mass 2 kg is attached to a point P at a height of 45 cm above a smooth horizontal table with a light inelastic string of length 53 cm. The particle is then set to rotate with a constant angular speed of $3\,\mathrm{rad\,s^{-1}}$ in a circular path on the table as in the diagram below.



   Find:

   (a) the tension in the string; (3)

   (b) the value of the normal reaction on the particle by the table; (3)

   (c) the tangential speed of the particle; (2)

   (d) the time taken for **ONE** complete revolution. (2)

   (**Total: 10 marks**)

Questions continue on next page.

10. A ship S is moving with a velocity of $20\,\mathbf{i}\,\mathrm{km\,h^{-1}}$ when the captain spots another ship R at a distance of $10\sqrt{2}\,\mathrm{km}$ in the positive $\mathbf{j}$ direction. The ship R is travelling at a speed of $20\,\mathrm{km\,h^{-1}}$ but is seen to move with a velocity of $v(\mathbf{i}-\mathbf{j})\,\mathrm{km\,h^{-1}}$ by the captain of the ship S. Find:

    (a) the value of v in $\mathrm{km\,h^{-1}}$; (2)

    (b) the distance of closest approach of the two ships; (2)

    (c) the time it takes for the two ships to be at a distance of $10\sqrt{2}\,\mathrm{km}$ again. (6)

    (**Total: 10 marks**)

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ADVANCED MATRICULATION LEVEL
2023 SECOND SESSION

SUBJECT:	**Applied Mathematics**
PAPER NUMBER:	II
DATE:	31st August 2023
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions. There are 10 questions in all.
Each question carries 15 marks.
Graphical calculators are **not** allowed. Scientific calculators can be used but all necessary working must be shown.
A booklet with mathematical formulae is provided.
In the paper, **i**, **j**, **k** are unit vectors along the x-, y-, z-axes of a Cartesian coordinate system.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

1. A uniform ladder has a length of 10 m and mass 20 kg and rests against a wall in a vertical plane at an angle of 70° to the horizontal. The coefficient of friction against both the wall and the ground is 1/4. A man weighing 100 kg starts climbing the steps until, when he reaches a point on the ladder which is at a distance d from the foot, the ladder is about to slide down.

   (a) Draw the diagram of forces acting on the ladder. (5)
   (b) Determine the frictional forces at both the wall and the ground. (5)
   (c) Calculate the value of d. (5)

   (**Total: 15 marks**)

2. A particle of mass m falls freely and vertically against an air resistance proportional to the particle's speed v, with constant of proportionality k.

   (a) Show that $\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{k}{m}v$. (2)
   (b) Solve the differential equation for v assuming the particle starts at rest. (5)
   (c) Considering only the limiting speed of the particle, what is the ratio of the time of vertical fall of two particles of masses m and $2m$ when they fall a height h? (3)
   (d) Show that if the particle of mass m is released from rest at $h = 0$, then its downwards vertical displacement after time t is given by
   $$h = \frac{mg}{k}\left[t - \frac{m}{k}(1 - e^{-kt/m})\right].$$ (5)

   (**Total: 15 marks**)

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3. A uniform beam of mass m and length l is clamped at one end A so that it is horizontal. A mass $2m$ hangs at the other end B.

   (a) Find the reaction and bending moment at the point A. (3)

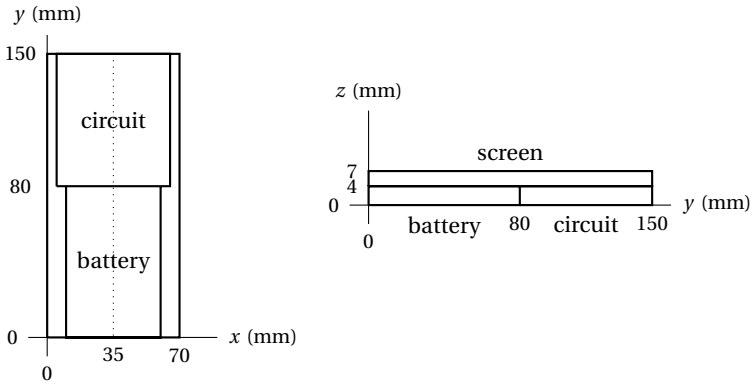
   (b) Find the shearing force and the bending moment at a general point x from A. (5)

   (c) Draw the graphs of the shearing force and bending moment for the beam. (5)

   (d) Find the maximum bending moment and the point on the beam where it occurs. (2)

   (**Total: 15 marks**)

4. A mobile phone is modelled as consisting of three components each of uniform density:
   - the screen taken as a solid cuboid of dimensions 70 mm × 150 mm × 3 mm and mass 90 g;
   - the battery of dimensions 50 mm × 80 mm × 4 mm and mass 35 g;
   - the electric circuit board of dimensions 60 mm × 70 mm × 4 mm and mass 20 g.



   All dimensions are in the x-, y-, and z-directions respectively, and all components are centred on the line $x = 35$ (in mm), with the screen on top of the battery and circuit board, as shown in the diagram above.

   (a) Find the x, y, z coordinates of the centre of mass of the phone. (8)

   (b) Where should a point mass of 10 g be placed so that the new centre of mass of the phone is exactly at the point $(35, 75, 5)$ mm? (4)

   (c) The phone is held by a light selfie stick of length 1 m with the phone making an angle of 45° with the stick while the phone screen is vertical. Find the moment of force at the end of the stick held by the hand. (3)

   (**Total: 15 marks**)

5. A rod OR of length a rotates in a plane about O with constant angular speed ω. A second rod RP of length $a/2$ connects a particle P to R, such that RP performs uniform circular motion in the same plane with angular speed 2ω relative to R. Initially, ORP is a straight line.

   (a) Write the position vector of P relative to O. (4)

   (b) Show that the distance $r = OP$ satisfies the equation $r^2 = a^2\left(\frac{5}{4} + \cos(\omega t)\right)$. (6)

   (c) Show that the magnitude of the acceleration of P is $2\omega^2 r$. (5)

   (**Total: 15 marks**)

6. A smooth elastic ball of mass 0.2 kg moves without rolling along a smooth horizontal surface with a constant speed of $6\,\mathrm{m\,s^{-1}}$. It hits a stationary vertical wall at an angle of 30° to the wall's normal, and rebounds from it at an angle of θ with the normal. The coefficient of restitution between the ball and the wall has a value of $\frac{1}{3}$.

   (a) Determine the speed of the ball just after the collision. (7)

   (b) Deduce that θ is 60°. (3)

   (c) If the time of impact is 2 ms, determine the average force acting on the ball during the collision, both in magnitude and direction. (5)

(Total: 15 marks)

7. A mass of 3 kg is attached at one end of a spring of natural length 1 m and modulus of elasticity 12 N, which is resting on a smooth horizontal surface. The other end of the spring is fixed so that the mass can perform horizontal simple harmonic oscillations with displacement x. The horizontal forces acting on the mass consist of a force $10\sin(2t)\,\mathrm{N}$ in the direction of x, where t is the time, the elastic spring force, and a dampening force $k\dot{x}$, where $k = 20\,\mathrm{N\,s\,m^{-1}}$ and a dot indicates a derivative with respect to time.

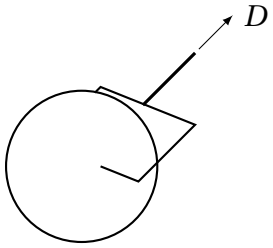
   (a) Show that the displacement x is governed by the equation,

   $$3\ddot{x} + 20\dot{x} + 12x = 10\sin(2t). \quad (3)$$

   (b) Find the general solution to this differential equation. (12)

(Total: 15 marks)

8. A child's toy consists of a solid sphere of mass $M = 2\,\mathrm{kg}$ and radius R mounted onto a light rod perpendicular to the sphere. A child pulls the rod with a force D of magnitude $6\sqrt{2}\,\mathrm{N}$ at an angle of 45° to the horizontal while walking at a uniform speed of $0.1\,\mathrm{m\,s^{-1}}$, so that the sphere rolls without slipping.



   (a) Calculate the moment of inertia of a solid sphere of mass M and radius R, given that the moment of inertia of a disc of mass m and radius r is $\frac{1}{2}mr^2$. (5)

   (b) Find the values of:

      (i) the total kinetic energy of the sphere; (2)

      (ii) the frictional force acting on the sphere. (3)

   (c) Determine the force needed at the same angle of 45° to the horizontal, in order that the sphere is about to slip, given that the coefficient of friction between the sphere and the ground is $\frac{1}{2}$. (5)

(Total: 15 marks)

Questions continue on next page.

9. A uniform disc of radius a and mass $2m$ is freely pivoted at a point P which is at a distance x from the centre of mass. The axis at P is normal to the disc.

   (a) Determine the moment of inertia at P. (2)

   (b) Show that the period for small oscillations is

   $$T = 2\pi\sqrt{\frac{a^2+2x^2}{2g\,x}}.$$ (2)

   (c) Find the distance x for which the period T has a stationary point with respect to x and show that this corresponds to a minimum by plotting T against x. (11)

(**Total: 15 marks**)

10. A uniform rod AB of mass 10 kg and length $3\sqrt{2}$ m is freely pivoted at A so that it can rotate in a vertical plane. The rod is released with AB horizontal, with an initial angular speed of $\sqrt{31}\,\mathrm{rad\,s^{-1}}$ in a downward direction. After turning through an angle of 135° the rod hits a nail N where $AN = 2$ m.

   (a) Find the angular velocity with which the rod hits the nail. (7)

   The rod rebounds inelastically off the nail and next comes to momentary rest at an angle of $\theta = \tan^{-1}\frac{17}{7}$ with the downward vertical. Find:

   (b) the angular velocity with which the rod rebounds; (5)

   (c) the impulse that the nail imparts on the rod. (3)

(**Total: 15 marks**)